ADDITIVE GROUP ACTIONS ON DANIELEWSKI VARIETIES AND THE CANCELLATION PROBLEM

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Abstract. The cancellation problem asks if two complex algebraic varieties X and Y of the same dimension such that $X \times \mathbb{C}$ and $Y \times \mathbb{C}$ are isomorphic are isomorphic. Iitaka and Fujita [15] established that the answer is positive for a large class of varieties of any dimension. In 1989, Danielewski [4] constructed a famous counter-example using smooth affine surfaces with additive group actions. His construction was further generalized by Fieseler [10] and Wilkens [22] to describe a larger class of affine surfaces. Here we construct higher dimensional analogues of these surfaces. We study algebraic actions of the additive group $\mathbb{C}_+$ on certain of these varieties, and we obtain counter-examples to the cancellation problem in every dimension $n \geq 2$.

Keywords: Danielewski varieties, Cancellation Problem, additive group actions, Makar-Limanov invariant.

Résumé. Le problème dit de simplification demande si deux variétés algébriques complexes X et Y telles $X \times \mathbb{C}$ et $Y \times \mathbb{C}$ soient isomorphes sont isomorphes. Iitaka et Fujita ont montré à la fin des années 70 que la réponse est affirmative pour une large classe de variétés. Les variétés affines-réglées ne font pas partie de cette classe, et, en 1989, Danielewski à construit un contre-exemple à partir de deux surfaces affines de ce type. Dans cet article, on généralise la construction de Danielewski pour obtenir des variétés affines qui sont les espaces totaux de fibrés principaux sous le groupe additif, de base un schéma non séparé, en l'occurrence, un espace affine dont les hyperplans de coordonnés on été multipliés. Grâce à une technique de déformation équivariante développée par Kaliman et Makar-Limanov, on détermine ensuite toutes les actions de groupes additifs sur certaines de ces variétés. Cela conduit finalement à des généralisations naturelles du contre-example de Danielewski, valables en toute dimension $n \geq 2$.

Mots clefs : variétés de Danielewski, Problème de Simplification, groupes additifs, invariant de Makar-Limanov.

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Introduction

The Cancellation Problem, which is sometimes referred to as Zariski's Problem although Zariski's original question was different (see e.g. [21]), has been already discussed in the early seventies as the question of uniqueness of coefficients rings. The problem at that time was to decide for which rings A and B an isomorphism of the polynomials rings $A[x]$ and $B[x]$ implies that A and B are isomorphic (see e.g. [8]). Using the fact that the tangent bundle of the real n-sphere is stably trivial but not trivial, Hochster [13] showed that this fails in general.

A geometric formulation of the Cancellation Problem asks if two algebraic varieties X and Y such that $Y \times \mathbb{A}^1$ is isomorphic to $X \times \mathbb{A}^1$ are isomorphic. Clearly, if either X or Y does not contain rational curves, for instance X or Y is an abelian variety, then every isomorphism $\Phi : X \times \mathbb{A} \xrightarrow{\sim} Y \times \mathbb{A}^1$ induces an isomorphism between X and Y. So the Cancellation Problem leads to decide if a given algebraic variety X contains a family of rational curves, where by a rational curve we mean the image of a nonconstant morphism $f : C \to X$, where C is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$. Iitaka and Fujita carried a geometric attack to this question using ideas from the classification theory of complete varieties. Every complex algebraic variety X embeds as an open subset of complete variety $\bar{X}$ for which the boundary $D = \bar{X} \setminus X$ is a divisor with normal crossing. By replacing the usual sheaves of forms $\Omega^q\left(\bar{X}\right)$ on $\bar{X}$ by the sheaves $\Omega^q(\log D)$ of rational q-forms having at worse logarithmic poles along D, Iitaka [14] introduced, among others invariants, the notion of logarithmic Kodaira dimension $\bar{\kappa}(X)$ of a noncomplete variety X, which is an analogue of the usual notion of Kodaira diamension for complete varieties. Iitaka anf Fujita [15] established the following result.

Theorem. *Let X and Y be two nonsingular algebraic varieties and assume that either $\bar{\kappa}(X) \geq 0$ or $\bar{\kappa}(Y) \geq 0$. Then every isomorphism $\Phi : X \times \mathbb{C} \xrightarrow{\sim} Y \times \mathbb{C}$ induces an isomorphism between X and Y.*

The hypothesis $\bar{\kappa}(X) \geq 0$ above guarantees that X cannot contain too many rational curves. For instance, there is no cylinder-like open subset $U \simeq C \times \mathbb{A}^1$ in X, for otherwise we would have $\bar{\kappa}(X) = -\infty$[1]. It turns out that this additional assumption is essential, as shown by the following example due to Danielewski [4].

Example. *The surfaces $S_1, S_2 \subset \mathbb{C}^3$ with equations $xz - y^2 + 1 = 0$ and $x^2z - y^2 + 1 = 0$ are not isomorphic but* $S_1 \times \mathbb{C}$ and $S_2 \times \mathbb{C}$ *are.* In the construction of Danielewski, these surfaces appear as the total spaces of principal homogeneous $\mathbb{C}_+$-bundles over $\tilde{\mathbb{A}}$, the affine line with a double origin, obtained by identifying two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$. The isomorphism $S_1 \times \mathbb{C} \simeq S_2 \times \mathbb{C}$ is obtained by forming the fiber product $S_1 \times_{\tilde{\mathbb{A}}} S_2$, which is a principal $\mathbb{C}_+$-bundle over both S_1 and S_2, and using the fact that every such bundle over an affine variety is trivial. On the other hand, S_1 and S_2 are not even homeomorphic when equipped with the complex topology. More precisely, Danielewski established that the fundamental groups at infinity of S_1 and S_2 are isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ respectively. Fieseler [10] studied and classified algebraic $\mathbb{C}_+$-actions on normal affine surfaces. As a consequence of his classification, he obtained many new examples of the same kind (see also [22]).

Here we construct higher dimensional analogues of Danielewski's counter-example. The paper is organized as follows. In the first section, we introduce a natural generalization of

[1]Actually, a nonsingular affine surface has logarithmic Kodaira dimension $-\infty$ if and only if its contains a cylinder-like open set (see e.g. [20]).

the surfaces S_1 and S_2 above in the form of affine varieties which are the total spaces of certain principal homogeneous $\mathbb{C}_+$-bundle over $\tilde{\mathbb{A}}^n$, the affine n-space with a multiple system of coordinate hyperplanes. We call them *Danielewski varieties.* For instance, for every multi-index $[m] = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$ the nonsingular hypersurface $X_{[m]} \subset \mathbb{C}^{n+2}$ with equation $x_1^{m_1} \cdots x_n^{m_n} z = y^2 - 1$ is a Danielewski variety. As a generalization of a result of Danielewski (see also [10]), we establish that the total space of a principal homogeneous $\mathbb{C}_+$-bundle over $\tilde{\mathbb{A}}^n$ is a Danielewski variety if and only if it is separated. This leads to a simple description of these varieties in terms of Čech cocycles (see Theorem 1.18).

In a second part, we study algebraic $\mathbb{C}_+$-actions on a certain class of varieties which contains the Danielewski varieties $X_{[m]}$ as above. In particular we compute the Makar-Limanov invariant [17] of these varieties, *i.e.* the set of regular functions invariant under *all* $\mathbb{C}_+$-actions. We obtain the following generalization of a result due to Makar-Limanov [19] for the case of surfaces (see Theorem 2.8).

Theorem. *If $(m_1, \ldots, m_n) \in \mathbb{Z}^n_{>1}$ then the Makar-Limanov invariant of a variety $X \subset \mathbb{C}^{n+2}$ with equation*

$$x_1^{m_1} \cdots x^{m_n} z = y^r + \sum_{i=0}^{r-1} a_i (x_1, \ldots, x_n) y^i, \quad \textit{where } r \geq 2,$$

is isomorphic to $\mathbb{C}[x_1, \ldots, x_n]$.

As a consequence, we obtain infinite families of counter-examples to the Cancellation Problem in every dimension $n \geq 2$.

Theorem. *Let $[m] = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>1}$ and $[m'] = (m'_1, \ldots, m'_n) \in \mathbb{Z}^n_{>1}$ be two multi-indices for which the subsets $\{m_1, \ldots, m_n\}$ and $\{m'_1, \ldots, m'_n\}$ of $\mathbb{Z}$ are distint, and let $\lambda_1, \ldots, \lambda_r$, where $r \geq 2$ be a collection of pairwise distinct complex numbers. Then the Danielewski varieties X and X' in $\mathbb{C}^{n+2}$ with equations*

$$x_1^{m_1} \cdots x_n^{m_n} z - \prod_{i=1}^{r} (y - \lambda_i) = 0 \quad \textit{and} \quad x_1^{m'_1} \cdots x_n^{m'_n} z - \prod_{i=1}^{r} (y - \lambda_i) = 0$$

are not isomorphic, but the varieties $X \times \mathbb{C}$ and $X' \times \mathbb{C}$ are isomorphic.

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1. Danielewski varieties

Danielewski's construction can be easily generalized to produce examples of affine varieties X and Y such that $X \times \mathbb{C}$ and $Y \times \mathbb{C}$ are isomorphic. Indeed, if we can equip two affine varieties X and Y with structures of principal homogeneous $\mathbb{C}_+$-bundle $\rho_X : X \to Z$ and $\rho_Y : Y \to Z$ over a certain scheme Z, then the fiber product $X \times_Z Y$ will be a principal homogeneous $\mathbb{C}_+$-bundle over X and Y, whence a trivial principal bundle $X \times \mathbb{C} \simeq X \times_Z Y \simeq Y \times \mathbb{C}$ as X and Y are both affine. The base scheme Z which arises in Danielewski's counter-example is the affine with a double origin. The most natural generalization is to consider an *affine space* $\mathbb{C}^n$ *with a multiple system of coordinate hyperplanes* as a base scheme.

Notation 1.1. In the sequel we denote the polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$ by $\mathbb{C}[\underline{x}]$, and the algebra $\mathbb{C}\left[x_1,x_1^{-1}\ldots,x_n,x_n^{-1}\right]$ of Laurent polynomials in the variables $x_1,\ldots,x_n$ by $\mathbb{C}\left[\underline{x},\underline{x}^{-1}\right]$. For every multi-index $[r]=(r_1,\ldots,r_n)\in\mathbb{Z}^n$, we let $\underline{x}^{[r]}=x_1^{r_1}\cdots x_n^{r_n}\in\mathbb{C}\left[\underline{x},\underline{x}^{-1}\right]$. We denote by $H_{\underline{x}}=V(x_1\cdots x_n)$ the closed subvariety of $\mathbb{C}^n$ consisting of the union of the n coordinate hyperplanes. Its open complement in $\mathbb{C}^n$, which is isomorphic to $(\mathbb{C}^*)^n$, will be denoted by $U_{\underline{x}}$.

Definition 1.2. We let $Z_{n,r}$ be the scheme obtained by gluing r copies $\delta_i : Z_i \stackrel{\sim}{\longrightarrow} \mathbb{C}^n$ of the affine space $\mathbb{C}^n=\operatorname{Spec}(\mathbb{C}[x_1,\ldots,x_n])$ by the identity along $(\mathbb{C}^*)^n$. We call $Z_{n,r}$ *the affine n-space with an r-fold system of coordinate hyperplanes.* We consider it as a scheme over $\mathbb{C}^n$ via the morphism $\delta : Z_{n,r}\to\mathbb{C}^n$ restricting to the δ_i's on the canonical open subset Z_i of $Z_{n,r}$, $i=1,\ldots,n$.

1.3. We recall that a principal homogeneous $\mathbb{C}_+$-bundle over a base scheme S is an S-scheme $\rho : X\to S$ equipped with an algebraic action of the additive group $\mathbb{C}_+$, such that there exists an open covering $\mathcal{U}=(S_i)_{i\in I}$ of S for which $\rho^{-1}(S_i)$ is equivariantly isomorphic to $S_i\times\mathbb{C}$, where $\mathbb{C}_+$ acts by translations on the second factor, for every $i\in I$. In particular, the total space of a principal homogeneous $\mathbb{C}_+$-bundle has the structure of an $\mathbb{A}^1$-bundle over S. The set $H^1(S,\mathbb{C}_+)$ of isomorphism classes of principal homogeneous $\mathbb{C}_+$-bundles over S is isomorphic to the first cohomology group $\check{H}^1(S,\mathcal{O}_S)\simeq H^1(S,\mathcal{O}_S)$.

Definition 1.4. A *Danielewski variety* is an affine variety of dimension $n\geq 2$ which is the total space $\rho : X\to Z_{n,r}$ of a principal homogeneous $\mathbb{C}_+$-bundle over $Z_{n,r}$ for a certain $r\geq 1$.

Example 1.5. The Danielewski surfaces $S_1=\left\{xz-y^2+1=0\right\}$ and $S_2=\left\{x^2z-y^2+1=0\right\}$ above are Danielewski varieties. Indeed, the projections $pr_x : S_i\to\mathbb{C}$, $i=1,2$, factor through structural morphisms $\rho_i : S_i\to Z_{2,1}$ of principal $\mathbb{C}_+$-bundles over the affine line with a double origin. More generally, the Makar-Limanov surfaces $S\subset\mathbb{C}^3$ with equations $x^nz-Q(x,y)=0$, where $n\geq 1$ and $Q(x,y)$ is a monic polynomial in y, such that $Q(0,y)$ has simple roots are Danielewski varieties.

Remark 1.6. The scheme $Z_{n,r}$ over which a Danielewski variety X becomes the total space of a principal homogeneous $\mathbb{C}_+$-bundle is unique up to isomorphism. Indeed, we have necessarily $n=\dim Z=\dim X-1$. On the other hand, it follows from 1.7 below that X is obtained by gluing r copies of $\mathbb{C}^n\times\mathbb{C}$ along $(\mathbb{C}^*)^n\times\mathbb{C}$. So we deduce by induction that $H_{n+1}(X,\mathbb{Z})$ is isomorphic to the direct sum of r copies of $H_n((\mathbb{C}^*)^n\times\mathbb{C},\mathbb{Z})\simeq H_n((\mathbb{C}^*)^n,\mathbb{Z})\simeq\mathbb{Z}$, whence to $\mathbb{Z}^r$. Therefore, if X admits another structure of principal homogeneous $\mathbb{C}_+$-bundle $\rho' : X\to Z_{n',r'}$ then $(n',r')=(n,r)$. However, we want to insist on the fact that *this does not imply that the structural morphism $\rho : X\to Z_{n,r}$ on a Danielewski variety is unique, even up to automorphisms of the base.* This question will be discussed in 1.13 below.

1.7. A principal homogeneous $\mathbb{C}_+$-bundle $\rho : X\to Z_{n,r}$ becomes trivial on the canonical open covering $\mathcal{U}$ of $Z_{n,r}$ be means of the open subsets $Z_i\simeq\mathbb{C}^n$, $i=1,\ldots,r$ (see definition 1.2 above). So there exists a Čech 1-cocycle

$$g=\{g_{ij}\}_{i,j=1,\ldots,g}\in C^1\left(\mathcal{U},\mathcal{O}_{Z_{n,r}}\right)\simeq\bigoplus_{i=1}^{r}\mathbb{C}\left[\underline{x},\underline{x}^{-1}\right]$$

representing the isomorphism class $[g]\in H^1\left(Z_{n,r},\mathcal{O}_{Z_{n,r}}\right)\simeq\check{H}^1\left(\mathcal{U},\mathcal{O}_{Z_{n,r}}\right)$ of X such that X is equivariantly isomorphic to the scheme obtained by gluing r copies $Z_i\times\mathbb{C}=\operatorname{Spec}(\mathbb{C}[\underline{x}][t_i])$ of

$\mathbb{C}^n \times \mathbb{C}$, equipped with $\mathbb{C}_+$-actions by translations on the second factor, outside $H_{\underline{x}} \times \mathbb{C} \subset Z_i \times \mathbb{C}$ by means of the equivariant isomorphisms

$$\phi_{ij} : \left(Z_j \setminus H_{\underline{x}}\right) \times \mathbb{C} \xrightarrow{\sim} \left(Z_i \setminus H_{\underline{x}}\right) \times \mathbb{C}, \qquad (\underline{x}, t_j) \mapsto \left(\underline{x}, t_j + g_{ij}\left(\underline{x}, \underline{x}^{-1}\right)\right), \quad i \neq j.$$

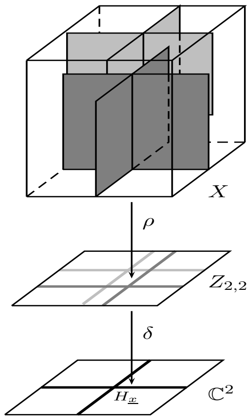

FIGURE 1.1. A Danielewski threefold X.

1.8. Since a Danielewski variety X is affine, the corresponding transition cocycle is not arbitrary. For instance, the trivial cocycle corresponds to the trivial $\mathbb{C}_+$-bundle $Z_{n,r} \times \mathbb{C}$ which is not even separated if $r \geq 2$. More generally, if one of the rational functions g_{ij} is regular at a point $\lambda = (\lambda_1, \ldots, \lambda_n) \in H_{\underline{x}} \subset \mathbb{C}^n$, then for every germ of curve $C \subset \mathbb{C}^n$ intersecting $H_{\underline{x}}$ transversely in λ, $(\rho \circ \delta)^{-1}(C) \subset X$ is a nonseparated scheme. On the other hand, Danielewski established that the total space of a principal homogeneous $\mathbb{C}_+$-bundle $\rho : X \to Z_{n,2}$ defined by a cocycle $g_{12} = \underline{x}^{-[r]} a(\underline{x})$, where $[r] \in \mathbb{Z}^n_{\geq 1}$, such that $a(\underline{x}) \mathbb{C}[\underline{x}] + \underline{x}^{[r]} \mathbb{C}[\underline{x}] = \mathbb{C}[\underline{x}]$ is affine, isomorphic to the variety $X \subset \mathbb{C}^{n+2}$ with equation $\underline{x}^{[r]} z - y^2 - a(\underline{x}) y = 0$. More generally, we have the following result.

Theorem 1.9. *For the total space of a principal $\mathbb{C}_+$-bundle $\rho : X \to Z_{n,r}$ defined by a transition cocycle $g = \left\{g_{ij}\left(\underline{x}, \underline{x}^{-1}\right)\right\}_{i,j=1,\ldots,r}$ the following are equivalent.*

(1) *For every $i \neq j$, $g_{ij} = x^{-[m_{ij}]} a_{ij}(\underline{x})$ for a certain multi-index $[m_{ij}] \in \mathbb{Z}^n_{>0}$ and a polynomial $a_{ij}(\underline{x})$ such that $a_{ij}(\underline{x}) \mathbb{C}[\underline{x}] + \underline{x}^{(1,\ldots,1)} \mathbb{C}[\underline{x}] = \mathbb{C}[\underline{x}]$,*

(2) *X is separated*

(3) *X is affine.*

Proof. We deduce from I.5.5.6 in [12] that X is separated if and only if $g_{ij} \in \mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right]$ generates $\mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right]$ as a $\mathbb{C}[\underline{x}]$-algebra for every $i \neq j$. Letting $g_{ij} = \underline{x}^{-[m]} a(\underline{x})$, where $[m] \in \mathbb{Z}^n_{\geq 0}$ and where $a(\underline{x}) \in \mathbb{C}[\underline{x}]$, this is the case if and only if $\underline{x}^{-[m]}$ generates $\mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right]$ as a $\mathbb{C}[\underline{x}]$-algebra and $a(\underline{x}) \mathbb{C}[\underline{x}] + \underline{x}^{[m]} \mathbb{C}[\underline{x}] = \mathbb{C}[\underline{x}]$. Indeed, the condition is sufficient as it guarantees that $\mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right] = \mathbb{C}[\underline{x}]\left[\underline{x}^{-[m]}\right] \subset \mathbb{C}[\underline{x}][g_{ij}]$. Conversely, if $\mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right] = \mathbb{C}[\underline{x}][g_{ij}]$ then $g_{ij} = x^{-[m]} a(\underline{x})$ for a certain multi-index $[m] = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 1}$ and a polynomial $a \in \mathbb{C}[\underline{x}]$ not divisible by x_i for every $i = 1, \ldots, r$. Indeed, if there exists an indice i such that $m_i \leq 0$ then $x_i^{-1} \notin \mathbb{C}[\underline{x}][g_{ij}]$ which contradicts our hypothesis. Furthermore, since

$x^{-[m]} \in \mathbb{C}[\underline{x}][g_{ij}]$, there exists polynomials $b_1, \ldots, b_s \in \mathbb{C}[\underline{x}]$ such that $x^{-[m]} = b_0 + b_1 a x^{-[m]} + \ldots + b_s a^{-s[m]} \in \mathbb{C}[\underline{x}][g_{ij}]$. This means equivalently that $\underline{x}^{(s-1)[m]} = b_0 \underline{x}^{s[m]} + ca$ for a certain $c \in \mathbb{C}[\underline{x}]$. If $s \neq 1$ then $c \in \underline{x}^{(s-1)[m]}\mathbb{C}[\underline{x}]$ as the x_i's do not divide a, and so, there exists $c' \in \mathbb{C}[\underline{x}]$ such that $1 = b_0 \underline{x}^{-[m]} + c'a$. This proves that (1) and (2) are equivalent.

Now it remains to show that if the $g_{ij} = x^{-[m_{ij}]} a_{ij}(\underline{x})$ satisfy (1), then X is affine. We first observe that there exists an indice i_0 such that $m_{1i_0,k} = \max\{m_{1i,k}\}$ for every $i = 2, \ldots, r$ and every $k = 1, \ldots, n$. Indeed, suppose on the contrary that there exists two indices $i \neq j$, say $i = 2$ and $j = 3$, and two indices $l \neq k$ such that $m_{12,k} < m_{13,k}$ but $m_{12,l} > m_{13,l}$. We let $[\mu] \in \mathbb{Z}^n_{\geq 0}$ be the multi-index with components $\mu_s = \max(m_{12,s}, m_{13,s})$, so that $\mu_k - m_{13,k} = 0$ and $\mu_k - m_{12,k} > 0$ whereas $\mu_l - m_{12,l} = 0$ and $\mu_l - m_{13,l} > 0$. It follows from the cocycle relation $g_{23} = g_{13} - g_{12}$ that

$$\underline{x}^{[\mu]-[m_{23}]} a_{23}(\underline{x}) \quad = \quad \underline{x}^{[\mu]-[m_{13}]} a_{13}(\underline{x}) - \underline{x}^{[\mu]-[m_{12}]} a_{12}(\underline{x}) \in (x_k, x_l)\,\mathbb{C}[\underline{x}] \subset \mathbb{C}[\underline{x}].$$

Since the x_i's do not divide the a_{ij}'s, it follows that neither x_k nor x_l divides the polynomial on the right. Thus $m_{23,l} = \mu_l$ and $m_{23,k} = \mu_k$. This implies that $a_{23}(\underline{x}) \in (x_k, x_l)\,\mathbb{C}[\underline{x}]$ which contradicts (1) above. Therefore, the subset of $\mathbb{Z}^n$ consisting of the multi-indices $[m_{1i}]$, $i = 2, \ldots, r$, is totally ordered for the restriction of the product ordering of $\mathbb{Z}^n$, and so, there exists an indice i_0 such that $m_{1i_0,k} = \max\{m_{1i,k}\}$ for every $i = 2, \ldots, r$ and every $k = 1, \ldots, n$. By construction, $\sigma_i(\underline{x}) = \underline{x}^{[m_{1i_0}]} g_{1i}(\underline{x}, \underline{x}^{-1})$ is a polynomial every $i = 2, \ldots, r$, and $\sigma_{i_0}(\underline{x})$ restricts to a nonzero constant $\lambda \in \mathbb{C}^*$ on $H_{\underline{x}} \subset \mathbb{C}^n$. Letting $\sigma_1(\underline{x}) = 0$, we deduce from the cocycle relation that $\underline{x}^{[m_{1i_0}]} g_{ij} = (\sigma_j(\underline{x}) - \sigma_i(\underline{x}))$ for every $i \neq j$. In turn, this implies that the local morphisms

$$\psi_i : Z_i \times \mathbb{C} = \operatorname{Spec}(\mathbb{C}[\underline{x}][t_i]) \longrightarrow \mathbb{C}^n \times \mathbb{C}, \quad (\underline{x}, t_i) \mapsto \left(\underline{x}, \underline{x}^{[m_{1i_0}]} t_i + \sigma_i(\underline{x})\right), \quad i = 1, \ldots, r$$

glue to a birational morphism $\psi : X \to \mathbb{C}^n \times \mathbb{C}$. By construction, the images by ψ of $H_{\underline{x}} \times \mathbb{C} \subset Z_{i_0} \times \mathbb{C}$ and $H_{\underline{x}} \times \mathbb{C} \subset Z_1 \times \mathbb{C}$ are disjoint, contained respectively in the closed subsets $V(\underline{x}, t - \lambda)$ and $V(\underline{x}, t)$ of $\mathbb{C}^n \times \mathbb{C} = \operatorname{Spec}(\mathbb{C}[\underline{x}][t])$. Therefore, $\psi^{-1}(\mathbb{C}^n \times \mathbb{C} \setminus V(\underline{x}, t))$ is contained in the complement V_1 in X of $H_{\underline{x}} \times \mathbb{C} \subset Z_1 \times \mathbb{C}$, whereas $\psi^{-1}(\mathbb{C}^n \times \mathbb{C} \setminus V(\underline{x}, t - \lambda))$ is contained in the complement V_{i_0} in X of $H_{\underline{x}} \times \mathbb{C} \subset Z_{i_0} \times \mathbb{C}$. Clearly, $\rho : X \to Z_{n,r}$ restricts on V_1 and V_{i_0} to the structural morphisms $\rho_1 : V_1 \to Z_{n,r-1}$ and $\rho_{i_0} : V_{i_0} \to Z_{n,r-1}$ of the principal homogeneous $\mathbb{C}_+$-bundles corresponding to the Čech cocycles $\{g_{ij}\}_{i,j=2,\ldots,r}$ and $\{g_{ij}\}_{i,j \neq i_0, i,j=1,\ldots,r}$. So we conclude by a similar induction argument as in Proposition 1.4 in [10] that V_1 and V_{i_0} are affine. In turn, this implies that $\psi : X \to \mathbb{C}^n \times \mathbb{C}$ is an affine morphism, and so, X is affine. □

The following example introduces a class of Danielewski varieties, which contains for instance the Makar-Limanov surfaces of example 1.5.

Example 1.10. Suppose given a collection σ of polynomials $\sigma_i(\underline{x}) \in \mathbb{C}[\underline{x}]$, $i = 1, \ldots, r$, with the following properties.

(1) $\sigma_i(0, \ldots, 0) \neq \sigma_j(0, \ldots, 0)$ for every $i \neq j$,

(2) $\sigma_i(\underline{x}) - \sigma_i(0, \ldots, 0) \in \underline{x}^{(1,\ldots,1)}\mathbb{C}[\underline{x}]$ for every $i = 1, \ldots, r$.

Then for every multi-index $[m] = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$ the variety $X_{[m],\sigma} \subset \mathbb{C}^{n+2}$ with equation

$$\underline{x}^{[m]} z - \prod_{i=1}^{r} (y - \sigma_i(\underline{x})) = 0$$

is a Danielewski variety.

Proof. Similarly as the Danielewski surfaces, a variety $X_{[m],\sigma}$ comes naturally equipped with a surjective morphism $\pi = pr_{\underline{x}} : X_{[m],\sigma} \to \mathbb{C}^n$, $(\underline{x}, y, z) \mapsto \underline{x}$ restricting to a trivial $\mathbb{A}^1$-bundle $\pi^{-1}\left((\mathbb{C}^*)^n\right) \simeq (\mathbb{C}^*)^n \times \mathbb{C}$ over $U_{\underline{x}} = (\mathbb{C}^*)^n$, with coordinate y on the second factor. On the other hand, it follows from our assumptions that the fiber

$$\pi^{-1}\left(H_{\underline{x}}\right) \simeq \mathrm{Spec}\left(\mathbb{C}\left[\underline{x}, y, z\right] / \left(\underline{x}^{(1,\ldots,1)}, \underline{x}^{[m]} z - \prod_{i=1}^{r}\left(y - \sigma_i\left(\underline{x}\right)\right)\right)\right)$$

decomposes as the disjoint union of r copies D_i of $H_{\underline{x}} \times \mathbb{C}$, with equations $\{x_1 \cdots x_n = 0, y = \sigma_i(0)\}$, and with coordinate z on the second factor. The open subsets $\pi^{-1}\left(U_{\underline{x}}\right) \cup C_i$ of $X_{[m],\sigma}$ are isomorphic to $\mathbb{C}^n \times \mathbb{C}$ with natural coordinates $\underline{x}$ and

$$t_i = \frac{y - \sigma_i\left(\underline{x}\right)}{\underline{x}^{[r]}} = \frac{z}{\prod_{j \neq i}\left(y - \sigma_j\left(\underline{x}\right)\right)}, \quad i = 1, \ldots, r,$$

and so, $X_{[m],\sigma}$ is isomorphic to the total space of the principal homogeneous $\mathbb{C}_+$-bundle defined by the transition cocycles $g_{ij} = \underline{x}^{-[r]}\left(\sigma_j\left(\underline{x}\right) - \sigma_i\left(\underline{x}\right)\right)$, $i, j = 1, \ldots, r$. □

As a consequence of the general principle discussed at the beginning of this section, Danielewski varieties are natural candidates for being counter-examples to the Cancellation problem.

Proposition 1.11. *If two Danielewski varieties X_1 and X_2 are the total spaces of $\mathbb{C}_+$-principal bundles over the same base $Z_{n,r}$ then $X_1 \times \mathbb{C}$ and $X_2 \times \mathbb{C}$ are isomorphic.*

Example 1.12. Given a polynomial $P(y) \in \mathbb{C}[y]$ with $r \geq 2$ simple roots, the varieties $\tilde{X}_{[m],P} \subset \mathbb{C}^{n+3} = \mathrm{Spec}\left(\mathbb{C}\left[\underline{x}, y, z, u\right]\right)$ with equations $\underline{x}^{[m]} z - P(y) = 0$, where $[m] \in \mathbb{Z}^n_{\geq 1}$ is an arbitrary multi-index, are all isomorphic. Indeed $\tilde{X}_{[m],P}$ is isomorphic to $X_{[m],P} \times \mathbb{C}$, where $X_{[m],P} \subset \mathbb{C}^{n+2} = \mathrm{Spec}\left(\mathbb{C}\left[\underline{x}, y, z\right]\right)$ denotes the Danielewski variety with equation $x^{[m]} z - P(y) = 0$, which has the structure of a principal homogeneous $\mathbb{C}_+$-bundle over $Z_{n,r}$ (see example 1.10).

1.13. This leads to the difficult problem of deciding which Danielewski varieties are isomorphic as abstract varieties. Things would be simpler if the structural morphism $\rho : X \to Z_{n,r}$ on a Danielewski variety were unique up to automorphisms of the base. However, this is definitely not the case in general, as shown by the Danielewski surface $S_1 = \left\{xz - y^2 + 1 = 0\right\} \subset \mathbb{C}^3$, which admits two such structures, due to the symmetry between the variables x and z. Actually, the situation is even worse since in general, a Danielewski variety admitting a second $\mathbb{C}_+$-action, whose general orbits are distinct from the general fibers of the structural morphism $\rho : X \to Z_{n,r}$, comes equipped with a one parameter family of distinct structures of principal homogeneous $\mathbb{C}_+$-bundles. Indeed, let $G_1 \simeq \mathbb{C}_+$ and $G_2 \simeq \mathbb{C}_+$ be one-parameter subgroups of $\mathrm{Aut}(X)$ corresponding respectively to a principal homogeneous $\mathbb{C}_+$-bundle structure on $\rho : X \to Z_{n,r}$ and another nontrivial $\mathbb{C}_+$-action on X with general orbits distinct from the ones of G_1. Then the subgroups $\phi_t^{-1} G_1 \phi_t \simeq \mathbb{C}_+$ of $\mathrm{Aut}(X)$, where $\phi_t \in G_2$, correspond to principal homogeneous $\mathbb{C}_+$-bundle structures on X, with pairwise distinct general orbits provided that the generators of G_1 and G_2 do not commute.

1.14. There exists a useful geometric criterion to decide if a smooth affine surface admits two $\mathbb{C}_+$-actions with distinct general orbits. As is well-known, a normal affine surface S admits a nontrivial algebraic $\mathbb{C}_+$-action if and only if it is equipped with a surjective flat morphism $q : S \to C$ over a nonsingular affine curve C, with general fiber isomorphic to $\mathbb{C}$. Indeed, these maps correspond exactly with algebraic quotient morphisms associated with $\mathbb{C}_+$-actions on S. In this context, Gizatullin [11] and Bertin [2] (see also [5] for the normal case) established successively that if a smooth surface S admits an $\mathbb{A}^1$-fibration $q : S \to C$ as above then this fibration is unique up to isomorphism of the base if and only if S does not admit a completion $S \hookrightarrow \bar{S}$ by a smooth projective surface $\bar{S}$ for which the boundary divisor $B = \bar{S} \setminus S$ is zigzag, that is, a chain of nonsingular rational curves. For instance, the fact that the Danielewski surface $S_1 = \left\{xz - y^2 + 1 = 0\right\}$ admits two $\mathbb{C}_+$-actions with distinct general orbits can be recovered from this result, as S_1 embeds as the complement of a diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ via the morphism

$$S_1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad (x, y, z) \mapsto ([x : y + 1], [y + 1 : z]) = ([z : y - 1], [x : y - 1]).$$

Bandman and Makar-Limanov [1] (see also [6] for a more general result) deduced from this criterion that a Danielewski surface $\rho : S \to Z_{1,r}$ admits two independent $\mathbb{C}_+$-actions if and only if it is isomorphic to a surface in $\mathbb{C}^3$ with equation $xz - P(y) = 0$, where P is a polynomial with r simple roots. Latter on, Daigle [3] established that all $\mathbb{C}_+$-actions on such a surface S are conjugated to a one whose general orbits coincide with the fibers of the principal homogeneous $\mathbb{C}_+$-bundle structure $\rho : S \to Z_{1,r}$ factoring the projection $pr_x : S \to \mathbb{C}$.

1.15. Unfortunately, there is no obvious generalization of Gizatullin criterion for higher dimensional varieties with $\mathbb{C}_+$-actions. However, it turns out that in certain situations such as the one described in Theorem 2.8 below, one can establish by direct computations that the structural morphism $\rho : X \to Z_{n,r}$ on a Danielewski variety is unique up to automorphisms of the base. If this holds, then it becomes easier to decide if another Danielewski variety is isomorphic to X as an abstract variety. Indeed, the group $\mathrm{Aut}(Z_{n,r}) \times \mathrm{Aut}(\mathbb{C}_+) \simeq \mathrm{Aut}(Z_{n,r}) \times \mathbb{C}^*$ acts on the set $H^1\left(Z_{n,r}, \mathcal{O}_{Z_{n,r}}\right)$ by sending a class $[g] \in H^1\left(Z_{n,r}, \mathcal{O}_{Z_{n,r}}\right)$ represented by a bundle $\rho : X \to Z_{n,r}$ with $\mathbb{C}_+$-action $\mu : \mathbb{C}_+ \times X \to X$ to the isomorphism class $(\phi, \lambda) \cdot [g]$ of the fiber product bundle $pr_2 : \phi^* X = X \times_{Z_{n,r}} Z_{n,r} \to Z_{n,r}$ equipped with the $\mathbb{C}_+$-action defined by $\mu_\lambda(t, (x, z)) \mapsto \left(\mu\left(\lambda^{-1} t, x\right), z\right)$. Similar arguments as in the proof of Theorem 1.1 in [22] imply the following characterization.

Proposition 1.16. *Let $\rho_1 : X_1 \to Z_{n,r}$ and $\rho_2 : X_2 \to Z_{n,r}$ be two Danielewski varieties. If ρ_1 is a unique $\mathbb{A}^1$-bundle structure on X_1 up to automorphisms of $Z_{n,r}$, then X_1 and X_2 are isomorphic as abstract varieties if their isomorphism classes as principal $\mathbb{C}_+$-bundles belong to the same orbit under the action of $Aut(Z_{n,r}) \times Aut(\mathbb{C}_+)$.*

1.17. Let us again consider the Danielewski varieties $X_{[m],\sigma} \subset \mathbb{C}^{n+2}$ with equations

$$\underline{x}^{[m]} z - \prod_{i=1}^{r} (y - \sigma_i(\underline{x})) \quad = \quad 0$$

where $[m] = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$ is a multi-index and where $\sigma = \{\sigma_i(\underline{x})\}_{i=1,\ldots,r}$ is collection of polynomials satisfying (1) and (2) in example 1.10. Again, we denote by $\pi = pr_{\underline{x}} : X_{[m],\sigma} \to \mathbb{C}^n$, $(\underline{x}, y, z) \mapsto \underline{x}$ the fibration which factors through the structural morphism of the principal homogeneous $\mathbb{C}_+$-bundle $\rho : X_{[m],\sigma} \to Z_{n,r}$ described in example 1.10 above. Suppose that one of the m_i's, say m_1 is equal to 1. Then $X_{[m],\sigma}$ admits a second fibration

$$\pi_1 : X_{[m],\sigma} \to \mathbb{C}^n, \quad (x_1, \ldots, x_n, y, z) \mapsto (x_2, \ldots, x_n, z)$$

restricting to the trivial $\mathbb{A}^1$-bundle over $(\mathbb{C}^*)^n$ and the same argument as in example 1.10 above shows that π_1 factors through the structural morphism of another principal homogeneous $\mathbb{C}_+$-bundle $\rho_1 : X_{[m],\sigma} \to Z_{n,r}$. On the hand, Makar-Limanov [19] established that for every integer $m \geq 2$ the $\mathbb{A}^1$-bundle structure $\rho : S \to Z_{1,r}$ above on a Danielewski surface $S \subset \mathbb{C}^3$ with equation $x^m z - P(y) = 0$, where $\deg P(y) = r \geq 2$, is unique up to isomorphism of the base. More generally, we have the following result.

Theorem 1.18. *Let $\sigma = \{\sigma_i(\underline{x})\}_{i=1,\ldots,r}$ be a collection of $r \geq 2$ polynomials satisfying* (1) *and* (2) *in example 1.10. Then for every multi-index $[m] \in \mathbb{Z}^n_{>1}$, $\rho : X_{[m],\sigma} \to Z_{n,r}$ is a unique structure of principal homogeneous $\mathbb{C}_+$-bundle structure on $X_{[m],\sigma}$ up to action of the group $Aut(Z_{n,r}) \times Aut(\mathbb{C}_+)$.*

Proof. This follows from Theorem 2.8 below which guarantees more generally that the algebraic quotient morphism $q : X_{[m],\sigma} \to X_{[m],\sigma}//\mathbb{C}_+$ associated with an arbitrary nontrivial $\mathbb{C}_+$-action on $X_{[m],\sigma}$ coincides with the projection $\pi = pr_{\underline{x}} : X_{[m],\sigma} \to \mathbb{C}^n$. □

It follows from 1.17 that every Danielewski variety $X_{[m],\sigma} \subset \mathbb{C}^{n+2}$ defined by a multi-index $[m'] \in \mathbb{Z}^n_{\geq 1} \setminus \mathbb{Z}^n_{>1}$ admits a second $\mathbb{C}_+$-action whose general orbits are distinct from the general fibers of the $\mathbb{A}^1$-bundle $\rho : X_{[m],\sigma} \to Z_{n,r}$. This leads to the following result.

Corollary 1.19. *For every collection $\sigma = \{\sigma_i(\underline{x})\}_{i=1,\ldots,r}$ of $r \geq 2$ polynomials satisfying* (1) *and* (2) *in example 1.10 and every pair of multi-index $[m] \in \mathbb{Z}^n_{>1}$ and $[m'] \in \mathbb{Z}^n_{\geq 1} \setminus \mathbb{Z}^n_{>1}$ the Danielewski varieties $X_{[m],\sigma}$ and $X_{[m'],\sigma}$ are not isomorphic.*

1.20. More generally, let $[m] = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>1}$ and $[m'] = (m'_1, \ldots, m'_n) \in \mathbb{Z}^n_{>1}$ be two multi-indices for which the subsets $\{m_1, \ldots, m_n\}$ and $\{m'_1, \ldots, m'_n\}$ of $\mathbb{Z}$ are distint. Then for every collection $\sigma = \{\sigma_i(\underline{x})\}_{i=1,\ldots,r}$ of $r \geq 2$ polynomials satisfying (1) and (2), the Čech cocycles

$$g_{ij} = \underline{x}^{-[m]}(\sigma_j(\underline{x}) - \sigma_i(\underline{x})) \quad \text{and} \quad g'_{ij} = \underline{x}^{-[m']}(\sigma_j(\underline{x}) - \sigma_i(\underline{x}))$$

in $C^1(\mathcal{U}, \mathcal{O}_{Z_{n,r}}) \simeq \mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right]^r$ are not cohomologous and do not belong to the same orbit under the action of $\mathrm{Aut}(Z_{n,r}) \times \mathrm{Aut}(\mathbb{C}_+)$ on $C^1(\mathcal{U}, \mathcal{O}_{Z_{n,r}})$. As a consequence of Proposition 1.16 and Theorem 1.18 above, we obtain the following result.

Corollary 1.21. *Under the hypothesis above, the Danielewski varieties $X_{[m],\sigma}$ and $X_{[m'],\sigma}$ are not isomorphic. In particular, there exists an infinite countable family of pairwise non-isomorphic Danielewski varieties $X_{[m],\sigma}$ with the property that all the varieties $X_{[m],\sigma} \times \mathbb{C}$ are isomorphic.*

Remark 1.22. Given a multi-index $[m] \in \mathbb{Z}^n_{>1}$, the problem of characterizing explicitly the collections $\sigma = \{\sigma_i(\underline{x})\}_{i=1,\ldots,r}$ which lead to isomorphic Danielewski varieties $X_{[m],\sigma}$ is more subtle in general. By virtue of Proposition 1.16, it is equivalent to describe the orbits of the associated cocycles $g_{ij} = \underline{x}^{-[m]}(\sigma_j(\underline{x}) - \sigma_i(\underline{x}))$ under the action of $\mathrm{Aut}(Z_{n,r}) \times \mathrm{Aut}(\mathbb{C}_+)$. In the case of surfaces, the question becomes simpler as $\mathrm{Aut}(Z_{1,r}) \simeq \mathbb{C}^* \times \mathfrak{S}_r$, where $\mathfrak{S}_r$ denotes the group of permutation of the origins $o_1, \ldots, o_r$ of $Z_{1,r}$. For instance, Makar-Limanov [19] obtained a complete classification of the Danielewski surfaces $S \subset \mathbb{C}^3$ with equation $x^n z - P(y) = 0$, where $n \geq 2$. More generally, we refer the interested reader to the forthcoming paper [7], in which we study Danielewski surfaces with equations $x^n z - Q(x, y) = 0$.

2. Additive group actions on Danielewski varieties

Makar-Limanov [18] observed that it is sometimes possible to obtain information on algebraic $\mathbb{C}_+$-actions on an affine variety X by considering homogeneous $\mathbb{C}_+$-actions on certain affine cones $\hat{X}$ associated with X. We recall that *the Makar-Limanov invariant* [17] of an affine variety $X = \mathrm{Spec}\,(B)$ is the subring $\mathrm{ML}\,(X)$ of B consisting of regular functions on B which are invariant under all $\mathbb{C}_+$-actions on X. Using associated homogeneous objects, he established in [18] that the Makar-Limanov invariant of the Russell cubic threefold, *i.e.* the hypersurface $X \subset \mathbb{C}^4$ with equation $x + x^2y + z^2 + t^3 = 0$, is not trivial $= \mathbb{C}$. He also computed in [19] the Makar-Limanov invariants of the affine surfaces $S = \{x^n z - P(y) = 0\}$, where $\deg(P) > 1$ and $n > 1$. Here we use a similar method, based on real-valued weight degree functions, to compute the Makar-Limanov invariants of the Danielewski varieties $X_{[m],\sigma}$, where $[m] \in \mathbb{Z}^n_{>1}$.

2.1. **Basic facts on locally nilpotent derivations.**

He we recall results on locally nilpotent derivations that will be used in the following subsections. We refer the reader to [9] and [17] for more complete discussions.

2.1. Algebraic $\mathbb{C}_+$-actions on a complex affine variety $X = \mathrm{Spec}\,(B)$ are in one-to-one correspondence with locally nilpotent $\mathbb{C}$-derivations of B, that is, derivations $\partial : B \to B$ such that every element b of B belongs to the kernel of ∂^m for a suitable $m = m(b)$. Indeed, for every algebraic $\mathbb{C}_+$-action on S with comorphism $\mu^* : B \to B \otimes_{\mathbb{C}} \mathbb{C}[t]$, $\partial_\mu = \frac{d}{dt}\mid_{t=0} \circ \mu^* : B \to B$ is a locally nilpotent derivation. Conversely, for every such derivation $\partial : B \to B$ the exponential map

$$\exp(t\partial) : B \to B[t], \quad b \mapsto \sum_{n \geq 0} \frac{\partial^n b}{n!} t^n = \sum_{n=0}^{m(b)-1} \frac{\partial^n b}{n!} t^n$$

coincides with the comorphism of an algebraic $\mathbb{C}_+$-action on X. To every locally nilpotent derivation ∂ of B, we associate a function

$$\deg_\partial : B \to \mathbb{N} \cup \{-\infty\}, \text{ defined by } \deg_\partial(b) = \begin{cases} -\infty & \text{if } b = 0 \\ \max\{m, \partial^m b \neq 0\} & \text{otherwise,} \end{cases}$$

which we call the *degree function generated by* ∂. We recall the following facts.

Proposition 2.2. *Let* ∂ *be a nontrivial locally nilpotent derivation of* B. *Then the following hold.*

(1) *B has transcendence degree one over $Ker(\partial)$. The field of fraction $Frac(B)$ of B is a purely transcendental extension of $Frac(Ker(\partial))$, and $Ker(\partial)$ is algebraically closed in B.*

(2) *For every $f \in Ker(\partial^2) \setminus Ker(\partial)$, the localization B_f of B at f is isomorphic to the polynomial ring in one variable $Ker(\partial)_{\partial(f)}[f]$ over the localization $Ker(\partial)_{\partial(f)}$ of $Ker(\partial)$ at $\partial(f)$. In particular, for every $b \in Ker(\partial^{m+1}) \setminus Ker(\partial^m)$, there exists $a', a_0, \ldots, a_m \in Ker(\partial)$, where $a', a_m \neq 0$, such that $a'b = \sum_{j=0}^m a_j f^j$.*

(3) *$\deg_\partial : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function, i.e.* $\deg_\partial(b + b') \leq \max(\deg_\partial(b), \deg_\partial(b'))$ *and* $\deg_\partial(bb') = \deg_\partial(b) + \deg_\partial(b')$.

(4) *If $b, b' \in B \setminus \{0\}$ and $bb' \in Ker(\partial)$, then $b, b' \in Ker(\partial)$.*

2.2. **Equivariant deformations to the cone following Kaliman and Makar-Limanov.** Here we review a procedure due to Kaliman and Makar-Limanov [18] and [16] which associates to a filtered algebra $(B,\mathcal{F})$ equipped with a locally nilpotent derivation ∂ a graded algebra equipped with an homogeneous locally nilpotent derivation induced by ∂.

2.3. We let B be a finitely generated algebra, equipped with an exhaustive, separated, ascending filtration $\mathcal{F}=\left\{F^tB\right\}_{t\in\mathbb{R}}$ by $\mathbb{C}$-linear subspaces F^tB of B. For every $t\in\mathbb{R}$, we let $F_0^tB=\bigcup_{s<t}F^sB$. We denote by

$$gr_{\mathcal{F}}B \quad = \quad \bigoplus_{t\in\mathbb{R}}\left(gr_{\mathcal{F}}B\right)^t, \quad \text{where } \left(gr_{\mathcal{F}}B\right)^t=F^tB/F_0^tB$$

the $\mathbb{R}$-graded algebra associated to the filtered algebra $(B,\mathcal{F})$, and we let $gr:B\rightarrow gr_{\mathcal{F}}B$ the natural map which sends an element $b\in F^tB\subset B$ to its image $gr(b)$ under the canonical map $F^tB\rightarrow F^tB/F_0^tB\subset gr_{\mathcal{F}}B$. Suppose further that $1\in F^0B\setminus F_0^0B$ and that

$$\left(F^{t_1}B\setminus F_0^{t_1}B\right)\left(F^{t_2}B\setminus F_0^{t_2}B\right)\subset\left(F^{t_1+t_2}B\setminus F_0^{t_1+t_2}B\right) \qquad \text{for every } t_1,t_2\in\mathbb{R}.$$

Then the filtration $\mathcal{F}$ is induced by a degree function $d_{\mathcal{F}}:B\rightarrow\mathbb{R}\cup\{-\infty\}$ on B. Indeed, the formulas $d_{\mathcal{F}}(0)=-\infty$ and $d_{\mathcal{F}}(b)=t$ if $b\in F^tB\setminus F_0^tB\subset B$ define a degree function on B such that $F^tB=\{b\in B,\, d(b)\leq t\}$ for every $t\in\mathbb{R}$. In what follows, we only consider filtrations induced by degree functions.

2.4. Given a nontrivial locally nilpotent derivation ∂ of B and a nonzero $b\in B$, we let $t(b)=d_{\mathcal{F}}(\partial b)-d_{\mathcal{F}}(b)\in\mathbb{R}$. By definition, if $b\in F^tB\setminus\left(\mathrm{Ker}\partial\cap F_0^tB\right)$ then $\partial b\in F^{t+t(b)}B\setminus F_0^{t+t(b)}B$. Since B is finitely generated, it follows that there exists a smallest $t_0\in\mathbb{R}$ such that $\partial F^tB\subset F^{t+t_0}B$. So ∂ induces a locally nilpotent derivation $gr\partial$ of the associated graded algebra $gr_{\mathcal{F}}B$ of $(B,\mathcal{F})$, defined by

$$gr\partial\left(gr\left(b\right)\right) \quad = \quad \begin{cases} gr\left(\partial b\right) & \text{if } d_{\mathcal{F}}\left(\partial\left(b\right)\right)-d_{\mathcal{F}}\left(b\right)=t_0 \\ 0 & \text{otherwise.}\end{cases}$$

By construction, $gr\partial$ sends an homogeneous component F^tB/F_0^tB of $gr_{\mathcal{F}}B$ into the homogeneous component $F^{t+t_0}B/F_0^{t+t_0}B$. We say that $gr\partial$ is the *homogeneous locally nilpotent derivation of $gr_{\mathcal{F}}B$ associated with ∂*. By construction, if $gr_{\mathcal{F}}B$ is a domain, then

$$\deg_{\partial}(b) \quad \geq \quad \deg_{gr\partial}\left(gr\left(b\right)\right) \tag{2.1}$$

for every $b\in B$. We will see below that this inequality plays a crucial role in the computation of the Makar-Limanov invariant of certain Danielewski varieties.

Remark 2.5. For integral-valued degree functions $d:B\rightarrow\mathbb{Z}\cup\{-\infty\}$, the above construction admits a simple geometric interpretation. Indeed, letting $\mathcal{F}=\{F^nB\}_{n\in\mathbb{Z}}$ be the filtration generated by d, we consider the Rees algebra

$$\mathcal{R}\left(B,\mathcal{F}\right)=\bigoplus_{n\in\mathbb{Z}}\mathcal{F}^ns^{-n}\subset B\left[s,s^{-1}\right].$$

Every locally nilpotent derivation ∂ of B canonically extends to a locally nilpotent derivation $\tilde{\partial}$ of $\mathcal{R}(B,\mathcal{F})$ with the property that $\tilde{\partial}(s)=0$. By construction, the inclusion $\mathbb{C}[s]\hookrightarrow\mathcal{R}(B,\mathcal{F})$ gives rise to a flat family $\rho:\mathcal{X}=\mathrm{Spec}(\mathcal{R}(B,\mathcal{F}))\rightarrow\mathbb{C}$ of affine varieties with $\mathbb{C}_+$-actions, such that for every $s\in\mathbb{C}^*$, the fiber $\mathcal{X}_s$ is isomorphic to X equipped with the $\mathbb{C}_+$-action defined by ∂, whereas the fiber $\mathcal{X}_0\simeq\mathrm{Spec}\left(\mathcal{R}(B,\mathcal{F})/s\mathcal{R}(B,\mathcal{F})\right)$ is canonically isomorphic to the spectrum of the graded algebra $gr_{\mathcal{F}}B$, equipped with the $\mathbb{C}_+$-action corresponding to the homogeneous locally nilpotent derivation $gr\partial$ of $gr_{\mathcal{F}}B$ defined above.

2.3. **On the Makar-Limanov invariants of Danielewski varieties $X_{[m],\sigma}$.**

Here we consider a class of affine varieties with $\mathbb{C}_+$-actions which contains the Danielewski varieties $X_{[m],\sigma}$ of example 1.10. We construct certain filtrations $\mathcal{F}_d$ of their coordinate rings induced by weight degree functions $d:\mathbb{C}[\underline{x}]\to\mathbb{R}$, and we determine the structure of the associated homogeneous objects. Finally we compute their Makar-Limanov invariants.

Definition 2.6. Given a monic polynomial $Q(\underline{x},y)=y^r+\sum_{i=0}^{r-1}a_i(\underline{x})\,y^i\in\mathbb{C}[\underline{x}][y]$ of degree $r\geq 2$ and a multi-index $[m]=(m_1,\ldots,m_n)\in\mathbb{Z}_{\geq 1}^n$, we denote by $X_{[m],Q}\subset\mathbb{C}^{n+2}$ the affine variety with equation $\underline{x}^{[m]}z-Q(\underline{x},y)=0$.

2.7. Clearly, the above class of affine varieties contains the Danielewski varieties $X_{[m],\sigma}\subset\mathbb{C}^{n+2}$ with equations $\underline{x}^{[m]}z-\prod_{i=1}^{r}(y-\sigma_i(\underline{x}))=0$. Again, the projection

$$\pi=pr_{\underline{x}}:X_{[m],Q}\to\mathbb{C}^n,\quad(\underline{x},y,z)\mapsto\underline{x}$$

is surjective, restricting to a trivial $\mathbb{A}^1$-bundle $(\mathbb{C}^*)^n\times\mathbb{C}=\operatorname{Spec}\left(\mathbb{C}\left[\underline{x},\underline{x}^{-1}\right][y]\right)$ over $(\mathbb{C}^*)^n\subset\mathbb{C}^n$. The locally nilpotent derivation ∂ of $\mathbb{C}[\underline{x},y,z]$ defined by

$$\partial(x_i)=0,\ i=1,\ldots,r,\quad\partial(y)=\underline{x}^{[m]}\quad\text{and }\partial(z)=\frac{\partial Q(x,y)}{\partial y}$$

annihilates the definning ideal $I=\left(\underline{x}^{[m]}z-Q(x,y)\right)$ of $X_{[m],Q}$, whence induces a nontrivial locally nilpotent derivation of the coordinate ring B of $X_{[m],Q}$. The general orbits of the corresponding $\mathbb{C}_+$-action coincide with the general fibers of π. Hence π coincides with the algebraic quotient morphism $q:X_{[m],Q}\to X_{[m],Q}//\mathbb{C}_+=\operatorname{Spec}\left(B^{\mathbb{C}_+}\right)$. This shows that $\operatorname{ML}\left(X_{[m],Q}\right)\subset\mathbb{C}[\underline{x}]$. Actually, a similar argument as in 1.17 above shows that $\operatorname{ML}\left(X_{[m],Q}\right)$ is a subring of $\mathbb{C}[x_{i_1},\ldots,x_{i_s}]$, where $i_1,\ldots,i_s$ denote the indices for which $m_{i_k}=1$. In particular, if $[m]=(1,\ldots,1)$, then $\operatorname{ML}\left(X_{[m],Q}\right)=\mathbb{C}$. In contrast, we have the following result.

Theorem 2.8. *If $[m]\in\mathbb{Z}_{>1}^n$ then the Makar-Limanov invariant of a variety $X_{[m],Q}$ is isomorphic to $\mathbb{C}[\underline{x}]$.*

2.9. It suffices to shows $\operatorname{Ker}\left(\partial^2\right)\subset\mathbb{C}[\underline{x},y]\subset B$ for every nontrivial locally nilpotent derivation ∂ on the coordinate ring B of $X_{[m],Q}$. Indeed, if ∂ is nontrivial, then it follows from (2) in Proposition 2.2 that there exists $f\in\operatorname{Ker}\left(\partial^2\right)\setminus\operatorname{Ker}(\partial)$ such that $z=\underline{x}^{-[m]}\left(y^2-1\right)\in B\subset\mathbb{C}\left[\underline{x},\underline{x}^{-1},y\right]$ satisfies a relation of the form $a'z=\sum_{j=1}^{m}a_jf^j$ for suitable elements $a',a_0,\ldots,a_m\in\operatorname{Ker}(\partial)$, where $a',a_m\neq 0$. Therefore, if $\operatorname{Ker}\left(\partial^2\right)\subset\mathbb{C}[\underline{x},y]$ then $z=r(\underline{x},y)/q(\underline{x},y)$ for a certain polynomial $q(\underline{x},y)\in\operatorname{Ker}(\partial)$. This implies that $\underline{x}^{[m]}$ divides $q(\underline{x},y)$ and so, by virtue of (3) in Proposition 2.2, $\mathbb{C}[\underline{x}]\subset\operatorname{Ker}(\partial)$ as $m_i\geq 1$ for every $i=1,\ldots,n$. To show that the inclusion $\operatorname{Ker}\left(\partial^2\right)\subset\mathbb{C}[\underline{x},y]$ holds for every nontrivial locally nilpotent derivation on B, we study in 2.10-2.16 below the homogeneous objects associated with certain filtrations on B induced by weight degree functions.

Definition 2.10. A *weight degree function* on a polynomial ring $\mathbb{C}[\underline{x}]$ is a degree function $d:\mathbb{C}[\underline{x}]\to\mathbb{R}$ defined by real weights $d_i=d(x_i)$, $i=1,\ldots,n$. The *d-degree* of a monomial $m=x^{[\alpha]}$ is $\alpha_1d_1+\ldots+\alpha d_n$, and the d-degree $d(p)$ of a polynomial $p\in\mathbb{C}[\underline{x}]$ is defined as the suppremum of the degrees $d(m)$, where m runs through the monomials of p. A weight degree function d defines a grading $\mathbb{C}[\underline{x}] = \bigoplus_{t\in\mathbb{R}} \mathbb{C}[\underline{x}]_t$, where $\mathbb{C}[\underline{x}]_t \setminus \{0\}$ consists of all the d-homogeneous polynomials of d-degree t. In what follows, we denote by $\bar{p}$ the *principal d-homogeneous component of p*, that is, the homogeneous component of p of degree $d(p)$. A degree function d on $\mathbb{C}[\underline{x}]$ naturally extends to a degree function on the algebra $\mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right]$ of Laurent polynomials.

2.11. Given a multi-index $[m] \in \mathbb{Z}^n_{>1}$ and a monic polynomial $Q(\underline{x}, y) \in \mathbb{C}[\underline{x}][y]$ as in Definition 2.6, we denote by $B = \mathbb{C}[\underline{x}, y, z]/I$, where $I = \left(\underline{x}^{[r]}z - Q(\underline{x}, y)\right)$, the coordinate ring of the corresponding variety $X_{[m],Q}$, and we denote by $\sigma : \mathbb{C}[\underline{x}, y, z] \rightarrow B$ the natural morphism. The polynomial ring $\mathbb{C}[\underline{x}, y]$ is naturally a subring of B. Moreover, by means of the localization homomorphism $B \hookrightarrow B_{\underline{x}} = B \otimes_{\mathbb{C}[\underline{x}]} \mathbb{C}\left[\underline{x}, \underline{x}^{-1}\right] \simeq \mathbb{C}\left[\underline{x}, \underline{x}^{-1}, y\right]$, B is itself identified to the subalgebra $\mathbb{C}\left[\underline{x}, y, \underline{x}^{-[m]}Q(\underline{x}, y)\right]$ of $\mathbb{C}\left[\underline{x}, \underline{x}^{-1}, y\right]$. Hence every weight degree function d on $\mathbb{C}\left[\underline{x}, \underline{x}^{-1}, y\right]$ induces an exhaustive separated ascending filtration $\mathcal{F}_d = \left\{F^t B\right\}_{t\in\mathbb{R}}$ of $B \subset \mathbb{C}\left[\underline{x}, \underline{x}^{-1}, y\right]$ by means of the subsets $F^t B = \{p \in B,\ d(p) \leq t\}$, $t \in \mathbb{R}$.

2.12. Since $Q(\underline{x}, y) = y^r + \sum_{i=0}^{r-1} a_i(\underline{x})\, y^i$ is monic, it follows that if the weight d_y of y is positive and sufficiently bigger that the weights d_i of the x_i's, then the principal d-homogeneous component of $Q(\underline{x}, y)$ is simply $\bar{Q}(\underline{x}, y) = y^r$. If this holds, then $gr_{\mathcal{F}}B$ is generated by $gr(x) = x$, $gr(y) = y$ and $gr(z) = x^{-[m]}y^r$, with the unique relation $x^{[m]}gr(z) = y^r$. Hence, letting $\tilde{d} : \mathbb{C}[\underline{x}, y, z] \rightarrow \mathbb{R}$ be the unique weight degree function restricting to d on $\mathbb{C}[\underline{x}, y] \subset \mathbb{C}[\underline{x}, y, z]$ and such that $\tilde{d}(z) = rd_y - (m_1 d_1 + \cdots + m_n d_n) \in \mathbb{R}$, we obtain an isomorphism of graded algebras

$$\phi : \hat{B} = \mathbb{C}[\underline{x}, y, z]/\hat{I} = \bigoplus_{t\in\mathbb{R}} \hat{B}^t \xrightarrow{\sim} gr_{\mathcal{F}_d}B = \bigoplus_{t\in\mathbb{R}} F^t B/F^t_0 B,$$

where $\hat{I} = \left(\underline{x}^{[m]}z - y^r\right) \subset \mathbb{C}[\underline{x}, y, z]$ denotes the $\tilde{d}$-homogeneous ideal generated by the principal components of the polynomials in $I = \left(\underline{x}^{[r]}z - Q(\underline{x}, y)\right)$, and where $\hat{B}^t = \hat{B}^t = \mathbb{C}[\underline{x}, y, z]_t/\hat{I} \cap \mathbb{C}[\underline{x}, y, z]_t$ for every $t \in \mathbb{R}$.

2.13. It follows from (1) and (4) in Proposition 2.2, that the kernel of an associated homogeneous locally nilpotent derivations $gr\partial$ of $gr_{\mathcal{F}_d}B$ contains n algebraically independent irreducible homogeneous elements. To make the study of these derivations easier, we need to make the set of these irreducible homogeneous elements as small as possible. For this purpose, we consider weight degree functions $d : \mathbb{C}[\underline{x}, y] \rightarrow \mathbb{R}$ satisfying the following properties :

(1) The weight d_y of y is positive, and $\bar{Q}(\underline{x}, y) = y^r$.
(2) The real weights $d_i = d(x_i)$ and d_y are linearly independent over $\mathbb{Z}$.

According to 2.12 above, the first condition guarantees that the graded algebra $gr_{\mathcal{F}}B$ of the filtered algebra $(B, \mathcal{F}_d)$ is isomorphic to the quotient $\hat{B}$ of $\mathbb{C}[\underline{x}, y, z]$ by the $\tilde{d}$-homogeneous ideal $\hat{I} = \left(\underline{x}^{[m]}z - y^r\right)$. The second one is motivated by the following result.

Lemma 2.14. *Under the hypothesis above, every homogeneous element of $\hat{B}$ is the image by the natural morphism $\hat{\sigma} : \mathbb{C}[\underline{x}, y, z] \rightarrow \hat{B}$ of a unique monomial of $\mathbb{C}[\underline{x}, y, z]$ not divisible by $\underline{x}^{[m]}z$. In particular, every irreducible homogeneous element of $\hat{B}$ is the image of a variable of $\mathbb{C}[\underline{x}, y, z]$.*

Proof. Since $\hat{I} = \left(\underline{x}^{[m]}z - y^r\right)$, every nonzero homogeneous element of $\hat{B}$ is the image by $\hat{\sigma}$ of a unique homogeneous polynomial $\bar{p} \in \mathbb{C}[\underline{x}, y, z]$ whose monomials are not divisible by $\underline{x}^{[m]}z$. On the other hand, the hypothesis on d, together with the fact that $\tilde{d}(z) = 2d_y - (m_1d_1 + \ldots + m_nd_n)$ implies that if $\bar{p}$ contains a pair of monomials $\mu_1 \neq \mu_2$, then there exists $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$ such that $\mu_1\mu_2^{-1} = \lambda\left(\underline{x}^{[m]}zy^{-r}\right)^k$. If $k \neq 0$, then $\underline{x}^{[m]}z$ divides one of the μ_i, which is impossible. Thus $\bar{p}$ is a monomial. □

Proposition 2.15. *If* $[m] \in \mathbb{Z}^n_{>1}$ *then* $Ker\left(\hat{\partial}\right) = \mathbb{C}[\underline{x}]$ *for every associated homogeneous locally nilpotent derivation* $\hat{\partial}$ *on* $\hat{B}$. *Furthermore* $\deg_{\hat{\partial}}(\hat{\sigma}(z)) \geq 2$.

Proof. By virtue of (1) and (4) in Proposition 2.2, the kernel of $\hat{\partial}$ contains n algebraically independent irreducible homogeneous elements $\xi_1, \ldots, \xi_n$. So it follows from Lemma 2.14 above that the ξ_i's are the images by $\hat{\sigma}$ of n distinct variables of $\mathbb{C}[\underline{x}, y, z]$. These functions ξ_i, $i = 1, \ldots, n$, define a morphism $q : \hat{X} = \mathrm{Spec}\left(\hat{B}\right) \to \mathbb{C}^n$ which is invariant for the $\mathbb{C}_+$-action defined by $\hat{\partial}$. In particular, for a general point $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, the $\mathbb{C}_+$-action on $\hat{X}$ specializes to a nontrivial $\mathbb{C}_+$-action on the fiber $q^{-1}(\lambda)$. Suppose that one of the ξ_i's, say ξ_1, is the image of y. Then, depending on the other variables inducing the ξ_i's, $i = 2, \ldots, n$, we would obtain, for a general $\mu \in \mathbb{C}$, a nontrivial $\mathbb{C}_+$-action on one of the curves $C \subset \mathbb{C}^2$ with equations $x_{i_1}^{m_{i_1}} x_{i_2}^{m_{i_2}} - \mu = 0$ or $x_{i_1}^{m_{i_1}} z - \mu = 0$, which is absurd. Similarly, is ξ_1 is the image of z then, for a general $\mu \in \mathbb{C}$, the $\mathbb{C}_+$-action on $\hat{X}$ would specialize to a nontrivial action on the curve with equation $\mu x_i^{m_i} - y^r = 0$ for a certain $i = 1, \ldots, n$. This impossible as $r > 1$ and $m_i > 1$ for every $i = 1, \ldots, n$ by hypothesis. This proves that $\mathrm{Ker}\left(\hat{\partial}\right)$ contains $\mathbb{C}[\underline{x}]$. Thus $\hat{\partial}$ naturally extends to a locally nilpotent derivation of $\hat{B}_{\underline{x}} \simeq \mathbb{C}\left[\underline{x}, \underline{x}^{-1}, y\right]$. In turn, this implies that $\deg_{\hat{\partial}}(y) = 1$ and $\deg_{\hat{\partial}}(\hat{\sigma}(z)) \geq 2$ as $\hat{\sigma}(z) \in \hat{B}$ coincides with $x^{-[m]}y^r \in \hat{B}_{\underline{x}}$ via the canonical injection $\hat{B} \hookrightarrow \hat{B}_{\underline{x}}$. Therefore, the projection $pr_{\underline{x}} : \hat{X} \to \mathbb{C}^n$ coincides with the algebraic quotient morphism of the associated $\mathbb{C}_+$-action. This proves that $\mathrm{Ker}\left(\hat{\partial}\right) = \mathbb{C}[\underline{x}]$. □

The following result completes the proof of Theorem 2.8.

Corollary 2.16. *For every nontrivial locally nilpotent* ∂ *of* B, $Ker\left(\partial^2\right)$ *is contained in* $\mathbb{C}[\underline{x}, y]$.

Proof. Recall that $b \in \mathrm{Ker}\left(\partial^2\right)$ if and only if $\deg_\partial(b) \leq 1$. Since I is generated by the polynomial $\underline{x}^{[m]}z - Q(\underline{x}, y)$, every $b \in \mathrm{Ker}\left(\partial^2\right)$ is the restriction to $X_{[m],Q}$ of a unique polynomial $p \in \mathbb{C}[\underline{x}, y, z]$ whose monomials are not divisible by $\underline{x}^{[m]}z$. Suppose that $p \notin \mathbb{C}[\underline{x}, y]$. Then there exists a weight degree function d on $\mathbb{C}[\underline{x}, y, z]$ as in 2.13 for which the principal d-homogeneous component $\bar{p}$ belongs to $\mathbb{C}[\underline{x}, y, z] \setminus \mathbb{C}[\underline{x}, y]$. We deduce from Lemma 2.14 above that $\bar{p} = \underline{x}^{[\alpha]}y^\beta z^\gamma$, where $\gamma \geq 1$ and $\underline{x}^{[m]}z$ does not divide $\underline{x}^{[\alpha]}z^\gamma$. Letting $\hat{\partial} = gr\partial$ be the homogeneous locally nilpotent derivation of $\hat{B} = gr_{\mathcal{F}}B$ associated with ∂, we have $\deg_{\hat{\partial}}(\hat{\sigma}(\bar{p})) \geq \deg_{\hat{\partial}}(\hat{\sigma}(z))$ and so (see (2.1)), $\deg_\partial(b) \geq \deg_{\hat{\partial}}(\hat{\sigma}(z))$ as $\hat{\sigma}(\bar{p})$ coincides via the isomorphism ϕ of 2.12 with the image $gr(b) \in gr_{\mathcal{F}}B$ of b. This is absurd as $\deg_{\hat{\partial}}(\hat{\sigma}(z)) \geq 2$ by virtue of Lemma 2.15. □

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