STEIN COMPACTS IN LEVI-FLAT HYPERSURFACES

FRANC FORSTNERIČ AND CHRISTINE LAURENT-THIÉBAUT

Prépublication de l'Institut Fourier n° 659 (2004)
http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract. We explore connections between geometric properties of the Levi foliation of a Levi-flat hypersurface M and holomorphic convexity of compact sets in M or bounded in part by M. We indicate applications to extendability of Cauchy-Riemann functions and solvability of the $\overline{\partial}_b$-equation.

1. Introduction

A compact set in a complex manifold is said to be a *Stein compact* if it admits a fundamental basis of open Stein neighborhoods. (For the theory of Stein manifolds see [22] and [27].) In this paper we consider the existence of Stein compacts in Levi-flat hypersurfaces, or bounded in part by such hypersurfaces. A real hypersurface M in an n-dimensional complex manifold is *Levi-flat* if it is foliated by complex manifolds of dimension $n-1$; this *Levi foliation* is as smooth as M itself according to Barrett and Fornæss [6].

Theorem 1.1 (**The main theorem).** *Asume that M is an orientable Levi-flat hypersurface of class $\mathcal{C}^3$ in a complex manifold X. Let ρ be a strongly plurisubharmonic $\mathcal{C}^2$ function in an open set $U \subset X$ such that $A \stackrel{\mathrm{def}}{=} \{x \in U \cap M \colon \rho(x) \le 0\}$ is compact. If M admits a $\mathcal{C}^3$ defining function v such that $dd^c v$ and its first order derivatives vanish at every point of M in a neighborhood of A then A is a Stein compact. This holds in each of the following cases:*

(a) *The Levi foliation $\mathcal{L}$ of M is defined in a neighborhood of A by a nonvanishing* closed *one-form of class $\mathcal{C}^2$.*
(b) *$\mathcal{L}$ is a* simple foliation *in a neighborhood of A (every point admits a local transversal to $\mathcal{L}$ intersecting each leaf at most once).*
(c) *M is real analytic and its fundamental group $\pi_1(M)$ is finite.*
(d) *M is compact and there exists a compact leaf $L \in \mathcal{L}$ with $H^1(L, \mathbb{R}) = 0$.*
(e) *$H^1(A, \mathcal{C}^2_{\mathcal{L}}) = 0$ where $\mathcal{C}^2_{\mathcal{L}}$ is the sheaf of real valued $\mathcal{C}^2$ functions on M which are constant on the leaves of $\mathcal{L}$.*

The manifold X in theorem 1.1 need not be Stein. The operator $dd^c = 2i\partial\overline{\partial}$ is the *pluricomplex Laplacian* annihilating local pluriharmonic functions (§2). The assumption in theorem 1.1 is that $M = \{v = 0\}$ for some $v \in \mathcal{C}^3(X)$ satisfying

Date: October 19, 2004.
2000 *Mathematics Subject Classification.* Primary 32D15, 32T20, 32T27, 32V05, 32V25; secondary 57R30.
Key words and phrases. Levi-flat hypersurfaces, Stein manifolds, pluriharmonic functions.
The first author was supported by grants P1-0291 and J1-6173, Republic of Slovenia.

$dv|_M \neq 0$ and $j^1(dd^c v)|_{M\cap U} = 0$ for an open set $U \subset X$ containing A, $j^1(dd^c v)$ being the total first order jet of (the components of) the $(1,1)$-form $dd^c v$. The existence of such *asymptotically pluriharmonic defining function* for M is equivalent to the existence of a nonvanishing closed one-form of class $\mathcal{C}^2$ which defines the Levi foliation of M (condition (a)); proposition 3.1. Each of the remaining conditions (b)–(e) implies (a). If 0 is a regular value of $\rho|_M$ then already the weaker assumption $j^1(dd^c v)|_A = 0$ gives a basis of Stein neighborhoods of A which admit strong deformation retractions onto A.

Another interesting point is that, with v as in theorem 1.1, the function $-\log v+\rho$ is strongly plurisubharmonic near A on the side $v > 0$ and blows up along A; the analogous conclusion holds for $-\log(-v) + \rho$ on the side $v < 0$ (remark 3.8).

The main motivation for theorem 1.1 were applications to the extendability of Cauchy-Riemann (CR) functions and local solvability of the $\overline{\partial}_b$-equation without shrinking the domain. Consider the following setup:

(1) X is an n-dimensional Stein manifold ($n \geq 2$),
(2) $D \subset\subset X$ is a smoothly bounded strongly pseudoconvex domain,
(3) M is a smooth Levi-flat hypersurface in X intersecting bD transversely,
(4) $A = M \cap \overline{D}$, and
(5) Ω is a connected component of $D\backslash M$ such that $b\Omega = A \cup \omega$, where ω is an open connected subset of bD.

In this setting we have the following addition to the main theorem.

Theorem 1.2. *Let X, M, D, A be as in (1)–(4), with bD of class $\mathcal{C}^2$. If one of the conditions in theorem 1.1 holds then the closure of every connected component of $D\backslash M$ is a Stein compact in X.*

Assuming the above setup we consider the following analytic conditions; (A2) and (A3) are relevant only if $n \geq 3$.

(A1): Every continuous CR function on ω extends to a (unique) continuous function on $E(\omega) \stackrel{\text{def}}{=} \overline{\Omega} \setminus A$ which is holomorphic in Ω.
(A2): For every integer q, $1 \leq q \leq n-3$, and every smooth differential $(0,q)$-form f on ω satisfying $\overline{\partial}_b f = 0$ there is a smooth $(0, q-1)$-form u on ω such that $\overline{\partial}_b u = f$.
(A3): For every smooth $(0, n-2)$-form f on ω for which $\int_\omega f \wedge \varphi = 0$ for every smooth, $\overline{\partial}$-closed $(n,1)$-form φ on $X \setminus A$ such that $\operatorname{supp}\varphi \cap \operatorname{supp} f$ is compact there is a smooth $(0, n-3)$-form u on ω satisfying $\overline{\partial}_b u = f$.

Theorem 1.3. ([31]) *If $\overline{\Omega}$ is a Stein compact then*

$$(A1), (A2) \text{ and } (A3) \text{ hold } \iff H^{n,q}(A) = 0 \text{ for all } 1 \leq q \leq n-1.$$

Since all Dolbeaut cohomology groups vanish on a Stein compact, theorems 1.1, 1.2 and 1.3 together imply

Corollary 1.4. *Assuming the setup (1)–(5), each of the conditions in theorem 1.1 implies that (A1), (A2) and (A3) hold.*

Results on extendability of CR functions from caps $\omega \subset bD$ have been obtained in [29], [30], [33], [34], [35]; for solutions of the $\overline{\partial}_b$-equation on ω without estimates up to the boundary see [32] and with L^p and Sobolev estimates see M.-C. Shaw [44], [45].

The sufficient conditions (a), (b) and (c) in theorems 1.1, 1.2 and corollary 1.4 are fairly close to optimal—the conclusions fail in general if M is real analytic but the fundamental group of A has an element of infinite order, or if the Levi foliation has nontrivial holonomy. In §7 we describe an oriented, real analytic, Levi-flat hypersurface $M \subset \mathbb{C}^* \times \mathbb{C}$ which divides a certain strongly pseudoconvex domain D in two connected components $D \backslash M = D_+ \cup D_-$ satisfying the following:

— the fundamental group of $A = M \cap \overline{D}$ is infinite cyclic,
— the Levi foliation of M is simple in the complement of a closed leaf $L_0 = \mathbb{C}^* \times \{0\} \subset M$, but is not simple in any neighborhood of L_0 due to nontrivial holonomy of L_0,
— none of the sets $A = M \cap \overline{D}$, $\overline{D}_+$ and $\overline{D}_-$ is a Stein compact, and
— there is a continuous CR function on $\omega = bD_+ \backslash A$ which does not admit a holomorphic extension to D_+.

The example, which is based on the *worm domain* of Diederich and Fornæss [13], was suggested to us by J.-E. Fornaess to whom we offer our sincere thanks.

In the proofs we combine complex analytic methods and the theory of codimension one foliations. The existence of a defining function satisfying $j^1(dd^c v)|_M = 0$ gives a Stein neighborhood basis of any compact subset $A \subset M$ satisfying the hypotheses of theorem 1.1, and also of the closure of any connected component of $D \backslash M$ in theorem 1.2 (theorems 3.4 and 3.6). Furthermore, M admits an asymptotically pluriharmonic defining function if and only if its Levi foliation is given by a nonvanishing *closed* one-form (proposition 3.1). This simple but important observation (which has been made in the real analytic case by D. Barrett [5]) ties the complex analytic properties of M with the geometric properties of its Levi foliation.

In sections §4–§6 we consider sufficient conditions for a transversely orientable codimension one foliation $\mathcal{L}$ to be defined by a closed one-form. In §4 we look at the case when $\mathcal{L}$ is a *simple foliation*. By Haefliger and Reeb [24] the space of leaves $Q = M/\mathcal{L}$ admits the structure of a smooth, one dimensional, possibly non-Hausdorff manifold such that the quotient projection $\pi \colon M \to Q$ is a smooth submersion. (In particular, all leaves of $\mathcal{L}$ are topologically closed.) Such Q need not admit any nonconstant smooth functions or one-forms. However, if $A \subset M$ is a compact subset with $\mathcal{C}^2$ boundary which is in general position with respect to $\mathcal{L}$ then $Q_A = A/\mathcal{L}$ admits a nowhere vanishing smooth one-form θ (theorem 4.2); its pull-back $\alpha = \pi^*\theta$ is then a smooth closed one-form defining the restricted foliation $\mathcal{L}_A$, thus establishing the implication (b)⇒(a) in theorem 1.1. (If A is simply connected then Q_A actually admits a function with nowhere vanishing derivative [24]). Even if M is real analytic, α cannot be chosen real analytic, not even on bounded contractible domains in M [17]. The restriction to compact subsets of M is essential (Wazewski [46]).

The implication (c)⇒(a) in theorem 1.1 relies on a classical theorem of Haefliger [23] (theorem 5.1), (d)⇒(a) follows from a stability theorem of Reeb and Thurston (theorem 4.8), and (e)⇒(a) employs an elementary argument (theorem 6.1).

In §8 we illustrate the scope of our results on a family of examples obtained by complexifiying piecewise real analytic codimension one foliations on real analytic manifolds. In this class of Levi-flat hypersurfaces one can perform certain standard operations such as the construction of Reeb components, turbulization along a closed transversal and spinning along a transverse boundary component.

2. Preliminaries

We recall some relevant notions concerning foliations and Levi-flat hypersurfaces; a knowledgeable reader should skip directly to §3. Our general references for the theory of foliations will be [11], [19], [26].

Let M be a real manifold of dimension m and class $\mathcal{C}^r$, $r \in \{1, 2, \ldots, +\infty, \omega\}$. A foliation $\mathcal{L}$ of codimension one and class $\mathcal{C}^r$ on M is given by a *foliation atlas* $\mathcal{U} = \{(U_j, \phi_j) \colon j \in J\}$ where $\{U_j\}_{j\in J}$ is a covering of M by open connected sets, for every $j \in J$ the map $\phi_j = (\phi'_j, h_j) \colon U_j \to P_j = P'_j \times I_j \subset \mathbb{R}^{m-1} \times \mathbb{R}$ is a $\mathcal{C}^r$ diffeomorphism, and the transition maps are of the form $\theta_{ij}(u,v) = \phi_i \circ \phi_j^{-1}(u,v) = (a_{ij}(u,v), b_{ij}(v))$ where b_{ij} is a diffeomorphism between a pair of intervals in $\mathbb{R}$. Each leaf $L \in \mathcal{L}$ intersects U_j in at most countably many *plaques* $\{x \in U_j \colon h_j(x) = c \in I_j\}$. The collection $\{b_{ij}\}$ is called a *holonomy cocycle*, or a *Haefliger cocycle* determining $\mathcal{L}$ [11], [23]. A foliation is *transversely orientable* (resp. *transversely real analytic*) if it admits a foliation atlas in which all diffeomorphism b_{ij} preserve the orientation of $\mathbb{R}$ (resp. are real analytic). A continuous function $u \colon M \to \mathbb{R}$ is a *first integral* for $\mathcal{L}$ if u has no local extrema and is constant on every leaf of $\mathcal{L}$; $u \in \mathcal{C}^1(M)$ is a *noncritical first integral* if in addition $du \neq 0$ on M.

A closed loop γ in a leaf $L \in \mathcal{L}$ determines the germ of a diffeomorphism ψ_γ (the *holonomy* of γ) on any local transversal $\ell \subset M$ at a point $x_0 \in \gamma$, depending only on the homotopy class $[\gamma] \in \pi_1(L, x_0)$. The induced map of $\pi_1(L, x_0)$ to the group of germs of diffeomorphisms of (ℓ, x_0) is called the *holonomy homomorphism* of L. The foliation $\mathcal{L}$ has *trivial leaf holonomy* if ψ_γ is the germ of the identity map for any loop γ contained in a leaf of $\mathcal{L}$. $\mathcal{L}$ admits *one-sided holonomy* if there exists $\psi = \psi_\gamma$ as above, defined on a local transversal ℓ at some $x_0 \in L \in \mathcal{L}$, which equals the identity map on one side of x_0 but not on the other side. A transversely real analytic foliation has no one-sided holonomy.

Let X be a complex manifold with the complex structure operator J. The operator d^c is defined on functions by $\langle d^c v, \xi\rangle = -\langle dv, J\xi\rangle$ for $\xi \in TX$. In local holomorphic coordinates $(z_1, \ldots, z_n)$, with $z_j = x_j + iy_j$ and $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$, we have $d^c v = \sum_{j=1}^n -\frac{\partial v}{\partial y_j}\, dx_j + \frac{\partial v}{\partial x_j}\, dy_j$. Then $d = \partial + \overline\partial$, $d^c = i(\overline\partial - \partial)$, and $dd^c = 2i\partial\overline\partial$ is the *pluricomplex Laplacian* annihilating pluriharmonic functions.

Let $M \subset X$ be a real hypersurface given by $M = \{v = 0\}$ for some function $v \in \mathcal{C}^r(X)$ $(r \geq 2)$ satisfying $dv \neq 0$ on M. The set $T^{\mathbb{C}}M = TM \cap J(TM) = TM \cap \ker d^c v$ is a real codimension one subbundle of TM on which J defines a complex structure. The *Levi form* of M is the operator $T^{\mathbb{C}}M \to \mathbb{R}$, $\xi \to \langle dd^c v, \xi \wedge J\xi\rangle$. M is *Levi-flat* if this form vanishes identically. This is equivalent to $dd^c v \wedge d^c v = 0$ on TM which is just the integrability condition for $T^{\mathbb{C}}M = \ker d^c v|_{TM}$. By Frobenius' integrability theorem the latter is equivalent to the existence of a codimension one foliation of M by manifolds tangent to $T^{\mathbb{C}}M$, that is, by complex hypersurfaces. (For Levi-flat hypersurfaces of low regularity see [2] and [41].)

3. Asymptotically pluriharmonic functions and Stein compacts

We denote by $j^r\alpha$ the r-jet extension of a function or (the components of) a differential form α. The following establishes the equivalence of the first two conditions in theorem 1.1.

Proposition 3.1. *Let M be an oriented closed Levi-flat hypersurface of class $\mathcal{C}^r$ ($r \ge 2$) in a complex manifold X. The following are equivalent:*

(i) *M admits a defining function $v \in \mathcal{C}^r(X)$ with $j^{r-2}(dd^c v)|_M = 0$. If $r = \omega$ then v can be chosen pluriharmonic in a neighborhood of M in X.*
(ii) *The Levi foliation of M is defined by a closed one-form of class $\mathcal{C}^{r-1}$.*

If M is simply connected then (i) and (ii) are further equivalent to

(iii) *There exists $f = u + iv \in \mathcal{C}^r(X)$ such that $M = \{v = 0\}$, $dv \neq 0$ on M, and $j^{r-1}(\overline{\partial} f)|_M = 0$. If $r = \omega$ then f can be chosen holomorphic in a neighborhood of M in X.*

A function f satisfying $j^{r-1}(\overline{\partial} f)|_M = 0$ is said to be *asymptotically holomorphic* of order $r-1$ on M, and a function v satisfying $j^{r-2}(dd^c v)|_M = 0$ is *asymptotically pluriharmonic* of order $r-2$ on M. The equivalence (i)$\Leftrightarrow$(ii) has been proved in the real analytic case by D. Barrett ([5], proposition 1, p. 461).

Proof. (i)$\Rightarrow$(ii). Denote by $\iota\colon M \hookrightarrow X$ the inclusion map. Assume that $v \in \mathcal{C}^r(X)$ satisfies $dv|_M \neq 0$ and $dd^c v|_M = 0$. From $0 = \iota^*(dd^c v) = d_M(\iota^* d^c v)$ we see that the one-form $\alpha = \iota^*(d^c v)$ is closed on M, and we have already remarked in §2 that $\ker\alpha = T^{\mathbb{C}}M = T\mathcal{L}$. (Proof: for $\xi \in T_x M$ we have $\langle \alpha, \xi\rangle = \langle d^c v, \xi\rangle = -\langle dv, J\xi\rangle$ which is zero if and only if $J\xi \in \ker dv_x = T_x M$, i.e., when $\xi \in T^c_x M$.) Hence α defines the Levi foliation of M.

(ii)$\Rightarrow$(i). Let $T\mathcal{L} = \ker\alpha$ where α is a closed one-form of class $\mathcal{C}^{r-1}$ on M. Locally near a point $x_0 \in M$ we have $\alpha = -du$ for some $\mathcal{C}^r$ function u which is unique up to an additive constant. Since u is constant on the leaves of $\mathcal{L}$ and hence a Cauchy-Riemann (CR) function on M, it admits a $\mathcal{C}^r$ extension $f = u + iv$ to an neighborhood $U \subset X$ of x_0 such that $j^{r-1}\overline{\partial} f|_{M\cap U} = 0$; such f is unique up to a term which is flat to order r along M ([10], p. 147, theorem 2 and the remark following it). Since $dd^c = 2i\partial\overline{\partial}$, the above implies $j^{r-2}dd^c(u+iv)|_{M\cap U} = 0$. As dd^c is a real operator, it follows that $j^{r-2}dd^c u|_{M\cap U} = 0$ and $j^{r-2}dd^c v|_{M\cap U} = 0$. Thus v satisfies condition (i) on $M \cap U$. Any two local functions v obtained in this way differ only by an r-flat term, and hence we obtain a global function v with these properties by patching with a smooth partition of unity along M.

If M is simply connected and $T\mathcal{L} = \ker\alpha$ with $d\alpha = 0$, we have $\alpha = -du$ for some $u \in \mathcal{C}^r(M)$. The same argument as above gives an asymptotically holomorphic extension $f = u + iv$ into a neighborhood of M in X (holomorphic in the real analytic case). This shows the equivalence of (iii) with (i) and (ii). ☐

Remark 3.2. The equivalent conditions in proposition 3.1 hold in a small neighborhood of any point $x_0 \in M$: take a local $\mathcal{C}^r$ first integral u of $\mathcal{L}$ and extend it to a $\overline{\partial}$-flat function $f = u+iv$ as above. Such f gives a local asymptotically holomorphic flattening of M. If M is smooth but not real analytic then in general such f cannot be chosen holomorphic on one side of M [9]. A real analytic M need not admit a holomorphic flattening on large domains, even on contractible ones [17].

Remark 3.3. Recall from §2 that integrability of the subbundle $T^{\mathbb{C}}M$ is equivalent to $d^c v \wedge dd^c v|_{TM} = 0$ for some (and hence any) defining function v for M. In particular, a hypersurface $M = \{v = 0\}$ with a defining function satisfying $dd^c v|_{TM} = 0$ is necessarily Levi-flat.

Theorem 3.4. *Assume that M is a Levi-flat hypersurface of class $\mathcal{C}^3$ in a complex manifold X. Let ρ be a strongly plurisubharmonic function of class $\mathcal{C}^2$ in an open set $U \subset X$ such that $A = \{x \in U \cap M \colon \rho(x) \le 0\}$ is compact. If M admits a defining function $v \in \mathcal{C}^3(V)$ in a neighborhood $V \subset M$ of A satisfying $j^1(dd^c v)|_{M\cap V} = 0$ then A is a Stein compact.*

Theorem 3.4 implies the main conclusion in theorems 1.1 and 1.2.

Proof. Choose a smooth Hermitean metric on X and denote by $|\xi|$ the associated norm of any tangent vector $\xi \in TX$. Shrinking U around A we may assume that v is defined on U and satisfies $dv \ne 0$ and $j^1(dd^c v)|_{M\cap U} = 0$. By further shrinking of U we may also assume that $|dv|$ and $dd^c\rho > 0$ are bounded and bounded away from zero on U. The distance of a point $x \in U$ to M is comparable to $|v(x)|$. Set $U^\pm = \{x \in U \colon \pm v(x) \ge 0\}$.

Let $c > 0$ be a regular value of ρ and $\rho|_{M\cap U}$ such that $\{x \in U \cap M \colon \rho(x) \le c\} \subset\subset U$. Then dv and $d\rho$ are $\mathbb{R}$-linearly independent at every point of $E = \{x \in M \cap U \colon \rho(x) = c\}$, and $D = \{x \in U \colon \rho(x) \le c\}$ has $\mathcal{C}^2$ strongly pseudoconvex boundary intersecting M transversely along E. For $\epsilon > 0$ and $x \in U$ set

$$\begin{aligned} v^\pm_\epsilon(x) &= \pm v(x) + \epsilon(\rho(x) - c), \\ \Gamma^\pm_\epsilon &= \{x \in U \colon v^\pm_\epsilon(x) = 0\}, \\ \Omega_\epsilon &= \{x \in U \colon v^+_\epsilon(x) < 0,\ v^-_\epsilon(x) < 0,\ \rho(x) < \epsilon\}. \end{aligned}$$

For sufficiently small $\epsilon > 0$ we have $dv^\pm_\epsilon = \pm dv + \epsilon d\rho \ne 0$ on U and hence $\Gamma^\pm_\epsilon$ are $\mathcal{C}^2$ hypersurfaces satisfying $\Gamma^\pm_\epsilon \cap D \subset U^\pm$ and $\Gamma^+_\epsilon \cap \Gamma^-_\epsilon = E$. The domain Ω_ϵ is bounded in part by $\Gamma^\pm_\epsilon$ and by $\{\rho = \epsilon\}$. As $\epsilon \to 0$, Ω_ϵ shrinks to A.

It remains to show that Ω_ϵ is a Stein domain for sufficiently small $\epsilon > 0$. To see this we first prove that $v^\pm_\epsilon$ are strongly plurisubharmonic on $\overline{\Omega}_\epsilon$ provided ϵ is sufficiently small. By the choice of U there is $C_0 > 0$ such that $\langle dd^c\rho(x), \xi \wedge J\xi\rangle \ge C_0|\xi|^2$ for $x \in U$ and $\xi \in T_x X$. Since $j^1(dd^c v)$ vanishes on $M \cap U$, we have $|dd^c v(x)| = o(|v(x)|)$ which gives

$$\begin{aligned} \langle dd^c v^\pm_\epsilon(x), \xi \wedge J\xi\rangle &= \pm\langle dd^c v(x), \xi \wedge J\xi\rangle + \epsilon\langle dd^c\rho(x), \xi \wedge J\xi\rangle \\ &\ge \big(-o(|v(x)|) + C_0\epsilon\big)\cdot|\xi|^2. \end{aligned}$$

For $x \in \overline{\Omega}_\epsilon$ we have $|v(x)| \le \epsilon|\rho(x) - c| \le C_1\epsilon$ with $C_1 = \sup\{|\rho(x) - c| \colon x \in D\}$. Inserting in the above inequality gives

$$\langle dd^c v^\pm_\epsilon(x), \xi \wedge J\xi\rangle \ge \big(C_0\epsilon - o(\epsilon)\big)\cdot|\xi|^2, \qquad x \in \overline{\Omega}_\epsilon$$

which is positive for sufficiently small $\epsilon > 0$ and $\xi \ne 0$. Choose a smooth strongly increasing convex function $h \colon (-\infty, 0) \to \mathbb{R}$ with $\lim_{t\to 0} = +\infty$. Then

$$\tau(x) = h(v^+_\epsilon(x)) + h(v^-_\epsilon(x)) + h(\rho(x) - \epsilon)$$

is a strongly plurisubharmonic exhaustion function on Ω_ϵ and hence Ω_ϵ is Stein ([27], Theorem 5.2.10, p. 127). ∎

Remark 3.5. If we assume in theorem 3.4 that 0 is a regular value of $\rho|_M$ (which means that $\{\rho = 0\}$ intersects M transversely along bA) then our proof gives the same conclusion under the weaker assumption that $j^1(dd^c v)$ vanishes on A. Indeed, transversality implies that the maximal distance of points in Ω_ϵ from A is comparable to ϵ and hence our estimates show as before that Ω_ϵ is a Stein domain.

Theorem 3.6. *Let X be a Stein manifold, $D \subset\subset X$ a strongly pseudoconvex domain and $M \subset X$ a $\mathcal{C}^3$-smooth Levi-flat hypersurface intersecting bD transversely. Set $A = M \cap \overline{D}$. If M admits a defining function $v \in \mathcal{C}^3(X)$ satisfying $j^1(dd^c v)|_A = 0$ the closure of any connected component of $D\backslash M$ is a Stein compact.*

Proof. Assume first that $D\backslash M = D^+ \cup D^-$ consists of two connected components $D^\pm$. Let $D = \{\rho < 0\}$ where ρ is strongly plurisubharmonic function in a neighborhood of $\overline{D}$. We may assume that $v > 0$ in D^+ locally near $M \cap D$. For sufficiently small $\epsilon > 0$ the hypersurface Γ_ϵ^-, constructed in the proof of theorem 3.4, intersects the hypersurface $\{\rho = \epsilon\}$ transversely and together they form the boundary of a domain Ω_ϵ containing $\overline{D^+}$. By a similar argument as in the proof of theorem 3.4 it can be shown that Ω_ϵ is Stein; as $\epsilon \to 0$, Ω_ϵ shrinks down to $\overline{D^+}$. A similar argument applies to D^-. If $D\backslash M$ consists of more than two connected components then one applies the above argument at each connected component of $M \cap D$. □

Remark 3.7. Theorem 3.6 can be generalized to compacts $K = \{x \in X : \rho(x) \le 0\}$ where ρ is strongly plurisubharmonic in a neighborhood of K. If we cut K by a Levi-flat hypersurface $M = \{v = 0\}$ where $v \in \mathcal{C}^3(X)$ is a defining function satisfying the conditions in theorem 3.4 in a neighborhood of $A = K \cap M$ then the sets $K^\pm = K \cap \{\pm v \ge 0\}$ are Stein compacts.

Remark 3.8. Under the conditions of theorem 3.4 the function $\tau = -\log v + \rho$, which is defined on $U \cap \{v > 0\}$ and tends to $+\infty$ along M, is strongly plurisubharmonic near M. Indeed we have

$$dd^c\tau = -\frac{1}{v}\, dd^c v + \frac{1}{v^2}\, dv \wedge d^c v + dd^c\rho.$$

From $j^1(dd^c v)|_M = 0$ we see that the first term vanishes on M, the second term is nonnegative for every v, and the last term is positive since ρ is strongly plurisubharmonic. The analogous observation holds for $-\log(-v) + \rho$ from the side $\{v < 0\}$.

4. Simple foliations

In this and the following section we consider sufficient conditions for a transversely orientable codimension one foliation $\mathcal{L}$ on a manifold M to be defined by a closed one-form. We begin by recalling some known facts.

1. $\mathcal{L}$ is defined by a closed one-form if and only if it admits a foliation atlas whose holonomy cocycle ([11], p. 28; [23]) consists of translations in $\mathbb{R}$.

2. A foliation defined by a closed one-form α has trivial leaf holonomy. Indeed, if γ is a loop contained in a leaf $L \subset \mathcal{L}$ then $\int_\gamma \alpha = 0$ (since α vanishes on TL) and hence $\alpha = du$ in a neighborhood of γ. Since $du = \alpha \neq 0$, u is a first integral of $\mathcal{L}$ which is injective on every local transversal to L, and the observation follows. Conversely, a transversely orientable codimension one foliation $\mathcal{L}$ without holonomy and of class $\mathcal{C}^r$ ($r \in \{2, \ldots, +\infty\}$) is topologically, but in general not diffeomorphically

conjugate to a $\mathcal{C}^r$ foliation defined by a closed nonvanishing one-form (Sacksteder [40]; [11], §9.2 and p. 218).

4. The *Godbillon-Vey class* of a foliation $\mathcal{L}$ given by a smooth one-form α is $gv(\mathcal{L}) = [\alpha\wedge\alpha_1\wedge\alpha_2] \in H^3(M,\mathbb{R})$ where α_1, α_2 are one-forms satisfying $d\alpha = \alpha_1\wedge\alpha$, $d\alpha_1 = \alpha_2 \wedge \alpha$ ([11], p. 38, [18], [20]). Clearly $d\alpha = 0$ implies $gv(\mathcal{L}) = 0$.

Definition 4.1. ([19], p. 79) A foliation $\mathcal{L}$ on a manifold M is *simple* if every point $x_0 \in M$ is contained in a local transversal $\ell \subset M$ to $\mathcal{L}$ which is not intersected more than once by any leaf of $\mathcal{L}$.

Theorem 4.2. *A transversely orientable simple foliation of codimension one and class $\mathcal{C}^r$ ($r \in \{2,\ldots,+\infty\}$) is defined in a neighborhood of any compact set by a closed nonvanishing one-form of class $\mathcal{C}^{r-1}$.*

Globally such one-form need not exist even if M is simply connected [46]. We begin by recalling from [24] some relevant results on leaf spaces of simple foliations.

Proposition 4.3. (Haefliger and Reeb [24]) *Let $\mathcal{L}$ be a simple foliation of codimension one and of class $\mathcal{C}^r$ ($r \geq 1$) on a connected manifold M. The space of leaves $Q = M/\mathcal{L}$ carries the structure of a connected, one dimensional $\mathcal{C}^r$ manifold (possibly non-Hausdorff) such that the quotient projection $\pi\colon M \to Q$ is a $\mathcal{C}^r$ submersion. Q is orientable if and only if $\mathcal{L}$ is transversely orientable, and is simply connected if and only if M is such. Any function on M which is constant on the leaves of $\mathcal{L}$ is of the form $f\circ\pi$ for some $f\colon Q \to \mathbb{R}$.*

The $\mathcal{C}^r$ structure on the leaf space $Q = M/\mathcal{L}$ is uniquely determined by the requirement that the restriction of the quotient projection $\pi\colon M \to Q$ to any local $\mathcal{C}^r$ transversal $\ell \subset M$ is a $\mathcal{C}^r$ diffeomorphism of ℓ onto $\psi(\ell) \subset Q$.

To prove theorem 4.2 it would suffice to find a nonvanishing one-form θ on the leaf space $Q = M/\mathcal{L}$; it's pull-back $\alpha = \pi^*\theta$ is then a nonvanishing one-form on M satisfying $\ker\alpha = T\mathcal{L}$. Unfortunately a non-Hausdorff manifold Q need not admit any nonconstant $\mathcal{C}^1$ functions or one-forms (Wazewski [46]; [24]). However, one can do this on compact subsets as we shall now explain.

A point q in a non-Hausdorff manifold Q is said to be a *branch point* if there exists another point $q' \in Q$, $q \neq q'$, such that q and q' have no pair of disjoint neighborhoods; such $\{q,q'\}$ is called a *branch pair*. (This relation is not transitive.) Branch points of $Q = M/\mathcal{L}$ correspond to *separatrices* of the foliation $\mathcal{L}$. For a given $q \in Q$ there may exist infinitely many $q' \in Q$ such that $\{q,q'\}$ is a branch pair. The main difficulty with constructing functions on non-Hausdorff manifolds is that, given a branch pair $\{q,q'\}$, a germ of a smooth function at q need not correspond to a smooth germ at q'. (An example [24] is obtained by gluing two copies $\mathbb{R}_1,\mathbb{R}_2$ of the real line $\mathbb{R}$ by identifying $t > 0$ in $\mathbb{R}_1$ with $t^\alpha \in \mathbb{R}_2$; no identifications are made for $t \leq 0$. The resulting space Q_α has the only branch points corresponding to $0 \in \mathbb{R}_1$ resp. $0 \in \mathbb{R}_2$. The manifolds Q_α for different values of $\alpha \geq 1$ are non-diffeomorphic. If $\alpha > 1$ then the germ at $0 \in \mathbb{R}_1$ of any $\mathcal{C}^1$ function on Q_α has vanishing derivative and hence Q_α is not regular (see below). If α is irrational then Q_α does not admit any nonconstant real analytic function in a neighborhood of the union of the two branch points.)

A one dimensional manifold Q of class $\mathcal{C}^r$ (not necessarily Hausdorff) is *regular* if any germ of a $\mathcal{C}^r$ function at any point $q \in Q$ is the germ at q of a global $\mathcal{C}^r$

function on Q (Definition 2 in [24], p. 116). Every Hausdorff manifold is regular. The following results are due to Haefliger and Reeb.

Proposition 4.4. ([24], Proposition, p. 123) *Let $\mathcal{L}$ be a simple $\mathcal{C}^r$ foliation of codimension one on a manifold M. For any open relatively compact domain $A \subset M$ the leaf space $Q_A = A/\mathcal{L}_A$ of the restricted foliation $\mathcal{L}_A$ is a regular one dimensional $\mathcal{C}^r$ manifold, possibly non-Hausdorff.*

Proposition 4.5. ([24], Proposition 1, p. 117) *A regular, simply connected, one dimensional manifold of class $\mathcal{C}^r$ $(r \geq 1)$ (possibly non-Hausdorff) admits a $\mathcal{C}^r$ function with nowhere vanishing differential.*

Combining propositions 4.3, 4.4, 4.5 one gets the following (known) corollary which was proved for smooth foliations of the plane $\mathbb{R}^2$ by Kamke [28].

Corollary 4.6. *A simple foliation of codimension one and class $\mathcal{C}^r$ $(r \geq 1)$ on a manifold M admits a noncritical first integral of class $\mathcal{C}^r$ on any relatively compact, simply connected open set in M.*

We emphasize that the manifold M in corollary 4.6 need not be simply connected, but of course one gets a first integral only on a simply connected subset.

Assume now that $(M, \mathcal{L})$ satisfies the hypotheses of theorem 4.2. Choose a compact set K in M. By lemma IV.1.6. in [19] there is a relatively compact domain $A \subset\subset M$, with $K \subset A$, whose boundary is of class $\mathcal{C}^r$ and *in general position* with respect to $\mathcal{L}$, meaning that any local defining function for $\mathcal{L}$ is a Morse function when restricted to bA and distinct critical points of these functions belong to distinct leaves of $\mathcal{L}$ ([19], Definition IV.1.4., p. 228).

A point $p \in bA$ is a critical point of a local defining function for $\mathcal{L}$ (restricted to bA) precisely when the leaf $L_0 \in \mathcal{L}$ through p meets bA tangentially at p, i.e., it is a *separatrix*. By the choice of A all but finitely many leaves intersect bA transversely and hence the leaf space $Q_A = A/\mathcal{L}_A$ has at most finitely many branch points. Furthermore, since distinct points of contact of leaves with bA belong to distinct leaves, Q_A has at most double branch points.

To complete the proof of theorem 4.2 it suffices to show the following.

Proposition 4.7. *An oriented, not necessarily Hausdorff one dimensional manifold of class $\mathcal{C}^r$ $(r \in \{1, 2, \ldots, +\infty\})$ which is regular and has at most finitely many branch points admits a nowhere vanishing differential one-form of class $\mathcal{C}^{r-1}$.*

Proof. Fix an orientation on Q. Let $x_j, y_j \in Q$, for $j = 1, \ldots, k$, be all pairs of branch points, i.e., x_j and y_j cannot be separated by neighborhoods but any other pair of distinct points of Q can be separated. (There are no multiple branch points by the choice of A.) Choose open coordinate neighborhoods $x_j \in U_j \subset Q$ and orientation preserving $\mathcal{C}^r$ diffeomorphisms $h_j : U_j \to I = (-1, 1) \subset \mathbb{R}$. By regularity of Q we can assume (after shrinking U_j) that h_j extends to a $\mathcal{C}^r$ function $h_j : Q \to \mathbb{R}$. Also by regularity of Q the extended function has nonzero derivative at the point y_j (which forms a branch pair with x_j); see [24]. Thus $\theta_j = dh_j$ is a nonvanishing one-form of class $\mathcal{C}^{r-1}$ in an open connected neighborhood $V_j \subset Q$ of the pair $\{x_j, y_j\}$. We may assume that the closures $\overline{V}_j$ for $j = 1, \ldots, k$ are pairwise disjoint.

Choose a smaller compact neighborhood $E_j \subset\subset V_j$ of $\{x_j, y_j\}$ such that $V_j \backslash E_j$ is a union of finitely many segments, none of them relatively compact in V_j. (In fact we have three segments for a suitable choice of E_j.) Set $V = \cup_{j=1}^k V_j$ and $E = \cup_{j=1}^k E_j$. Then $Q_0 = Q \backslash E$ is an open, one dimensional, paracompact, oriented Hausdorff manifold, hence a union of open segments and one-spheres. Any one-sphere in Q_0 is a connected component of Q; choosing a nonvanishing one-form on it (in the correct orientation class) does not affect any choices that we shall make on the rest of the set. We do the same on any open segment of Q_0 which is a connected component of Q.

It remains to consider open segments of Q_0 which intersect at least one of the sets $V_j \backslash E_j$. Choose such a segment J and an orientation preserving parametrization $\phi \colon I = (-1, 1) \to J$. Let $I' = \{t \in I \colon \phi(t) \in V\}$. Then I' consists of either one or two subintervals of I, each of them having an end point at -1 or $+1$. Each of these subintervals is mapped by ϕ onto a segment in some $V_j \backslash E_j$. (Other possibilities would require a branch point of Q_0, a contradiction.)

Consider the case when $I' = I_0 \cup I_1$ where $I_0 = (-1, a)$, $I_1 = (b, 1)$ for a pair of points $-1 < a \le b < 1$. Let j and l be chosen such that $\phi(I_0) \subset V_j$, $\phi(I_1) \subset V_l$. Note that $\phi(a)$ is an endpoint of V_j and similarly $\phi(b)$ is an endpoint of V_l (we might have $j = l$). Then $\phi^*\theta_j = d(h_j \circ \phi)$ resp. $\phi^*\theta_l = d(h_l \circ \phi)$ are nonvanishing one-forms on I_0 resp. on I_1, both positive with respect to the standard orientation of $\mathbb{R}$. Choose a one-form τ on $(-1, 1)$ which agrees with the above forms near the respective end points -1 and $+1$ (obviously such τ exists). Then $(\phi^{-1})^*\tau$ is a one-form on $J = \phi(I) \subset Q_0$ which agrees with θ_j in a neighborhood of E_j and with θ_l in a neighborhood of E_l. Similarly we deal with the case that J intersects only one of the sets V_j. Performing this construction for each of the finitely many segments $J \subset Q_0$ which intersect V we obtain a nonvanishing one-form on Q. $\square$

Suppose now that M is a compact Levi-flat hypersurface, either without boundary or with boundary consisting of Levi leaves. In this case the stability theorems of Reeb and Thurston ([19], II.3; [11], §6.2) imply the following.

Theorem 4.8. *Suppose that M is a compact, connected, oriented, Levi-flat hypersurface of class $\mathcal{C}^r$ in a complex manifold X ($r \in \{2, 3, \dots, +\infty\}$) whose boundary bM is either empty or a union of Levi leaves. If there exists a compact Levi leaf L with $H^1(L, \mathbb{R}) = 0$ then the Levi foliation of M is simple and hence is defined by a closed one-form of class $\mathcal{C}^{r-1}$.*

Proof. The hypotheses imply that all leaves of $\mathcal{L}$ are compact and $\mathcal{L}$ consists of the level sets of a fibration $M \to Q$, where Q is either the circle S^1 (if $bM = \emptyset$) or the interval $[0, 1]$ ([11], theorem 6.2.1, p. 142). Thus $\mathcal{L}$ is simple and hence defined by a closed one-form. $\square$

Remark 4.9. If M is as in theorem 4.8 and $\mathcal{L}$ is assumed to be without holonomy (but we do not assume the existence of a compact leaf), one has the following dichotomy [11], p. 206: (i) either $\mathcal{L}$ has a compact leaf, and then all leaves are compact and $\mathcal{L}$ is given by a submersion of M onto S^1 or $[0, 1]$ (so theorem 4.8 holds); or (ii) all leaves are dense in M, and in this case $\mathcal{L}$ need not be given by a closed one-form (see the examples in Chapter 9 of [11]).

5. Levi-flat hypersurfaces with a finite fundamental group

The following result, together with theorem 3.4, establishes theorem 1.1 in case that hypothesis (c) holds.

Theorem 5.1. *Let M be a smooth orientable Levi-flat hypersurface with a finite fundamental group in a complex manifold X. If the Levi foliation $\mathcal{L}$ of M has no nontrivial one-sided holonomy (this holds in particular if M is real analytic) then the restriction of $\mathcal{L}$ to any open relatively compact subset $\omega \subset M$ is defined by a closed smooth one-form, and M admits a smooth defining function v such that $dd^c v$ is flat on ω. If in addition ω is simply connected then there are an open set $U \subset X$ with $U \cap M = \omega$ and a function $f = u + iv \in \mathcal{C}^\infty(U)$ such that v is a defining function for ω, $u|_\omega$ is a first integral of the Levi foliation $\mathcal{L}|_\omega$, and the forms $\overline{\partial} f$, $dd^c u$ and $dd^c v$ are flat on ω.*

The conclusion of theorem 5.1 may fail if the Levi foliation admits nontrivial one-sided holonomy, or if M is real analytic and $\pi_1(M)$ contains an element of infinite order (see the example in §7).

Proof. Let $\mathcal{L}$ be a smooth, transversely orientable, codimension one foliation of a connected smooth manifold M. By a theorem of Haefliger [23] (see also Theorem 1.3 in [19], p. 228) the existence of a closed null-homotopic transversal $\gamma \subset M$ to $\mathcal{L}$ implies the existence of a leaf $L \in \mathcal{L}$ with nontrivial one-sided holonomy along some closed curve in L. (The proof of Haefliger's theorem relies on the Poincaré-Bendixson theorem concerning the dynamics of foliations with singularities on the two-disc which are transverse to the boundary circle.) Hence the conditions in theorem 5.1 imply that $\mathcal{L}$ does not admit any closed transversal.

Let $\ell \subset M$ be a smooth embedded arc transverse to $\mathcal{L}$. We claim that each leaf $L \in \mathcal{L}$ intersects ℓ in at most one point (and hence $\mathcal{L}$ is a simple foliation). If not, we find a subarc $\tau \subset \ell$ whose endpoints x_0 and x_1 belong to a L. Connecting x_1 to x_0 by an arc $\tau' \subset L$ we get a closed loop $\lambda = \tau \cdot \tau' \subset M$. Using the triviality of the normal bundle to $\mathcal{L}$ along τ' one can modify λ in a small tubular neighborhood of τ' into a closed transversal $\widetilde{\lambda}$ to $\mathcal{L}$ ([19], p. 228, 1.2. (iii)). Since we have seen that a closed transversal does not exist, this contradiction proves the claim and shows that $\mathcal{L}$ is a simple foliation. By theorem 4.2 $\mathcal{L}$ is defined on any relatively compact subset $\omega \subset M$ by a smooth closed one-form α, and proposition 3.1 gives a smooth defining function v with $dd^c v$ flat on ω. If ω is simply connected then $\alpha = du$ for some $u \in \mathcal{C}^\infty(\omega)$. Clearly u is constant on the leaves of $\mathcal{L}$, i.e., a first integral of $\mathcal{L}|_\omega$. Its asymptotic complexification $f = u + iv$ is a smooth function in an open set $U \subset X$ with $U \cap M = \omega$ such that v is a defining function for ω and the forms $\overline{\partial} f$, $dd^c u$ and $dd^c v$ are flat on ω (see the proof of proposition 3.1).

Note that, by [17], there is no pluriharmonic defining function for ω even if M is real analytic and $\overline{\omega}$ is contractible. □

6. Vanishing of foliation cohomology

Let $(M, \mathcal{L})$ be a $\mathcal{C}^r$ foliated manifold. Denote by $\mathcal{C}^r_{\mathcal{L}}$ the sheaf of real $\mathcal{C}^r$ functions on M which are constant on the leaves. If $\mathcal{L}$ is a Levi foliation of a Levi-flat hypersurface M then $\mathcal{C}^r_{\mathcal{L}}$ is the sheaf of real valued CR functions of class $\mathcal{C}^r$ on M.

Theorem 6.1. *Let $\mathcal{L}$ be a transversely orientable codimension one foliation of class $\mathcal{C}^r$ on a manifold M ($r \in \{2, 3, \cdots, \infty, \omega\}$). If $H^1(M, \mathcal{C}^{r-1}_{\mathcal{L}}) = 0$ then $\mathcal{L}$ is given by a closed, nowhere vanishing one-form of class $\mathcal{C}^{r-1}$.*

Theorem 6.1 establishes the implication (e)⇒(a) in theorem 1.1. The cohomology group $H^1(M, \mathcal{C}^r_{\mathcal{L}})$ may be understood as a Čech or a de Rham group [36].

Proof. Transverse orientability of $\mathcal{L}$ implies the existence of a $\mathcal{C}^r$ vector field ν on M which is transverse to $\mathcal{L}$. Choose a transversely oriented $\mathcal{C}^r$ foliation atlas $\{(U_j, \phi_j) \colon j \in J\}$ on M defining $\mathcal{L}$. Write $\phi_j = (\phi'_j, h_j)$ where h_j maps U_j onto an open interval $I_j \subset \mathbb{R}$ and $\{h_j = c\}$ are the plaques of $\mathcal{L}|_{U_j}$. For any $i, j \in J$ with $U_{ij} := U_i \cap U_j \neq \emptyset$ we have $h_i = \alpha_{ij} \circ h_j$ on U_{ij} where $\alpha_{ij} \colon h_j(U_{ij}) \to h_i(U_{ij})$ is a $\mathcal{C}^r$ diffeomorphism with positive derivative. (The collection $\{\alpha_{ij}\}$ is a *Haefliger cocycle* defining $\mathcal{L}$ [11], [23].) We may assume $\nu(h_j) > 0$ for every j. Differentiation gives

$$\nu(h_i) = (\alpha'_{ij} \circ h_j)\, \nu(h_j) \quad \text{on } U_{ij}.$$

The collection of positive functions $b_{ij} = \alpha'_{ij} \circ h_j \in \mathcal{C}^{r-1}(U_{ij})$ is a one-cocycle on the covering $\{U_j\}$ with values in the sheaf $\mathcal{B}^{r-1}_{\mathcal{L}}$ of positive functions of class $\mathcal{C}^{r-1}$ which are constant on the leaves of $\mathcal{L}$. The exponential map, $\exp \colon \mathcal{C}^{r-1}_{\mathcal{L}} \to \mathcal{B}^{r-1}_{\mathcal{L}}$, $b \to e^b$, defines an isomorphism between the two sheaves (the group operation is additive on the first sheaf and multiplicative on the second). The hypothesis $H^1(M, \mathcal{C}^{r-1}_{\mathcal{L}}) = 0$ therefore implies that, after passing to a finer $\mathcal{L}$-atlas, the cocycle b_{ij} is a coboundary, $b_{ij} = b_j / b_i$ for some $b_j \in \Gamma(U_j, \mathcal{B}^{r-1}_{\mathcal{L}})$. This gives

$$b_i\, \nu(h_i) = b_j\, \nu(h_j) \quad \text{on } U_{ij}.$$

Since b_j is constant on the plaques $\{h_j = c\} \subset U_j$, we have $b_j = \beta_j \circ h_j$ for a unique $\mathcal{C}^{r-1}$ function $\beta_j \colon h_j(U_j) \to \mathbb{R}$. Setting $\alpha_j = \int \beta_j$ and $u_j = \alpha_j \circ h_j$ we have

$$\nu(u_j) = (\alpha'_j \circ h_j)\, \nu(h_j) = (\beta_j \circ h_j)\, \nu(h_j) = b_j\, \nu(h_j).$$

We have thus obtained functions $u_j \in \mathcal{C}^r_{\mathbb{R}}(U_j)$ ($j \in J$) which are constant on the plaques in U_j and satisfy $\nu(u_i) > 0$ on U_i, $\nu(u_i) = \nu(u_j) > 0$ on U_{ij}. Assuming as we may that the sets U_{ij} are connected, it follows that the differences $c_{ij} = u_j - u_i$ on U_{ij} are real constants and hence the differentials du_i define a closed nowhere vanishing one-form α on M with $\ker \alpha = T\mathcal{L}$. □

Problem 6.2. Find sufficient conditions for $H^1(M, \mathcal{C}^r_{\mathcal{L}}) = 0$.

7. A Levi-flat hypersurface with a worm domain on each side

The example in this section has been suggested to us by J.-E. Fornæss. The failure of theorems 1.1, 1.2 and corollary 1.4 in this example justifies the hypotheses (a), (b) and (c) in theorem 1.1.

Denote by (z, w) the coordinates on $\mathbb{C}^* \times \mathbb{C}$. Let $M \subset \mathbb{C}^* \times \mathbb{C}$ be the real-analytic hypersurface defined by the following equivalent equations:

$$M \colon \quad \Im(w e^{i \log z}) = 0 \Longleftrightarrow \Im(w e^{i \log |z|}) = 0.$$

Indeed, the functions under parentheses differ only by the positive multiplicative factor $e^{\arg z}$. The first function is multivalued pluriharmonic and hence M is Levi-flat. Introducing the holomorphic map $\Phi \colon \mathbb{C}^2 \to \mathbb{C}^* \times \mathbb{C}$, $\Phi(\zeta, t) = \big(e^{\zeta}, t e^{-i\zeta}\big)$, one sees that $M = \Phi(\mathbb{C}\times\mathbb{R})$. The restriction $\Phi(\cdot\,,t)$ to a leaf $\mathbb{C}\times\{t\}\subset\mathbb{C}\times\mathbb{R}$ gives a parametrization of the corresponding leaf

$$L_t = \{(e^{\zeta}, te^{-i\zeta})\colon \zeta\in\mathbb{C}\}\subset M,\quad t\in\mathbb{R}$$

in the Levi foliation $\mathcal{L}$ of M. This parametrization is biholomorphic if $t\neq 0$ (so $L_t\simeq\mathbb{C}$) while for $t=0$ it is the covering map $\mathbb{C}\to L_0=\mathbb{C}^*\times\{0\}$, $\zeta\to(e^{\zeta},0)$.

The line $E=\{(1,s)\colon s\in\mathbb{R}\}\subset M$ is a global transversal for $\mathcal{L}$, and for every $t\in\mathbb{R}$ we have $L_t\cap E=\{(1,te^{2k\pi})\colon k\in\mathbb{Z}\}$. The only closed leaf is $L_0=\mathbb{C}^*\times\{0\}$ to which all other leaves L_t approach spirally. Identifying E with $\mathbb{R}$ we see that the holonomy of L_0 along the positively oriented circle $|z|=1$ is $s\to se^{-2\pi}$. The space of leaves $M/\mathcal{L}$ is the union of a point representing L_0 and two closed circles, each representing the leaves L_t for $t>0$ resp. for $t<0$. The foliation is simple in the complement of L_0.

Writing $h(z,w)=we^{i\log z}$ we see that the restriction to $T(M\backslash L_0)$ of the closed holomorphic one-form

$$\alpha=\frac{dh}{h}=i\frac{dz}{z}+\frac{dw}{w}$$

on $\mathbb{C}^*\times\mathbb{C}^*$ defines the Levi foliation of $M\backslash L_0$. There is no such closed one-form in any neighborhood of L_0 due to nontrivial holonomy.

The hypersurface M divides $\mathbb{C}^*\times\mathbb{C}$ in two connected components

$$M_\pm=\{(z,w)\in\mathbb{C}^*\times\mathbb{C}\colon\ \pm\Im(we^{i\log|z|})>0\}$$

which have the essential properties of a *worm domain* (Diederich and Fornæss [13]). Consider the family of complex annuli

$$R_s=\{(z,is)\colon e^{-\pi/2}<|z|<e^{\pi/2}\},\quad s\in\mathbb{R}.$$

A calculation shows that

— $R_0\subset L_0\subset M$,
— $bR_s\subset M$ for all $s\in\mathbb{R}$,
— $R_s\subset M_+$ if $s>0$ and $R_s\subset M_-$ if $s<0$.

If f is a holomorphic function in a small neighborhood U of the annular set

$$A_0=\{(z,w)\in M\colon e^{-\pi/2}\le|z|\le e^{\pi/2},\ |w|\le 1\}$$

then by analytic continuation along the family of annuli R_s, $s\in[-1,1]$, we obtain a holomorphic extension of f to a neighborhood of the Levi-flat hypersurface $R=\cup_{s\in[-1,1]}R_s$ which therefore belongs to the holomorphic hull of A_0. Since R intersects both M_+ and M_-, we see that theorem 1.1 fails, and likewise theorem 1.2 fails for any strongly pseudoconvex domain $D\subset\mathbb{C}^*\times\mathbb{C}$ containing A_0.

Corollary 1.4 fails as well, which is seen as follows. With D as above set $D_\pm=D\cap M_\pm$, $A=\overline{D}\cap M\supset A_0$, $\omega=bD_+\backslash A$, $K=\overline{D}_-$ and $\Omega=D\backslash\widehat{K}\subset D_+$, where $\widehat{K}$ denotes the $\mathcal{O}(\overline{D}_+)$-hull of K. The above discussion shows that $R\subset\widehat{K}$ and hence Ω is a proper subset of D_+. By [29] and [34] every continuous CR function on ω extends holomorphically to Ω. Since Ω is pseudoconvex [43], there exists a function $f\in\mathcal{O}(\Omega)\cap\mathcal{C}(\Omega\cup\omega)$ which does not extend holomorphically to D_+.

Since the Levi foliation of M is simple in the complement of the leaf $L_0=\mathbb{C}^*\times\{0\}$, our results apply on $M\backslash L_0=M\cap(\mathbb{C}^*\times\mathbb{C}^*)$.

We mention that (non)continuation of holomorphic functions across Levi-flat hypersurfaces with singularities was also considered by E. Bedford [8].

8. Piecewise real analytic Levi-flat hypersurfaces

In this section we show that the standard constructions with codimension one foliations can be performed in the class of piecewise real analytic Levi-flat hypersurfaces. We also discuss the compact Levi-flat hypersurfaces.

8.1. **Complexification of a piecewise real analytic foliation.** Let Σ be a real analytic manifold (open or closed, with or without boundary, not necessarily orientable). Suppose that $\Sigma = D_1 \cup \cdots \cup D_m$ where $D_j \subset \Sigma$ are closed domains with real analytic boundaries bD_j such that for $j \neq k$ the intersection $D_j \cap D_k$ is a union of connected components of bD_j and bD_k (possibly empty). Assume that $\mathcal{F}_j$ is a real analytic foliation of codimension one in an open neighborhood of D_j in Σ such that bD_j is a union of leaves of $\mathcal{F}_j$. The foliation $\mathcal{F}$ of Σ whose restriction to D_j equals $\mathcal{F}_j|_{D_j}$ has real analytic leaves and is continuous in the transverse direction. If the foliations on two adjacent domains match up to order r along the common boundary components then $\mathcal{F}$ is a *piecewise real analytic foliation of class* $\mathcal{C}^r$ of Σ. (This is a special case of the *tangential gluing* of foliations; [11], §3.4.) A pair of codimension one foliations match up to order r along a common boundary leaf L if and only if their holonomies along any closed loop $\gamma \subset L$ are tangent to the identity map to order r on their respective sides of a local transversal to L at a point $x_0 \in \gamma$ (Proposition 3.4.2 in [11], p. 91).

Suppose now that Σ is embedded as a real analytic *maximal real submanifold* in a complex manifold (X, J), meaning that $T_xX = T_x\Sigma \oplus J(T_x\Sigma)$ for all $x \in \Sigma$. (Every real analytic manifold Σ can be realized as a maximal real submanifold of a Stein complexification. If $\dim \Sigma = 3$ and Σ is orientable then it admits a maximal real embedding in $\mathbb{C}^3$ [3], [16], [21].) If $\mathcal{F}$ is a real analytic codimension one foliation of Σ then by complexifying its leaves we obtain a Levi foliation on a real analytic Levi-flat hypersurface $M \subset X$ which is defined in an open neighborhood of Σ in X. Along Σ we have $TM = T\mathcal{F} \oplus J(T\mathcal{F}) \oplus N$ where J is the complex structure on X and $N \simeq T\Sigma/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$ in Σ.

The same applies to a piecewise real analytic foliation $\mathcal{F}$ of Σ described above. If $\mathcal{F}$ is of class $\mathcal{C}^r$ then the complexifications of the individual foliations $\mathcal{F}_j$ on D_j match to order r along the common boundary components and define the Levi foliation of a piecewise real analytic $\mathcal{C}^r$ hypersurface $M = M_1 \cup \cdots \cup M_m$ such that the Levi foliation $\mathcal{L}_j$ of M_j is the complexification of $\mathcal{F}_j$. The common boundary of M_j and M_k (if not empty) is a union of complex leaves. The foliated manifolds $(M, \mathcal{L})$ and $(\Sigma, \mathcal{F})$ have the same structure (in particular, the same space of leaves).

8.2. **A complexified Reeb foliation.** (For the standard $\mathcal{C}^\infty$ Reeb foliation see examples 1.1.12 and 3.3.11 in [11].) Choose integers $n, r \geq 1$. Let (x, t) be coordinates on $\mathbb{R}^n \times S^1$. Choose a strictly increasing polynomial function $\lambda\colon \mathbb{R} \to \mathbb{R}$ satisfying $\lambda(0) = -\pi/2$, $\lambda(1) = 0$, $\lambda(4) = \pi/2$, whose derivatives up to order r vanish at $0, 1, 4$. The one-form

$$\omega = \cos\lambda(\rho) d\rho + \sin\lambda(\rho) dt \qquad (\rho = |x|^2 \in \mathbb{R}_+)$$

is real analytic and nonvanishing on $\mathbb{R}^n \times S^1$. From $d\omega \wedge \omega = 0$ we infer that ω determines a codimension one foliation $\mathcal{F}$ of $\mathbb{R}^n \times S^1$. Its restriction to $T = D^n \times S^1$ ($D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$) is a Reeb foliation of T (a *Reeb component*) with the only closed leaf $F_0 = bD^n \times S^1 = S^{n-1} \times S^1$ to which all other leaves spirally approach.

This foliation is not defined by a closed one-form in any neighborhood of F_0 due to nontrivial holonomy along the loops $\{x\} \times S^1$ ($|x| = 1$); this holonomy is flat to order r. The restriction of $\mathcal{F}$ to $\{|x| < 1\} \times S^1$ is a simple foliation given by a fibration over S^1.

We decompose the three-sphere $S = S^3$ in a union $T_1 \cup T_2$ of two solid tori diffeomorphic to $D^2 \times S^1$, with $T_1 \cap T_2 = bT_1 = bT_2 = F_0 \simeq S^1 \times S^1$. Endowing each T_j with a Reeb foliation $\mathcal{F}_j$ described above one obtains a piecewise real analytic Reeb foliation $\mathcal{F}$ of S whose Reeb components match to order $r \in \mathbb{N}$ along the boundary leaf F_0 (for the $\mathcal{C}^\infty$ case see Example 3.4.4 in [11], p. 93). Embedding S as a real analytic totally real submanifold of $\mathbb{C}^3$ [1] and complexifying $\mathcal{F}$ we obtain a piecewise real analytic Levi-flat hypersurface $M \subset \mathbb{C}^3$ whose Levi foliation has the same structure (and the same leaf space) as the Reeb foliation of S^3. M admits an asymptotically defining function in the complement of the leaf L_0 which is the complexification of the torus leaf $F_0 \subset S$ since the foliation is simple on $M \backslash L_0$; hence our results apply on $M \backslash L_0$. However, there is no such function in any neighborhood of L_0 due to nontrivial holonomy.

Problem 8.1. Let $U \subset \mathbb{C}^3$ be a strongly pseudoconvex tubular neighborhood of S. Does $A = M \cap \overline{U}$ admit a basis of Stein neighborhoods?

If U is sufficiently thin then a positive answer is obtained by observing that M admits a transverse holomorphic vector field in a neighborhood of S. Indeed, there is a smooth vector field ν on S which is transverse to $\mathcal{F}$; since S is totally real in $\mathbb{C}^3$, we can approximate ν by a holomorphic vector field w defined in a neighborhood of S in $\mathbb{C}^3$, and Jw is then transverse to M near S.

In this connection we mention a theorem of Novikov [38] to the effect that *every* $\mathcal{C}^2$ foliation of S^3 by surfaces contains a Reeb component. According to Barrett [4] the $\mathcal{C}^\infty$ Reeb foliation on S^3 itself cannot be realized as the Levi foliation of a smooth Levi-flat hypersurface [4].

8.3. **Turbulization and spinning.** Let $\mathcal{F}$ be the foliation in §8.2 but considered now on $\{|x| \le 2\} \times S^1$. Since $\lambda(4) = \pi/2$ and the derivatives of λ up to order r vanish at 4, the form ω is tangent to dt to order r along the torus $T' = \{|x| = 2\} \times S^1$, and hence $\mathcal{F}$ matches along T' to order r with the trivial (horizontal) foliation $\mathcal{F}_0$ of $\mathbb{R}^n \times S^1$ with leaves $\{t = c\}$. Let $\mathcal{F}_{turb}$ denote the foliation of $\mathbb{R}^n \times S^1$ which equals $\mathcal{F}$ on $\{|x| \le 2\} \times S^1$ and equals $\mathcal{F}_0$ on $\{|x| \ge 2\} \times S^1$. This deformation of $\mathcal{F}_0$, known as *turbulization* (*tourbillonement*), can be made in a small tubular neighborhood of any closed transversal γ in a codimension one foliation and produces a new Reeb component along γ. In the real analytic case it can be made by a piecewise real analytic deformation which is suitable for complexification, thus given Levi-flat realizations of turbulization.

A similar *spinning* modification can be made at a boundary component $S \subset bM$ of a foliated manifold $(M, \mathcal{F})$ provided that every leaf of $\mathcal{F}$ intersects S transversely and the induced foliation $\{F \cap S \colon F \in \mathcal{F}\}$ is determined by a closed one-form on S (Example 3.3.B in [11], p. 84). This modification changes S into a closed leaf of a new foliation $\mathcal{F}_{spin}$ which is then suitable for tangential gluing along S. If all data are real analytic then by a simple modification of the standard construction $\mathcal{F}_{spin}$ can be made piecewise real analytic and smooth to a given finite order. The Reeb component $\mathcal{F}$ on $D^n \times S^1$ in §8.2 is obtained by spinning the trivial foliation with leaves $D^n \times \{t\}$ along the boundary $bD^n \times S^1$.

8.4. **A Levi-flat $M^5 \subset \mathbb{C}^3$ with an exceptional minimal set.** Let Σ_2 denote the closed connected surface of genus 2 with a real analytic structure. There is a real analytic foliation $\mathcal{F}$ of $\Sigma_2 \times S^1$ obtained by suspending a pair of real analytic diffeomorphisms $h_0, h_1 \in \mathrm{Diff}_\omega(S^1)$ of the circle over a pair of generators of the fundamental group $\pi_1(\Sigma_2)$ (Example 4.1.6 in [11], p. 104). A suitable choice of h_0 and h_1 insures that $\mathcal{F}$ has an *exceptional minimal set* K (which is the unique minimal set of $\mathcal{F}$). The intersection of such K with any transverse arc to $\mathcal{F}$ gives a Cantor set in that arc. Such K always contains a leaf with nontrivial holonomy and hence the foliation is not given by a closed one-form. Embed $\Sigma_2 \times S^1$ as a real analytic totally real submanifold of $\mathbb{C}^3$ [15] and complexify $\mathcal{F}$ to obtain a real analytic Levi-flat hypersurface $M^5 \subset \mathbb{C}^3$, diffeomorphic to $\Sigma_2 \times S^1 \times D^2$, whose Levi foliation contains an exceptional minimal set.

8.5. **Compact real analytic Levi-flat hypersurfaces.** Every orientable real analytic foliation $\mathcal{L}$ of codimension one on a real analytic three-manifold M can be realized as the Levi foliation on an embedded real analytic hypersurface $M \subset X$ in a complex surface X [5]. (The corresponding result in higher dimension requires that $T^{\mathbb{C}}M$ admits a complex structure operator J satisfying the Newlander-Nirenberg integrability condition on the leaves, [25].)

Examples of compact, real analytic, three-dimensional Levi-flat hyersurfaces include boundaries of holomorphic disc bundles over compact Riemann surfaces [14], manifolds obtained as quotients of real hyperplanes in $\mathbb{C}^2$ under a group action, and hypersurfaces obtained by intersecting a compact projective surface $X \subset \mathbb{P}^n$ by a generic real hyperplane $\Gamma \subset \mathbb{P}^n$ [37]. ($\mathbb{P}^n$ itself does not admit any compact smooth Levi-flat hypersurfaces [12], [42].) In the latter case it may happen that both connected components $\Omega_\pm$ of $X \backslash M$ (where $M = X \cap \Gamma$) are Stein domains. Such M cannot admit a pluriharmonic defining function v since the level sets $\{v = c\}$ for small $c \neq 0$ would be compact Levi-flat hypersurfaces in a Stein domain, contradicting the maximum principle for strongly plurisubharmonic functions. (A two dimensional Stein domain with compact Levi-flat boundary does not even admit a plurisubharmonic defining function [37], Corollary, p. 4.)

In [5] Barrett discussed real analytic Levi-flat structures on the three-torus $\mathbb{T}^3$, motivated by Ohsawa's example [39]. An interesting feature of the class which he investigated is that the existence of a pluriharmonic defining function is equivalent to triviality of the holonomy of its Levi foliation (Theorem 1 in [5]). Barrett also gave an example of a real analytic Levi-flat three-manifold with trivial holonomy and without a pluriharmonic defining function (Theorem 3 in [5]); its construction uses a real analytic diffeomorphism of the circle which is topologically but not diffeomorphically conjugate to a rotation.

In [7] the authors proved that that every $\mathcal{C}^\infty$ Levi-flat three-manifold is diffeomorphic to $S^2 \times S^1$ or to a compact quotient of $\mathbb{R}^3$ by a lattice. Their methods do not apply directly to foliations of a finite smoothness class, and it does not seem clear what (if any) topological obstruction on M can be inferred from the existence of a piecewise real analytic Levi-flat structure on M.

Acknowledgements. We thank J.-E. Fornæss, J. Mrčun, T. Ohsawa and M.-C. Shaw for helpful remarks and suggestions.

References

1. AHERN, P., RUDIN, W., Hulls of 3-spheres in $\mathbb{C}^3$, in *The Madison Symposium on Complex Analysis* (Madison, WI, 1991, 1–27), *Contemp. Math.*, 137, Amer. Math. Soc., Providence, RI, 1992.
2. AIRAPETIAN, R. A., Extending CR functions from piecewise smooth CR manifolds. *Math. Sb.*, 134 (1987), 108–118 (Russian).
3. AUDIN, M., Fibrés normaux d'immersions en dimension double, points doubles d'immersions lagragiennes et plongements totalement réels. *Comment. Math. Helv.*, 63 (1988), 593–623.
4. BARRETT, D. E., Complex analytic realization of Reeb's foliation of S^3. *Math. Z.*, 203 (1990), 355–361.
5. ———, Global convexity properties of some families of three-dimensional compact Levi-flat hypersurfaces. *Trans. Amer. Math. Soc.*, 332 (1992), 459–474.
6. BARRETT, D. E., FORNÆSS, J.-E., On the smoothness of Levi-foliations. *Publ. Mat.*, 32 (1988), 171–177.
7. BARRETT, D. E., INABA, T., On the topology of compact smooth three-dimensional Levi-flat hypersurfaces. *J. Geom. Anal.*, 2 (1992), 489–497.
8. BEDFORD, E., Holomorphic continuation of smooth functions over Levi-flat hypersurfaces. *Trans. Amer. Math. Soc.*, 232 (1977), 323–341.
9. BEDFORD, E., DE BARTOLOMEIS, P., Levi flat hypersurfaces which are not holomorphically flat. *Proc. Amer. Math. Soc.*, 81 (1981), 575–578.
10. BOGGESS, A., *CR Manifolds and the Tangential Cauchy-Riemann Complex.* CRC Press, Bocca Raton, 1991.
11. CANDEL, A., CONLON, L., *Foliations I.* Grad. Studies in Math., 23, Amer. Math. Soc., Providence, Rhode Island, 2000.
12. CAO, J., SHAW, M.-C., WANG, L., Estimates for the $\overline{\partial}$-Neumann problem and nonexistence of $\mathbb{C}^2$ Levi-flat hypersurfaces in $\mathbb{CP}^n$, *Math. Z.*, 247 (2004), 183–221; Erratum, 223–225.
13. DIEDERICH, K., FORNÆSS, J.-E., Pseudoconvex domains: an example with nontrivial Nebenhlle. *Math. Ann.*, 225 (1977), 275–292.
14. DIEDERICH, K., OHSAWA, T., Harmonic maps and disc bundles over compact Kähler manifolds. *Publ. Res. Inst. Math. Sci. Kyoto Univ.*, 21 (1985), 819–833.
15. FORSTNERIČ, F., Some totally real embeddings of three–manifolds. *Manuscripta Math.*, 55 (1986), 1–7.
16. ———, On totally real embeddings into $\mathbb{C}^n$. *Expo. Math.*, 4 (1986), 243–255.
17. ———, A contractible Levi-flat hypersurface in $\mathbb{C}^2$ which is a determining set for pluriharmonic functions. Preprint, June 2004. [arXiv: mathCV/0406572]
18. GHYS, É., L'invariant de Godbillon-Vey. *Astérisque*, 177-178 (1989), 155–181.
19. GODBILLON, C., *Feuilletages, études géométriques.* Birkhäuser, Basel-Boston-Berlin, 1991.
20. GODBILLON, C., VEY, J., Un invariant des feuilletages de codimension 1. *C. R. Acad. Sci. Paris, Sér. A*,
21. GROMOV, M., *Partial Differential Relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 9. Springer, Berlin–New York, 1986.
22. GUNNING, R. C., ROSSI, H., *Analytic Functions of Several Complex Variables.* Prentice–Hall, Englewood Cliffs, 1965.
23. HAEFLIGER, A., Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. *Comment. Math. Helv.*, 32 (1958), 248–329.
24. HAEFLIGER, A., REEB, G., Variétés (non séparées) à un dimension et structures feuilletées du plan. *L'Enseignement Math.*, 3 (1957), 107–125.
25. HAEFLIGER, A., SUNDARARAMAN, D., Complexifications of transversely holomorphic foliations. *Math. Ann.*, 272 (1985), 23–27.
26. HECTOR, G., HIRSCH, U., *Introduction to the geometry of foliations, Parts A & B.* Second ed., Friedr. Vieweg & Sohn, Braunschweig, 1986 & 1987.
27. HÖRMANDER, L. *An Introduction to Complex Analysis in Several Variables,* 3rd ed. North Holland, Amsterdam, 1990.

28. KAMKE, E., Über die partielle Differentialgleichung $fz_x + gz_y = h$. *Math. Z.*, 41 (1936), 56–66; 42 (1936), 287–300.
29. LAURENT–THIÉBAUT, C., Sur l'extension des fonctions CR dans une variété de Stein. *Ann. Mat. Pura Appl.*, 150 (1988), 141–151.
30. ——— , Extension de formes différentielles CR. *C. R. Acad. Sci. Paris Sér. I Math.*, 306 (1988), 539–542.
31. ——— , Sur l'équation de Cauchy-Riemann tangentielle dans une calotte strictement pseudoconvexe. Preprint, Institut Fourier, no. 645 (2004). [http://www-fourier.ujf-grenoble.fr/prepublications.html]
32. LAURENT-THIÉBAUT, C., LEITERER, J., On the Hartogs-Bochner extension phenomenon for differential forms *Math. Ann.*, 284 (1989), 103–119.
33. LAURENT-THIÉBAUT, C., PORTEN, E., Analytic extension from non-pseudoconvex boundaries and $A(D)$-convexity. *Ann. Inst. Fourier (Grenoble)*, 53 (2003), 847–857.
34. LUPACCIOLU, G., Characterization of removable sets in strongly pseudoconvex boundaries. *Arkiv Math.*, 32 (1994), 455–473.
35. LUPACCIOLU, G., TOMASSINI, G., An extension theorem for CR-functions. *Ann. Mat. Pura Appl.*, 137 (1984), 257–263.
36. MASA, X., Alexander-Spanier cohomology of foliated manifolds. *Illinois J. Math.*, 46 (2002), 979–998.
37. NEMIROVSKI, S., Stein domains with Levi-flat boundaries on compact complex surfaces. *Mat. Zametki*, 66 (1999), 632–634; English transl. in *Math. Notes*, 66 (1999), 522–525.
38. NOVIKOV, S. P., Topology of foliations. *Trans. Moscow Math. Soc.*, 14 (1965), 268–305.
39. OHSAWA, T., A Stein domain with smooth boundary which has a product structure. *Publ. Res. Inst. Math. Sci. Kyoto Univ.*, 18 (1982), 1185–1186.
40. SACKSTEDER, R., Foliations and pseudogroups. *Amer. J. Math.*, 87 (1965), 79–102.
41. SHCHERBINA, N. V., Decomposition of a common boundary of two domains of holomorphy into analytic curves. *Iszvestia Akad. Nauk SSSR Ser. Mat*, 46 (1982), 1106–1123; *Math. USSR Iszvestia*, 21 (1983), 399–413.
42. SIU, Y.-T., $\overline{\partial}$-regularity for weakly pseudoconvex domains in hermitian symmetric spaces with respect to invariant metrics. *Ann. Math.*, 156 (2002), 595–621.
43. SLODKOWSKI, Z., Analytic set-valued functions and spectra. *Math. Ann.*, 256 (1981), 363–386.
44. SHAW, M.-C., L^p estimates for local solutions of $\overline{\partial}_{\rm b}$ on strongly pseudo-convex CR manifolds. *Math. Ann.*, 288 (1990), 35–62.
45. ——— , Local existence theorems with estimates for $\overline{\partial}_b$ on weakly pseudo-convex CR manifolds. *Math. Ann.*, 294 (1992), 677–700.
46. WAZEWSKY, T., Sur l'équation $Pp + Qq = 0$. *Mathematica*, 8 (1934), 103–116; 9 (1935), 179–182.

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: franc.forstneric@fmf.uni-lj.si

Institut Fourier, UMR 5582 CNRS/UJF, BP 74, 38402 St Martin d'Hères Cedex, France

E-mail address: Christine.Laurent@ujf-grenoble.fr