The optional stopping theorem for quantum martingales

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Résumé

In classical probability theory, a random time T is a stopping time in a filtration $(\mathcal{F}_t)_{t\geq 0}$ if and only if the optional sampling holds at T for all bounded martingales. Futhermore, if a process $(X_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$, then X_T is $\mathcal{F}_T$-measurable. Unfortunatly, this is not the case in non commutative probability with the definition of stopped process used until now. It is shown in this article that we can define the stopping of non commutative processes in Fock space in such a way that all the bounded martingales can be stopped at any stopping time T, are adapted to the filtration of the past before T and satisfy the optional stopping theorem.

1 Introduction

Stopping times have been invented by Doob and are a basic feature of Probability Theory. They are very important in the martingale theory because of the property of optional stopping. In classical probability, if $(M_t)_{t\in\mathbb{R}^+}$ is a martingale in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$ and if S and T are $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-stopping times with $S \leq T$, then the optional stopping theorem says that under good conditions $E[X_T|\mathcal{F}_S] = X_S$.
Futhermore, F.B. Knight and B. Maisonneuve have proved in [12] that optional sampling gives a characterization of stopping times : if R is a random time in $\mathcal{F}_\infty$, the terminal element of a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ such that optional sampling holds at R for all bounded martingales, then R is a stopping time.
Optional sampling implies for example this useful result :

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A cadlag process X is a martingale if and only if for every bounded stopping time T, X_T is integrable and $E[X_T] = E[X_0]$.
So the stopped process $M^T = (M_{T\wedge t})_{t\in\mathbb{R}^+}$ is a $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$-martingale.
A quantum stopping time is a quantum random variable positive which satisfies some adaptedness property. Then the notions of the classical theory have a counterpart in the quantum context. The space of events anterior to a stopping time T is defined.
The stopping of non commutative processes is studied in [4, 7, 10] for example. But we will see below that these definitions don't allow a process stopped at time T to be adapted with respect to the space before T. The object of this article is to define the stopping of processes so that stopped processes are adapted with respect to the space before T and that the optional stopping theorem is satisfied.

2 Preliminaries

2.1 A review of classical definitions and results

In classical probability theory a stopping time T on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$ is a random variable such that for all $t \in \mathbb{R}^+$, $\{T \leq t\} \in \mathcal{F}_t$.
The σ-field of events anterior to T noted $\mathcal{F}_T$ is the set of events A such that for all $t \in \mathbb{R}^+$, $A \cap \{T \leq t\} \in \mathcal{F}_t$. Then T is $\mathcal{F}_T$-measurable.
If S and T are two stopping times on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$ with $S \leq T$ then $\mathcal{F}_S \subset \mathcal{F}_T$.
If $X = (X_t)_{t\geq 0}$ is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, then X_T which is defined on $\{\omega/T(\omega) < +\infty\}$ by $X_T(\omega) = X_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable on $\{\omega/T(\omega) < +\infty\}$.
With a stopping time T and a process X, we can associate the stopped process X^T defined by $X^T_t(\omega) = X_{t\wedge T}(\omega)$. If X is progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, X^T is progressively measurable with respect to the filtration $(\mathcal{F}_{t\wedge T})_{t\geq 0}$.
An uniformly integrable martingale X satisfies the Optional Stopping Theorem :
If $S \leq T$, then $X_S = E[X_T|\mathcal{F}_S]$ p.s..
This theorem implies the following proposition : A cadlag adapted process X is a martingale if and only if for every bounded stopping time T, X_T is in L^1 and

$$E[X_T] = E[X_0]$$

Then we have the corollary : If M is a martingale and T a stopping time, the stopped process M^T is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

2.2 Quantum stopping times

The notion of quantum stopping times has been studied by several authors : [1, 3, 8, 9, 4, 15],...
In all these papers we work with $(\mathcal{U}, (\mathcal{U}_t)_{t\in\mathbb{R}^+}, (M_t)_{t\in\mathbb{R}^+}, \Omega)$ where $\mathcal{U}$ is a Von Neumann algebra acting on an Hilbert space $\mathcal{H}$, $(\mathcal{U}_t)_{t\in\mathbb{R}^+}$ is an increasing sequence of Von Neumann subalgebras which generates $\mathcal{U}$, M_t is a ω-invariant conditional expectation from $\mathcal{U}$ to $\mathcal{U}_t$ where $\omega = \langle\Omega, .\Omega\rangle$, Ω being a cyclic element of $\mathcal{H}$. We denote $\mathcal{H}_t$ the closed subspace generated by $\{X\Omega,\ X \in \mathcal{U}_t\}$ and E_t the orthogonal projection on $\mathcal{H}_t$.
There are two fondamental examples of such family :

Example 1 : $(\mathcal{U}, (\mathcal{U}_t)_{t\in\mathbb{R}^+}, (M_t)_{t\in\mathbb{R}^+}, \Omega)$ is a non commutative base where Ω is separating. In this case $\mathcal{H}_t$ is noted $L^2(\mathcal{U}_t)$. This setup includes the Ito-Clifford (fermion) theory [5] and the quasi-free CAR and CCR theories [6].

Example 2 : The symmetric Fock space over $L^2(\mathbb{R}^+)$. We denote $\Phi = \Gamma(L^2(\mathbb{R}^+)$. If we define $\Phi_{t]} = \Gamma(L^2([0,t])$ and $\Phi_{[t} = \Gamma(L^2([t,+\infty[)$, we then have the well-known "continuous tensor product" property of Fock spaces :

$$\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$$

In the framework of quantum stochastic calculus [11] a bounded operator H on Φ is said to be adapted at time t if it is of the form $H = K \otimes I$ for some $K : \Phi_{t]} \to \Phi_{t]}$.
In this case $\mathcal{U} = \mathcal{B}(\Phi)$, $\mathcal{U}_t$ is the algebra of t-adapted bounded operator, E_t is the orthogonal projection on $\Phi_{t]}$, $M_t(X) = E_t X E_t \otimes I$ and Ω is the vacuum.

We consider now a non commutative space $(\mathcal{U}, (\mathcal{U}_t)_{t\in\mathbb{R}^+}, (M_t)_{t\in\mathbb{R}^+}, \Omega)$ and the Hilbert space $\mathcal{H}$ associated. A stopping time T is a (right continuous) spectral measure on $\mathbb{R}^+ \cup \{+\infty\}$ such that for all $t \in \mathbb{R}^+$, $T([0,t]) \in \mathcal{U}_t$.
In the following we adopt probabilistic-like notations : for every Borel subset $E \subset \mathbb{R}^+ \cup \{+\infty\}$ we write $1_{T\in E}$ instead of $T(E)$. In the same way $1_{T\leq t}$ means $T([0,t])$, $1_{T=t}$ means $T(\{t\})$,....
A point $t \in \mathbb{R}^+$ is a continuity point for T if $1_{T=t} = 0$. Note that as $\mathcal{H}$ is separable, then any stopping time T admits an at most countable set of points which are not of continuity for T.
A stopping time T is discrete if there exists a finite set $E = \{0 \leq t_1 < t_2 < \cdots < t_n \leq +\infty\}$ in $\mathbb{R}^+ \cup \{+\infty\}$ such that $1_{T\in E} = I$.
A sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ is said to converge to a stopping time T if $1_{T_n\leq t}$ strongly converges to $1_{T\leq t}$ for all continuity point t of T.

Definition 2.1- We say that $(E_n)_{n\geq 0}$ is a sequence of T-refining partitions of $\mathbb{R}^+$ if $E_n = \{0 = t_0^n < t_1^n < t_2 < \cdots < t_{N_n}^n < +\infty\}$ is a sequence of partition of $\mathbb{R}^+$ such that :

i) all the t_j^n are continuity points for T, $n \in \mathbb{N}$, $j \geq 1$;

ii) $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$;

iii) the diameter $\delta_n = \sup\{t_{i+1}^n - t_i^n \ ; \ i \in \mathbb{N}\}$ of E_n tends to 0 when n tends to $+\infty$;

iv) $\sup E_n$ tends to $+\infty$ as n tends to $+\infty$.

Then, if we define T_n by

$$\begin{cases} T_n(\{t_i^n\}) = T([t_{i-1}^n, t_i^n[) \\ T_n(+\infty) = T([t_{N_n}^n, +\infty]) \end{cases}$$

for all $n \geq 0$, T_n is a discrete stopping time, $T_0 \geq T_1 \geq \cdots \geq T_n \geq T$ and $(T_n)_{n\geq 0}$ converges to T.
Mimicking the definition of $\mathcal{F}_T$, we define the space of events anterior to T by

$$\mathcal{H}_T = \{f \in \mathcal{H} \ ; \ 1_{T\leq t} f \in \mathcal{H}_t \text{ for all } t \in \mathbb{R}^+\}$$

If $S \leq T$, then $\mathcal{H}_S \subset \mathcal{H}_T$.
We denote by E_T the orthogonal projection onto $\mathcal{H}_T$ which is a closed subspace of $\mathcal{H}$.

By using the preceding notations, we have :

$$E_T = \mathrm{s} - \lim \sum_{i=1}^{N_n} 1_{T\in[t_{i-1}^n, t_i^n[} E_{t_i} + 1_{T\geq t_{N_n}}$$

Questions : How to define X_T for a process of operators $X = (X_t)_{t\geq 0}$ with $X_t \in \mathcal{U}_t$?
Can we have $X_T \in \mathcal{U}$ and X_T adapted in the sense that X_T maps $\mathcal{H}_T$ in $\mathcal{H}_T$?
If $X \in \mathcal{U}$ and $(X_t)_{t\geq 0}$ is the associated martingale defined by $X_t = M_t(X)$, have we the Optional Stopping Theorem ?
Stopping process is studied in the two preceding examples.
In the first example, Ω is separating and we can identify X_t with $X_t\Omega$. So stopping $X = (X_t)_{t\geq 0}$ is done by stopping $x = (x_t)_{t\geq 0}$ where $x_t = X_t\Omega$. This is studied in [4, 7] for example.
If T is a discrete time $x_T = \sum_i 1_{T=t_i} x_{t_i}$ and in the general case x_T is the limit if it exists of $\sum_i 1_{T\in[t_{i-1},t_i[} x_{t_i} = \sum_i 1_{T\in[t_{i-1},t_i[} X_{t_i}\Omega$. So if T is discrete as Ω is separating, we have no choice to define X_T, $X_T = \sum_i 1_{T=t_i} X_{t_i}$ if we want $X_T \in \mathcal{U}$.
The second example was studied by [1, 3, 10, 13, 15]. In [15], the value at any stopping time of some Weyl processes in the Fock space is computed.
In [13] and [3], process of vectors are stopped with the preceding definition : x_T is the limit if it exists of $\sum_i 1_{T\in[t_{i-1},t_i[} x_{t_i}$. They show that we can stop a large class of processes of vectors, in particular they stop processes of semi-martingales.
The way of stopping process defined in these articles is one the following in the case of discrete time :

left-stopping : $T \circ X = \sum_i 1_{T=t_i} X_{t_i}$

right-stopping : $X \circ T = \sum_i X_{t_i} 1_{T=t_i}$

Note that, since $(T\circ X)^* = X^* \circ T$ for discrete time, the study of one case is sufficient.
Then it's necessary to establish for some process X, the existence of the limit of $T_n \circ X$ when T_n is defined in definition 2.1. This was done for Weyl process.
So in the two examples, left-stopping is studied.

But in this case, we do not have in general the "T-adaptedness property", namely X_T does not map $\mathcal{H}_T$ in $\mathcal{H}_T$. This property implies that $X_T E_T = E_T X_T E_T$ and this is not true as we can see below.
Let T be a discrete stopping time which takes two values $t_1 < t_2$. Let $P = 1_{T=t_1}$ and so $1_{T=t_2} = I - P$.
We have $X_T = PX_{t_1} + (I-P)X_{t_2}$ and $E_T = PE_{t_1} + (I-P)E_{t_2}$.
So $X_T E_T = E_T X_T E_T$ if and only if $PX_{t_1}(I-P)E_{t_2} = PX_{t_1}(I-P)E_{t_1}$.
If we require this for all process and all stopping time this implies that for all t, for all $X_t \in \mathcal{U}_t$, for all projection P in $\mathcal{U}_t$, $(PX_t - PX_tP) = (PX_t - PX_tP)E_t$.
This relation has no reason in general to be satisfied unless $\mathcal{U}_t$ is commutative or equal to $\mathcal{U}$.

In the Fock space case, this relation implies that $PX_t - PX_tP = 0$ and so X_t is a multiple of Identity.

Look at now the martingale's properties.
We define for X in $\mathcal{U}$ the martingale $(X_t)_{t\geq 0}$ by $X_t = M_t(X)$. Let T be a discrete stopping time.
In the first example, the conditions $X_TE_T = E_TXE_T$ and $X_T \in \mathcal{U}$ implies that $X_T\Omega = E_TX\Omega = \sum_i 1_{T=t_i}E_{t_i}(X\Omega) = \sum_i 1_{T=t_i}M_{t_i}(X)\Omega$. So X_T is obtained by left stopping and is not adapted to the past before T.
In the second example for left stopping, the proof of non-adaptness shows that this martingale's property is not satisfied in general.
Anothers ways of stopping can be proposed in the Fock's space case :
Double-stopping : $T \circ X \circ T = \sum_i 1_{T=t_i}X_{t_i}1_{T=t_i}$
In this case X_T is adapted but doesn't satisfy the martingale's property.

Strong tensor product : Parthasarathy and Sinha in [15] have proved the strong factorisability of Φ. For a finite stopping time T, they showed that Φ is canonically isomorphic to the tensor product of Φ_T and a "post-T" Hilbert space Φ^T. This space is the image of Φ under an isometry U^T. If T is discrete, U^T is given by $\sum_i 1_{T=t_i}\Gamma(\theta_{t_i})$ where θ_t is the right shift operator on $L^2(\mathbb{R}_+)$ and $\Gamma(\theta_t)$ is its second quantisation, so is an isometry.
There exists an unique unitary isomorphism J_T from $\Phi_T \otimes \Phi^T$ onto Φ such that :

$$J_T\left(\sum_i 1_{T=t_i}E_{t_i}x \;\otimes\; \sum_i 1_{T=t_i}\Gamma(\theta_{t_i})y\right) = \sum_i 1_{T=t_i}E_{t_i}x \otimes_{t_i} \Gamma(\theta_{t_i})y$$

So for $X \in \mathcal{B}(\Phi)$, we can define $M_T(X)$ by $J_T \circ (E_TXE_T \otimes Id) \circ J_T^{-1}$.
With this definition, $M_T(X)$ is adapted to the strong tensor product. If $X_t = M_t(X)$, we can define X_T by $X_T = M_T(X)$. So clearly, a part of the optional stopping theorem is satisfied.
But unfortunalty, some others important properties are not satisfied.
For example, the process $(X_{T\wedge t})_{t\geq 0}$ is not adapted as we can see below :
Let T takes two values $t_1 < t_2$ and let $t > t_2$. then $T \wedge t = T$ and $X_{T\wedge t} = X_T$.
Let $g \in L^2(\mathbb{R}^+)$. We have that :

$$U^Te(g) = 1_{T=t_1}1 \otimes_{t_1} \Gamma(\theta_{t_1})e(g) + 1_{T=t_2}1 \otimes_{t_2} \Gamma(\theta_{t_2})e(g)$$

So

$$\begin{aligned} E_tU^Te(g) &= 1_{T=t_1}1 \otimes_{t_1} \Gamma(\theta_{t_1})e(g1_{[0,t-t_1]}) + 1_{T=t_2}1 \otimes_{t_2} \Gamma(\theta_{t_2})e(g1_{[0,t-t_2]}) \\ &= J_T((1_{T=t_1}1) \otimes_T U^Te(g1_{[0,t-t_1]})) + J_T((1_{T=t_2}1) \otimes_T U^Te(g1_{[0,t-t_2]})) \end{aligned}$$

And

$$\begin{aligned} X_TE_tU^Te(g) &= J_T((E_TX1_{T=t_1}1) \otimes_T U^Te(g1_{[0,t-t_1]})) + J_T((E_TX1_{T=t_2}1) \otimes_T U^Te(g1_{[0,t-t_2]})) \\ &= 1_{T=t_1}E_{t_1}X1_{T=t_1}1 \otimes_{t_1} \Gamma(\theta_{t_1})e(g1_{[0,t-t_1]}) \\ &\quad +1_{T=t_2}E_{t_2}X1_{T=t_1}1 \otimes_{t_2} \Gamma(\theta_{t_2})e(g1_{[0,t-t_1]}) \\ &\quad +1_{T=t_1}E_{t_1}X1_{T=t_2}1 \otimes_{t_1} \Gamma(\theta_{t_1})e(g1_{[0,t-t_2]}) \\ &\quad +1_{T=t_2}E_{t_2}X1_{T=t_2}1 \otimes_{t_2} \Gamma(\theta_{t_2})e(g1_{[0,t-t_2]}) \end{aligned}$$

So

$$(E_t - I_d) X_T E_t U^T e(g) = 1_{T=t_2} E_{t_2} X 1_{T=t_1} 1 \otimes_{t_2} e(g(.-t_2) 1_{[t_2,t]}) \otimes_t (1 - e(g(.-t_2) 1_{[t,t+t_2-t_1]})) \neq 0$$

So is it possible to find a definition for X_T for which we have "good properties" ?
Is it possible to construct some algebra $\mathcal{U}_T \subset \mathcal{B}(\mathcal{H})$ for all stopping times T such that if $X \in \mathcal{U}_T$, $XE_T = E_T X E_T$ and if $S \leq T$, $\mathcal{U}_S \subset \mathcal{U}_T$ and if T is constant equal to t, $\mathcal{U}_T = \mathcal{U}_t$.

Proposition 2.2 : This is impossible in the Fock space.

Proof : Let $S = s$ and $X \in \mathcal{U}_s$.
Let P be an orthogonal projection in $\mathcal{U}_s$ and let T defined by $1_{T=s} = P$ and $1_{T=+\infty} = I - P$.
So $S \leq T$ and $E_T = PE_s + (I - P)$.
Suppose that X satifies $XE_T = E_T X E_T$.
So $XPE_s + X(I-P) = PE_s X E_s P + PE_s X(I-P) + (I-P)XPE_s + (I-P)X(I-P)$.
It follows that $E_s(PX - PXP) = PX - PXP$. But $PX - PXP \in \mathcal{U}_s$, so in the Fock space this implies that $PX - PXP = 0$ and so X is a multiple of identity.

We will show in the next section that it is possible to stop a martingale such that the optional stopping theorem is satisfied, but now we are going to recall some facts and definitions about Fock space.

2.3 Background material on the Fock space

For any complex separable Hilbert space h, we denote by $\Gamma(h)$ the boson Fock space over h.
We write $\Phi = \Gamma(L^2(\mathbb{R}_+))$.
We denote for f in $L^2(\mathbb{R}_+)$ by $e(f)$ the associated coherent or exponential vector in Φ ; the exponential domain is denoted $\mathcal{E}$ (see [14] for more details). Recall that $e(0)$ is the vacuum vector in Φ. We denote it $\mathbf{1}$.
We denote $\Phi_{t]} = \Gamma\big(L^2([0,t])\big)$, $\Phi_{[s,t]} = \Gamma\big(L^2([s,t])\big)$ and $\Phi_{[t} = \Gamma\big(L^2([t,+\infty[)\big)$.
For all $f \in L^2(\mathbb{R}_+)$, let for all $s \leq t$,

$$\begin{cases} f_{t]} = f\mathbf{1}_{[0,t]} \\ f_{[t} = f\mathbf{1}_{[t,+\infty[} \\ f_{[s,t]} = f\mathbf{1}_{[s,t]} \end{cases}$$

We have the well known "continuous tensor product" structure

$$\Phi \simeq \Phi_{t]} \otimes \Phi_{[t} \ .$$

The annihilation, creation and conservation operators are defined for f, g in $L^2(\mathbb{R}_+)$ and $T \in \mathcal{B}(L^2(\mathbb{R}_+))$, the algebra of all bounded operators on $L^2(\mathbb{R}_+)$ on the domain $\mathcal{E}$, by the relations

$$\begin{cases} a^-(f)e(g) & = \langle f, g\rangle e(g) \\ a^+(f)e(g) & = \frac{d}{d\lambda} e(g + \lambda f)\big|_{\lambda=0} \\ \lambda(T)e(g) & = \frac{d}{d\lambda} e(e^{\lambda T} g)\big|_{\lambda=0} \end{cases}$$

the derivations being understood in the strong sense.

The operators $a^-(f)$ and $a^+(f)$ are adjoint to each others on $\mathcal{E}$. If T^* is the adjoint of T then $\lambda(T^*)$ and $\lambda(T)$ are adjoint to each other on $\mathcal{E}$.
If $f = 1_{[0,t]}$ and if T is the operator of multiplication by f, then $a^-(f)$, $a^+(f)$ and $\lambda(T)$ are respectively denoted by a_t^-, a_t^+ and a_t^0. We put $a_t^\times = tI$.

A family of operators $(X_t)_{t\geq 0}$ defined on $\mathcal{E}$ is called an *adapted process* if the following conditions are fulfilled :

i) for all $t > 0$, $X_t\big(e(f_{t]})\big) \in \Phi_{t]}$ and $X_t(e(f)) = X_t\big(e(f_{t]})\big) \otimes e(f1_{[t})$ in $\Phi_{t]} \otimes \Phi_{[t}$;
ii) for all $f \in L^2(\mathbb{R}_+)$, the map $L^2(\mathbb{R}_+)_+ \to \Phi : t \mapsto X_t(e(f))$ is strongly measurable.

Let us now recall some elements of the Hudson-Parthasarathy's quantum stochastic calculus ([11, 14]).
Let $(H_t^\varepsilon)_{t\geq 0}$, $\varepsilon \in \{-,+,\times,0\}$ be adapted processes such that for all $f \in L^2(\mathbb{R}_+)$ and for all $t > 0$

$$(2.3) \qquad \int_0^t \Big\{|f(s)|^2 \|H_s^0(e(f))\|^2 + \|H_s^+(e(f))\|^2 + \|H_s^\times(e(f))\| + |f(s)|\ \|H_s^-(e(f))\|\Big\}\, ds < +\infty\ .$$

Then the stochastic integral $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ is defined as the unique adapted process satisfying the relation :

$$(2.4) \qquad \langle e(f), T_t(e(g))\rangle = \int_0^t \langle e(f), \big\{H_s^\times(e(g)) + g(s)H_s^-(e(g)) + \overline{f(s)}H_s^+(e(g)) + \overline{f(s)}g(s)H_s^0(e(g))\big\}\rangle ds\ .$$

Let E_t be, for $t \geq 0$, the orthogonal projection onto $\Phi_{t]}$.
We now recall the theorem giving the Ito formula for composition of processes ([11]). Let $I_t = \sum_\epsilon \int_0^t H_s^\epsilon da_s^\epsilon$ and $I'_t = \sum_{\epsilon'} \int_0^t K_s^{\epsilon'} da_s^{\epsilon'}$ be two stochastics integrals satisfying (2.3), ϵ, ϵ' running over $\{+,0,-,\times\}$.
Then if for all t, I_t and I'_t are bounded

$$I_t I'_t = \sum_\epsilon \int_0^t H_s^\epsilon I'_s da_s^\epsilon + \sum_{\epsilon'} \int_0^t I_s K_s^{\epsilon'} da_s^{\epsilon'} + \sum_{\epsilon,\epsilon'} \int_0^t H_s^\epsilon K_s^{\epsilon'} da_s^{\epsilon.\epsilon'}$$

where $da_s^{\epsilon.\epsilon'}$ is given by

$$\begin{cases} da^{0.0} = da^0 \\ da^{-.0} = da^- \\ da^{0.+} = da^+ \\ da^{-.+} = dt \end{cases}$$

the other products being equal to 0.

We need some others integral representations for elements of Φ.
Let $(\chi_t)_{t\geq 0}$ be the process of elements of Φ which satisfies :

i) For all $t \geq 0$, $\chi_t \in \Phi_{t]}$.

ii) For all $t \geq 0$, $f \in L^2(\mathbb{R}_+)$, $\langle \chi_t, e(f) \rangle = \int_0^t f(s)ds$.

If $(g_t)_{t\in\mathbb{R}^+}$ is a family of elements of Φ such that

a) $g_t \in \Phi_{t]}$ for all t

b) $t \mapsto g_t$ is measurable

c) $\int_0^\infty \|g_t\|^2 \, dt < \infty$

then $(g_t)_{t\in\mathbb{R}^+}$ is said to be *Ito-integrable*. In this case we write $\int_0^\infty g_t \, d\chi_t$ for the element h of Φ given for all $f \in L^2(\mathbb{R}_+)$ by $\langle h, e(f) \rangle = \int_0^{+\infty} f(t) \langle g_t, e(f_{t]}) \rangle dt$.
This element h of Φ is called the *Ito integral* of $(g_t)_{t\in\mathbb{R}^+}$ and we have $\|h\|^2 = \int_0^\infty \|g_t\|^2 \, dt$.
Futhermore for all $f \in L^2(\mathbb{R}_+)$,

$$e(f) = \mathbf{1} + \int_0^{+\infty} f(t) e(f_{t]}) d\chi_t$$

The following theorem is proved in [3], Proposition 6.

Theorem 2.5 : Let T be a stopping time on Φ. Then for all $f \in L^2(\mathbb{R}_+)$ we have

$$E_T e(f) = \mathbf{1} + \int_0^{+\infty} f(s) 1_{T>s} e(f_{s]}) d\chi_s$$

3 A new definition of stopping processes in the Fock space

We start as before with $X \in \mathcal{U}$ and $(X_t)_{t\geq 0}$ the associated martingale.
We want X_T T-adapted so it has to satisfy $X_T E_T = E_T X E_T$. Namely if T is discrete :

$$\begin{aligned} X_T E_T &= \sum_{i,j} 1_{T=t_i} E_{t_i} X 1_{T=t_j} E_{t_j} \\ &= \sum_{i,j} 1_{T=t_i} E_{t_i} X_{t_i \vee t_j} 1_{T=t_j} E_{t_j} \\ &= \sum_{i,j} E_{t_i} (1_{T=t_i} X_{t_i \vee t_j} 1_{T=t_j}) E_{t_j} \end{aligned}$$

So by analogy with the definition of M_t, we put

$$M_T(X) = \sum_{i,j} E_{t_i} (1_{T=t_i} X_{t_i \vee t_j} 1_{T=t_j}) E_{t_j} \otimes_{t_i \vee t_j} I = \sum_{i,j} M_{t_i \vee t_j} (E_{t_i} 1_{T=t_i} X_{t_i \vee t_j} 1_{T=t_j} E_{t_j})$$

Denote for $s \leq t$, $E_{s,t} = M_t(E_s)$, so $E_{s,s} = I$ and if $s \leq t \leq u$, $E_{s,t} E_{t,u} = E_{s,u}$ and $E_{s,u} E_t = E_s$.

Definitions 3.1 :

1. Let $X \in \mathcal{U}$ and T a discrete stopping time. We define $M_T(X)$ by

$$M_T(X) = \sum_{i,j} E_{t_i, t_i \vee t_j} 1_{T=t_i} X_{t_i \vee t_j} 1_{T=t_j} E_{t_j, t_i \vee t_j}$$

   where $(X_t)_{t\geq 0}$ is the martingale associated with X, namely $X_t = M_t(X)$.

2. Let $(X_t)_{t\geq 0}$ be a process of bounded operators and T a discrete stopping time. We define X_T by :

$$X_T = \sum_{i,j} E_{t_i,t_i\vee t_j} 1_{T=t_i} X_{t_i\vee t_j} 1_{T=t_j} E_{t_j,t_i\vee t_j}$$

Remark : This definition is not valid in example 1 of a non commutative base with a separating element because E_t doesn't belong to $\mathcal{U}$ ($E_t\Omega = \Omega$ and $E_t \neq I$). So we can't define $M_t(E_s)$. We use here the factorisation property of the Fock space.

Properties 3.2 :

1. This stopping define T-adapted process in the sense that it satisfies $X_T E_T = E_T X_T E_T$.
2. The process of operators $(X_{T\wedge t})_{t\geq 0}$ is adapted in the original sense and $1_{T\leq t} X_T 1_{T\leq t} = 1_{T\leq t} X_{T\wedge t} 1_{T\leq t}$.
3. If $(X_t)_{t\geq 0}$ is a martingale, then $X_{T\wedge t} = M_t(X_T)$ and so $X^T = (X_{T\wedge t})_{t\geq 0}$ is a martingale.

Proofs :

1.
$$\begin{aligned} X_T E_T &= \sum_{i,j,k} 1_{T=t_i} E_{t_i,t_i\vee t_j} X_{t_i\vee t_j} E_{t_j,t_i\vee t_j} 1_{T=t_j} 1_{T=t_k} E_{t_k} \\ &= \sum_{i,j} 1_{T=t_i} E_{t_i,t_i\vee t_j} X_{t_i\vee t_j} E_{t_j,t_i\vee t_j} 1_{T=t_j} E_{t_j} \\ &= E_T X_T E_T \end{aligned}$$

   We have used the fact that

$$\begin{aligned} & E_{t_i,t_i\vee t_j} X_{t_i\vee t_j} E_{t_j} \\ &= E_{t_i,t_i\vee t_j} X_{t_i\vee t_j} E_{t_i\vee t_j} E_{t_j} \\ &= E_{t_i,t_i\vee t_j} E_{t_i\vee t_j} X_{t_i\vee t_j} E_{t_j} \\ &= E_{t_i} X_{t_i\vee t_j} E_{t_j} \end{aligned}$$

   By the same calculus, we prove that $E_T X_T = E_T X_T E_T$.

2. We see easily that if $T \leq t$, X_T is t-adapted because $M_s(X)$ is t-adapted if $s \leq t$ for any process X. If T takes the values $\{t_i\}$, $T \wedge t$ takes the values $\{t_i / t_i \leq t\}$ and the value t with $1_{T\wedge t = t} = 1_{T>t}$, so

$$\begin{aligned} X_{T\wedge t} &= 1_{T\leq t} X_T 1_{T\leq t} + 1_{T>t} X_t \sum_{j/t_j\leq t} 1_{T=t_j} E_{t_j,t} \\ &+ \sum_{i/t_i\leq t} 1_{T=t_i} E_{t_i,t} X_t 1_{T>t} + 1_{T>t} X_t 1_{T>t} \\ &= 1_{T\leq t} X_T 1_{T\leq t} + 1_{T>t} X_t M_t(E_T) 1_{T\leq t} \\ &+ M_t(E_T) 1_{T\leq t} X_t 1_{T>t} + 1_{T>t} X_t 1_{T>t} \end{aligned}$$

   This implies as $1_{T\leq t} X_T 1_{T\leq t} = 1_{T\leq t} X_{T\wedge t} 1_{T\leq t}$ is t-adapted, that $X_{T\wedge t}$ is t-adapted.

3. By using the preceding formula, we see that

$$X_{T\wedge t} E_t = 1_{T\leq t} E_t X_T E_t 1_{T\leq t} + 1_{T>t} X_t E_T 1_{T\leq t} + E_T 1_{T\leq t} X_t 1_{T>t} + 1_{T>t} X_t 1_{T>t} E_t$$

But as $(X_t)_{t\geq 0}$ is a martingale

$$\begin{aligned}
E_tX_TE_t &= 1_{T\leq t}E_tX_TE_t1_{T\leq t} + \sum_{(i,j)/t_i\leq t, t_j>t} E_t1_{T=t_i}E_{t_i,t_j}X_{t_j}1_{T=t_j}E_t \\
&+ \sum_{(i,j)/t_i>t, t_j\leq t} E_t1_{T=t_i}X_{t_i}1_{T=t_j}E_{t_j,t_i}E_t \\
&+ \sum_{(i,j)/t_i>t, t_j>t} E_t1_{T=t_i}E_{t_i,t_i\vee t_j}X_{t_i\vee t_j}1_{T=t_j}E_{t_j,t_i\vee t_j}E_t \\
&= 1_{T\leq t}E_tX_TE_t1_{T\leq t} + 1_{T\leq t}E_T\sum_{j/t_j>t}E_{t_j}X_{t_j}E_{t_j}1_{T=t_j}E_t \\
&+E_t\sum_{i/t_i>t}1_{T=t_i}E_{t_i}X_{t_i}E_{t_i}E_T1_{T\leq t} + E_t\sum_{(i,j)/t_i>t,t_j>t}1_{T=t_i}E_{t_i}X_{t_i\vee t_j}1_{T=t_j}E_{t_j}E_t \\
&= 1_{T\leq t}E_tX_TE_t1_{T\leq t} + 1_{T\leq t}E_TX\sum_{j/t_j>t}E_{t_j}1_{T=t_j}E_t \\
&+E_t\sum_{i/t_i>t}1_{T=t_i}E_{t_i}XE_T1_{T\leq t} + E_t\sum_{(i,j)/t_i>t,t_j>t}1_{T=t_i}E_{t_i}X1_{T=t_j}E_{t_j}E_t \\
&= 1_{T\leq t}E_tX_TE_t1_{T\leq t} + 1_{T\leq t}E_TXE_t1_{T>t} + 1_{T>t}E_tXE_T1_{T\leq t} + E_t1_{T>t}X1_{T>t}E_t \\
&= 1_{T\leq t}E_tX_TE_t1_{T\leq t} + 1_{T\leq t}E_TX_tE_t1_{T>t} + 1_{T>t}E_tX_tE_T1_{T\leq t} + E_t1_{T>t}X_t1_{T>t}E_t
\end{aligned}$$

So $X_{T\wedge t}E_t = E_tX_TE_t$ and thus $X_{T\wedge t}E_t = M_t(X_T)E_t$ and by t-adaptation, $X_{T\wedge t} = M_t(X_T)$.

Now we have to extend the definition for non discrete stopping times.

Definition 3.3 : Let $(X_t)_{t\geq 0}$ be an adapted process of bounded operators.
Let T be a stopping time.
If for all $(E_n)_{n\geq 1}$ sequence of refining T-partitions of $\mathbb{R}^+$, for all $F \in \mathcal{E}$, $X_{T_n}F$ converges then we denote the limit X_TF and called X_T the stopped process of $(X_t)_{t\geq 0}$ by T.

Remark : $(\varphi(t)I)_{t\geq 0}$ is stoppable by T and the stopped process is given by $\varphi(T)$.

Theorem 3.4 : Let $(X_t)_{t\geq 0}$ be a martingale with a final value $X \in \mathcal{U}$.
Then this martingale admits a stopped process X_T which satisfies on $\mathcal{E}$:

$$\begin{aligned}
X_T &= E_TXE_T + \int_0^{+\infty} 1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}\, da_s^0 \\
&= E_TXE_T + \int_0^{+\infty} 1_{T\leq s}M_s(E_TXE_T)1_{T\leq s}\, da_s^0
\end{aligned}$$

Futhermore, X_T is bounded and $\|X_T\| \leq \|X\|$.

For the proof of this result we need several lemmata.

Lemma 3.5 : Let $(X_t)_{t\geq 0}$ be a martingale with a final value $X \in \mathcal{U}$.
For all $t \geq 0$, we have on $\mathcal{E}$:

$$X_t = E_tXE_t + \int_t^{+\infty} E_{t,s}X_sE_{t,s}\, da_s^0$$

Remark : This lemma is the version of the theorem for $T = t$.

Proof of lemma 3.5 : Let f and g be in $L^2(\mathbb{R}_+)$. By using formula 2.4

$$\begin{aligned}
&\langle e(f), (E_t X E_t + \textstyle\int_t^{+\infty} E_{t,s} X_s E_{t,s}\ da_s^0) e(g)\rangle \\
&= \langle E_t e(f), X E_t e(f)\rangle + \textstyle\int_t^{+\infty} \overline{f}(s) g(s) \langle E_t e(f), X_s E_t e(g)\rangle \langle e(f_{[s}), e(g_{[s})\rangle ds \\
&= \langle E_t e(f), X E_t e(f)\rangle (1 + \textstyle\int_t^{+\infty} \overline{f}(s) g(s) \langle e(f_{[s}), e(g_{[s})\rangle ds \\
&= \langle E_t e(f), X E_t e(f)\rangle e^{\int_t^{+\infty} \overline{f} g} \\
&= \langle e(f), X_t e(g)\rangle
\end{aligned}$$

We have used the fact that $E_t X_s E_t = X_t E_t$ if $s \geq t$.

Lemma 3.6 : Let $(H_t)_{t\geq 0}$ be an adapted process of bounded operators such that $\sup \|H_s\| < +\infty$ and $X_s \in \mathcal{U}_s$, then if $t > s$ and $\int_s^t H_\tau da_\tau^0$ bounded,

$$\int_s^t H_\tau\ da_\tau^0\ X_s = \int_s^t H_\tau X_s\ da_\tau^0$$

$$X_s \int_s^t H_\tau\ da_\tau^0 = \int_s^t X_s H_\tau\ da_\tau^0$$

$$X_s \int_s^t H_\tau\ da_\tau^0\ X_s = \int_s^t X_s H_\tau X_s\ da_\tau^0$$

Proof of lemma 3.6 : Let $x_s = \sum_i \lambda_i e(f_{i,s]})$ and $g \in L^2(\mathbb{R}_+)$.

$$\begin{aligned}
\langle e(f), \left(\textstyle\int_s^t H_\tau da_\tau^0\right) x_s \otimes e(g_{[s})\rangle &= \sum_i \lambda_i \int_s^t \overline{f}(\tau) g(\tau) \langle e(f), H_\tau(e(f_{i,s]}) \otimes e(g_{[s}))\rangle d\tau \\
&= \textstyle\int_s^t \overline{f}(\tau) g(\tau) \langle e(f), H_\tau(x_s \otimes e(g_{[s}))\rangle d\tau
\end{aligned}$$

If $x_s \in \Phi_{s]}$, it exists $x_s^{(n)} \in \mathcal{E} \cap \Phi_{s]}$ such that $x_s^{(n)}$ converges to x_s. As $\int_s^t H_\tau da_\tau^0$ and H_τ are bounded,

$$\langle e(f), \left(\int_s^t H_\tau da_\tau^0\right) x_s \otimes e(g_{[s})\rangle = \int_s^t \overline{f}(\tau) g(\tau) \langle e(f), H_\tau(x_s \otimes e(g_{[s}))\rangle d\tau$$

But X_s is s-adapted so if $x_s = X_s e(g_{s]})$, we have

$$\begin{aligned}
&\langle e(f), \left(\textstyle\int_s^t H_\tau da_\tau^0\right) X_s e(g)\rangle = \langle e(f), \left(\textstyle\int_s^t H_\tau da_\tau^0\right) x_s \otimes e(g_{[s})\rangle \\
&= \textstyle\int_s^t \overline{f}(\tau) g(\tau) \langle e(f), H_\tau(x_s \otimes e(g_{[s}))\rangle d\tau = \textstyle\int_s^t \overline{f}(\tau) g(\tau) < e(f), H_\tau X_s e(g) > d\tau \\
&= \langle e(f), \left(\textstyle\int_s^t H_\tau X_s da_\tau^0\right) e(g)\rangle
\end{aligned}$$

So we have proved that $\int_s^t H_\tau da_\tau^0 X_s = \int_s^t H_\tau X_s da_\tau^0$.
The others formulae are proved by similar methods.

Lemma 3.7 : Let $F = \sum_i \lambda_i e(f_i)$ be an element of $\mathcal{E}$. Then

$$\int_0^{+\infty} \|1_{T\leq s} M_s(E_T)\left(\sum_i f_i(s)\lambda_i e(f_i)\right)\|^2 ds = \|F\|^2 - \|E_T F\|^2$$

Proof of lemma 3.7 :

$$\begin{aligned}&\|1_{T\leq s} M_s(E_T)(\sum_i f_i(s)\lambda_i e(f_i))\|^2 \\ &\quad = \|M_s(E_T)(\sum_i f_i(s)\lambda_i e(f_i))\|^2 - \|1_{T>s} M_s(E_T)(\sum_i f_i(s)\lambda_i e(f_i))\|^2\end{aligned}$$

but $1_{T>s} M_s(E_T) = 1_{T>s}$ and the theorem 2.5 implies that

$$\langle M_s(E_T)e(f_i), M_s(E_T)e(f_j)\rangle = e^{\int_s^{+\infty} \overline{f}_i f_j}\left[1 + \int_0^s \overline{f}_i f_j \langle 1_{T>u} e(f_{iu]}), e(f_{ju]})\rangle du\right]$$

and so

$$\begin{aligned}&\textstyle\int_0^{+\infty} \|M_s(E_T)\left(\sum_i f_i(s)\lambda_i e(f_i)\right)\|^2 ds \\ &= \sum_{i,j} \overline{\lambda}_i \lambda_j \int_0^{+\infty}\left(\overline{f}_i(s) f_j(s) e^{\int_s^{+\infty} \overline{f}_i f_j}\left[1 + \int_0^s \overline{f}_i f_j \langle 1_{T>u} e(f_{iu]}), e(f_{ju]})\rangle du\right]\right) ds\end{aligned}$$

By an integration by parts, we obtain

$$\int_0^{+\infty} \|M_s(E_T)\left(\sum_i f_i(s)\lambda_i e(f_i)\right)\|^2 ds = \int_0^{+\infty} \|1_{T>s}\left(\sum_i f_i(s)\lambda_i e(f_i)\right)\|^2 ds - \|E_T F\|^2 + \|F\|^2$$

Proof of theorem 3.4 : Let $(E_n)_{n\geq 0}$ be a sequence of refining T-partitions of $\mathbb{R}^+$ as defined in 2.1, we have by using lemma 3.5 and 3.6 :

$$\begin{aligned}X_{T_n} &= \sum_{i,j} 1_{T_n=t_i^n} E_{t_i^n, t_i^n\vee t_j^n} X_{t_i^n\vee t_j^n} E_{t_j^n, t_i^n\vee t_j^n} 1_{T_n=t_j^n} \\ &= \sum_{i,j} 1_{T_n=t_i^n} E_{t_i^n} X E_{t_j^n} 1_{T_n=t_j^n} \\ &+ \sum_{i,j} 1_{T_n=t_i^n} E_{t_i^n, t_i^n\vee t_j^n}\left(\int_{t_i^n\vee t_j^n}^{+\infty} E_{t_i^n\vee t_j^n, s} X_s E_{t_i^n\vee t_j^n, s} da_s^0\right) E_{t_j^n, t_i^n\vee t_j^n} 1_{T_n=t_j^n} \\ &= E_{T_n} X E_{T_n} + \sum_{i,j}\int_{t_i^n\vee t_j^n}^{+\infty} 1_{T_n=t_i^n} E_{t_i^n, t_i^n\vee t_j^n} E_{t_i^n\vee t_j^n, s} X_s E_{t_i^n\vee t_j^n, s} E_{t_j^n, t_i^n\vee t_j^n} 1_{T_n=t_j^n} da_s^0 \\ &= E_{T_n} X E_{T_n} + \textstyle\int_0^{+\infty} 1_{T_n\leq s} M_s(E_{T_n}) X_s M_s(E_{T_n}) 1_{T_n\leq s}\, da_s^0\end{aligned}$$

$E_{T_n} X E_{T_n}$ converges strongly to $E_T X E_T$.
Let f be in $L^2(\mathbb{R}_+)$. Denote

$$A_n = \|(\int_0^{+\infty} 1_{T_n\leq s} M_s(E_{T_n}) X_s M_s(E_{T_n}) 1_{T_n\leq s}\, da_s^0 - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s}\, da_s^0) e(f)\|^2$$

$$\begin{array}{ll} A_n & \leq 2\| \int_0^{+\infty}(1_{T_n\leq s}M_s(E_{T_n})-1_{T\leq s}M_s(E_T))X_sM_s(E_T)1_{T\leq s}da^0_s\ e(f)\|^2 \\ & +2\| \int_0^{+\infty} 1_{T_n\leq s}M_s(E_{T_n})X_s(1_{T_n\leq s}M_s(E_{T_n})-1_{T\leq s}M_s(E_T))da^0_s\ e(f)\|^2 \end{array}$$

So by [14] p 134, formula (7.8), it exists a constant $c(f)$ such that

$$\begin{array}{ll} A_n & \leq c(f)\Big\{ \int_0^{+\infty}|f(s)|^2\|X_s\|^2\|(1_{T_n\leq s}M_s(E_{T_n})-1_{T\leq s}M_s(E_T))e(f)\|^2ds \\ & + \int_0^{+\infty}|f(s)|^2\|(1_{T_n\leq s}M_s(E_{T_n})-1_{T\leq s}M_s(E_T))X_s1_{T\leq s}M_s(E_T)e(f)\|^2ds\Big\} \end{array}$$

The adaptation of the operators implies that :

$$\begin{array}{ll} A_n & \leq c(f)\int_0^{+\infty}|f(s)|^2\left(\|E_{T_n\wedge s}e(f)-E_{T\wedge s}e(f)\|^2+\|(1_{T_n\leq s}-1_{T\leq s})E_se(f)\|^2\right. \\ & \left.+\|(E_{T_n\wedge s}-E_{T\wedge s})X_s1_{T\leq s}E_{T\wedge s}e(f)\|^2+\|(1_{T_n\leq s}-1_{T\leq s})(X_s1_{T\leq s}E_{T\wedge s}e(f)\|^2\right)ds \end{array}$$

But for almost all $s\in\mathbb{R}^+$, all these quantities converge to 0 and are bounded by $2\|e(f)\|^2+2\|X\|^2\|e(f)\|^2$, so by the theorem of dominated convergence A_n converges to 0.
So $X_{T_n}e(f)$ converges to $E_TXE_Te(f)+\int_0^{+\infty}1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}\ da^0_s\ e(f)$ and on $\mathcal{E}$,

$$X_T=E_TXE_T+\int_0^{+\infty}1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}\ da^0_s$$

We remark by property 3.2, 1 that $E_{T_n}X_{T_n}=E_{T_n}XE_{T_n}$, so on $\mathcal{E}$, $E_TX_T=E_TXE_T$ and

$$E_T\int_0^{+\infty}1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}\ da^0_s=0$$

Denote $\Gamma_t=\int_0^t 1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}\ da^0_s$. When we apply the preceding relation to $T\wedge t$, we obtain that for all $t>0$, $M_t(E_T)\Gamma_t=0$.
To see that X_T is bounded, we are going to calculate for f and g in $L^2(\mathbb{R}_+)$, $\langle X_Te(f),X_Te(g)\rangle$. By using the Ito formula, we have

$$\begin{array}{l} \langle X_Te(f),X_Te(g)\rangle=\langle E_TXE_Te(f),E_TXE_Te(g)\rangle \\ \quad+\int_0^{+\infty}\overline{f}(s)g(s)\left\{\langle\Gamma_se(f),1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}e(g)\rangle\right. \\ \quad+\langle 1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}e(f),\Gamma_se(g)\rangle \\ \quad\left.+\langle 1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}e(f),1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}e(g)\rangle\right\}ds \\ \quad=\langle E_TXE_Te(f),E_TXE_Te(g)\rangle \\ \quad+\int_0^{+\infty}\langle 1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}(f(s)e(f)),1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}(g(s)e(g))\rangle ds \end{array}$$

So if $F=\sum_i\lambda_ie(f_i)$,

$$\begin{array}{ll} \|X_TF\|^2 & =\langle E_TXE_TF,E_TXE_TF)\rangle \\ & +\int_0^{+\infty}\|1_{T\leq s}M_s(E_T)X_sM_s(E_T)1_{T\leq s}(\sum_i f_i(s)\lambda_ie(f_i))\|^2ds \\ & \leq\|E_TXE_TF\|^2+\|X\|^2\int_0^{+\infty}\|1_{T\leq s}M_s(E_T)(\sum_i f_i(s)\lambda_ie(f_i))\|^2ds \\ & \leq\|X\|^2\|E_TF\|^2+\|X\|^2(\|F\|^2-\|E_TF\|^2) \\ & \leq\|X\|^2\|F\|^2 \end{array}$$

where we have used lemma 3.7.

Proposition 3.8 : The Optional Stopping Theorem Let S and T be two stopping times such that $S \leq T$ and let $(X_t)_{t\geq 0}$ be a martingale with a final value X. We suppose that X_T is bounded then :

$$M_S(X_T) = X_S$$

Futhermore $(X_t^T)_{t\geq 0}$ defined by $X_t^T = X_{T\wedge t}$ is a martingale.

Proof : Let $(X_t)_{t\geq 0}$ be a martingale with a final value X. Define the martingale $(Y_t)_{t\geq 0}$ by $Y_t = M_t(X_T)$. This martingale has a final value given by X_T which is bounded by theorem 3.4. Futhermore :

$$M_S(X_T) = Y_S = E_S X_T E_S + \int_0^{+\infty} 1_{S\leq s} M_s(E_S X_T E_S) 1_{S\leq s}\, da_s^0$$

But as $S \leq T$, $E_S E_T = E_S$ and by consequence

$$E_S X_T E_S = E_S E_T X_T E_T E_S = E_S E_T X E_T E_S = E_S X E_S$$

So

$$M_S(X_T) = E_S X E_S + \int_0^{+\infty} 1_{S\leq s} M_s(E_S X E_S) 1_{S\leq s}\, da_s^0 = X_S$$

We have to verify that $X_{T\wedge t} = M_t(X_T)$.
But $X_{T\wedge t} = M_t(E_T) X_t M_t(E_T) + \int_0^t 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s}\, da_s^0$ thus is t-adapted.
$M_t(X_T)E_t = E_t X_T E_t = E_t E_T X E_T E_t + E_t(\int_0^{+\infty} 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s}\, da_s^0) E_t = X_{T\wedge t} E_t$.

Remark : Theorem 3.4 and proposition 3.8 implies that M_T satisfies $M_T \circ M_T = M_T$ and $\|M_T\| \leq 1$.

The following property is satisfied by martingale as in the classical case :

Proposition 3.9 : Let $(X_t)_{t\geq 0}$ be an adapted bounded process. Then $(X_t)_{t\geq 0}$ is a martingale if and only if for all T stopping time $E[X_T] = E[X_0]$, where $E[X] = \langle \mathbf{1}, X\mathbf{1}\rangle$.

Proof : It's clear that if $(X_t)_{t\geq 0}$ is a martingale then

$$E[X_T] = \langle \mathbf{1}, X_T \mathbf{1}\rangle = \langle \mathbf{1}, E_T X_\infty E_T \mathbf{1}\rangle = \langle \mathbf{1}, X\mathbf{1}\rangle = \langle \mathbf{1}, X_0 \mathbf{1}\rangle$$

Conversely, suppose that for all T stopping time, $E[X_T] = E[X_0]$. By taking $T = t$, we have that for all t, $E[X_t] = E[X_0]$.
We have to prove that for all $s < t$, $E_s X_t E_s = X_s E_s$. Let T be a discrete stopping time taking the values s and t. We suppose that $1_{T=s} = P$ and $1_{T=t} = I - P$ with P a projection belonging to $\mathcal{U}_s$.
We have $E[X_T] = E[PX_sP] + E[PX_t(I-P)] + E[(I-P)X_tP] + E[(I-P)X_t(I-P)]$.
So $E[X_T] = E[X_t]$ implies that $E[P(X_t - X_s)P] = 0$ for all P projection in $\mathcal{U}_s$.
As $P\mathbf{1}$ generated $\Phi_{s]}$, then $E_s(X_t - X_s)E_s = 0$.

Proposition 3.10 : One denote $Z_t^T = M_t(E_T)$. Then $(Z_t^T)_{t\geq 0}$ is a bounded martingale which satisfies on Φ :

$$Z_t^T = I - \int_0^t 1_{T\leq s} Z_s^T \ da_s^0 = I + \int_0^t (1_{T>s} - Z_s^T) \ da_s^0$$

In particular $E_T = I - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) da_s^0$.

Proof : By theorem 2.5, for all $f \in L^2(\mathbb{R}_+)$,

$$M_t(E_T)E_t e(f) = E_t E_T E_t e(f) = E_{T\wedge t} e(f) = \mathbf{1} + \int_0^t f(s) 1_{T>s} E_s e(f) d\chi_s$$

Let f and g be in $L^2(\mathbb{R}_+)$. We have :

$$\begin{aligned} \langle e(f), M_t(E_T)(e(g))\rangle &= e^{\int_t^{+\infty} \overline{f} g} \langle e(f_{t]}), E_{T\wedge t} E_t e(g)\rangle \\ &= e^{\int_t^{+\infty} \overline{f} g} \left(1 + \int_0^t \overline{f}(s) g(s) \langle e(f_{s]}), 1_{T>s} e(g_{s]})\rangle ds\right) \end{aligned}$$

If we denote $\varphi(t) = \langle e(f), M_t(E_T)(e(g))\rangle$, we have

$$\varphi'(t) = -\overline{f}(t) g(t) \varphi(t) + \overline{f}(t) g(t) \langle e(f), 1_{T>t} e(g)\rangle$$

and $\varphi(0) = \langle e(f), e(g)\rangle$, so

$$\varphi(t) = \langle e(f), e(g)\rangle + \int_0^t \overline{f}(s) g(s) \langle e(f), (1_{T>s} - M_s(E_T)) e(g)\rangle ds$$

So on $\mathcal{E}$, $M_t(E_T) = I + \int_0^t (1_{T>s} - M_s(E_T)) da_s^0$. The processes $((1_{T>t} - M_t(E_T))_{t\geq 0}$ and $(M_t(E_T))_{t\geq 0}$ being bounded implies that we can extend the integration over all Φ

Remark : $(M_t(E_T))_{t\geq 0}$ is a martingale so by theorem 3.4, it's stoppable and

$$M_T(E_T) = E_T - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) da_s^0 = I$$

Proposition 3.11 : Let $(H_s)_{s\geq 0}$ be a bounded process of operators such that $\sup_s \|H_s\| < +\infty$ then for all $t \geq 0$,

$$M_t(E_T) \int_0^t 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} da_s^0 = 0$$

Proof : We have already seen this in the particular case of $H_s = M_s(X)$ with X a bounded operator.
We denote $A_t = \int_0^t 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} da_s^0$ and use proposition 3.10 and the Ito formula to obtain :

$$\begin{aligned} M_t(E_T) A_t &= \int_0^t [-1_{T\leq s} M_s(E_T) A_s + M_s(E_T) 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} \\ &\quad -1_{T\leq s} M_s(E_T) 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s}] \, da_s^0 \\ &= -\int_0^t 1_{T\leq s} M_s(E_T) A_s da_s^0 \end{aligned}$$

So for all $f \in L^2(\mathbb{R}_+)$, if $Y_t = M_t(E_T)A_t$ we obtain with the Ito formula :

$$\|Y_t e(f)\|^2 = -\int_0^t |f(s)|^2 \|1_{T\leq s} Y_s E_s e(f)\|^2 ds$$

So $\|Y_t e(f)\|^2 = 0$ and $Y_t = 0$.

Definition 3.12 : An adapted process $(X_t)_{t\geq 0}$ of bounded operator is a bounded semi-martingale if it exists an adapted process $(H_t)_{t\geq 0}$ such that $\int_0^{+\infty} \|H_s\| \; ds < +\infty$ and $X_t - \int_0^t H_s \; ds$ defines a martingale with a final value .

Proposition 3.13 : Let $(X_t)_{t\geq 0}$ be a bounded semi-martingale with the notation of definition 3.12 and denote M the finale value of the martingale $X_t - \int_0^t H_s \; ds$. If $X = M + \int_0^{+\infty} H_s \; ds$, then the semi-martingale $(X_t)_{t\geq 0}$ admits a stopped process given on $\mathcal{E}$ by :

$$X_T = E_T X E_T + \int_0^{+\infty} 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s} \; da_s^0 - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} \; ds$$

Futhermore X_T is a bounded process.

Proof : Let $(Y_t)_{t\geq 0}$ the process defined by $Y_t = \int_0^t H_s ds$ and $Y = \int_0^{+\infty} H_s ds$.
We have the following relation that we can prove as in lemma 3.5 :

$$Y_t = E_t Y E_t + \int_t^{+\infty} E_{t,s} Y_s E_{t,s} da_s^0 - \int_t^{+\infty} E_{t,s} H_s E_{t,s} ds$$

So by taking $(E_n)_{n\geq 0}$ a sequence of refining T-partitions of $\mathbb{R}^+$ as in definition (2.1), we have using lemma 3.6 :

$$\begin{aligned}
Y_{T_n} &= \sum_{i,j} 1_{T_n = t_i^{(n)}} E_{t_i^{(n)}, t_i^{(n)} \vee t_j^{(n)}} Y_{t_i^{(n)} \vee t_j^{(n)}} E_{t_j^{(n)}, t_i^{(n)} \vee t_j^{(n)}} 1_{T_n = t_j^{(n)}} \\
&= E_{T_n} Y E_{T_n} + \sum_{i,j} \int_{t_i^{(n)} \vee t_j^{(n)}}^{+\infty} 1_{T_n = t_i^{(n)}} E_{t_i^{(n)}, s} Y E_{t_j^{(n)}, s} 1_{T_n = t_j^{(n)}} da_s^0 \\
&- \sum_{i,j} \int_{t_i^{(n)} \vee t_j^{(n)}}^{+\infty} 1_{T_n = t_i^{(n)}} E_{t_i^{(n)}, s} H_s E_{t_j^{(n)}, s} 1_{T_n = t_j^{(n)}} ds \\
&= E_{T_n} Y E_{T_n} + \int_0^{+\infty} 1_{T_n \leq s} M_s(E_{T_n}) Y_s M_s(E_{T_n}) 1_{T_n \leq s} da_s^0 \\
&- \int_0^{+\infty} 1_{T_n \leq s} M_s(E_{T_n}) H_s M_s(E_{T_n}) 1_{T_n \leq s} ds
\end{aligned}$$

As in the proof of theorem 3.4, for all $f \in L^2(\mathbb{R}_+)$,

$$\int_0^{+\infty} 1_{T_n \leq s} M_s(E_{T_n}) Y_s M_s(E_{T_n}) 1_{T_n \leq s} da_s^0 e(f) \to \int_0^{+\infty} 1_{T\leq s} M_s(E_T) Y_s M_s(E_T) 1_{T\leq s} da_s^0 e(f)$$

and for all $F \in \Phi$,

$$\int_0^{+\infty} 1_{T_n \leq s} M_s(E_{T_n}) H_s M_s(E_{T_n}) 1_{T_n \leq s} ds F \to \int_0^{+\infty} 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} ds F$$

So on $\mathcal{E}$,

$$Y_T = E_T Y E_T + \int_0^{+\infty} 1_{T\leq s} M_s(E_T) Y_s M_s(E_T) 1_{T\leq s} da_s^0 - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} ds$$

and by consequence :

$$\begin{aligned} X_T &= M_T + Y_T \\ &= E_T X E_T + \int_0^{+\infty} 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s} da_s^0 - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} ds \end{aligned}$$

If we denote $A_t = \int_0^t 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s} da_s^0$, we have by proposition 3.11 that $A_t M_t(E_T) = M_t(E_T) A_t = 0$ for all $t \in \mathbb{R}^+ \cup \{+\infty\}$, so

$$\begin{aligned} &\langle A_\infty e(f), A_\infty e(g)\rangle \\ &= \int_0^{+\infty} \langle 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s}(f(s)e(f)), 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s}(g(s)e(g))\rangle ds \end{aligned}$$

and so for all $F \in \mathcal{E}$ by lemma 3.7,

$$\|A_\infty F\|^2 \leq (\sup_s \|X_s\|^2)(\|F\|^2 - \|E_T F\|^2)$$

So A_∞ is bounded and $X_T = E_T X E_T + A_\infty - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} ds$ also.

Proposition 3.14 : Let $(X_t)_{t\geq 0}$ be a bounded adapted process given by

$$X_t = \sum_\epsilon \int_0^t H_s^\epsilon \, da_s^\epsilon$$

with $(H_t^\epsilon)_{t\geq 0}$ being some bounded adapted processes which satisfy :

$$\int_0^{+\infty} \|H_s^{+,-}\|^2 \, ds < +\infty \ , \ \int_0^{+\infty} \|H_s^\times\| \, ds < +\infty \ , \ \sup_s \|H_s^0\| < +\infty$$

then $(X_t)_{t\geq 0}$ admits a stopped process by T given on $\mathcal{E}$ by :

$$X_T = \sum_\epsilon \int_0^{+\infty} R_s^\epsilon \, da_s^\epsilon$$

where

$$\begin{aligned} R_s^\times &= M_s(E_T) H_s^\times 1_{T>s} + 1_{T>s} H_s^\times M_s(E_T) - 1_{T>s} H_s^\times 1_{T>s} \\ R_s^+ &= 1_{T>s} H_s^+ M_s(E_T) \\ R_s^- &= M_s(E_T) H_s^- 1_{T>s} \\ R_s^0 &= 1_{T>s} H_s^0 1_{T>s} - 1_{T\leq s} M_s(E_T) X_s 1_{T>s} - 1_{T>s} X_s 1_{T\leq s} M_s(E_T) \end{aligned}$$

Proof : We know by proposition 3.13 that

$$X_T = E_T X E_T + \int_0^{+\infty} 1_{T\leq s} M_s(E_T) X_s M_s(E_T) 1_{T\leq s} \, da_s^0 - \int_0^{+\infty} 1_{T\leq s} M_s(E_T) H_s^\times M_s(E_T) 1_{T\leq s} \, ds$$

where $X = \sum_\epsilon \int_0^{+\infty} H_s^\epsilon \, da_s^\epsilon$. So we have only to find a formula for $E_T X E_T$ which is the limit when t goes to infinity of $M_t(E_T) X_t M_t(E_T)$.
We have just to apply proposition 3.10 and twice the Ito formula to calculate $M_t(E_T) X_t M_t(E_T)$.

4 Future and adaptability to the future

The problem is composed of the existence of a future Φ^T which has to satisfy :

- $\Phi_T \otimes \Phi^T$ is isomorphic to Φ
- The stopped process X_T of $(X_t)_{t\geq 0}$ verifies :

$$X_T(A \otimes_T B) = X_T A \otimes_T B$$

Unfortunately we have seen that it will be impossible with the definition of [15]. So the question is : how define Φ^T or Φ_T ?
Let $f \in L^2(\mathbb{R}_+)$ and $(R_t)_{t\geq 0}$ be defined by $R_t = \Gamma(\theta_t)\Gamma(\theta_t)^*$ where θ_t is the right shift operator on $L^2(\mathbb{R}_+)$ and $\Gamma(\theta_t)$ is its second quantisation. So $R_t e(f) = e(f_{[t})$ and $R_t = M_t(E_0)$ is thus a martingale. By theorem 3.4, R_T is a contraction and satisfies :

$$R_T = E_0 + \int_0^{+\infty} 1_{T\leq s} R_s 1_{T\leq s} da_s^0$$

Definition 4.1 : Let T be a stopping time.
The space of the future after T is the closure of the image of Φ by R_T. We denote it Φ^T.

Remark : In [15], they define the future by stopping $\Gamma(\theta_t)$ instead of R_t.

Now have we a strong tensor product ?
If the answer is yes, we must have for any stopped process X_T :

$$X_T R_T e(g) = \text{“}X_T \mathbf{1} \otimes R_T e(g)\text{”} \tag{4.2}$$

Proposition 4.3 : Let $X \in \mathcal{U}$ and $X_T = M_T(X)$.
Let $x = (x_s)_{s\geq 0}$ be the process of adapted vectors defined by $x_s = 1_{T\leq s} X_T 1_{T\leq s}\mathbf{1}$ and $(L_s(x))_{s\geq 0}$ be the process of bounded adapted operators defined by

$$L_s(x)E_s F = \langle 1_{T\leq s}\mathbf{1}, F\rangle x_s$$

Then, we have :

$$X_T R_T = E_T X E_0 + \int_0^{+\infty} L_s(x) da_s^0$$

Proof : The theorem 3.4 implies :

$$X_T = E_T X E_T + \int_0^{+\infty} 1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} da_s^0$$

By using proposition 3.11, we have for all $t > 0$,

$$\begin{aligned} &M_t(E_T)\textstyle\int_0^t 1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} da_s^0 = \int_0^t 1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} da_s^0 M_t(E_T) \\ &= M_t(E_T)\textstyle\int_0^t 1_{T\leq s} R_s 1_{T\leq s} da_s^0 = 0 \end{aligned}$$

This implies :

$$X_T R_T = E_T X E_0 + \Gamma_\infty A_\infty$$

where $\Gamma_t = \int_0^t 1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} da_s^0$ and $A_t = \int_0^t 1_{T\leq s} R_s 1_{T\leq s} da_s^0$.
We can use Ito formula :

$$X_T R_T = E_T X E_0 + \int_0^{+\infty} \begin{array}{l}(1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} A_s \\ +\Gamma_s 1_{T\leq s} R_s 1_{T\leq s} + 1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} R_s 1_{T\leq s})\, da_s^0\end{array}$$

As $1_{T\leq s}$ and $M_s(E_T)$ commute, we have :

$$X_T R_T = E_T X E_0 + \int_0^{+\infty} 1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} R_s 1_{T\leq s} da_s^0$$

But for all $F \in \Phi$, $1_{T\leq s} M_s(E_T X E_T) 1_{T\leq s} R_s 1_{T\leq s} E_s F = \langle 1_{T\leq s}\mathbf{1}, F\rangle 1_{T\leq s} E_T X 1_{T\leq s}\mathbf{1} = L_s(x) E_s F$.

This proposition shows that $X_T R_T F$ depends not only of $X_T\mathbf{1}$ but also of $(1_{T\leq s} X_T 1_{T\leq s}\mathbf{1})_{s\geq 0}$. So we need to modify the definition of Φ_T.

Definition 4.4 : A process of vectors $x = (x_t)_{t\geq 0}$ belongs to $\mathcal{A}_T$ if for all $t \geq 0$, $x_t \in \Phi_{T\wedge t}$, $1_{T\leq t} x_t = x_t$, $x_t \to x_\infty$ and $\|x\|_T = \sup_t \frac{\|x_t\|}{\|1_{T\leq t}\mathbf{1}\|} < +\infty$.
Let $\widehat{\Phi}_T$ be the subspace of $L^\infty(\mathbb{R}^+, \Phi)$ defined by the closure of $\mathcal{A}_T$ for $\|\ \|_T$.

Remark : We identify $\mathbf{1}$ with $(1_{T\leq t}\mathbf{1})_{t\geq 0}$ and denote it $\widehat{\mathbf{1}}$.
$\widehat{\Phi}_T$ contains all the process of vectors $(1_{T\leq s} X_T 1_{T\leq s}\mathbf{1})_{s\geq 0}$ for bounded stopped process X_T.
The map i from Φ_T to $\widehat{\Phi}_T$ defined by $x \mapsto (\|1_{T\leq t}\mathbf{1}\| 1_{T\leq t} x)_{t\geq 0}$ is an isometry if T is finite.

Definition 4.5 : Let X_T be the bounded stopped operator of a process $(X_t)_{t\geq 0}$. We define an operator $\widehat{X}_T$ of $\mathcal{B}(\widehat{\Phi}_T)$ by

$$\widehat{X}_T((x_t)_{t\geq 0}) = (1_{T\leq t} X_T x_t))_{t\geq 0}$$

Remark : The proposition 4.3 implies that

$$X_T R_T = E_T X E_0 + \int_0^{+\infty} L_s(\widehat{X}_T(\widehat{\mathbf{1}})) da_s^0$$

Proposition 4.6 : Let x and z be some elements of $\widehat{\Phi}_T$ and f and g be elements of $L^2(\mathbb{R}^+)$.

$$\begin{array}{l}\langle \int_0^{+\infty} L_s(x) da_s^0 e(f), \quad \int_0^{+\infty} L_s(z) da_s^0 e(g)\rangle \\ \qquad = \int_0^{+\infty} \overline{f}(s) g(s) e^{\int_s^{+\infty} \overline{f} g} \langle x_s, z_s\rangle \overline{\langle 1_{T\leq s}\mathbf{1}, e(f)\rangle} \langle 1_{T\leq s}\mathbf{1}, e(g)\rangle ds\end{array}$$

Proof : As $L_s(x) = 1_{T\leq s} M_s(E_T) L_s(x) M_s(E_T) 1_{T\leq s}$, we first see by proposition 3.11 that $M_t(E_T) \int_0^t L_s(x) da_s^0 = 0$.
The Ito formula gives the result.

By consequence, if $F = \sum_\alpha \lambda_\alpha e(f_\alpha)$ and if $x \in \widehat{\Phi}_T$,

$$\begin{aligned}\| \int_0^{+\infty} L_s(x) da_s^0 F\|^2 &= \int_0^{+\infty} \|x_s\|^2 \| \sum_\alpha \lambda_\alpha f_\alpha(s) \langle 1_{T\leq s}\mathbf{1}, e(f_\alpha)\rangle e(f_{\alpha[s]})\|^2 ds \\ &\leq \|x\|_T^2 \int_0^{+\infty} \|1_{T\leq s}\mathbf{1}\|^2 \| \sum_\alpha \lambda_\alpha f_\alpha(s) \langle 1_{T\leq s}\mathbf{1}, e(f_\alpha)\rangle e(f_{\alpha[s]})\|^2 ds \\ &\leq \|x\|_T^2 \| \int_0^{+\infty} L_s(\widehat{\mathbf{1}}) da_s^0 F\|^2 \\ &\leq \|x\|_T^2 \|R_T F - E_0 F\|^2\end{aligned}$$

The operator $\int_0^{+\infty} L_s(x) da_s^0$ is thus a bounded operator and we have proved the following proposition :

Proposition 4.7 : The application $\widehat{J}$ from $\widehat{\Phi}_T \times \Phi^T$ to Φ defined by

$$(x, R_T F) \mapsto \langle \mathbf{1}, R_T F\rangle x_\infty + \left(\int_0^{+\infty} L_s(x) da_s^0 \right) F$$

is a continuous bilinear operator with a norm less than 1.

So we can define a continuous contraction J from $\widehat{\Phi}_T \otimes \Phi^T$ to Φ by $J(x \otimes B) = \widehat{J}((x, B))$.

We can prove as in the proposition 4.3 that

$$\mathbf{X_T \circ J(x \otimes B) = J(\widehat{X_T}(x) \otimes B)}$$

So X_T is adapted to this factorisation.

Have we $\overline{J(\widehat{\Phi}_T \otimes \Phi^T)} = \Phi$?
In all generality, the answer is no : Let T be a stopping time taking two values t_0 and t_1 such that $1_{T=t_0}\mathbf{1} = 0$ but $1_{T=t_0} \neq 0$. We verify easily that $\Phi^T = \Phi_{[t_1}$ and $\Phi_T \subset \Phi_{t_1]}$ but $\Phi_T \neq \Phi_{t_1]}$. Let $a \in \Phi_{t_1]}$ and $a \in \Phi_T^\perp$. Then for all $x \in \widehat{X}_T$, for all $g \in L^2(\mathbb{R}^+)$,

$$\langle a, J(x \otimes R_T e(g))\rangle = \langle a, x_\infty\rangle + \langle a, \left(\int_0^{+\infty} L_s(x) da_s^0 \right) e(g)\rangle$$

$\langle a, x_\infty\rangle = 0$ because $a \in \Phi_T^\perp$.
Futhermore for all $k \in L^2(\mathbb{R}^+)$,

$$\langle E_{t_1} e(k), \left(\int_0^{+\infty} L_s(x) da_s^0 \right) e(g)\rangle = \int_0^{t_1} \overline{k}(s) g(s) \langle e(k), x_s\rangle \langle 1_{T\leq s}\mathbf{1}, e(g)\rangle ds = 0$$

because $1_{T\leq s}\mathbf{1} = 0$ if $s < t_1$. So as $a \in \Phi_{t_1]}$, $\langle a, \left(\int_0^{+\infty} L_s(x) da_s^0 \right) e(g)\rangle = 0$.

New hypothesis 4.8 : For all $t \geq 0$, $1_{T\leq t}\mathbf{1} = 0$ implies $1_{T\leq t} = 0$.
This is a condition of "separating " for $\mathbf{1}$ on T.

Theorem 4.9 : If 4.8 is satisfied, then $\overline{J(\widehat{\Phi}_T \otimes \Phi^T)} = \Phi$.

Proof : We are going to prove that for all $f \in L^2(\mathbb{R}^+)$, $e(f)$ belongs to $\overline{J(\widehat{\Phi}_T \otimes \Phi^T)}$.
Let $b > a \geq 0$ such that $1_{T \leq b}\mathbf{1} \neq 0$. We define the element x of $\Phi_{T\wedge b}$ by $x = E_T(1_{T\in[a,b]}E_b e(f))$.
Let $(\mathcal{S}_n)_{n\geq 0}$ a refining sequence of $[b, +\infty[$ given by $b = t_0^{(n)} < t_1^{(n)} < \cdots < t_k^{(n)} < \cdots$.
For all n and k, we define the element $x^{(n,k)}$ of $\widehat{\Phi}_T$ by

$$x_t^{(n,k)} = \begin{cases} x & \text{si } t_k^{(n)} \leq t < t_{k+1}^{(n)} \\ 0 & \text{sinon} \end{cases}$$

$\|x^{(n,k)}\|_T \leq \frac{\|x\|}{\|1_{T\leq b}\mathbf{1}\|}$. One define the element $f^{(n,k)}$ of $L^2(\mathbb{R}^+)$ by $f^{(n,k)} = f1_{[t_k^{(n)}}$.
Let z_n the element of $\overline{J(\widehat{\Phi}_T \otimes \Phi^T)}$ define by

$$z_n = \sum_{k\geq 0} \frac{1}{\|1_{T\leq t_k^{(n)}}\mathbf{1}\|^2} J(x^{(n,k)} \otimes R_T e(f^{(n,k)}))$$

The proposition 4.6 implies that the sequence $(z_n)_{n\geq 0}$ is uniformly bounded.
For all $l \in L^2(\mathbb{R}^+)$,

$$\langle e(l), z_n\rangle = \langle e(l), x \otimes_b e(f_{[b})\rangle - \langle e(l), x\rangle + \langle e(l), x\rangle A_n$$

where $A_n = \displaystyle\sum_{k\geq 0} \int_{t_k^{(n)}}^{t_{k+1}^{(n)}} \bar{l}(s) f(s) e^{\int_s^{+\infty} \bar{l} f} \left(\frac{\langle 1_{T\leq s}\mathbf{1}, e(f_{[t_k^{(n)},s]})\rangle}{\|1_{T\leq t_k^{(n)}}\mathbf{1}\|^2} - 1 \right) ds.$
We verify easily that A_n goes to 0.
This implies that for all $l \in L^2(\mathbb{R}^+)$, $\langle e(l), z_n\rangle$ goes to $\langle e(l), x \otimes_b e(f_{[b}) - x\rangle$ and thus that $x \otimes_b e(f_{[b}) - x$ belongs to $\overline{J(\widehat{\Phi}_T \otimes \Phi^T)}$. As $x = J(x^{(n,k)} \otimes R_T\mathbf{1})$, then for all $a < b$ such that $1_{T\leq b}\mathbf{1} \neq 0$ $E_T(1_{T\in[a,b]}E_b e(f)) \otimes_b e(f_{[b})$ belongs to $\overline{J(\widehat{\Phi}_T \otimes \Phi^T)}$.

Let b such that $\mathbf{1}_{T\leq b}\mathbf{1} = 0$ and for all $t > b$, $\mathbf{1}_{T\leq t}\mathbf{1} \neq 0$. We use a refining partition of $[b, +\infty[$ as before.
So $b_n = \displaystyle\sum_{k\geq 0} E_T(1_{T\in[t_k^{(n)}, t_{k+1}^{(n)}[} E_{t_{k+1}^{(n)}} e(f)) \otimes e(f_{[t_{k+1}^{(n)}})$ belongs to $\overline{J(\widehat{\Phi}_T \otimes \Phi^T)}$.
By the hypothesis 4.8, $e(f) = \displaystyle\sum_{k\geq 0} 1_{T\in[t_k^{(n)}, t_{k+1}^{(n)}[} E_{t_{k+1}^{(n)}} e(f) \otimes e(f_{[t_{k+1}^{(n)}})$. We verify easely that b_n goes to $e(f)$. That gives the result.

5 Some examples

5.1 Projection on chaoses

This example was studied in [2]. The Fock space Φ can be advantageously understood as the space $L^2(\mathcal{P})$ where $\mathcal{P}$ is the set of finite subsets of $\mathbb{R}^+$ equipped with the Guichardet symmetric measure. That is, an element f of $\Phi = L^2(\mathcal{P})$ is a measurable function $f : \mathcal{P} \to \mathbb{C}$ such that

$$\begin{aligned} \|f\|^2 &= \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma \\ &= |f(\emptyset)|^2 + \sum_{n=1}^{\infty} \int_{0<s_1<\cdots<s_n} |f(\{s_1 \cdots s_n\})|^2 ds_1 \cdots ds_n < \infty \ . \end{aligned}$$

For all $f \in \Phi$, if we define $D_t f$ by

$$[D_t f](\sigma) = f(\sigma \cup \{t\}) 1_{\sigma \subset [0,t]}$$

we then have that $D_t f$ belongs to Φ for a.a.t and

$$\|f\|^2 = |f(\emptyset)|^2 + \int_0^\infty \|D_t f\|^2 \, dt \ .$$

For every $f \in \Phi$, the family $(D_t f)_{t\in\mathbb{R}^+}$ is Ito integrable and we have

$$f = f(\emptyset)\mathbf{1} + \int_0^\infty D_t f \, d\chi_t \ .$$

For all $t \in \mathbb{R}^+ \setminus \{0\}$ we have

$$E_t f = f(\emptyset)\mathbf{1} + \int_0^t D_s f \, d\chi_s.$$

For every $n \in \mathbb{N}$, we denote by C_n the space of $f \in \Phi$ such that $f(\sigma) = 0$ unless $\#\sigma = n$. It is a closed subspace of Φ and we have

$$\Phi = \oplus_{n\in\mathbb{N}} C_n \ .$$

The space C_n is called the *n-th chaos* of Φ. We denote by Q_n the orthogonal projection from Φ onto $\oplus_{i=0}^n C_i$, that is

$$[Q_n f](\sigma) = f(\sigma) 1_{\#\sigma \leq n}$$

and by $Q_{n,t}$ the operator

$$[Q_{n,t}, f](\sigma) = f(\sigma) 1_{\#(\sigma\cap[0,t])\leq n} \ .$$

The operator $Q_{n,t}$ is t-adapted and equal to

$$Q_{n|\Phi_{t]}} \otimes I_{|\Phi_{[t}} \ .$$

It is an orthogonal projection also and $Q_{n,t} \leq Q_{n,s}$ if $s \leq t$. We define a stopping time T_n by putting

$$\begin{cases} 1_{T_n > t} = Q_{n,t} \\ 1_{T_n = +\infty} = Q_n \end{cases}$$

We clearly have $T_n \leq T_{n+1}$ for all $n \in \mathbb{N}$. Note that for all $s, t \in \mathbb{R}^+$ we have

$$1_{T_n \leq s} 1_{T_{n+1} \leq t} = 1_{T_{n+1} \leq t} 1_{T_n \leq s} \ .$$

We also have

$$\begin{aligned} E_{T_n} f &= f(\emptyset)\mathbf{1} + \textstyle\int_0^\infty 1_{T_n > t} D_t f \, d\chi_t \\ &= f(\emptyset)\mathbf{1} + \textstyle\int_0^\infty Q_n D_t f \, d\chi_t \\ &= f(\emptyset)\mathbf{1} + Q_{n+1} \textstyle\int_0^\infty D_t f \, d\chi_t \\ &= Q_{n+1} f \end{aligned}$$

Thus $\Phi_{T_n} = \oplus_{i=0}^{n+1} C_i$.

To any f in $L^2(\mathbb{R}^+)$ and φ in $L^\infty_{\mathbb{R}}(\mathbb{R}^+)$, we associate the Weyl operator $W(f,\varphi)$ given for any g element of $L^2(\mathbb{R}^+)$ by

$$W(f,\varphi)e(g) = e^{-\frac{1}{2}\|f\|^2 - \langle f, e^{i\varphi}g\rangle} e(f + e^{i\varphi}g)$$

For $t \geq 0$, we define the process of adapted operators $(W_t(f,\varphi))_{t\geq 0}$ by $W_t(f,\varphi) = W(f_{t]}, \varphi_{t]})$. This process satisfies the equation :

$$dW_t = \left(f da_t^+ - e^{i\varphi}\overline{f} da_t^- + (e^{i\varphi} - 1) da_t^0 - \frac{1}{2}|f|^2 dt \right) W_t$$

To calculate W_{T_n}, we can use proposition 3.13 which give us :

$$\begin{aligned} W_{T_n} = \; & E_{T_n} W E_{T_n} + \int_0^{+\infty} 1_{T_n \leq s} M_s(E_{T_n}) W_s M_s(E_{T_n}) 1_{T_n \leq s} da_s^0 \\ & + \tfrac{1}{2} \int_0^{+\infty} |f(s)|^2 1_{T_n \leq s} M_s(E_{T_n}) W_s M_s(E_{T_n}) 1_{T_n \leq s} ds \end{aligned}$$

but $1_{T_n \leq s} M_s(E_{T_n}) = Q_{n+1,s} - Q_{n,s}$ and so

$$\begin{aligned} W_{T_n} & = Q_{n+1} W Q_{n+1} + \int_0^{+\infty} (Q_{n+1,s} - Q_{n,s}) W_s (Q_{n+1,s} - Q_{n,s}) da_s^0 \\ & + \tfrac{1}{2} \int_0^{+\infty} |f(s)|^2 (Q_{n+1,s} - Q_{n,s}) W_s (Q_{n+1,s} - Q_{n,s}) ds \end{aligned}$$

If $\varphi = 0$ and $f = i\lambda h$ with h real, $W_t = e^{i\lambda(a_t^+(h) + a_t^-(h))}$.
So $\frac{W_t - I}{i\lambda}$ goes to $B_t(h) = a_t^+(h) + a_t^-(h)$.
If $f = 0$ and $\varphi = \lambda\varphi$, $W_t = e^{i\lambda a^0(\varphi)}$.
So we can deduce the following result :

$$B_{T_n}(h) = Q_{n+1} B(h) Q_{n+1}$$

This is just the restriction of $B(h)$ to the $n+1$ first chaoses.

$$a^0_{T_n}(\varphi) = a^0(\varphi) Q_{n+1} + \int_0^{+\infty} a_s^0(\varphi)(Q_{n+1,s} - Q_{n,s}) da_s^0$$

For $\sigma \in \mathcal{P}$ and $F \in \Phi$, $a^0(\varphi)(\sigma) = \sum_{s\in\sigma} \varphi(\sigma)F(\sigma)$. Denote for $\sigma \in \mathcal{P}$, $\sigma = \{t_1 < t_2 < \cdots < t_r\}$,

$$\sigma_{n+1} = \begin{cases} \sigma \quad \text{si } \#\sigma \leq n+1 \\ \{t_1, t_2, \ldots, t_{n+1}\} \quad \text{si } \#\sigma \geq n+1 \end{cases}$$

Then we have : $a^0_{T_n}(\varphi)F(\sigma) = (\sum_{s\in\sigma_{n+1}} \varphi(s))F(\sigma)$.

5.2 Stopping the brownian motion at some stopping time

5.2.1 Stopping the brownian motion at the first jumping time of the Poisson process

It's another example studied in [2].
Let $N_t = a_t^+ + a_t^- + a_t^0 + tI$ be the Poisson process on Φ.

The first jumping time T of the Poisson process is defined by :

$$1_{T>t} = I - \int_0^t 1_{T>s} dN_s$$

We can show that for all $F \in \Phi$, $1_{T>t} E_t F = \langle e(-1_{[0,t]}), F\rangle x_t$ where $x_t = e^{-t} e(-1_{[0,t]})$.
So F belongs to Φ_T if it exists φ such that $\int_0^{+\infty} |\varphi(s)|^2 e^{-s} ds < +\infty$ and λ in $\mathbb{R}$ with $F = \lambda \mathbf{1} + \int_0^{+\infty} \varphi(s) x_s d\chi_s$. Futhermore $E_T F = E_0 F - \int_0^{+\infty} \frac{\partial}{\partial s}\langle e(-1_{[0,s]}), F\rangle\ x_s d\chi_s$.
So we can calculate the moments of B_T given by $(B_T)^n \mathbf{1}$.
If $(B_T)^n \mathbf{1} = \lambda_n \mathbf{1} + \int_0^{+\infty} \varphi_n(s) x_s d\chi_s$, we obtain the folowing relations :

$$\begin{cases} \lambda_{n+1} = \int_0^{+\infty} \varphi_n(s) e^{-s} ds \\ \varphi_{n+1}(s) = \lambda_n - \left(2s\varphi_n(s) + \int_0^s \varphi_n(\tau) d\tau + e^s \int_s^{+\infty} \varphi_n(\tau) e^{-\tau} d\tau\right) \end{cases}$$

Unfortunatly, I don't recognize the law given by these numbers λ_n.

5.2.2 Martingales of brownian motion

We know that for all $n \geq 1$, the processes $(H_n(B_t, t))_{t\geq 0}$ are martingales if $H_n(x,a) = a^{\frac{n}{2}} h_n(\frac{x}{\sqrt{a}})$ where h_n are the Hermite's polynoms.
Let $(B_t = a_t^- + a_t^+)_{t\geq 0}$ be the brownian motion on Φ.
Then for all $n \geq 1$, the processes $(H_n(B_t, t))_{t\geq 0}$ are martingales and if we can stop then we obtain new martingales.
For example, $(B_{T\wedge t})_{t\geq 0}$, $(B^2_{T\wedge t} - T \wedge t)_{t\geq 0}$ are martingales.
For $n = 3$, $H_3(x,a) = x^3 - 3ax$, so we have to stop $(tB_t)_{t\geq 0}$.
Thus we have to stop process like $(\psi(t) M_t)_{t\geq 0}$ where $(M_t)_{t\geq 0}$ is a martingale.
By using proposition 3.13, we obtain if $Y_t = \psi(t) M_t$ and $H_t = \psi'(t) M_t$:

$$\begin{aligned} Y_{T\wedge t} &= M_t(E_T) Y_t M_t(E_T) + \int_0^t 1_{T\leq s} M_s(E_T) Y_s M_s(E_T) 1_{T\leq s} da_s^0 \\ &\quad - \int_0^t 1_{T\leq s} M_s(E_T) H_s M_s(E_T) 1_{T\leq s} da_s^0 \\ &= \psi(t) M_{T\wedge t} - \int_0^t \psi'(s) 1_{T\leq s} M_{T\wedge s} 1_{T\leq s} ds \\ &= \psi(T\wedge t) M_{T\wedge t} + \int_0^t \psi'(s) 1_{T\leq s} M_{T\wedge s} 1_{T>s} ds + \text{martingale} \end{aligned}$$

So for example, $((B^3)_{T\wedge t} - 3(T\wedge t) B_{T\wedge t} - 3\int_0^t 1_{T\leq s} B_{T\wedge s} 1_{T>s} ds)_{t\geq 0}$ is a martingale.
We see in this formula the non commutativity in the last term.

5.3 Strictly smaller stopping times

In [2], we define the property for a stopping time of being strictly smaller than an other one. Two stopping times S and T are said to satisfy $S < T$ if and only if one has that the expression

$$\sum_{i=1}^{N_n} 1_{T>t_i} 1_{S\in[t_{i-1}, t_i[}$$

weakly converges to I when $\{t_i,\ i = 1 \cdots N_n\}$ follows a sequence of refining S-partitions of $\mathbb{R}^+$.
Denote R_n the process $\sum_{i=1}^{N_n} 1_{T>t_i} 1_{S\in[t_{i-1}, t_i[}$.

Here we say that $S < T$ if R_n and R_n^* converges strongly to I.
We consider the bounded adapted process $X_t = 1_{T>t}$.
Then $S < T$ if $X_S = I$.
In fact, it's easy to prove that if S is a discrete stopping time and $(X_t)_{t\geq 0}$ an adapted bounded process then

$$X_S = X_S E_S + \int_0^{+\infty} 1_{S\leq s} M_s(E_S) M_s(X_S) M_s(E_S) 1_{S\leq s} da_s^0$$

So if $(S_n)_{n\geq 0}$ is the sequence associated with a refining S-partitions of $\mathbb{R}^+$, we have that for all n,

$$X_{S_n} = X_{S_n} E_{S_n} + \int_0^{+\infty} 1_{S_n\leq s} M_s(E_{S_n}) M_s(X_{S_n}) M_s(E_{S_n}) 1_{S_n\leq s} da_s^0$$

But $X_{S_n} E_{S_n} = E_{S_n} R_n^* R_n E_{S_n}$ and so converge strongly to E_S.
By the same way $1_{S_n\leq s} M_s(E_{S_n}) M_s(X_{S_n}) M_s(E_{S_n}) 1_{S_n\leq s}$ converge strongly to $1_{S\leq s} M_s(E_S)$ and on $\mathcal{E}$,

$$\int_0^{+\infty} 1_{S_n\leq s} M_s(E_{S_n}) M_s(X_{S_n}) M_s(E_{S_n}) 1_{S_n\leq s} da_s^0 \to \int_0^{+\infty} 1_{S\leq s} M_s(E_S) da_s^0$$

So on $\mathcal{E}$, X_{S_n} converge to $E_S - \int_0^{+\infty} 1_{S\leq s} M_s(E_S) da_s^0 = I$.

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