ON COVERING AND QUASI-UNSPLIT FAMILIES OF RATIONAL CURVES

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Abstract. We study extremality properties of covering families of rational curves on projective varieties. Among others, we show that on a normal and $\mathbb{Q}$-factorial projective variety X with $\dim(X) \leq 4$, every covering and quasi-unsplit family of rational curves generates a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$ of classes of effective 1-cycles.

1. Introduction

Let X be a normal and uniruled complex projective variety. Consider an irreducible and closed subset V of $\mathrm{Chow}(X)$ such that:

- ◦ any element of V is a cycle whose irreducible components are rational curves;
- ◦ V is covering (which means that for any point $x \in X$, there exists an element of V passing through x).

We call such a V a *covering family of rational 1-cycles* on X. If moreover, all irreducible components of the cycles parametrized by V are numerically proportional, we call V a *covering and quasi-unsplit family of rational 1-cycles* on X (see [CO04, Definition 2.13]).

For any covering family V of rational 1-cycles on X, we will denote by $[V]$ the numerical class in $\mathrm{NE}(X)$ of the general cycle of the family V and by $\mathbb{R}_{\geq 0}[V]$ the half-line generated by $[V]$.

A *geometric extremal ray* of the Mori cone $\overline{\mathrm{NE}}(X)$ is a half-line $R \subseteq \overline{\mathrm{NE}}(X)$ such that if $\gamma_1 + \gamma_2 \in R$ for some $\gamma_1, \gamma_2 \in \overline{\mathrm{NE}}(X)$, then $\gamma_1, \gamma_2 \in R$ (see Section 2 for precise definitions and notation).

Question. Let V be a covering and quasi-unsplit family of rational 1-cycles on X. Is $\mathbb{R}_{\geq 0}[V]$ a geometric extremal ray of $\overline{\mathrm{NE}}(X)$?

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Note that this question is natural, since any family of rational 1-cycles such that the general member generates a geometric extremal ray of $\overline{\mathrm{NE}}(X)$ is quasi-unsplit. The converse is not true if the family is not covering (just think of a smooth blow-down of a smooth projective variety to a non projective one).

Let V be any covering family of rational 1-cycles on X. Then V defines set-theoretically an equivalence relation on X: two points x, x' are *V-equivalent* if there exist $v_1, \dots, v_m \in V$ such that some connected component of $C_{v_1} \cup \dots \cup C_{v_m}$ contains x and x', where $C_v \subset X$ is the curve corresponding to $v \in V$.

In this situation, after Campana's results (see Section 2), there exists an almost holomorphic map $q\colon X \dashrightarrow Y$, to a projective algebraic variety, whose general fibers are V-equivalence classes.

We first prove the following result, which involves the dimension of the general fiber of q.

Theorem 1. *Let X be a normal and $\mathbb{Q}$-factorial complex projective variety of dimension n. Let V be a covering and quasi-unsplit family of rational 1-cycles on X, and let f_V be the dimension of a general V-equivalence class.*

If $f_V \geq n-3$, then $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$.

We then immediately get the following.

Corollary 1. *Let X be a normal and $\mathbb{Q}$-factorial complex projective variety of dimension $n \leq 4$. Let V be a covering and quasi-unsplit family of rational 1-cycles on X. Then $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$.*

As previously recalled, one can associate a rational map $q\colon X \dashrightarrow Y$ to any covering family of rational 1-cycles on X. We call a *geometric quotient* for V a morphism $q'\colon X \to Y'$, onto a normal projective variety Y', such that *every* fiber of q' is a V-equivalence class. If such a quotient exists, then it is clearly unique up to isomorphism. On the other hand, even if X is smooth, a geometric quotient for V does not necessarily exist (see example 1).

The study of the extremal contraction given by the previous result leads to the following.

Theorem 2. *Let X be a normal and $\mathbb{Q}$-factorial complex projective variety, having canonical singularities, of dimension n. Let V be a covering and quasi-unsplit family of rational 1-cycles on X, and let f_V be the dimension of a general V-equivalence class.*

If $f_V \geq n-3$, then the Mori contraction of $\mathbb{R}_{\geq 0}[V]$, $\mathrm{cont}_{[V]}\colon X \to Y'$, is the geometric quotient for V. If moreover $f_V \geq n-2$, then $\mathrm{cont}_{[V]}$ is equidimensional.

We finally consider the toric case, where we can prove both extremality and existence of the geometric quotient for a quasi-unsplit family in any dimension.

Theorem 3. *Let X be a toric and $\mathbb{Q}$-factorial complex projective variety, and let V be a quasi-unsplit covering family of rational 1-cycles in X. Then $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$, and the Mori contraction of $\mathbb{R}_{\geq 0}[V]$, $\mathrm{cont}_{[V]}\colon X \to Y'$, is the geometric quotient for V.*

The following is an immediate application of Theorems 1 and 3.

Corollary 2. *Let $X \subset \mathbb{P}^N$ be a normal and $\mathbb{Q}$-factorial variety, covered by lines. Assume either that X is toric, or that X has canonical singularities and $\dim X \leq 4$. Let V be an irreducible family of lines covering X.*

Then there exists a morphism $q' \colon X \to Y'$, onto a normal, $\mathbb{Q}$-factorial, projective variety Y' with $\rho_{Y'} = \rho_X - 1$,[2] such that all lines of V are contracted by q'.

2. Set-up on families of rational 1-cycles

Let X be a normal, irreducible, n-dimensional complex projective variety. We denote by $\mathcal{N}_1(X)_{\mathbb{R}}$ (respectively, $\mathcal{N}_1(X)_{\mathbb{Q}}$) the vector space of 1-cycles in X with real (respectively, rational) coefficients, modulo numerical equivalence. In $\mathcal{N}_1(X)_{\mathbb{R}}$, let $\overline{\mathrm{NE}}(X)$ be the closure of the cone generated by classes of effective 1-cycles in X.

Recall that the existence of a covering family V of rational 1-cycles on X is equivalent to X being uniruled [Kol96, Proposition IV.1.3].

For such family V, we have a diagram given by the incidence variety $\mathcal{C}$ associated to V:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & X \\ \downarrow{\scriptstyle \pi} & & \\ V & & \end{array} \tag{1}$$

where π and F are proper and surjective. We set $C_v := F(\pi^{-1}(v))$ for any $v \in V$.

The relation of V-equivalence on X induced by such a family was introduced and studied in [Cam81]; we refer the reader to [Cam04], [Deb01, §5.4] or [Kol96, §IV.4] for more details. In particular, there exists a rational map $q \colon X \dashrightarrow Y$ associated to V, whose main properties are recalled now. By [Deb01, Theorem 5.9], there exists a closed and irreducible subset of $\mathrm{Chow}(X)$ whose normalization Y satisfies the following properties:

(a) let $Z \subset Y \times X$ be the restriction of the universal family,

$$\begin{array}{ccc} Z & \xrightarrow{e} & X \\ \downarrow{\scriptstyle p} & \swarrow{\scriptstyle q} & \\ Y & & \end{array} \tag{2}$$

then e is birational and $q = p \circ e^{-1}$ is almost holomorphic (which means that the indeterminacy locus of q does not dominate Y);

(b) a general fiber of q is a V-equivalence class,

(c) a general fiber of q, hence of p, is irreducible.

As a consequence of the existence of this map q, a general V-equivalence class is a closed subset of X. We denote by f_V its dimension, so that $\dim Y = n - f_V$. Moreover, it is well known that any V-equivalence class is a countable union of closed subsets of X.

[2] We denote by ρ_Z the Picard number of an algebraic variety Z.

Definition 1. We say that a subset Z of X is *V-rationally connected* if every connected component of Z is contained in some V-equivalence class.

Lemma 1. *Let X be a normal projective variety and V be a covering family of rational 1-cycles on X. Consider the diagram* (2) *above. Then $e(p^{-1}(y))$ is V-rationally connected for any $y \in Y$.*

Proof. Let $\mathcal{R} \subset X \times X$ the graph of the equivalence relation defined by V: it is a countable union of closed subvarieties since V is proper. The fiber product $Z \times_Y Z$ is irreducible and thus $(e \times e)(Z \times_Y Z) \subset \mathcal{R}$ thanks to properties (a) and (b) above. Therefore, for any $x \in e(p^{-1}(y))$, the cycle $e(p^{-1}(y))$ is contained in the V-equivalence class of x. ■

The following well known remark will be of constant use (see [Kol96, Proposition IV.3.13.3], or [ACO04, Corollary 4.2]).

Remark 1. If $Z \subset X$ is V-rationally connected, every curve contained in Z is numerically equivalent in X to a linear combination with rational coefficients of irreducible components of cycles in V. In particular, if V is quasi-unsplit, the numerical class of every curve contained in a V-rationally connected subset Z of X belongs to $\mathbb{R}_{\geq 0}[V]$.

Finally, we will need the following.

Lemma 2. *Let X be a normal projective variety and V be a covering and quasi-unsplit family of rational 1-cycles on X. Then there exists a covering and quasi-unsplit family V' of rational 1-cycles on X such that:*

- *the general cycle of V' is reduced and irreducible;*
- *for any $v' \in V'$ there exists $v \in V$ such that $C_{v'} \subseteq C_v$; in particular $\mathbb{R}_{\geq 0}[V] = \mathbb{R}_{\geq 0}[V']$.*

Proof. Let $\mathcal{C}$ be the incidence variety associated to V as in (1).

It is well-known that every irreducible component of $\mathcal{C}$ dominates V, let $\mathcal{C}''$ be an irreducible component of $\mathcal{C}$ which dominates X too. Let $\mathcal{C}'$ be the normalization of $\mathcal{C}''$ and $\mathcal{C}' \to V'$ be the Stein factorization of the composite map $\mathcal{C}' \to \mathcal{C}'' \to V$. Since $\mathcal{C}' \to V'$ has connected fibers and $\mathcal{C}'$ is normal, the general fiber of $\mathcal{C}' \to V'$ is irreducible. Moreover, the image in X of every fiber of $\mathcal{C}' \to V'$ is contained in a cycle of V.

Since V' is normal, there is a holomorphic map $V' \to \mathrm{Chow}(X)$. Then after replacing V' by its image in $\mathrm{Chow}(X)$ and $\mathcal{C}'$ by its image in $\mathrm{Chow}(X) \times X$, we get the desired family. ■

3. Properties of the base locus and extremality

Let V be a covering family of rational 1-cycles on X, and recall the diagram (2) associated to V.

Let $E \subset Z$ be the exceptional locus of e, and $B := e(E) \subset X$. Observe that since X is normal, $\dim B \leq n - 2$.

Proposition 1. *Let X be a normal and $\mathbb{Q}$-factorial projective variety, and V be a covering and quasi-unsplit family of rational 1-cycles on X. Consider the associated diagram as in* (2)*. Then:*

(i) $e(p^{-1}(y))$ is a V-equivalence class of dimension f_V for every $y \in Y \setminus p(E)$;
(ii) B is the union of all V-equivalence classes of dimension bigger than f_V.

Proof. Set $X^0 := X \setminus B$ and $Y^0 := Y \setminus p(E) = q(X^0)$. Choose a very ample line bundle L on Y, and let $U \subset |L|$ be the open subset of divisors H that are irreducible and such that $H \cap Y^0 \neq \emptyset$. For any H in U, we define $\widehat{H} := \overline{q^{-1}(H \cap Y^0)}$, which is a Weil divisor in X. Since X is $\mathbb{Q}$-factorial, some multiple of $\widehat{H}$ defines a line bundle $\widehat{L}$ on X.

Let now $N := h^0(L)$, and let $s_1, \dots, s_N$ be general global sections generating L. For each $i = 1, \dots, N$, let $H_i \in |L|$ be the divisor of zeros of s_i and $\widehat{H}_i$ in X as defined above.

Let's show that $\widehat{H}_1 \cap \cdots \cap \widehat{H}_N = B$. If $x \notin B$, then q is defined in x and there is some $i_0 \in \{1, \dots, N\}$ such that $q(x) \notin H_{i_0}$, so $x \notin \widehat{H}_{i_0}$. Conversely, let $x \in B$ and fix $i \in \{1, \dots, N\}$. Then $e^{-1}(x)$ has positive dimension; let $C \subset Z$ be an irreducible curve such that $e(C) = x$. Then $p(C)$ is a curve in Y, hence $H_i \cap p(C) \neq \emptyset$ and $p^{-1}(H_i) \cap C \neq \emptyset$. Now observe that $p^{-1}(H_i)$ does not contain any component of E, hence $e(p^{-1}(H_i))$ is a divisor in X which coincides with $\widehat{H}_i$ over $X \setminus B$. Then $\widehat{H}_i = e(p^{-1}(H_i))$ and $x \in \widehat{H}_i$.

Let $i \in \{1, \dots, N\}$. Observe that $\widehat{H}_i \cdot [V] = 0$, because $[V]$ is quasi-unsplit and any irreducible component of general cycle of the family is contained in a fiber of q disjoint from $\widehat{H}_i$. This implies that $\widehat{H}_i$ is closed with respect to V-equivalence. In fact, let C be an irreducible component of a cycle of V such that $C \cap \widehat{H}_i \neq \emptyset$. Since V is quasi-unsplit, we have $\widehat{H}_i \cdot C = 0$, which implies $C \subseteq \widehat{H}_i$.

Now since $B = \widehat{H}_1 \cap \cdots \cap \widehat{H}_N$ and all $\widehat{H}_i$'s are closed with respect to V-equivalence, we see that B is a union of V-equivalence classes.

Observe that if $C \subset X \setminus B$ is an irreducible curve such that $\widehat{H} \cdot C = 0$ for some $H \in U$, then $q(C)$ is a point. In fact, if $q(C)$ is a curve, there exists $H_0 \in U$ such that H_0 intersects $q(C^0)$ in a finite number of points. Then $\widehat{H}_0$ intersects C without containing it, a contradiction, because $\widehat{H}$ and $\widehat{H}_0$ are numerically equivalent, so $C \cdot \widehat{H} > 0$.

Now fix $y_0 \in Y^0$. We know by Lemma 1 that $e(p^{-1}(y_0))$ is contained in a V-equivalence class F. Since B is closed with respect to V-equivalence, we have $F \subset X^0$. Consider an irreducible component C of a cycle of V such that $C \subseteq F$. Since V is quasi-unsplit, we have $\widehat{H} \cdot C = 0$, hence $q(C)$ is a point by what we proved above. Therefore $q(F) = y_0$ and $F = e(p^{-1}(y_0))$, so we have (i).

For any $x \in X$, let $Y_x := p(e^{-1}(x))$ be the family of cycles parametrized by Y and passing through x, and $\mathrm{Locus}(Y_x) := e(p^{-1}(Y_x))$. Observe that for any $y \in Y_x$, the subset $e(p^{-1}(y))$ contains x and is V-rationally connected by Lemma 1. Hence $\mathrm{Locus}(Y_x)$ is V-rationally connected for any $x \in X$.

Since $Z \subset X \times Y$, we have $\dim Y_x = \dim e^{-1}(x)$. Thus $\dim Y_x > 0$ if and only if $x \in B$, by Zariski's main Theorem. If so, $\text{Locus}(Y_x)$ has dimension at least $f_V + 1$.

Now let F be a V-equivalence class contained in B, and $x \in F$. Then $\text{Locus}(Y_x)$ has dimension at least $f_V + 1$ and is contained in F, hence $\dim F \geq f_V + 1$. ∎

Let us remark that in general, if V is not quasi-unsplit, B is not closed with respect to V-equivalence.

Example 1. In $\mathbb{P}^2$ fix two points x, y and the line $L = \overline{xy}$. Consider $\mathbb{P}^2 \times \mathbb{P}^2$ with the projections π_1, π_2 on the two factors, and fix three curves R_x, R_y, L' such that:

- R_x is a line in $\mathbb{P}^2 \times x$ and R_y is a line in $\mathbb{P}^2 \times y$;
- $\pi_1(R_x) \cap \pi_1(R_y)$ is a point $z \in \mathbb{P}^2$;
- $L' := z \times L$ is the unique line dominating L via π_2 and intersecting both R_x and R_y.

Let $\sigma\colon W \to \mathbb{P}^2 \times \mathbb{P}^2$ be the blow-up of R_x and R_y. In W, the strict transform of L' is a smooth rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$. Let X be the variety obtained by "flipping" this curve. Then X is a smooth toric Fano 4-fold with $\rho_X = 4$ (this is Z_2 in Batyrev's list, see [Bat99, Proposition 3.3.5]).

$$\begin{array}{ccc} X & \dashrightarrow & W \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle \pi_2\circ\sigma} \\ & & \mathbb{P}^2 \end{array}$$

The strict transform of a general line in a fiber of π_2 gives a covering family V of rational curves on X. The birational map $X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ is an isomorphism over $\mathbb{P}^2 \times (\mathbb{P}^2 \setminus L)$; if $U \subset X$ is the corresponding open subset, then U is closed with respect to V-equivalence and every fiber of $q\colon U \to \mathbb{P}^2 \setminus L$ is a V-equivalence class isomorphic to $\mathbb{P}^2$. Thus $f_V = 2$.

Let T_x and T_y be the images in X of the exceptional divisors of σ in W. These two divisors are V-rationally connected, and they can not be contained in B because $\dim B \leq 2$. Moreover, $P := T_x \cap T_y$ is the $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ obtained under the flip. The map $q\colon X \dashrightarrow \mathbb{P}^2$ can not be defined over P, so $P \cap B \neq \emptyset$. Therefore B can not be closed with respect to V-equivalence.

Observe that the numerical class of V lies in the interior of $\overline{\text{NE}}(X)$, hence the unique morphism, onto a projective variety, which contracts curves in V, is $X \to \{pt\}$.

A key observation is the following.

Proposition 2. *Let X be a normal and $\mathbb{Q}$-factorial projective variety, and V a covering and quasi-unsplit family of rational 1-cycles on X.*

If B is V-rationally connected, then $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of $\overline{\text{NE}}(X)$.

Proof. Let $X^0 := X \setminus B$ and $Y^0 := Y \setminus p(E) = q(X^0)$. Let L be a very ample line bundle on Y. Let $U \subset |L|$ be the open subset of divisors H that are irreducible and such that $H \cap Y^0 \neq \emptyset$. For any H in U, we define $\widehat{H} := \overline{q^{-1}(H \cap Y^0)}$ as in the proof of Proposition 1. Recall that $\widehat{H} \cdot [V] = 0$.

Let's show that $\widehat{H}$ is nef. Assume by contradiction that there exists an irreducible curve C with $C \cdot \widehat{H} < 0$.

Claim. $C \subseteq B$.

Actually, either C is contained in a fiber of q, hence it is numerically proportional to $[V]$ which contradicts $C \cdot \widehat{H} < 0$. Or $C \cap X^0 =: C^0$ is an open subset of C, $\dim q(C^0) = 1$, hence there exists $H_0 \in U$ such that H_0 intersects $q(C^0)$ in a finite number of points. Then $\widehat{H}_0$ intersects C without containing it, a contradiction, because $\widehat{H}$ and $\widehat{H}_0$ are numerically equivalent, so $C \cdot \widehat{H} > 0$.

Since B is V-rationally connected, C must be numerically proportional to V, impossible.

Let's finally show that $C \cdot \widehat{H} = 0$ if and only if C is numerically proportional to $[V]$: actually, if $C \cdot \widehat{H} = 0$, the previous arguments show that either $C \subset B$ or C is contained in a fiber of q, both are V-rationally connected, hence C is numerically proportional to V. ∎

Unfortunately, B is not V-rationally connected in general as shown by the following example.

Example 2 (see [Kac97] Example 11.1 and references therein). Fix a point p_0 in $\mathbb{P}^3$ and let

$$P_0 := \{\Pi \in (\mathbb{P}^3)^* \mid p_0 \in \Pi\} \simeq \mathbb{P}^2$$

be the variety of 2-planes in $\mathbb{P}^3$ containing p_0. Consider the variety $X \subset \mathbb{P}^3 \times P_0$ defined as

$$X := \{(p, \Pi) \in \mathbb{P}^3 \times P_0 \mid p \in \Pi\}.$$

Then X is a smooth Fano 4-fold, with Picard number 2 and pseudo-index 2. The two elementary extremal contractions are given by the projections on the two factors.

The morphism $X \to P_0$ is a fibration in $\mathbb{P}^2$: the fiber over a point is the plane corresponding to that point.

Consider the morphism $X \to \mathbb{P}^3$. If $p \neq p_0$, the fiber over p is the $\mathbb{P}^1$ of planes containing p and p_0. But the fiber F_0 over p_0 is naturally identified with P_0, hence it is isomorphic to $\mathbb{P}^2$. We have $\mathcal{N}_{F_0/X} = \Omega^1_{\mathbb{P}^2}(1)$ and $(-K_X)_{|F} = \mathcal{O}_F(2)$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F=e} & X \\ {\scriptstyle \pi=p}\downarrow & \overset{q}{\swarrow} & \downarrow{\scriptstyle q'} \\ V & \xrightarrow[\psi]{} & \mathbb{P}^3 \end{array}$$

Here $V \to \mathbb{P}^3$ is the blow-up of p_0 and $\mathcal{C} \to X$ is the blow-up of F_0. Observe that V is a family of extremal irreducible rational curves of anticanonical degree 2.

If we consider $X \times \mathbb{P}^1$ with the same family of curves, we have $\dim Y = 4$, $f_V = 1$ and $B = F_0 \times \mathbb{P}^1$ which is not V-rationally connected.

We finally get the following result: if B has the smallest possible dimension, then it is V-rationally connected.

Lemma 3. *Let X be a normal and $\mathbb{Q}$-factorial projective variety, and V be a covering and quasi-unsplit family of rational 1-cycles on X.*

If $\dim B = f_V + 1$, then every connected component of B is a V-equivalence class.

Proof. By Proposition 1, we know that B is the union of all V-equivalence classes whose dimension is $f_V + 1$. Since each of these equivalence classes must contain an irreducible component of B, they are in a finite number, and each is contained in a connected component of B.

So if B_0 is a connected component of B, we have $B_0 = F_1 \cup \cdots \cup F_r$, where each F_i is a V-equivalence class. We want to show that $r = 1$.

Assume by contradiction that $r > 1$. Observe that the F_i's are disjoint and B_0 is connected, hence at least one F_i is not a closed subset of X, assume it is F_1.

Then F_1 is a countable union of closed subsets. Considering the decomposition of B_0 as a union of irreducible components, we find an irreducible component T of B_0 such that

$$T = \bigcup_{m \in \mathbb{N}} K_m$$

where each K_m is a non empty proper closed subset of T. Since T is an irreducible complex projective variety, this is impossible. ∎

We then reformulate in a single result what we proved so far, and show that it implies Theorem 1.

Proposition 3. *Let X be a normal and $\mathbb{Q}$-factorial projective variety, and V a covering and quasi-unsplit family of rational 1-cycles on X. Then:*

- (i) *either $B = \emptyset$ or $\dim(B) \geq f_V + 1$,*
- (ii) *if $B = \emptyset$ or if $\dim(B) = f_V + 1$ then $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$.*

Proof of Theorem 1. Just notice that if $f_V \geq n-3$ and B is not empty, Proposition 3 (i) gives $\dim B \geq f_V + 1 \geq n - 2$, so $\dim B = n - 2 = f_V + 1$. Then Proposition 3 (ii) gives that $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$. ∎

4. Existence of a geometric quotient

Let V be a covering and quasi-unsplit family of rational 1-cycles on X, and assume that there exists a geometric quotient $q' \colon X \to Y'$ for V.

Observe that q' has the following property: *for any irreducible curve C in X, $q'(C)$ is a point if and only if $[C]$ is proportional to $[V]$.*

Conversely, we show that a morphism with the property above is quite close to be a geometric quotient.

Proposition 4. *Let X be a normal and $\mathbb{Q}$-factorial projective variety, and V a covering and quasi-unsplit family of rational 1-cycles on X.*

Assume that there exists a morphism with connected fibers $q'\colon X \to Y'$, onto a complete and normal algebraic variety Y', such that for any irreducible curve C in X, $q'(C)$ is a point if and only if $[C]$ is proportional to $[V]$.

Then there exists a birational morphism $\psi\colon Y \to Y'$ that fits into the commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{e} & X \\ {\scriptstyle p}\downarrow & \overset{q}{\swarrow} & \downarrow{\scriptstyle q'} \\ Y & \xrightarrow[\psi]{} & Y' \end{array} \tag{3}$$

Moreover, if $B' := q'(B)$, we have $(q')^{-1}(B') = B$, and

$$B' = \{y \in Y' \mid \dim(q')^{-1}(y) > f_V\} = \{y \in Y' \mid \dim \psi^{-1}(y) > 0\}.$$

In particular, every fiber of q' over $Y' \setminus B'$ is a V-equivalence class.

Observe that in example 2, ψ is not an isomorphism.

Proof. Let's show first of all that $(q')^{-1}(B') = B$.

If $C \subset X$ is an irreducible curve contained in a fiber of q', then either $C \cap B = \emptyset$, or $C \subseteq B$. In fact, assume that $C \cap B \neq \emptyset$. Let $\widehat{H}_0, \ldots, \widehat{H}_N$ be as in the proof of Proposition 1. Then for any $i = 0, \ldots, N$, we have $C \cdot \widehat{H}_i = 0$ and $C \cap \widehat{H}_i \neq \emptyset$, hence $C \subseteq \widehat{H}_i$ so $C \subseteq B$.

Since q' has connected fibers, we see that for every fiber F of q', either $F \cap B = \emptyset$, or $F \subseteq B$. This means that $(q')^{-1}(q'(B)) = B$.

The existence of ψ as in (3) follows easily from the normality of Y and the fact that q' contracts all curves in V, hence all V-equivalence classes. Observe that ψ is surjective with connected fibers.

Let's show that p contracts to a point any fiber of $q' \circ e$ over $Y' \setminus B'$.

Let F be a fiber of q' over $Y' \setminus B'$, then we have $F \subset X \setminus B$. Let $C \subset F$ be an irreducible curve, and choose an irreducible curve $C' \subset X \setminus B$ which is a component of a cycle of the family V. Since $q'(C)$ is a point, there exists $\lambda \in \mathbb{Q}_{>0}$ such that $C \equiv \lambda C'$ in X.

Set $X^0 := X \setminus B$. Notice that e is an isomorphism over X^0, so X^0 can be viewed also as an open subset of Z; in the same way the curves C and C' can be viewed also as a curves in Z. Let's show that $C \equiv \lambda C'$ still holds in Z.

Let $L \in \operatorname{Pic} Z$, and write $L_{|X^0} = \mathcal{O}_{X^0}(D)$, D a Cartier divisor in X^0. Let $\overline{D}$ be the closure of D in X (meaning, if $D = \sum_i a_i V_i$, that $\overline{D} = \sum_i a_i \overline{V}_i$) and let $m \in \mathbb{Z}_{>0}$ be such that $m\overline{D}$ is Cartier in X. Then set $M := e^*(\mathcal{O}_X(m\overline{D})) \in \operatorname{Pic} Z$.

By construction, $M \otimes L^{\otimes(-m)}$ is trivial on X^0, so we can write $L^{\otimes m} = M \otimes \mathcal{O}_Z(G)$, where G is a Cartier divisor in Z with $\operatorname{Supp} G \subseteq E$.

Now observe that $C \cdot G = C' \cdot G = 0$, because both curves are disjoint from E, and that $C \cdot M = \lambda C' \cdot M$ by the projection formula. Then we have $C \cdot L = \lambda C' \cdot L$, so $C \equiv \lambda C'$ in Z.

Then p must contract C to a point, because Y is projective. Since $e^{-1}(F)$ is connected, we have shown that p contracts $e^{-1}(F)$ to a point. Since Y and Y' are normal, this implies that ψ is an isomorphism over $Y' \setminus B'$.

Finally, let $y \in B'$ and let $F' = (q')^{-1}(y)$. Then $F' \subseteq B$, so e has positive dimensional fibers on F', and $\dim e^{-1}(F') > \dim F' \geq f_V$. Since $e^{-1}(F') = p^{-1}(\psi^{-1}(y))$ and p has all fibers of dimension f_V, we must have $\dim \psi^{-1}(y) > 0$. ■

We can finally prove our results.

Theorem 4. *Let X be a normal and $\mathbb{Q}$-factorial complex projective variety of dimension n, having canonical singularities. Let V be a covering and quasi-unsplit family of rational 1-cycles on X.*

If $\dim B \leq f_V + 1$, then $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of the Mori cone $\overline{\mathrm{NE}}(X)$ and the Mori contraction of $\mathbb{R}_{\geq 0}[V]$, $\mathrm{cont}_{[V]} \colon X \to Y'$, is the geometric quotient for V.

Proof. If B is empty, then the statement is clear. Assume that B is not empty. Then Proposition 3 and Lemma 3 yield that $\dim B = f_V + 1$, every connected component of B is a V-equivalence class, and $\mathbb{R}_{\geq 0}[V]$ is a geometric extremal ray of $\overline{\mathrm{NE}}(X)$.

We have to show that $-K_X \cdot [V] > 0$. Let V' be the covering family of rational 1-cycles on X given by Lemma 2, and consider a resolution of singularities $f \colon X' \to X$. The family V' determines a covering family V'' of rational 1-cycles in X'. If $C_0 \subset X$ is a general element of the family V', then $C' := \overline{f^{-1}(C_0 \setminus \mathrm{Sing}(X))}$ is a general element of V'', and $C_0 = f_*(C')$.

Since C_0 is reduced and irreducible, so is C'. Moreover V'' is covering, so C' is a free curve in X', and it has positive anticanonical degree.

Let $m \in \mathbb{Z}_{>0}$ be such that mK_X is Cartier. Since X has canonical singularities, we have

$$mK_{X'} = f^*(mK_X) + \sum_i a_i E_i,$$

where E_i are exceptional divisors of f and $a_i \in \mathbb{Z}_{\geq 0}$. Then

$$-mK_X \cdot C_0 = -f^*(mK_X) \cdot C' = -mK_{X'} \cdot C' + \sum_i a_i E_i \cdot C' > 0.$$

This gives $-K_X \cdot [V'] > 0$ and thus $-K_X \cdot [V] > 0$.

Since X has canonical singularities, the cone theorem and the contraction theorem hold for X (see [Deb01, Theorems 7.38 and 7.39]). Moreover, the extremal ray $\mathbb{R}_{\geq 0}[V]$ lies in the K_X-negative part of the Mori cone, hence it can be contracted.

Let $\mathrm{cont}_{[V]} \colon X \to Y'$ be the extremal contraction; then Y' is a normal, projective variety, and it is $\mathbb{Q}$ factorial by [Deb01, Proposition 7.44].

Applying Proposition 4, we see that all fibers of $\mathrm{cont}_{[V]}$ over $Y' \setminus \mathrm{cont}_{[V]}(B)$ are V-equivalence classes. Since connected components of B are V-equivalence classes, they are exactly the fibers of $\mathrm{cont}_{[V]}$ over $\mathrm{cont}_{[V]}(B)$, and we have the statement. ■

Observe that Theorem 2 is a straightforward consequence of Theorem 4.

5. The toric case: proof of Theorem 3

In the case $\rho_X = 1$, the statement is true for $q' \colon X \to \{pt\}$. In fact, using Proposition 4, we see that the geometric quotient Y must be a point.

Assume that $\rho_X > 1$. Recall the diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & X \\ \downarrow{\scriptstyle \pi} & & \\ V & & \end{array}$$

Recall also that if $D \subset X$ is a prime invariant Weil divisor, there is a natural inclusion $i_D \colon \mathcal{N}_1(D)_{\mathbb{R}} \hookrightarrow \mathcal{N}_1(X)_{\mathbb{R}}$.

Step 1: let $D \subset X$ be a prime invariant Weil divisor such that $D \cdot [V] = 0$. Then there exists a covering and quasi-unsplit family V_D of rational 1-cycles in D such that $\mathbb{R}_{\geq 0}(i_D[V_D]) = \mathbb{R}_{\geq 0}[V]$.

Choose an irreducible component W of $F^{-1}(D)$ which dominates D. Set $V'_D := \pi(W)$, and let $\mathcal{C}'_D$ be an irreducible component of $\pi^{-1}(V'_D)$ containing W. Consider the normalization $\mathcal{C}_D$ of $\mathcal{C}'_D$, and let $\pi_D \colon \mathcal{C}_D \to V_D$ be the Stein factorization of the composite map $\mathcal{C}_D \to \mathcal{C}'_D \to V'_D$. Finally let $F_D \colon \mathcal{C}_D \to X$ be the induced map.

For $v \in V_D$, set $G_v := F_D(\pi_D^{-1}(v))$. Then $G_v \cap D \neq \emptyset$, G_v is connected, and $G_v \cdot D = 0$ because V is quasi-unsplit. This implies $G_v \subseteq D$, hence $F_D(\mathcal{C}_D) \subseteq D$. Moreover, since W dominates D, we have $F_D(\mathcal{C}_D) = D$.

Since V_D is normal, there is a holomorphic map $V_D \to \mathrm{Chow}(D)$. Then after replacing V_D by its image in $\mathrm{Chow}(D)$ and $\mathcal{C}_D$ by its image in $\mathrm{Chow}(D) \times X$, we get the desired family.

Step 2: there exists an invariant prime Weil divisor having intersection zero with $[V]$.

In fact, let $q \colon X \dashrightarrow Y$ be the rational map associated to V. Since $\rho_X > 1$, Y is not a point. Let D be a prime divisor in Y intersecting $q(X^0)$ and set $D' := \overline{q^{-1}(D)}$. Since there are curves of the family V disjoint from D', we have $D' \cdot [V] = 0$. Moreover, D' is linearly equivalent to $\sum_i a_i D_i$, where $a_i \in \mathbb{Q}_{>0}$ and D_i are invariant prime Weil divisors. Hence the statement.

Step 3: we prove the statement.

Let Σ_X be the fan of X in $N \cong \mathbb{Z}^n$, and let G_X be the set of primitive generators of one dimensional cones in Σ_X. It is well known that G_X is in bijection with the set of invariant prime divisors of X; for any $x \in G_X$, we denote D_x the associated divisor. Recall that for any class $\gamma \in \mathcal{N}_1(X)_{\mathbb{Q}}$, we have

$$\sum_{x \in G_X} (\gamma \cdot D_x)x = 0 \quad \text{in } N \otimes_{\mathbb{Z}} \mathbb{Q},$$

and that the association $\gamma \mapsto \sum_{x \in G_X} (\gamma \cdot D_x)x$ gives a canonical identification of $\mathcal{N}_1(X)_{\mathbb{Q}}$ with the $\mathbb{Q}$-vector space of linear relations with rational coefficients among G_X.

Let $m_1x_1 + \cdots + m_hx_h = 0$ be the relation corresponding to $[V]$, with $x_i \in G_X$ and m_i non zero rational numbers for all i. Since V is covering and quasi-unsplit, all m_i's must be positive. For $y \in G_X$, we have $D_y \cdot [V] = 0$ if and only if y is different from $x_1, \ldots, x_h$. So by Step 2, we know that $G_X \setminus \{x_1, \ldots, x_h\}$ is non empty.

The following two statements are equivalent (see [Rei83, Theorem 2.4] and [Cas03, Theorem 2.2]):

(a) there exists a $\mathbb{Q}$-factorial, projective toric variety Y', and a flat, equivariant morphism $q' \colon X \to Y'$, such that for any curve C in X, $q'(C)$ is a point if and only if $[C]$ is proportional to $[V]$;

(b) for any $\tau \in \Sigma_X$ such that $x_1, \ldots, x_h \notin \tau$, we have

$$\tau + \langle x_1, \ldots, \check{x}_i, \ldots, x_h \rangle \in \Sigma_X \quad \text{for all } i = 1, \ldots, h. \tag{4}$$

Let's show (b) by induction on the dimension of X.

Clearly, it is enough to check (4) for any maximal τ in Σ_X not containing any x_i. Since $\{x_1, \ldots, x_h\} \subsetneq G_X$, such a maximal τ will have positive dimension.

Let $y \in G_X \cap \tau$. We have $D_y \cdot [V] = 0$, so by Step 1 there exists a quasi-unsplit, covering family V_{D_y} in D_y such that $i_{D_y}[V_{D_y}]$ is proportional to $[V]$.

Set $\overline{N} := N/\mathbb{Z} \cdot y$ and for any $z \in N$, write $\overline{z}$ for its image in $\overline{N}$. The fan Σ_{D_y} of D_y is given by the projections in $\overline{N} \otimes_{\mathbb{Z}} \mathbb{Q}$ of all cones of Σ_X containing y. The relation corresponding to $[V_{D_y}]$ is $\lambda m_1\overline{x}_1 + \cdots + \lambda m_h\overline{x}_h = 0$, for some $\lambda \in \mathbb{Q}_{>0}$. By induction, we know that (b) holds for V_{D_y} in D_y. In particular, the projection $\overline{\tau}$ of τ is in Σ_{D_y}, so we have

$$\overline{\tau} + \langle \overline{x}_1, \ldots, \check{\overline{x}}_i, \ldots, \overline{x}_h \rangle \in \Sigma_{D_y} \quad \text{for all } i = 1, \ldots, h.$$

This yields (4).

Finally, since q' is flat, all fibers must be V-equivalence classes and $B = \emptyset$.

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