ON POLYAKOV'S NOTION OF REGULAR q-CONCAVE CR MANIFOLDS

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1. INTRODUCTION

Let M be a $\mathcal{C}^\infty$-submanifold of codimension k in $\mathbb{C}^n$, $1 \le k \le n-1$. A collection $(U;\rho_1,\dots,\rho_k)$ will be called a **local defining collection** of M if $U \subseteq \mathbb{C}^n$ is open, $\rho_1,\dots,\rho_k$ are real $\mathcal{C}^\infty$-functions on U, $U \cap M = \{\zeta \in U \mid \rho_1(\zeta) = \ldots = \rho_k(\zeta) = 0\} \neq \emptyset$ and $d\rho_1(\zeta) \wedge \ldots \wedge d\rho_k(\zeta) \neq 0$ for all $\zeta \in U \cap M$.

For $a \in M$, let $T_a^{\mathbb{C}}(M)$ be the complex tangent space of M at a. We assume that M is a **generic CR-submanifold** of $\mathbb{C}^n$ which means, by definition, that

$$\dim_{\mathbb{C}} T_a^{\mathbb{C}}(M) = n-k \qquad \text{for all } a \in M.$$

Further, for $a \in M$, let $G\big(p, T_a^{\mathbb{C}}(M)\big)$, $1 \le p \le \dim_{\mathbb{C}} T_a^{\mathbb{C}}(M)$, be the Grassmann manifold of complex p-dimensional subspaces of $T_a^{\mathbb{C}}(M)$. Set $S^{k-1} = \{x \in \mathbb{R}^k \mid |x| = 1\}$.

If $a \in M$ and $\rho = (U;\rho_1,\dots,\rho_k)$ is a defining collection of M at a, then, for all $x \in S^{k-1}$ we set $\rho(x) := x_1\rho_1 + \ldots + x_k\rho_k$, and, for $1 \le p \le \dim_{\mathbb{C}} T_a^{\mathbb{C}}(M)$, we denote by

$$G^+_{\rho(x)}\big(p, T_a^{\mathbb{C}}(M)\big)$$

the set of all $T \in G\big(p, T_a^{\mathbb{C}}(M)\big)$ such that, the Levi form of $\rho(x)$ at a restricted to T is positive definite.

We assume that, for some $1 \le q \le (n-k)/2$, M is **q-concave** in the sense of G. Henkin [He]. This means that, for each $a \in M$, there exists a defining collection ρ of M at a such that

$$G^+_{\rho(x)}\big(q, T_a^{\mathbb{C}}(M)\big) \neq \emptyset \qquad \text{for all } x \in S^{k-1}. \tag{1.1}$$

In the constructions of integral formulas for $\overline{\partial}_M$ started by G. Henkin in [He] and then continued by R. Airapetjan and G. Henkin in [AiHe1, AiHe2], the hypothesis is used that, for each $a \in M$ and any defining collection ρ of M at a satisfying condition (1.1), there exists a continuous map

$$T : S^{k-1} \to G\big(q, T_a^{\mathbb{C}}(M)\big) \tag{1.2}$$

with $T(x) \in G^+_{\rho(x)}\big(q, T_a^{\mathbb{C}}(M)\big)$ for all $x \in S^{k-1}$. However, so far as we know, it is not clear whether such a map always exists, although by proposition 3.3.1 in [AiHe1]

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this is claimed.[1] Therefore, P. Polyakov [P1, P2, P3], developing the theory of Airapetian and Henkin, calls M **regular q-concave** if such a map always exists.

The question whether M is always regular q-concave or not remains open also in the present paper. Instead we observe that the answer to this question is not important, as long as the map (1.2) is used only for the integral representations of [He, AiHe1, AiHe2, P1, P2, P3] or for those of [BaLa][2]. Namely, instead of q, for each point $a \in M$ and any defining collection ρ of M at a, we consider the number

$$p_{\max}(a,\rho) := \max\Big\{ p \in \mathbb{N}^* \;\Big|\; G^+_{\rho(x)}\big(p, T^{\mathbb{C}}_a(M)\big) \neq \emptyset \text{ for all } x \in S^{k-1} \Big\}.$$

This number has by definition the property that there exists at least one point $x_0 \in S^{k-1}$ such that the Levi form of $\rho(x_0)$ at a has precisely $p_{\max}(a,\rho)$ positive eigenvalues, and not more. This implies that $G^+_{\rho(x_0)}\big(p_{\max}(a), T^{\mathbb{C}}_a(M)\big)$ is contractible, in distinction to $G^+_{\rho(x_0)}\big(q, T^{\mathbb{C}}_a(M)\big)$ if $q < p_{\max}(a,\rho)$. This enables us to prove the following

1.1. **Theorem.** *For each $a \in M$ and any defining collection ρ of M at a, there exists a continuous map*

$$T : S^{k-1} \to G\big(p_{max}(a), T^{\mathbb{C}}_a(M)\big) \tag{1.3}$$

such that $T(x) \in G^+_{\rho(x)}\big(p_{max}(a), T^{\mathbb{C}}_a(M)\big)$ for all $x \in S^{k-1}$.

1.2. **Remark.** Since the complex tangent bundle of M is locally trivial, theorem 1.1 immediately implies the following statement:

For each $a_0 \in M$ and any defining collection ρ of M at a_0, there exist a neighborhood $U \subseteq M$ of a_0 and a continuous map $T(a,x)$ defined for $a \in U$ and $x \in S^{k-1}$ such that $T(a,x) \in G^+_{\rho(x)}\big(p_{max}(a_0,\rho), T^{\mathbb{C}}_a(M)\big)$ for all $a \in U$ and $x \in S^{k-1}$.

Since $p_{\max}(a) \geq q$, in the constructions of [He, AiHe1, AiHe2, P1, P2, P3, BaLa], instead of the map (1.2) always the map (1.3) can be used. If $p_{\max}(a) > q$, then the results become even better.

Theorem 1.1 is a special case of theorem 4.1 proved at the end of this paper. First we have to prove theorem 3.2, which in particular implies the following

1.3. **Corollary.** *For each $a \in M$ and any defining collection ρ of M at a and for each compact subset Q of S^{k-1} which is homeomorphic to the cube $[0,1]^{k-1}$ (for example $Q = S^{k-1} \setminus B$ where B is a small open ball centered at some point of S^{k-1}), there exists a continuous map*

$$T : Q \to G\big(q, T^{\mathbb{C}}_a(M)\big)$$

[1] The proof given for this proposition in [AiHe1] is not correct, because the authors use the assertion that $G^+_{\rho(x)}\big(q, T^{\mathbb{C}}_a(M)\big)$ is contractible, which is not true, except for the case when the Levi form of $\rho(x)$ at a has precisely q positive eigenvalues, and not more. Indeed, assume the number of positive eigenvalues of the Levi form of $\rho(x)$ at a is $q' > q$. Let $T_{q'}$ be the q'-dimensional space spanned by the eigenvectors associated to the positive eigenvalues of the Levi form of $\rho(x)$ at a. Then the Grassmann manifold $G\big(q, T_{q'}\big)$ of q-dimensional subspaces of $T_{q'}$ is contained in $G^+_{\rho(x)}\big(q, T^{\mathbb{C}}_a(M)\big)$. In particular, there exists a projective line $S \subseteq G^+_{\rho(x)}\big(q, T^{\mathbb{C}}_a(M)\big)$. Consider $G^+_{\rho(x)}\big(q, T^{\mathbb{C}}_a(M)\big)$ as a subset of $\mathbb{P}^N$ via the Plücker imbedding of $G\big(q, T^{\mathbb{C}}_a(M)\big)$. Then S becomes a complex line in $\mathbb{P}^N$. Since S cannot be deformed to a point inside $\mathbb{P}^N$, it follows that S cannot be deformed to a point inside $G^+_{\rho(x)}\big(q, T^{\mathbb{C}}_a(M)\big)$. Hence $G^+_{\rho(x)}\big(q, T^{\mathbb{C}}_a(M)\big)$ is not contractible.

[2] Although not mentioned in [BaLa], also there the existence of such a map is used. The first author wishes to thank an unknown referee for this remark.

such that $T(x)\in G^+_{\rho(x)}\big(q,T^{\mathbb{C}}_a(M)\big)$ *for all* $x\in Q$.

Earlier we believed that this corollary can be proved by elementary and very simple arguments using the contractibility and compactness of $[0,1]^{k-1}$ (cp. page 396 in [LaLe]). However, we have to confess that we are unable to do this using only these two topological properties. Our proof of theorem 3.2 indeed is completely elementary, but not very simple, and we use much more properties of $[0,1]^{k-1}$ than only contractibility and compactness. For example, we do not know whether the assertion of corollary 1.3 remains true if Q is the subset of the Hilbert space l_2 which consists of the sequences $\{\xi_j\}_{j\in\mathbb{N}^*}$ with $|\xi_j|\le 1/j$.

From now on, in this paper, we use the following notations:

- $n\in\mathbb{N}^*$, and $\mathcal{H}(n)$ is the space of Hermitian $n\times n$-matrices.
- For $H\in\mathcal{H}(n)$ we denote by $\mu_1(H),\dots,\mu_n(H)$ the eigenvalues of H such that $\mu_1(H)\ge\dots\ge\mu_n(H)$. We set $\mu_{n+1}(H)=-\infty$.
- For $H\in\mathcal{H}$ and $\lambda\in\mathbb{R}$ we denote by $E_\lambda(H)$ the subspace of $\mathbb{C}^n$ spanned by all eigenvectors of H which associated to eigenvalues $\ge\lambda$, i.e. if $e_1,\dots,e_n$ is a basis of $\mathbb{C}^n$ with $He_j=\mu_j(H)e_j$, $1\le j\le n$, and if $j(\lambda)$ is the index with $\mu_{j(\lambda)}\ge\lambda$ and $\mu_{j(\lambda)+1}<\lambda$, then $E_\lambda(H)$ is spanned by $e_1,\dots,e_{j(\lambda)}$.
- For $0\le q\le n$, we denote by $G(q,n)$ the Grassmann manifold of complex q-dimensional subspaces of $\mathbb{C}^n$.
- $(\cdot,\cdot)$ is the Euclidean scalar product, and $\|\cdot\|$ the Euclidean norm in $\mathbb{C}^n$.
- If V is a subspace of $\mathbb{C}^n$ and $H\in\mathcal{H}(n)$, then we set

$$\mu(H,V)=\inf_{v\in V\,,\,\|v\|=1}(Hv,v).$$

2. A gluing lemma

2.1. **Definition.** A pair (K_0,K_1) will be called a **bump in** $\mathbb{R}^k$ if K_0, K_1 are compact subsets of $\mathbb{R}^k$ such that there exists a neighborhood U_1 of $\overline{K_1\setminus K_0}$ with

$$(\overline{K_0\setminus K_1})\cap\overline{U}_1=\emptyset. \tag{2.1}$$

If this neighborhood U_1 can be chosen so that, moreover, $(K_0\cap K_1)\setminus U_1$ is contractible, then (K_0,K_1) will be called **simple**.

2.2. **Lemma.** *Let* (K_0,K_1) *be a bump in* $\mathbb{R}^k$, $k\in\mathbb{N}^*$, *and let*

$$A:K_0\cup K_1\to\mathcal{H}(n)\,,\qquad T:K_1\to G(n,q)$$

be continuous maps, $n\in\mathbb{N}^*$, $1\le q\le n$. *Set*

$$\mu_q(x)=\mu_q\big(A(x)\big)\qquad \textit{for } x\in K_0\cup K_1.$$

Let $\varepsilon,\delta>0$ *and suppose that*

$$\mu\big(A(x),T(x)\big)\ge\mu_q(x)-\varepsilon\qquad \textit{for all } x\in K_0\cap K_1, \tag{2.2}$$

$$\|A(x)-A(y)\|\le\delta\qquad \textit{for all } x,y\in K_0, \tag{2.3}$$

and

$$\big|\mu_q(x)-\mu_q(y)\big|\le\delta\qquad \textit{for all } x,y\in K_0, \tag{2.4}$$

Moreover we assume that at least one of the following two conditions is fulfilled:

(i) (K_0,K_1) *is simple.*

(ii) $\dim E_{\mu_q(x)-\varepsilon-\delta}\big(A(x)\big) = q$ *for all* $x \in K_0 \cap K_1$.[3]

Then there exists a continuous map $\widetilde{T} : K_0 \cup K_1 \to G(n,q)$ *such that*

$$\widetilde{T}(x) = T(x) \qquad \text{if } x \in K_1 \setminus K_0, \tag{2.5}$$

$$\mu\Big(A(x), \widetilde{T}(x)\Big) \geq \mu_q(x) - 2\delta \qquad \text{if } x \in K_0 \setminus K_1, \tag{2.6}$$

$$\mu\Big(A(x), \widetilde{T}(x)\Big) \geq \mu_q(x) - \varepsilon - 3\delta \qquad \text{if } x \in K_0 \cap K_1. \tag{2.7}$$

Proof. Let U_1 be a neighborhood of $\overline{K_1 \setminus K_0}$ with (2.1). Fix $x_0 \in (K_0 \cap K_1) \setminus U_1$ and set

$$E = E_{\mu_q(x_0)-\varepsilon-\delta}\big(A(x_0)\big).$$

Let E' be the orthogonal complement of E in $\mathbb{C}^n$. Then

$$E' \cap T(x) = \{0\} \qquad \text{for all } x \in K_0 \cap K_1. \tag{2.8}$$

Indeed, assume there exists $0 \neq v \in E' \cap T(x)$ where $x \in K_0 \cap K_1$. Since v is orthogonal to E, then v is a non-trivial linear combination of eigenvectors of $A(x_0)$ associated to eigenvalues $< \mu_q(x_0) - \varepsilon - \delta$. Therefore

$$\big(A(x_0)v, v\big) < \big(\mu_q(x_0) - \varepsilon - \delta\big)\,\|v\|^2.$$

By (2.3) this implies

$$\big(A(x)v, v\big) \leq \delta\|v\|^2 + \big(A(x_0)v, v\big) < \big(\mu_q(x_0) - \varepsilon\big)\|v\|^2.$$

Since $v \in T(x)$, this contradicts (2.2).

Let P be the orthogonal projection from $\mathbb{C}^n$ to E, and let $Q(x)$ be the orthogonal projection from C^n to $T(x)$, $x \in K_0 \cap K_1$. Set

$$B(t,x) = tPQ(x) + (1-t)Q(x) \qquad \text{for } 0 \leq t \leq 1 \text{ and } x \in K_0 \cap K_1.$$

Since $Q(x)$ is a projection onto $T(x)$ and, by (2.8), $\operatorname{Ker} P \cap \operatorname{Im} Q(x) = \{0\}$, $B(t,x)\big|_{T(x)}$ is injective. Therefore

$$M(t,x) := B(t,x)\Big(T(x)\Big)\,, \qquad 0 \leq t \leq 1\,,\ x \in K_0 \cap K_1.$$

defines a continuous map

$$M : [0,1] \times \big(K_0 \cap K_1\big) \longrightarrow G(n,q).$$

Let $v \in M(t,x)$, $0 \leq t \leq 1$, $x \in K_0 \cap K_1$, and let $w \in T(x)$ be the vector with $v = B(t,x)w$. Since $\operatorname{Im} Q(x) = T(x)$, then

$$v = tPw + (1-t)w = Pw + (1-t)P'w$$

where P' is the orthogonal projection from $\mathbb{C}^n$ onto E'. This implies that

$$\|v\|^2 = \|Pw\|^2 + (1-t)^2\|P'w\|^2 \tag{2.9}$$

and

$$\begin{aligned}\Big(A(x_0)v, v\Big) &= \Big(A(x_0)Pw + (1-t)A(x_0)P'w\,,\ Pw + (1-t)P'w\Big)\\ &= \Big(A(x_0)Pw\,,\ Pw\Big) + (1-t)^2\Big(A(x_0)P'w\,,\ P'w\Big).\end{aligned}$$

[3] Here the interesting point is that $\dim E_{\mu_q(x)-\varepsilon-\delta}\big(A(x)\big) \leq q$. It is clear that always $\dim E_{\mu_q(x)-\varepsilon-\delta}\big(A(x)\big) \geq \dim E_{\mu_q(x)}\big(A(x)\big) \geq q$ (by definition of $\mu_q(x)$).

Since

$$\Big(A(x_0)P'w\,,\,P'w\Big)=\Big(A(x_0)w-A(x_0)Pw\,,\,w-Pw\Big)$$
$$=\big(A(x_0)w,w\big)-\big(A(x_0)Pw,w\big)-\big(A(x_0)w,Pw\big)+\big(A(x_0)Pw,Pw\big)$$
$$=\big(A(x_0)w,w\big)-\big(A(x_0)w,Pw\big)=\big(A(x_0)w,w\big)-\big(A(x_0)Pw,Pw\big),$$

the second equation implies

$$\Big(A(x_0)v,v\Big)=\Big(A(x_0)Pw\,,\,Pw\Big)+(1-t)^2\Big(\Big(A(x_0)w\,,\,w\Big)-\Big(A(x_0)Pw\,,\,Pw\Big)\Big)$$
$$=\Big(1-(1-t)^2\Big)\Big(A(x_0)Pw\,,\,Pw\Big)+(1-t)^2\Big(A(x_0)w\,,\,w\Big).$$

By definition of $E=\operatorname{Im}P$, it follows that

$$\Big(A(x_0)v,v\Big)\geq\Big(1-(1-t)^2\Big)\Big(\mu_q(x_0)-\varepsilon-\delta\Big)\|Pw\|^2+(1-t)^2\Big(A(x_0)w\,,\,w\Big).$$

Since, by (2.3) and (2.2),

$$\Big(A(x_0)w\,,\,w\Big)\geq\Big(A(x)w\,,\,w\Big)-\delta\|w\|^2\geq\Big(\mu_q(x)-\varepsilon-\delta\Big)\|w\|^2,$$

this yields

$$\Big(A(x_0)v,v\Big)\geq\Big(1-(1-t)^2\Big)\Big(\mu_q(x_0)-\varepsilon-\delta\Big)\|Pw\|^2$$
$$+(1-t)^2\Big(\mu_q(x)-\varepsilon-\delta\Big)\|w\|^2$$

and further, by (2.4),

$$\Big(A(x_0)v,v\Big)\geq\Big(\mu_q(x)-\varepsilon-2\delta\Big)\Big(\Big(1-(1-t)^2\Big)\|Pw\|^2+(1-t)^2\|w\|^2\Big)$$
$$=\Big(\mu_q(x)-\varepsilon-2\delta\Big)\Big(\|Pw\|^2+(1-t)^2\Big(\|w\|^2-\|Pw\|^2\Big)\Big)$$
$$=\Big(\mu_q(x)-\varepsilon-2\delta\Big)\Big(\|Pw\|^2+(1-t)^2\|P'w\|^2\Big).$$

In view of (2.9) this means

$$\Big(A(x_0)v,v\Big)\geq\Big(\mu_q(x)-\varepsilon-2\delta\Big)\|v\|^2.$$

Using again (2.3), we obtain

$$\Big(A(x)v,v\Big)\geq\Big(A(x_0)v,v\Big)-\delta\|v\|^2\geq\Big(\mu_q(x)-\varepsilon-3\delta\Big)\|v\|^2.$$

i.e. we have proved that

$$\mu\Big(A(x),M(t,x)\Big)\geq\mu_q(x)-\varepsilon-3\delta \tag{2.10}$$

for all $0\leq t\leq 1$ and $x\in K_0\cap K_1$.

Since U_1 is a neighborhood of $\overline{K_1\setminus K_0}$, we can find a continuous map $\chi_1:\mathbb{R}^k\to[0,1]$ such that $\chi_1\equiv 0$ in a neighborhood of $\overline{K_1\setminus K_0}$ and $\chi_1\equiv 1$ outside a compact subset of U_1. Set

$$T'(x)=\begin{cases}M\big(\chi_1(x),x\big) & \text{if } x\in U_1\cap K_1\\ M(1,x) & \text{if } x\in K_1\setminus U_1.\end{cases}$$

Then T' is a continuous map from K_1 to $G(n,q)$ such that

$$T'(x) = M\big(0,x\big) = T(x) \qquad \text{if } x \in K_1 \setminus K_0, \tag{2.11}$$

$$T'(x) = M\big(1,x\big) = P\Big(T(x)\Big) \subseteq E \qquad \text{if } x \in K_1 \setminus U_1, \tag{2.12}$$

and, by (2.10),

$$\mu\Big(A(x), T'(x)\Big) \geq \mu_q(x) - \varepsilon - 3\delta \qquad \text{for all } x \in K_0 \cap K_1. \tag{2.13}$$

If condition (ii) is fulfilled, then $E = E_{\mu_q(x_0)}$ and it follows from (2.12) that

$$T'(x) = E_{\mu_q(x_0)} \qquad \text{for all } x \in K_1 \setminus U_1. \tag{2.14}$$

Setting

$$\widetilde{T}(x) = \begin{cases} T'(x) & \text{if } x \in K_1 \\ E_{\mu_q(x_0)} & \text{if } x \in K_0 \setminus K_1, \end{cases}$$

then we complete the proof.

Now we assume that condition (i) is satisfied (and, possibly, (ii) is not). Then U_1 can be chosen so that $(K_1 \cap K_0) \setminus U_1$ is contractible, and, since $x_0 \in (K_1 \cap K_0) \setminus U_1$, we can find a continuous map

$$\phi : [0,1] \times \Big((K_1 \cap K_0) \setminus U_1\Big) \longrightarrow (K_1 \cap K_0) \setminus U_1$$

such that, for all $x \in (K_1 \cap K_0) \setminus U_1$,

$$\phi(0,x) = x \qquad \text{and} \qquad \phi(1,x) = x_0.$$

Since $(\overline{K_0 \setminus K_1}) \cap \overline{U}_1 = \emptyset$, we can find a continuous map $\chi_2 : \mathbb{R}^k \to [0,1]$ such that $\chi_2 \equiv 0$ in a neighborhood of $\overline{U}_1$ and $\chi_2 \equiv 1$ in a neighborhood U_0 of $\overline{K_0 \setminus K_1}$. Then, setting

$$T''(x) = \begin{cases} T'(x) & \text{if } x \in U_1 \cap K_1 \\ T'\Big(\phi\Big(\chi_2(x), x\Big)\Big) & \text{if } x \in K_1 \setminus U_1 \\ T'(x_0) & \text{if } x \in K_0 \cap U_0 \end{cases}$$

we obtain a continuous map $T'' : K_0 \cup K_1 \to G(n,q)$.

Since $x_0 \in (K_0 \cap K_1) \setminus U_1$, it follows from (2.12) that $T'(x_0) \subseteq E$. Since the Grassmann manifold $G(E,q)$ of q-dimensional subspaces of E is connected, we can find a continuous map $S : [0,1] \to G(E,q)$ such that $S(0) = T'(x_0)$ and $S(1) \subseteq E_{\mu_q(x_0)}\big(A(x_0)\big)$. Then

$$\mu\big(A(x_0), S(1)\big) \geq \mu_q(x_0) \tag{2.15}$$

and

$$\mu\big(A(x_0), S(t)\big) \geq \mu_q(x_0) - \varepsilon - \delta \qquad \text{for all } 0 \leq t \leq 1. \tag{2.16}$$

Choose a continuous function $\chi_3 : \mathbb{R}^k \to [0,1]$ such that $\chi_3 \equiv 1$ in a neighborhood of $\overline{K_0 \setminus K_1}$ and $\chi_3 \equiv 0$ outside a compact subset of U_0. Setting

$$\widetilde{T}(x) = \begin{cases} T''(x) & \text{if } x \in K_1 \setminus U_0 \\ S\big(\chi_3(x)\big) & \text{if } x \in K_0 \cap U_0 \end{cases}$$

we get a continuous map $\widetilde{T} : K_0 \cup K_1 \to G(n,q)$ with

$$\widetilde{T}(x) = \begin{cases} T'(x) & \text{if } x \in U_1 \cap K_1 \\ T'\Big(\phi\Big(\chi_2(x), x\Big)\Big) & \text{if } x \in K_1 \setminus (U_0 \cup U_1) \\ S\big(\chi_3(x)\big) & \text{if } x \in U_0 \cap K_0 \\ S(1) & \text{if } x \in K_0 \setminus K_1 \end{cases}$$

If $x \in K_1 \setminus K_0$, then it follows from (2.11) that

$$\widetilde{T}(x) = T'(x) = T(x),$$

i.e. we have (2.5). If $x \in K_0 \setminus K_1$, then it follows from (2.3), (2.15) and (2.4) that

$$\mu\Big(A(x), \widetilde{T}(x)\Big) = \mu\Big(A(x), S(1)\Big) \geq \mu\Big(A(x_0), S(1)\Big) - \delta \geq \mu_q(x_0) - \delta \geq \mu_q(x) - 2\delta,$$

i.e. we have (2.6).

Finally we prove (2.7). Consider an arbitrary $x \in K_0 \cap K_1$. Then, we are in at least one of the following three cases:

1) $x \in U_0 \cap K_0$, 2) $x \in U_1 \cap K_1$ or 3) $x \in K_1 \setminus (U_0 \cup U_1)$.

In case 1) it follows from (2.3), (2.16) and (2.4) that

$$\begin{aligned} \mu\big(A(x), \widetilde{T}(x)\big) = \mu\Big(A(x), S\big(\chi_3(x)\big)\Big) &\geq \mu\Big(A(x_0), S\big(\chi_3(x)\big)\Big) - \delta \\ &\geq \mu_q(x_0) - \varepsilon - 2\delta \geq \mu_q(x) - \varepsilon - 3\delta. \end{aligned}$$

In case 2) we get from (2.13) that

$$\mu\big(A(x), \widetilde{T}(x)\big) = \mu\Big(A(x), T'(x)\big)\Big) \geq \mu_q(x) - \varepsilon - 3\delta,$$

and in case 3) it follows from (2.12) that

$$\mu\big(A(x), \widetilde{T}(x)\big) = \mu\bigg(A(x), T'\Big(\phi\Big(\chi_2(x), x\Big)\Big)\bigg) \geq \mu_q(x) - \varepsilon - 3\delta.$$

□

3. Continuous sections over cubes

Let $I = [0,1]$ be the closed unit interval and

$$I^k := \underbrace{I \times \ldots \times I}_{k \text{ times}} \qquad \text{for } k \in \mathbb{N}^*.$$

For $N \in \mathbb{N}^*$ and $0 \leq j \leq N-1$ we set

$$I(N,j) = \left[\frac{j}{N}, \frac{j+1}{N}\right] \quad \text{and} \quad I'(N,j) = \left[\frac{j}{N} - \frac{1}{3N}, \frac{j+1}{N} + \frac{1}{3N}\right].$$

For $N, k \in \mathbb{N}$, we denote by $P(k,N)$ the set of all k-tuples $(j_0, \ldots, j_{k-1})$ with $j_0, \ldots, j_{k-1} \in \{0, \ldots, N-1\}$. We give $P(k,N)$ the lexicographical order, i.e.

$$(j_0, \ldots, j_{k-1}) < (m_0, \ldots, m_{k-1})$$

if and only if there exists $0 \leq s \leq k-1$ such that $j_s < m_s$ and (if $s > 0$) $j_\nu = m_\nu$ for $0 \leq \nu < s$. For $\mathcal{J} = (j_0, \ldots, j_{k-1}) \in P(k,N)$ with $\mathcal{J} \neq (0, \ldots, 0)$, we set

$$I(N, \mathcal{J}) = I(N, j_0) \times \ldots \times I(N, j_{k-1}),$$

$$I'(N, \mathcal{J}) = I'(N, j_0) \times \ldots \times I'(N, j_{k-1}),$$

$$K_1(N,\mathcal{J}) = \bigcup_{\mathcal{M}\in P(k,N)\,,\,\mathcal{M}<\mathcal{J}} I(N,\mathcal{M})$$

and

$$K_0(N,\mathcal{J}) = I(N,\mathcal{J}) \cup \Big(I'(N,\mathcal{J}) \cap K_1(N,\mathcal{J})\Big).$$

3.1**. Lemma.** *Let $k, N \in \mathbb{N}^*$ and $\mathcal{J} \in P(k,N)$ with $\mathcal{J} \neq (0,\dots,0)$. Then $\Big(K_0(N,\mathcal{J})\,,\,K_1(N,\mathcal{J})\Big)$ is a simple bump in $\mathbb{R}^k$.*

Proof. Let $\mathcal{J} = (j_0,\dots,j_{k-1})$. Set

$$I''(N,j) = \left[\frac{j}{N} - \frac{1}{6N}, \frac{j+1}{N} + \frac{1}{6N}\right], \qquad 0 \le j \le N-1,$$

$$I''(N,\mathcal{J}) = I''(N,j_0) \times \ldots \times I''(N,j_{k-1}) \quad \text{and} \quad U_1 = \mathbb{R}^k \setminus I''(N,\mathcal{J}).$$

Then it is clear that U_1 is a neighborhood of $K_1(N,\mathcal{J}) \setminus K_0(N,\mathcal{J})$ and

$$\Big(\overline{K_0(N,\mathcal{J}) \setminus K_1(N,\mathcal{J})}\Big) \cap \overline{U}_1 = \emptyset.$$

It remains to prove that

$$\Big(K_0(N,\mathcal{J}) \cap K_1(N,\mathcal{J})\Big) \setminus U_1$$

is contractible. First note that

$$\begin{aligned}\text{(3.1)} \quad \Big(K_0(N,\mathcal{J}) \cap K_1(N,\mathcal{J})\Big) \setminus U_1 &= K_0(N,\mathcal{J}) \cap K_1(N,\mathcal{J}) \cap I''(N,\mathcal{J}) \\ &= K_1(N,\mathcal{J}) \cap I''(N,\mathcal{J}).\end{aligned}$$

Consider the indices $0 \le \nu_1 < \ldots < \nu_\lambda \le k-1$ such that $j_{\nu_s} > 0$ for $s = 1,\dots,\lambda$ and $j_\mu = 0$ if $\mu \in \{0,\dots,k-1\} \setminus \{\nu_1,\dots,\nu_\lambda\}$. (Since $\mathcal{J} \neq (0,\dots,0)$, such indices exist.) Now we set

$$A_s = \left[0, \frac{j_0+1}{N}\right] \times \ldots \times \left[0, \frac{j_{\nu_s-1}+1}{N}\right] \times \left[0, \frac{j_{\nu_s}}{N}\right] \times I^{k-1-\nu_s}\,, \quad 1 \le s \le \lambda,$$

and prove that

$$\text{(3.2)} \qquad K_1(N,\mathcal{J}) = \bigcup_{s=1}^{\lambda} A_s.$$

To prove "$\subseteq$", let $\mathcal{M} = (m_0,\dots,m_{k-1}) \in P(k,N)$ with $\mathcal{M} < \mathcal{J}$. Then there exists $1 \le s \le \lambda$ with $m_{\nu_s} + 1 \le j_{\nu_s}$ and $m_\mu = j_\mu$ for $0 \le \mu \le j_{\nu_s} - 1$, and it follows that

$$\begin{aligned} I(N,\mathcal{M}) = &\left[\frac{j_0}{N}, \frac{j_0+1}{N}\right] \times \ldots \times \left[\frac{j_{\nu_s-1}}{N}, \frac{j_{\nu_s-1}+1}{N}\right] \times \left[\frac{m_{\nu_s}}{N}, \frac{m_{\nu_s}+1}{N}\right] \\ &\times \left[\frac{m_{\nu_s+1}}{N}, \frac{m_{\nu_s+1}+1}{N}\right] \times \ldots \times \left[\frac{m_{k-1}}{N}, \frac{m_{k-1}+1}{N}\right] \\ &\subseteq \left[0, \frac{j_0+1}{N}\right] \times \ldots \times \left[0, \frac{j_{\nu_s-1}+1}{N}\right] \times \left[0, \frac{m_{\nu_s}+1}{N}\right] \times I^{k-1-\nu_s} \\ &\subseteq \left[0, \frac{j_0+1}{N}\right] \times \ldots \times \left[0, \frac{j_{\nu_s-1}+1}{N}\right] \times \left[0, \frac{j_{\nu_s}}{N}\right] \times I^{k-1-\nu_s} = A_s. \end{aligned}$$

To prove "$\supseteq$", we consider $1 \le s \le \lambda$ and $x = (x_0, \ldots, x_{k-1}) \in A_s$. Then

$$0 \le x_\mu \le \frac{j_\mu + 1}{N} \quad \text{for} \quad 0 \le \mu \le \nu_s - 1 \qquad \text{and} \qquad 0 \le x_{\nu_s} \le \frac{j_{\nu_s}}{N}.$$

Hence, there exists $\mathcal{I} = (i_0, \ldots, i_{k-1}) \in P(k, N)$ with $x \in I(N, \mathcal{I})$ such that the following condition is satisfied:

$$0 \le i_\mu \le j_\mu \quad \text{for} \quad 0 \le \mu \le \nu_s - 1 \qquad \text{and} \qquad 0 \le i_{\nu_s} < j_{\nu_s}.$$

As this condition yields $\mathcal{I} < \mathcal{J}$, it follows that $x \in K_1(N, \mathcal{J})$. Hence (3.1) is proved.

Further we set

$$I'''(N, j) = I \cap I''(N, j)\,, \qquad 0 \le j \le N - 1.$$

Then

$$A_s \cap I''(N, \mathcal{J}) = I'''(N, j_0) \times \ldots \times I'''(N, j_{\nu_s - 1}) \times \left[\frac{j_{\nu_s}}{N} - \frac{1}{6N}, \frac{j_{\nu_s}}{N}\right] \times I'''(N, j_{\nu_s + 1}) \times \ldots \times I'''(N, j_{k-1})\,, \qquad 1 \le s \le \lambda. \tag{3.3}$$

Setting for $0 \le r \le k - 1$,

$$I^*(r) = \begin{cases} I'''(N, 0) & \text{if } j_r = 0 \\ \left[\frac{j_{\nu_s}}{N} - \frac{1}{6N}, \frac{j_{\nu_s}}{N}\right] & \text{if } r = \nu_s \text{ for some } 1 \le s \le \lambda, \end{cases}$$

it follows that

$$\bigcap_{s=1}^{\lambda} A_s \cap I''(N, \mathcal{J}) = I^*(0) \times \ldots I^*(k-1) \ne \emptyset. \tag{3.4}$$

Conclusion: By (3.1) and (3.2), the set $\Big(K_0(N, \mathcal{J}) \cap K_1(N, \mathcal{J})\Big) \setminus U_1$ is the union of the sets $A_s \cap I''(N, \mathcal{J})$, $1 \le s \le \lambda$, where each of these sets is a cube (by (3.3)) and the intersection of these sets is not empty (by (3.4)). Hence $\Big(K_0(N, \mathcal{J}) \cap K_1(N, \mathcal{J})\Big) \setminus U_1$ is the union of a family of convex sets with non-empty intersection. Such sets are contractible. $\square$

3.2. **Theorem.** *Let $A : I^k \to \mathcal{H}(n)$ be a continuous map, $k, n \in \mathbb{N}^*$, $1 \le q \le n$. Then for each $\varepsilon > 0$, there exists a continuous map*

$$T : I^k \to G(n, q)$$

such that

$$\mu\Big(A(x), T(x)\Big) \ge \mu_q\Big(A(x)\Big) - \varepsilon \qquad \textit{for all } x \in I^k. \tag{3.5}$$

Proof. Set

$$\mu_q(x) = \mu_q\big(A(x)\big) \qquad \text{for } x \in I^k,$$
$$\delta = \frac{\varepsilon}{3^{k+1}},$$

and choose $N \in \mathbb{N}^*$ so big that, for each $\mathcal{J} \in P(k, N)$ and all $x, y \in I'(N, \mathcal{J})$,

$$\|A(x) - A(y)\| \le \delta \tag{3.6}$$

and

$$|\mu_q(x) - \mu_q(y)| \le \delta. \tag{3.7}$$

Let $\mathcal{J}_1, \dots, \mathcal{J}_{N^k}$ be the elements of $P(k, N)$ numbered by the lexicographical order of $P(k, N)$. For $x \in I^k$ and $1 \le m \le N^k$ we denote by $\theta_m(x)$ the number of numbers $\mu \in \{1, \dots, m\}$ with $x \in I'(N, \mathcal{J}_\mu)$. Note that

$$1 \le \theta_m(x) \le 3^k \qquad \text{for all } x \in I^k \text{ and } 1 \le m \le N^k$$

and

$$\theta_m(x) \le \theta_{m+1}(x) \le \theta_m(x) + 1 \qquad \text{for all } x \in I^k \text{ and } 1 \le m \le N^k - 1.$$

Now we prove by induction over m that, for $1 \le m \le N^k$, the following statement holds:

$S(m)$: *There exists a continuous map*

$$T_m : I(N, \mathcal{J}_1) \cup \dots \cup I(N, \mathcal{J}_m) \to G(n, q)$$

such that

$$\mu\Big(A(x), T_m(x)\Big) \ge \mu_q(x) - 3\theta_m(x)\delta \tag{3.8}$$

for all $x \in I(N, \mathcal{J}_1) \cup \dots \cup I(N, \mathcal{J}_m)$.

Then assertion of the theorem then follows from $S(N^k)$, because $I^k = I(N, \mathcal{J}_1) \cup \dots \cup I(N, \mathcal{J}_{N^k})$ and $3\theta_{N^k}(x)\delta \le \varepsilon$ for all $x \in I^k$.

Proof of $S(1)$: Choose a point $x_0 \in I(N, J_0)$, a space $T_0 \in G(n, q)$ with $\mu\big(A(x_0), T_0\big) = \mu_q(x_0)$ and set $T_1(x) = T_0$ for $x \in I(N, J_0)$. Then it follows from (3.6) and (3.7) that

$$\begin{aligned}\mu\Big(A(x), T_1(x)\Big) &\ge \mu\Big(A(x), T_0\Big) \ge \mu\Big(A(x_0), T_0\Big) - \delta \\ &= \mu_q(x_0) - \delta \ge \mu_q(x) - 2\delta > \mu_q(x) - 3\delta = \mu_q(x) - 3\theta_1(x)\delta.\end{aligned}$$

Proof of $S(m) \Rightarrow S(m+1)$: By lemma 3.1,

$$\bigg(I(N, \mathcal{J}_{m+1}) \cup \Big(I'(\mathcal{J}_{m+1}) \cap \Big(I(N, J_1) \cup \dots \cup I(N, J_m)\Big)\Big)\ ,\ I(N, J_1) \cup \dots \cup I(N, J_m)\bigg)$$

is a simple bump in $\mathbb{R}^k$. Applying lemma 2.2 to this bump and the map T_m, we obtain a continuous map

$$T_{m+1} : I(N, \mathcal{J}_1) \cup \dots \cup I(N, \mathcal{J}_{m+1}) \to G(n, q)$$

such that $T_{m+1}(x) = T_m(x)$ and therefore, by (3.8),

$$\mu\Big(A(x), T_{m+1}(x)\Big) \ge \mu_q(x) - 3\theta_m(x)\delta \tag{3.9}$$

if $x \in \Big(I(N, \mathcal{J}_1) \cup \dots \cup I(N, \mathcal{J}_m)\Big) \setminus I'(N, \mathcal{J}_{m+1})$,

$$\mu\Big(A(x), T_{m+1}(x)\Big) \ge \mu_q(x) - 2\delta \tag{3.10}$$

if $x \in I(N, \mathcal{J}_{m+1})$ and

$$\mu\Big(A(x), T_{m+1}(x)\Big) \ge \mu_q(x) - 3\theta_m(x)\delta - 3\delta \tag{3.11}$$

if $x \in I'(\mathcal{J}_{m+1}) \cap \Big(I(N, J_1) \cup \dots \cup I(N, J_m)\Big)$.

Since $\theta_{m+1}(x) = \theta_m(x)$ if $x \in \Big(I(N,\mathcal{J}_1) \cup \ldots \cup I(N,\mathcal{J}_m)\Big) \setminus I'(N,\mathcal{J}_{m+1})$, it follows from (3.9) that

$$\mu\Big(A(x), T_{m+1}(x)\Big) \geq \mu_q(x) - 3\theta_{m+1}(x)\delta$$

if $x \in \Big(I(N,\mathcal{J}_1) \cup \ldots \cup I(N,\mathcal{J}_m)\Big) \setminus I'(N,\mathcal{J}_{m+1})$. Since always $3\theta_{m+1}(x) \geq 3 > 2$, it follows from (3.10) that

$$\mu\Big(A(x), T_{m+1}(x)\Big) \geq \mu_q(x) - 3\theta_{m+1}(x)\delta$$

if $x \in I(N,\mathcal{J}_{m+1})$. Since $\theta_{m+1}(x) = \theta_m(x) + 1$ if $x \in I'(\mathcal{J}_{m+1}) \cap \Big(I(N,J_1) \cup \ldots \cup I(N,J_m)\Big)$, it follows from (3.11) that

$$\mu\Big(A(x), T_{m+1}(x)\Big) \geq \mu_q(x) - 3\theta_{m+1}(x)\delta$$

if $x \in I'(\mathcal{J}_{m+1}) \cap \Big(I(N,J_1) \cup \ldots \cup I(N,J_m)\Big)$. The last three inequalities together mean that

$$\mu\Big(A(x), T_{m+1}(x)\Big) \geq \mu_q(x) - 3\theta_{m+1}(x)\delta$$

holds for all $x \in I(N,J_1) \cup \ldots \cup I(N,J_{m+1})$. Thus $S(m+1)$ is proved. □

4. Continuous sections over spheres

4.1. **Theorem.** *Let S^k be the unit sphere in $\mathbb{R}^{k+1}$, $k \geq 2$, and let $A : S^k \to \mathcal{H}(n)$ be a continuous map, $n \in \mathbb{N}^*$. We use the abbreviation*

$$\mu_j(x) = \mu_j\Big(A(x)\Big) \qquad \text{for } x \in S^k \text{ and } 1 \leq j \leq n.$$

Assume $1 \leq q \leq n$ is an integer such that

$$\mu_q(x) > 0 \qquad \text{for all } x \in S^k \tag{4.1}$$

and either $q = n$ or there exists at least one point $x_0 \in S^k$ with

$$\mu_{q+1}(x_0) \leq 0.$$

Then there exists a continuous map

$$T : S^k \to G(n,q)$$

such that

$$\mu\Big(A(x), T(x)\Big) > 0 \qquad \text{for all } x \in S^k. \tag{4.2}$$

Moreover, for each $\varepsilon > 0$, this map can be chosen so that

$$\mu\Big(A(x), T(x)\Big) \geq \mu_q\Big(A(x)\Big) - \varepsilon \qquad \text{for all } x \in S^k. \tag{4.3}$$

Proof. Let $\varepsilon > 0$ be given. Set

$$\delta = \min\left\{\frac{\varepsilon}{10}, \frac{1}{10}\min_{x \in S^k} \mu_q(x)\right\}.$$

Choose two open balls B and B' in $\mathbb{R}^{k+1}$ centered at x_0 such that $\overline{B}' \subseteq B$ and B is so small that, for

$$K_0 := \overline{B} \cap S^k,$$

we have the estimates

$$\mu_{q+1}(x) \leq \delta\,, \qquad x \in K_0,$$
$$\|A(x) - A(y)\| \leq \delta\,, \qquad x, y \in K_0,$$
$$\|\mu_q(x) - \mu_q(y)\| \leq \delta\,, \qquad x, y \in K_0,$$

and set $K_1 = S^{k+1} \setminus B'$. Since K_1 is homeomorphic to the unit cube I^k, by theorem 3.2, there exists a continuous map

$$T_1 : K_1 \to G(n, q)$$

such that

$$\mu\Big(A(x), T_1(x)\Big) \geq \mu_q(x) - \delta \qquad \text{for all } x \in K_1.$$

Since $\mu_q(x) - 2\delta \geq 8\delta$ and $\mu_{q+1}(x) \leq \delta$ for $x \in K_0 \cap K_1$, it is clear that

$$\dim E_{\mu_q(x)-2\delta} = q \qquad \text{for } x \in K_0 \cap K_1.$$

Hence, for $\varepsilon = \delta$, the hypotheses of lemma 2.2 are satisfied (with condition (ii)), and we obtain from this lemma a continuous map $T : S^k = K_0 \cup K_1 \to G(n, q)$ such that

$$\mu\Big(A(x), T(x)\Big) = \mu\Big(A(x), T_1(x)\Big) \geq \mu_q(x) - \delta > 0 \qquad \text{if } x \in K_1 \setminus K_0,$$
$$\mu\Big(A(x), T(x)\Big) \geq \mu_q(x) - 2\delta > 0 \qquad \text{if } x \in K_0 \setminus K_1,$$
$$\mu\Big(A(x), T(x)\Big) \geq \mu_q(x) - 4\delta > 0 \qquad \text{if } x \in K_0 \cap K_1.$$

Since $4\delta < \varepsilon$, this completes the proof. □

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