Splitting formulae for the Kontsevich-Kuperberg-Thurston invariant of rational homology 3-spheres

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Abstract

The Kontsevich-Kuperberg-Thurston invariant Z of rational homology 3-spheres was constructed by M. Kontsevich using configuration space integrals. G. Kuperberg and D. Thurston have proved that it is a universal finite type invariant for integral homology spheres in the sense of Ohtsuki, Goussarov and Habiro.

We discuss the behaviour of Z under rational homology handlebodies replacements. The explicit formulae that we present generalize a sum formula obtained by the author for the Casson-Walker invariant in 1994. They allow us to identify the degree one term of Z with the Walker invariant for rational homology spheres.

Keywords: 3-manifolds, configuration space integrals, homology spheres, finite type invariants, Jacobi diagrams, clovers, claspers, Casson-Walker invariant
A.M.S. subject classification: 57M27 57N10 55R80 57R20

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Contents

1 Introduction

In 1995 in [O], Tomotada Ohtsuki introduced a notion of finite type invariants for homology 3-spheres (that are compact oriented 3-manifolds with the same homology with integral coefficients as the standard 3-sphere S^3), following the model of the notion of Vassiliev invariants for links in the ambient space $\mathbb{R}^3$. He defined a filtration of the real vector space freely generated by homology 3-spheres and began the study of the associated graded space. In [Le], Thang Le finished identifying this graded space to an algebra of Jacobi diagrams called $\mathcal{A}(\emptyset)$ whose definition is recalled in Subsection 2.1. To do this, Le proved that the LMO invariant of 3-manifolds Z_{LMO} that he constructed with the help of J. Murakami and Ohtsuki in [LMO] induces an isomorphism from the Ohtsuki graded space to $\mathcal{A}(\emptyset)$. In [KT], following Witten, Axelrod, Singer, Kontsevich, Bott and Cattaneo, Greg Kuperberg and Dylan Thurston constructed another (possibly equal) invariant Z_{KKT} of rational homology 3-spheres valued in $\mathcal{A}(\emptyset)$, and they proved that Z_{KKT} also induces the already mentioned Le isomorphism. All real-valued finite type invariants in the Ohtsuki sense factor through $Z = Z_{KKT}$ (or Z_{LMO}). Therefore Z_{KKT} and Z_{LMO} are called universal finite type invariants of homology 3-spheres. They play the same role as the Kontsevich integral does play in the theory of Vassiliev link invariants.

In this article, we prove explicit formulae on the behaviour of Z_{KKT} under Lagrangian-preserving rational homology handlebodies replacements. The precise statement is given in Theorem 2.4 after the needed definitions. This behaviour had been observed in the case of Torelli replacements by Kuperberg and Thurston in [KT], it is the key point in their proof of universality for their invariant.

The obtained formulae described below generalize the formulae obtained in [L1] for the Casson-Walker invariant. They enlight the relationships between finite type invariants, Jacobi diagrams, intersection forms and linking forms. They also allow us to identify the degree one part of Z_{KKT} with the Walker invariant for any rational homology sphere in Theorem 2.6.

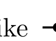
In the case of integral homology spheres, it is proved in [AL] that the splitting formulae obtained in this article follow from the Kuperberg-Thurston formulae for Torelli replacements.

The detailed proofs of the formulae in the general case are given in Sections 3 to 5. Their sketch, that is given in Section 3, is the now-standard sketch in this kind of proofs. But filling in the details in the general case was surprisingly complicated to me. The detailed proofs are given here with full generality, they substantially simplify in the case of Torelli replacements. Since the proofs heavily rely on the Kuperberg-Thurston construction, this construction has been recalled in [L2, Section 1] and all the precise statements that are needed in our proof are given and proved in [L2]. I hope that the technical work contained here will help finding other properties for the invariant Z_{KKT}.

I thank Dylan Thurston for very useful and pleasant conversations.

2 Statement of the main result

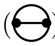
2.1 Jacobi diagrams

Here, a *Jacobi diagram* Γ is a trivalent graph Γ without simple loop like . The set of vertices of such a Γ will be denoted by $V(\Gamma)$, its set of edges will be denoted by $E(\Gamma)$. A *half-edge* c of Γ is an element of

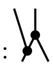
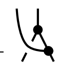
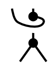
$$H(\Gamma) = \{c = (v(c); e(c)) | v(c) \in V(\Gamma); e(c) \in E(\Gamma); v(c) \in e(c)\}.$$

An *automorphism* of Γ is a permutation b of $H(\Gamma)$ such that for any $c, c' \in H(\Gamma)$,

$$v(c) = v(c') \Longrightarrow v(b(c)) = v(b(c')) \text{ and } e(c) = e(c') \Longrightarrow e(b(c)) = e(b(c')).$$

The number of automorphisms of Γ will be denoted by $\sharp\mathrm{Aut}(\Gamma)$. For example, $\sharp\mathrm{Aut}($$) = 12$. *An orientation* of a vertex of such a diagram Γ is a cyclic order of the three half-edges that meet at that vertex. A Jacobi diagram Γ is *oriented* if all its vertices are oriented (equipped with an orientation). The *degree* of such a diagram is half the number of its vertices.

Let $\mathcal{A}_n(\emptyset)$ denote the real vector space generated by the degree n oriented Jacobi diagrams, quotiented out by the following relations AS and IHX:

AS :  +  $= 0$, and IHX :  +  +  $= 0$.

Each of these relations relate diagrams which can be represented by planar immersions that are identical outside the part of them represented in the pictures. Here, the orientation of vertices is induced by the counterclockwise order of the half-edges. For example, AS identifies the sum of two diagrams which only differ by the orientation at one vertex to zero. $\mathcal{A}_0(\emptyset)$ is equal to $\mathbb{R}$ generated by the empty diagram.

2.2 The Kontsevich-Kuperberg-Thurston universal finite type invariant Z

Let Λ be $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Q}$. A Λ-*sphere* is a compact oriented 3-manifold M such that $H_*(M;\Lambda) = H_*(S^3;\Lambda)$. A $\mathbb{Z}$-sphere is also called a homology sphere while a rational homology sphere is a $\mathbb{Q}$-sphere. Following Witten, Axelrod, Singer, Kontsevich, Bott and Cattaneo, Greg Kuperberg and Dylan Thurston constructed invariants $Z_n = (Z_{KKT})_n$ of oriented $\mathbb{Q}$-spheres valued in $\mathcal{A}_n(\emptyset)$ and they proved that their invariants have the following property:

Theorem 2.1 (Kuperberg-Thurston [KT]) *An invariant ν of $\mathbb{Z}$-spheres valued in a real vector space X is of degree $\leq n$ if and only if there exist linear maps*

$$\phi_k(\nu) : \mathcal{A}_k(\emptyset) \longrightarrow X,$$

for any $k \leq n$, such that

$$\nu = \sum_{k=0}^{n} \phi_k(\nu) \circ Z_k.$$

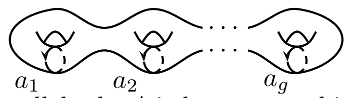
A *real finite type invariant* of $\mathbb{Z}$-spheres is a topological invariant of $\mathbb{Z}$-spheres valued in a real vector space X which is of degree less than some natural integer n. The Kontsevich-Kuperberg-Thurston construction is recalled in [L2, Section 1]. In this article, Theorem 2.1 is used as a definition of degree $\leq n$ real-valued invariants of $\mathbb{Z}$-spheres.

A degree $\leq n$ invariant ν is of *degree n* if $\phi_n(\nu) \neq 0$. In this case, $\phi_n(\nu)$ is the *weight system* of ν and is denoted by W_ν.

Remark 2.2 The above definition coincides with the Ohtsuki definition of real finite type invariants [O]. The Ohtsuki degree (that is always a multiple of 3) is three times the above degree. We shall not discuss the concept of finite-type invariants any further here. See [O, GGP, Ha, AL] and references therein.

2.3 Generalized clovers

Unless otherwise mentioned, manifolds are compact and oriented. Boundaries are oriented with the outward normal first convention. A *genus g rational homology handlebody or $\mathbb{Q}$-handlebody* (resp. a *genus g integral homology handlebody or $\mathbb{Z}$-handlebody*) is an (oriented, compact) 3-manifold A with the same homology with rational (resp. integral) coefficients as the standard (solid) handlebody H_g below.

$H_g =$ 

Note that the boundary of such a $\mathbb{Q}$-handlebody A is homeomorphic to the boundary ($\partial H_g = \Sigma_g$) of H_g. The intersection form on a surface Σ is denoted by $\langle , \rangle_\Sigma$. For a (compact, oriented) 3-manifold A with boundary ∂A, $\mathcal{L}_A$ denotes the kernel of the map induced by the inclusion:

$$H_1(\partial A; \mathbb{Q}) \longrightarrow H_1(A; \mathbb{Q}).$$

It is a Lagrangian of $(H_1(\partial A; \mathbb{Q}), \langle , \rangle_{\partial A})$, we call it the *Lagrangian* of A.

A *rational generalised clover* is a 4-tuple

$$D = (M; k; (A^i)_{i=1,\dots,k}; (B^i)_{i=1,\dots,k})$$

where

1. M is a rational homology sphere,

2. for any $i = 1, 2, \dots k$, A^i and B^i are $\mathbb{Q}$-handlebodies whose boundaries are identified by implicit diffeomorphisms (we shall write $\partial B^i = \partial A^i$) so that $\mathcal{L}_{B^i} = \mathcal{L}_{A^i}$,

3. the disjoint union of the A^i is embedded in M. We shall write

$$\sqcup_{i=1}^k A^i \subset M.$$

The integral number k is called the *degree* of D. Such a rational generalised clover D is an *integral generalised clover* if furthermore M is an integral homology sphere, and if B_1, B_2, $\dots$, B_k are integral homology handlebodies.

For such a rational generalised clover D, if J is a subset of $\{1, \dots, k\}$, $M_J(D)$ denotes the rational homology sphere obtained by replacing A^i by B^i for every element i of J.

$$M_J(D) = \left(M \setminus \cup_{i \in J} \mathrm{Int}(A^i)\right) \cup_{(\cup_{i \in J} \partial A^i)} \left(\cup_{i \in J} B^i\right)$$

If I is a topological invariant of integral (resp. rational) homology spheres valued in an abelian group, and if D is an integral (resp. rational) generalised clover, then we define $I(D)$ as $I(D) = \sum_{J \subseteq \{1,\dots,k\}} (-1)^{\sharp J} I(M_J(D))$.

Remark 2.3 The terminology *generalised clover* may not be a very happy one. I use it for the following reasons. The *generalised clovers* generalise the [GGP] clovers. In [Ha], Habiro independently developed a *clasper calculus* that encloses the *clover calculus* and also allows for more general modifications. In the Habiro terminology, *clovers* are called *allowable graph claspers.* I feel that the terminology *generalised clasper* cannot be used for something that does not generalise all the Habiro claspers, and I do not feel like saying *generalised allowable graph claspers.*

2.4 Generalised clovers and Jacobi diagrams

Let Γ be an oriented degree n Jacobi diagram. Let $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices of Γ and the set of edges of Γ, respectively. The set of half-edges of Γ is denoted by $H(\Gamma)$ and its two natural projections onto $V(\Gamma)$ and $E(\Gamma)$ are denoted by v and e, respectively.

Let $D = (M; 2n; (A^i)_{i=1,\dots,2n}; (B^i)_{i=1,\dots,2n})$ be a rational generalised clover. Let $\sigma : V(\Gamma) \longrightarrow \{1, 2, \dots, 2n\}$ be a bijection. Let us define the *linking number* $\ell(D; \Gamma; \sigma)$ *of* D *with respect to* Γ *and* σ.

The Mayer-Vietoris boundary map

$$\partial_{i,MV} : H_2(A^i \cup_{\partial A^i} -B^i) \longrightarrow \mathcal{L}_{A^i}$$

that maps the homology class of an oriented surface to the oriented boundary of its intersection with A^i is an isomorphism. This isomorphism carries the intersection form of the closed 3-manifold $(A^i \cup_{\partial A^i} -B^i)$ on $\otimes^3 H_2(A^i \cup_{\partial A^i} -B^i)$ to a linear form $\mathcal{I}(A^i, B^i)$ on $\otimes_{j=1}^3 \mathcal{L}_{A^i}^{(j)}$ which is antisymmetric with respect to the permutation of two factors, where $\mathcal{L}_{A^i}^{(j)}$ denote the j^{th} copy of $\mathcal{L}_{A^i}$. The linear form $\mathcal{I}(A^i, B^i)$ may be seen canonically as an element of $\otimes_{j=1}^3 \left(\mathcal{L}_{A^i}^{(j)}\right)^*$ where $\left(\mathcal{L}_{A^i}^{(j)}\right)^*$ denotes the dual $\mathrm{Hom}(\mathcal{L}_{A^i}^{(j)}; \mathbb{Q})$ of $\mathcal{L}_{A^i}^{(j)}$.

For each vertex w, number the three half-edges that contain w with a bijection

$$b(w) : v^{-1}(w) \longrightarrow \{1, 2, 3\}$$

that induces the given cyclic order of these half-edges.

Let c be a half-edge. Assign it the space

$$X(c) = \left(\mathcal{L}_{A^{\sigma(v(c))}}^{(b(v(c))(c)}\right)^*.$$

The linear form $\mathcal{I}(A^i, B^i)$ belongs to $\otimes_{c \in H(\Gamma); \sigma(v(c))=i} X(c)$. The tensor product of all the $\mathcal{I}(A^i, B^i)$, for $i = 1, 2, \dots, 2n$, belongs to

$$\otimes_{c \in H(\Gamma)} X(c).$$

For $\{i, j\} \subseteq \{1, 2, \dots, 2n\}$, the linking number in M induces a bilinear form on $H_1(A^i; \mathbb{Q}) \times H_1(A^j; \mathbb{Q})$, where $H_1(A^i)$ is canonically isomorphic to $\frac{H_1(\partial A^i; \mathbb{Q})}{\mathcal{L}_{A^i}}$. Furthermore, the intersection form $\langle , \rangle_{\partial A^i}$ induces the map

$$\begin{array}{cccc} \langle , . \rangle : & H_1(\partial A^i; \mathbb{Q}) & \longrightarrow & \mathcal{L}_{A^i}^* \\ & x & \mapsto & \langle ., x \rangle \end{array}$$

that in turn induces an isomorphism from $\frac{H_1(\partial A^i; \mathbb{Q})}{\mathcal{L}_{A^i}}$ to $\mathcal{L}_{A^i}^*$.

Thus, for each edge $f \in E(\Gamma)$ made of two half-edges c and d, (so that $e^{-1}(f) = \{c, d\}$) the linking number yields a contraction

$$\ell_f : X(c) \otimes X(d) \longrightarrow \mathbb{Q}.$$

Applying all these contractions to our big tensor maps this tensor to the *linking number* $\ell(D; \Gamma; \sigma)$ *of* D *with respect to* Γ *and* σ.

Finally, we define the *linking number* $\ell(D; \Gamma)$ *of* D *with respect to* Γ as the sum running over all the bijections σ from $V(\Gamma)$ to $\{1, 2, \dots, 2n\}$ of the $\ell(D; \Gamma; \sigma)$. Note that the product $\ell(D; \Gamma)[\Gamma]$ does not depend on the vertex-orientation of Γ.

2.5 Statement of the theorem

The main theorem of this article is the following one.

Theorem 2.4 *Let n and k be two integers such that $k \geq 2n \geq 0$. Let D be a degree k rational generalised clover.*

$$\begin{array}{rcll} Z_n(D) & = & 0 & \textit{if } k > 2n, \\ Z_n(D) & = & \sum_\Gamma \frac{\ell(D;\Gamma)}{\sharp Aut(\Gamma)}[\Gamma] & \textit{if } k = 2n, \end{array}$$

where the sum runs over all degree n Jacobi diagrams Γ.

The above theorem has the following immediate corollary.

Corollary 2.5 *Let n and k be two integral numbers such that $k \geq 2n$. Let ν be a degree n invariant of homology spheres valued in a real vector space. Let D be a degree k integral generalised clover.*

$$\begin{array}{rcll} \nu(D) & = & 0 & \textit{if } k > 2n, \\ \nu(D) & = & \sum_\Gamma \frac{\ell(D;\Gamma)}{\sharp Aut(\Gamma)} W_\nu(\Gamma) & \textit{if } k = 2n, \end{array}$$

where the sum runs over all degree n Jacobi diagrams Γ. The product $\ell(D;\Gamma)W_\nu(\Gamma)$ does not depend on the vertex-orientation.

Note that this corollary applies to $(Z_{LMO})_n$.

The following corollary of Theorem 2.4 is proved in Section 6.

Theorem 2.6 *For any rational homology sphere M, if λ_W denotes the Walker invariant normalized as in [W], then*

$$Z_1(M) = \frac{\lambda_W(M)}{4}[\ominus]$$

As another corollary, we could describe the generalised clovers in the setting of the Habiro-Goussarov filtration of integral homology spheres, and give an algebraic version of the clover calculus for integral homology 3-spheres over $\mathbb{Q}$. This is done in [AL], where an algebraic version of the clover calculus for integral homology 3-spheres over $\mathbb{Z}$ is also given.

Theorem 2.4 was observed by D. Thurston and G. Kuperberg when the rational homology handlebodies B^i are obtained from the A^i by composition of the identification of the boundaries by a Torelli diffeomorphism that is a diffeomorphism that induces the identity in homology in [KT]. This particular case is enough to conclude that Z is a universal finite type invariant of integral homology spheres. Together with the fact that $Z(S^3) = 1$, it implies Theorem 2.6 for integral homology spheres that is also due to D. Thurston and G. Kuperberg.

The proof of Theorem 2.4 strongly relies on the Kuperberg-Thurston construction of Z that is given in [L2] and not repeated here.

3 Sketch of the proof of Theorem 2.4

We refer to the construction of Z given in [L2, Section 1]. However we choose the homogeneous volume form ω_{S^2} on S^2 with total volume 1 once for all, and we only consider forms on $C_2(M)$ that are antisymmetric (with respect to the exchange of two points). In particular, in this article, we say that a 2-form ω_M on $C_2(M)$ is *fundamental* with respect to a trivialisation τ_M of $T(M \setminus \infty)$ that is standard near ∞ if it is antisymmetric and fundamental with respect to τ_M and ω_{S^2} in the sense of Definition 1.4 in [L2]. Similarly, here, a two-form ω_M on $C_2(M)$ or on $\partial C_2(M)$ is *admissible* if
• its restriction to $\partial C_2(M) \setminus ST(B_M)$ is $p_M(\tau_M)^*(\omega_{S^2})$ for some trivialisation τ_M of $T(M \setminus \infty)$ standard near ∞, and,
• it is closed, and antisymmetric.
In particular, all fundamental or admissible forms coincide on $\partial C_2(M) \setminus ST(B_M)$.

Fix a rational generalised clover $D = (M; \sharp N; (A^i)_{i=1,\dots,\sharp N}; (B^i)_{i=1,\dots,\sharp N})$.
For $I \subseteq N = \{1, 2, \dots, \sharp N\}$, set $M_I = M_I(D)$.

For any $i \in N = \{1, \dots, \sharp N\}$, fix disjoint simple closed curves $(a^i_j)_{j=1,\dots,g_i}$ and simple closed curves $(z^i_j)_{j=1,\dots,g_i}$ on ∂A^i, such that

$$\mathcal{L}_{A^i} = \oplus_{j=1}^{g_i} [a^i_j],$$

so that

$$\langle a^i_j, z^i_k \rangle_{\partial A^i} = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Let τ_M be a trivialisation of $M \setminus \infty$ that is standard near ∞. Define $\omega(\tau_M) = p_M(\tau_M)^*(\omega_{S^2})$ on $\partial C_2(M)$.

For $j \in N$, define an antisymmetric closed 2-form ω_j on $ST(B^j)$ that coincides with $\omega(\tau_M)$ on $ST(B^j)_{|\partial B^j}$, and define a trivialisation $\tau_j^{\mathbb{C}}$ of $TB^j \otimes \mathbb{C}$ that is the complexification of τ_M on ∂B^j as follows.

When the restriction of τ_M to ∂B^j extends to B^j as a trivialisation τ_j, simply set $\omega_j = p_{M_j}(\tau_j)^*(\omega_{S^2})$, and $\tau_j^{\mathbb{C}} = \tau_j \otimes \mathbb{C}$.

When $\tau_{M|\partial B^j}$ does not extend to B^j, there exists a curve c_j^0 in ∂B^j such that the twist $\mathcal{T}_{c_j^0} \circ \tau_{M|\partial B^j}$ across c_j^0 of $\tau_{M|\partial B^j}$ (see Definition 4.2) extends to B^j. The curve c_j^0 inherits a framing from ∂B^j. Let $N(c_j) = [a,b] \times c_j \times [-1,1]$ denote a neighborhood of a curve c_j parallel to c_j^0 inside B^j such that $\{b\} \times c_j \times [-1,1] \subset \partial B^j$ and $c_j^0 = \{b\} \times c_j \times \{0\}$.

Then $\tau_{M|\partial B^j}$ extends to the closure of $(B^j \setminus N(c_j))$ as a trivialisation τ_j, and τ_M extends to $N(c_j)$ so that

$$\tau_{M|\partial N(c_j)} = \begin{cases} \tau_j & \text{over } \partial N(c_j) \setminus (\{a\} \times c_j \times [-1,1]) \\ \mathcal{T}_{c_j}^{-1} \circ \tau_j & \text{over } \{a\} \times c_j \times [-1,1]. \end{cases}$$

Then with the notation of Subsection 4.3, set $\omega_j = \omega(c_j; \tau_j, \tau_{M|N(c_j)})$ and $\tau_j^{\mathbb{C}} = \tau_{\mathbb{C}}(c_j; \tau_j, \tau_{M|N(c_j)})$.

Remark 3.1 When A is a compact oriented 3-manifold bounded by a compact connected surface, set

$$\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}} = \mathrm{Ker}(H_1(\partial A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(A; \mathbb{Z}/2\mathbb{Z}))$$

and set

$$\mathcal{L}_A^{\mathbb{Z}} = \mathrm{Ker}(H_1(\partial A; \mathbb{Z}) \longrightarrow H_1(A; \mathbb{Z}))$$

so that $\mathcal{L}_A = \mathcal{L}_A^{\mathbb{Z}} \otimes \mathbb{Q}$. When A is a $\mathbb{Z}$-handlebody, $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}} = \mathcal{L}_A^{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z}$. If $\mathcal{L}_{A^j}^{\mathbb{Z}/2\mathbb{Z}} = \mathcal{L}_{B^j}^{\mathbb{Z}/2\mathbb{Z}}$, then the restriction of τ_M to ∂B^j extends to B^j. This would always be the case, if only $\mathbb{Z}$-handlebodies were involved. This would also be the case if B^j were obtained from A^j by twisting the identification of ∂A^j

by a *Torelli homeomorphism* of ∂A^j, that is a homeomorphism that induces the identity in homology. However, in the general case, $\tau_{M|\partial B^j}$ may fail to extend to B^j. See Example 4.7.

Nevertheless, in a first approach to the proof, the reader can assume that $\tau_{M|\partial B^j}$ extends to B^j and forget about Subsections 4.3 and 4.4 and the previous paragraph that are useless in the case when $\tau_{M|\partial B^j}$ extends to B^j.

For $I \subseteq N$, equip $\partial C_2(M_I)$ with the admissible 2-form $\omega(M_I)$ that coincides with $\omega(\tau_M)$ on $ST(B_{M_I}) \setminus \cup_{j\in I} ST(B^j)$ and that is equal to ω_j on $ST(B^j)$. Similarly, equip $T(M_I \setminus \infty) \otimes \mathbb{C}$ with the trivialisation $\tau_I^{\mathbb{C}}$ that is the complexification of τ_M over $M \setminus (\infty \cup (\cup_{j\in I} B^j))$, and that equals $\tau_j^{\mathbb{C}}$ over B^j. Define $Z(\omega(M_I)) = (Z_n(\omega(M_I))_{n\in\mathbb{N}}$ as in Proposition 2.14 in [L2]. Then according to Proposition 4.8, (or to [L2, Theorem 1.9] when $\tau_{M|\partial B^j}$ extends to B^j),

$$Z(M_I) = Z(\omega(M_I)) \exp(\frac{p_1(\tau_I^{\mathbb{C}})}{4}\xi).$$

Furthermore, the $p_1(\tau_I^{\mathbb{C}})$ are related by the following lemma.

Lemma 3.2 *Let $p(i) = p_1(\tau(M_{\{i\}})) - p_1(\tau(M))$. Then, for any subset I of N,*

$$p_1(\tau_I^{\mathbb{C}}) = p_1(\tau(M)) + \sum_{i\in I} p(i).$$

PROOF: Fix $j \in N$. Let Y be a rational homology handlebody with boundary $-\partial A^j$ and with the same lagrangian $\mathcal{L}_Y$ as $(M_I \setminus A^j)$ for any $I \subseteq (N \setminus \{j\})$. Embed the standard neighborhood $(S^3 \setminus B^3)$ of ∞ into Y. Let τ_Y be a trivialisation of $T(Y \setminus \infty) \otimes \mathbb{C}$ that coincides with $\tau_j^{\mathbb{C}}$ on ∂A^j and that is standard near ∞. Let $\tau(\tau_Y, \tau_j^{\mathbb{C}})$ denote the trivialisation of $T(B^j \cup Y \setminus \infty) \otimes \mathbb{C}$ that coincides with τ_Y over $Y \setminus \infty$ and with $\tau_j^{\mathbb{C}}$ over B^j. Similarly define the trivialisation $\tau(\tau_Y, \tau_{M|A^j})$ of $T(A^j \cup Y \setminus \infty) \otimes \mathbb{C}$. Then it is enough to prove that

$$p_1(\tau(\tau_Y, \tau_j^{\mathbb{C}})) - p_1(\tau(\tau_Y, \tau_{M|A^j}))$$

is independent of the rational homology handlebody Y with boundary $-\partial A^j$ and with lagrangian $\mathcal{L}_Y$, and that it is independent of τ_Y.

As any 3-manifold, the manifold $\left(-A^j \cup ([0,1] \times \partial A^j) \cup B^j\right)$ bounds a 4-manifold W_j. Then

$$W_j \cup_{[0,1]\times\partial A^j} [0,1] \times \overline{Y \setminus (S^3 \setminus B^3)}$$

is a cobordism between $B_{Y\cup A^j}$ and $B_{Y\cup B^j}$ whose signature is independent of the rational homology handlebody Y with boundary $-\partial A^j$ and with lagrangian $\mathcal{L}_Y$ and can be adjusted to zero for any such after a connected sum with copies of $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ in the interior of W_j. After this adjustment, $p_1(\tau(\tau_Y, \tau_j^{\mathbb{C}})) - p_1(\tau(\tau_Y, \tau_{M|A^j}))$ is just the obstruction to extend the trivialisation of TW_j on ∂W_j induced by the given trivialisations $\tau_j^{\mathbb{C}}$ and $\tau_{M|A^j}$. Therefore it is independent of Y and τ_Y.

$\diamond$

Let $[-4,4] \times (\coprod_{i\in N} \partial A^i)$ be a tubular neighborhood of $(\coprod_{i\in N} \partial A^i)$ in M. This neighborhood intersects A^i as $[-4,0] \times \partial A^i$. Let $[-4,0] \times \partial A^i$ be a neighborhood of $\partial B^i = \partial A^i$ in B^i. The manifold $M_{\{i\}} = M_i$ is obtained from M by removing $\left(A^i \setminus (]-4,0] \times \partial A^i)\right)$ and by gluing back B^i along $(]-4,0] \times \partial A^i)$.

Let $C_I^i \subset M_I$, $C_I^i = A^i$ if $i \notin I$, $C_I^i = B^i$ if $i \in I$. Let $\eta_{[-1,1]}$ be a one-form with compact support in $]-1,1[$ such that $\int_{[-1,1]} \eta_{[-1,1]} = 1$. Let $(a_j^i \times [-1,1])$ be a tubular neighborhood of a_j^i in ∂A^i.

Let $\eta(a^i_j)$ be a closed one-form on C^i_I such that the support of $\eta(a^i_j)$ intersects $[-4,0]\times\partial A^i$ inside $[-4,0]\times(a^i_j\times[-1,1])$, where $\eta(a^i_j)$ can be written

$$\eta(a^i_j)=\pi^*_{[-1,1]}(\eta_{[-1,1]}).$$

Note that the forms $\eta(a^i_j)$ on A^i and B^i induce closed one-forms still denoted by $\eta(a^i_j)$ on $(A^i\cup_{\partial A^i}-B^i)$ that restrict to the previous ones.

The following innocent-looking proposition 3.3 is the key proposition. As of yet, I do not have a simple proof for it. Fix a degree $\sharp N$ generalised clover D, and, for $I\subseteq N=\{1,2,\dots,\sharp N\}$, set $M_I=M_I(D)$. Let

$$p_{12}: C_2(M_I)\longrightarrow (M_I)^2$$

be the natural projection. For $X\subset M_I$, $C_2(X)$ denotes $p_{12}^{-1}(X^2)\subset C_2(M_I)$.

Proposition 3.3 *There exist admissible 2-forms $\omega(M_I)$ on $C_2(M_I)$, that extend the 2-forms $\omega(M_I)$ defined on $\partial C_2(M_I)$ such that:*

- *For any $I,J\subseteq N$, $\omega(M_I)=\omega(M_J)$ wherever it makes sense in $C_2(M)$, that is on $C_2\left(\left(M\setminus\bigcup_{i\in I\cup J}\mathit{Int}(A^i)\right)\cup\bigcup_{j\in I\cap J}B^j\right)$.*
- *On $C^i_I\times C^k_I$,*

$$\omega(M_I)=\sum_{\substack{j=1,\dots,g_i\\ \ell=1,\dots,g_k}}\ell(z^i_j,z^k_\ell)\pi^*_{C^i_I}(\eta(a^i_j))\wedge\pi^*_{C^k_I}(\eta(a^k_\ell))$$

where $\ell(.,.)$ stands for the linking number.

Using these forms, we can easily prove the following lemma.

Lemma 3.4 *Let $n\in\mathbb{N}$. Let Γ be a degree n oriented Jacobi diagram:*

$$\begin{array}{rcll}\sum_{I\subseteq N}(-1)^{\sharp I}I_\Gamma(\omega(M_I)) & = & 0 & \text{if } \sharp N>2n,\\ & = & \ell(D;\Gamma) & \text{if } \sharp N=2n.\end{array}$$

Proof: Let E be the set of edges of Γ. We want to compute

$$\Delta=\sum_{I\subseteq N}(-1)^{\sharp I}I_\Gamma(\omega(M_I)).$$

Number the vertices of Γ so that $V(\Gamma)=\{1,2,\dots,2n\}$ and

$$\breve{C}_{V(\Gamma)}(M_I)=(M_I\setminus\infty)^{2n}\setminus\{\text{all diagonals}\}=\mathrm{Int}(C_{2n}(M_I)).$$

Fix $i\in N=\{1,2,\dots,\sharp N\}$, the contributions to Δ of the integrals of $\bigwedge_{e\in E}p_e^*(\omega(M_I))$ over

$$\mathrm{Int}(C_{2n}(M_I))\cap(M_I\setminus C^i_I)^{2n}$$

are identical for $I=K$ and $I=K\cup\{i\}$ for any $K\subseteq N\setminus\{i\}$. Since they enter the sum with opposite signs, they cancel each other. This argument allows us to get rid of all the contributions of the integrals over

$$\bigcup_{i\in N}\left(\mathrm{Int}(C_{2n}(M))\cap(M_I\setminus C^i_I)^{2n}\right).$$

Thus, we are left with the contributions of the integrals over the subsets P_I of $\mathrm{Int}(C_{2n}(M_I))\subset(M_I)^{2n}$ such that:

For any $i \in N$, any element of P_I projects onto C_I^i under at least one of the $(2n)$ projections onto M_I. These subsets P_I are clearly empty if $\sharp N > 2n$, and the lemma is proved in this case. Otherwise, P_I is equal to

$$\cup_{\sigma \in \Sigma_N} \prod_{i=1}^{2n} C_I^{\sigma(i)}$$

where Σ_N is the set of permutations of N. We get

$$\Delta = \sum_{\sigma \in \Sigma_N} \Delta_\sigma$$

with

$$\Delta_\sigma = \sum_{I \subseteq N} (-1)^{\sharp I} \int_{\prod_{i=1}^{2n} C_I^{\sigma(i)}} \bigwedge_{e \in E} p_e^*(\omega(M_I)).$$

It is enough to prove that $\Delta_\sigma = \ell(D;\Gamma;\sigma)$. Recall that the vertices are numbered. For any $i \in N$,

$$p_i : \mathrm{Int}(C_{2n}(M_I)) \longrightarrow M_I$$

denotes the projection onto the i^{th} factor. When e is an oriented edge from the vertex $x(e) \in V(\Gamma)$ to $y(e) \in V(\Gamma)$.

$$p_e^*(\omega(M_I))_{|\prod_{i=1}^{2n} C_I^{\sigma(i)}} =$$

$$\sum_{\substack{j=1,\dots,g_{\sigma(x(e))} \\ \ell=1,\dots,g_{\sigma(y(e))}}} \ell(z_j^{\sigma(x(e))}, z_\ell^{\sigma(y(e))}) p_{x(e)}^*(\eta(a_j^{\sigma(x(e))})) \wedge p_{y(e)}^*(\eta(a_\ell^{\sigma(y(e))}))$$

Recall that when c is a half-edge, $v(c)$ denotes the label of the vertex contained in that half-edge, and that $H(\Gamma)$ denotes the set of half-edges. We shall also use the notation $x(e)$ and $y(e)$ for the corresponding halves of an edge e. Let F denote the set of maps f from $H(\Gamma)$ to $\mathbb{N}$ such that for any $c \in H(\Gamma)$, $f(c) \in \{1, 2, \dots, g_{\sigma(v(c))}\}$.

$$\Delta_\sigma = \sum_{f \in F} \left(\prod_{e \in E} \ell(z_{f(x(e))}^{\sigma(x(e))}, z_{f(y(e))}^{\sigma(y(e))}) \right) I(f)$$

with

$$I(f) = \int_{\prod_{i=1}^{2n} (A^{\sigma(i)} \cup -B^{\sigma(i)})} \bigwedge_{c \in H} p_{v(c)}^*(\eta(a_{f(c)}^{\sigma(v(c))}))$$

Recall that the set of half-edges has been ordered in two equivalent ways (up to an even permutation) in the beginning of Subsection 1.4 in [L2]. The above exterior product must be computed with this order. In particular, thinking of this order as being given by the order of the vertices, we get:

$$I(f) = \prod_{i=1}^{2n} \int_{A^{\sigma(i)} \cup (-B^{\sigma(i)})} \bigwedge_{c \in H \cap v^{-1}(i)} \eta(a_{f(c)}^{\sigma(i)}).$$

with the order of the three half-edges given by the vertex-orientation. Now,

$$\int_{A^{\sigma(i)} \cup (-B^{\sigma(i)})} \bigwedge_{c \in H \cap v^{-1}(i)} \eta(a_{f(c)}^{\sigma(i)}) = \pm \mathcal{I}(A^{\sigma(i)}, B^{\sigma(i)})(\bigotimes_{c \in H \cap v^{-1}(i)} a_{f(c)}^{\sigma(i)}),$$

where the sign $\pm$ does not depend on i. It is easy to conclude that $\Delta_\sigma = \ell(D;\Gamma;\sigma)$. $\diamond$

Proof of Theorem 2.4: This lemma easily implies

$$\begin{array}{rcll} \sum_{I \subseteq N} (-1)^{\sharp I} Z_n(M_I, \omega(M_I)) & = & 0 & \text{if } \sharp N > 2n, \\ & = & \sum_\Gamma \frac{\ell(D;\Gamma)}{\sharp \mathrm{Aut}(\Gamma)} [\Gamma] & \text{if } \sharp N = 2n. \end{array}$$

and it suffices to deal with the framing corrections by proving that if $\sharp N \geq 2n$, then

$$\sum_{I \subseteq N} (-1)^{\sharp I} Z_n(M_I) = \sum_{I \subseteq N} (-1)^{\sharp I} (Z_n(M_I, \omega(M_I))$$

By definition, we have that

$$\sum_{I \subseteq N} (-1)^{\sharp I} Z_n(M_I) =$$

$$\sum_{I \subseteq N} (-1)^{\sharp I} \left(Z_n(M_I, \omega(M_I)) + \sum_{j<n} Z_j(M_I, \omega(M_I)) P_{n-j}(I) \right)$$

where $P_{n-j}(I)$ stands for an element of $\mathcal{A}_{n-j}(\emptyset)$ that is a combination of $m[\Gamma]$ where the m are monomials in $p_1(\tau(M))$ and in the $p(i)$ of degree at most $(n-j)$, for degree $(n-j)$ Jacobi diagrams Γ. Furthermore, such an $m[\Gamma]$ appears in $P_{n-j}(I)$ if and only if m is a monomial in the variables $p_1(\tau(M))$ and $p(i)$ for $i \in I$. Therefore, we can rewrite the sum of the annoying terms by factoring out the $m[\Gamma]$. Let $K \subset N$ be the subset of the i such that $p(i)$ appears in m. ($\sharp K \leq n-j$). The factor of $m[\Gamma]$ reads

$$\sum_{K \subseteq I \subseteq N} (-1)^{\sharp I} Z_j(M_I, \omega(M_I)).$$

Therefore, the sum runs over the subsets of $N \setminus K$ whose cardinality is at least $\sharp N + j - n$. Since $\sharp N \geq 2n$ and $j < n$, $\sharp N - n \geq n > j$, hence $\sharp N + j - n > 2j$ and the preceeding lemma ensures that the above sum is zero. This concludes the reduction of the proof of Theorem 2.4 to the construction of *special admissible forms* in Subsection 4.3, and to the proofs of Proposition 4.8 in Subsection 4.4 and Proposition 3.3 in Section 5. $\diamond$

4 Preliminaries to the simultaneous normalization of the forms

In this section, Subsection 4.1 is useful even in the case when $\tau_{M|\partial B^j}$ extends to B^j for any j, while Subsections 4.3 and 4.4 are useless in that case.

4.1 Preliminaries

Let (e_1, e_2, e_3) denote the standard basis $((1,0,0),(0,1,0),(0,0,1))$ of $\mathbb{R}^3$, and let $v_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$ denote the $i^{\rm th}$ coordinate with respect to this basis. Let $R_\theta = R_{\theta,e_1}$ denote the rotation of $\mathbb{R}^3$ with axis directed by e_1 and with angle θ. Let ω_{S^2} denote the homogeneous two-form on S^2 with total area 1. When $\alpha \in S^2$, and when v and w are two tangent vectors of S^2 at α,

$$\omega_{S^2}(v \wedge w) = \frac{1}{4\pi} det(\alpha, v, w)$$

where $\alpha \wedge v \wedge w = det(\alpha, v, w) e_1 \wedge e_2 \wedge e_3$ in $\bigwedge^3 \mathbb{R}^3$.

Lemma 4.1 *Let*

$$\begin{array}{llll} \mathcal{T}_k : & \mathbb{R} \times S^2 & \longrightarrow & S^2 \\ & (\theta, \alpha) & \mapsto & R_{k\theta}(\alpha). \end{array}$$

Then with the notation above,

$$\mathcal{T}_k^*(\omega_{S^2}) = \mathcal{T}_0^*(\omega_{S^2}) + \frac{k}{4\pi} d\theta \wedge dv_1$$

PROOF: Since R_θ preserves the area, the restrictions of $\mathcal{T}_k^*(\omega_{S^2})$ and $\mathcal{T}_0^*(\omega_{S^2})$ coincide on $\bigwedge^2 T_{(\theta,\alpha)}(\{\theta\} \times S^2)$. Therefore, we are left with the computation of $(\mathcal{T}_k^*(\omega_{S^2}) - \mathcal{T}_0^*(\omega_{S^2}))(u \wedge v)$ when $u \in T_{(\theta,\alpha)}(\mathbb{R} \times \{\alpha\})$ and $v \in T_{(\theta,\alpha)}(\{\theta\} \times S^2)$, where of course, $\mathcal{T}_0^*(\omega_{S^2})(u \wedge v) = 0$, and by definition,

$$\mathcal{T}_k^*(\omega_{S^2})(u \wedge v) = \frac{1}{4\pi} det(R_{k\theta}(\alpha), D\mathcal{T}_k(u), D\mathcal{T}_k(v)).$$

Since $D\mathcal{T}_k(v) = R_{k\theta}(v)$, and since $R_{k\theta}$ preserves the volume in $\mathbb{R}^3$,

$$\mathcal{T}_k^*(\omega_{S^2})(u \wedge v) = \frac{1}{4\pi} det(\alpha, R_{-k\theta}(D\mathcal{T}_k(u)), v).$$

Now, let α_i stand for $v_i(\alpha)$. $D\mathcal{T}_k(u) = kd\theta(u) R_{k\theta+\pi/2}(\alpha_2 e_2 + \alpha_3 e_3)$. Therefore, $\mathcal{T}_k^*(\omega_{S^2})(u \wedge v) = \frac{kd\theta(u)}{4\pi} det(\alpha, -\alpha_3 e_2 + \alpha_2 e_3, v)$, and,

$$\begin{array}{ll} \mathcal{T}_k^*(\omega_{S^2})(u \wedge .) & = \frac{kd\theta(u)}{4\pi} det \left(\begin{array}{ccc} \alpha_1 & 0 & dv_1 \\ \alpha_2 & -\alpha_3 & dv_2 \\ \alpha_3 & \alpha_2 & dv_3 \end{array} \right) \\ & = \frac{kd\theta(u)}{4\pi} \left((1-\alpha_1^2) dv_1 - \alpha_1\alpha_2 dv_2 - \alpha_1\alpha_3 dv_3 \right) \\ & = \frac{kd\theta(u)}{4\pi} dv_1. \end{array}$$

$\diamond$

Definition 4.2 Let S be an oriented surface, let T be an $\mathbb{R}^3$-bundle over S, and let $\tau : T \longrightarrow S \times \mathbb{R}^3$ be a trivialisation of T. Let $\varepsilon > 0$ be a small positive real number. Let $\theta : [-1,1] \longrightarrow [0, 2\pi]$ be a smooth map that maps $[-1, -1+\varepsilon]$ to 0, that increases from 0 to 2π on $[-1+\varepsilon, 1-\varepsilon]$, and such that $\theta(-x) + \theta(x) = 2\pi$ for any $x \in [-1,1]$. When $c \times [-1,1]$ is an oriented neighborhood of an oriented curve c in S, let

$$\begin{array}{llll} \theta(c) : & S & \longrightarrow & \frac{[0,2\pi]}{0 \sim 2\pi} \\ & x \notin c \times [-1,1] & \mapsto & 0 \\ & (\gamma, u) \in c \times [-1,1] & \mapsto & \theta(u). \end{array}$$

A *twist of* τ across an oriented curve c with oriented neighborhood $c \times [-1,1]$ is the trivialisation

$$\begin{array}{rlcl} \mathcal{T}_c \circ \tau : & T & \longrightarrow & S \times \mathbb{R}^3 \\ & \tau^{-1}(x,v) & \mapsto & (x, R_{\theta(c)(x)}(v)) \end{array}$$

where R_θ is defined in the beginning of Subsection 4.1.

Definition 4.3 Let S be a non-necessarily oriented compact surface with possible boundary, and let v be a nonzero vector field of TS on the boundary ∂S of S. Then the *Euler number* $\chi(TS; v_{|\partial S})$ of v is the obstruction to extend the vector field v of $TS_{|\partial S}$ to a nonzero section of TS. More precisely, if S is a disk then its unit tangent bundle $S(TS)$ is isomorphic to $S^1 \times S$, and $\chi(TS; v_{|\partial S})$ is the degree of the composite map

$$\partial S \xrightarrow{v} S(TS) \cong S^1 \times S \xrightarrow{\pi_{S^1}} S^1.$$

Note that the orientation of S does not matter for this definition (because changing it changes both orientations of ∂S and of the fiber S^1). When S is connected, then the vector field $v_{|\partial S}$ can be extended as a nonzero section v outside the interior of a disk D in the interior of S, and $\chi(TS; v_{|\partial S}) = \chi(TD; v_{|\partial D})$. When S is not connected, then $\chi(TS; v_{|\partial S})$ is the sum of the numbers associated to the different components of S.

Note that when the boundary of S is empty, $\chi(TS)$ is the Euler characteristic $\chi(S)$ of S. The following lemma is left to the reader.

Lemma 4.4 *Let S be an oriented surface and let v be a nonzero vector field of TS on ∂S. Use the two orthogonal unit vector fields on ∂S, "outward normal to S" $\vec{N}(\partial S)$ and "tangent vector to ∂S" $\vec{T}(\partial S)$ to trivialise $TS_{|\partial S}$ by mapping $(\vec{N}(\partial S), \vec{T}(\partial S))$ to the basis (e_1, e_2) of $\mathbb{R}^2$. Using this trivialisation,*

$$S(TS)_{|\partial S} = S(\mathbb{R}e_1 \oplus \mathbb{R}e_2) \times \partial S = S^1 \times \partial S.$$

Let d be the degree of the composite map

$$\partial S \xrightarrow{v} S(TS)_{|\partial S} = S^1 \times \partial S \xrightarrow{\pi_{S^1}} S^1.$$

Then

$$\chi(TS; v_{|\partial S}) = d + \chi(S).$$

Lemma 4.5 *Let $S \times [-1,1]$ denote a collar of a connected oriented surface S with possible boundary. Let T denote the restriction of the tangent bundle of $S \times [-1,1]$ to $S = S \times \{0\}$. Let n denote a nonzero vector field of T that is tangent to $\{x\} \times [-1,1]$ at $(x,0)$, and let $S(n)$ denote the corresponding section of the spherical bundle $S(T)$ over S. Let $\tau : T \longrightarrow S \times \mathbb{R}^3$ be a trivialisation that maps $n(x)$ to (x, e_1) for any $x \in \partial S$. Use τ to identify $S(T)$ to $S \times S^2$, and let $\pi_{S^2}(\tau)$ denote the associated projection from $S(T)$ to S^2. Then*

$$\int_{S(n)} \pi_{S^2}(\tau)^*(\omega_{S^2}) = \frac{1}{2}\chi(TS; \tau^{-1}(.,e_2)_{|\partial S}).$$

Proof: We first prove the result for a special trivialisation $\tau(S)$ of T.

If the boundary of S is non-empty then TS is trivialisable, and we fix $\tau(S)$ such that $\tau(S)(n(x)) = (x, e_1)$ for $x \in S$. Then both sides of the equality to be shown vanish.

If the boundary of S is empty, we trivially embed $S \times [-1,1]$ in $\mathbb{R}^3$ and we pull-back the standard trivialisation of $\mathbb{R}^3$ through this embedding. Then we compute the left-hand side as the degree of the map "direction of the positive normal" from S to S^2. This degree is $(1 - g(S) = \chi(S)/2)$. Therefore it coincides with the right-hand side that is $(\chi(TS)/2 = \chi(S)/2)$.

Of course, both sides of the equality are unchanged under a homotopy of τ such that $n=\tau^{-1}(e_1)$ on ∂S. Now, up to this kind of homotopy, any such trivialisation is obtained from $\tau(S)$ by a twist across a curve c with possible boundary with a neighborhood $c\times[-1,1]$ properly embedded in S. Thus, it is enough to prove that both sides vary in the same way under such a twist. It is clear that the right-hand side varies like half the degree of the map $\tau^{-1}(.,e_2)$ from ∂S to the S^1 fiber of $S(TS)_{|\partial S}$ equipped with an arbitrary fixed trivialisation. Therefore, the variation of the right-hand side is

$$\frac{1}{2}\deg\left(R_{-\theta(c)(.)}(e_2):\partial S\longrightarrow S^1=S(\mathbb{R}e_2\oplus\mathbb{R}e_3)\right)=-\frac{\langle c,\partial S\rangle}{2}.$$

For the left-hand side, consider

$$S(n)\hookrightarrow S(TS)\xrightarrow{\tau(S)}S\times S^2\xrightarrow{\theta(c)\times\mathrm{Id}}\frac{[0,2\pi]}{0\sim 2\pi}\times S^2\xrightarrow{\mathcal{T}_1}S^2.$$

$\int_{S(n)}\left(\pi_{S^2}(\mathcal{T}_c\circ\tau(S))^*\right)(\omega_{S^2})-\int_{S(n)}\left(\pi_{S^2}(\tau(S))^*\right)(\omega_{S^2})=$

$$\int_{S(n)}(\theta(c)\times\mathrm{Id})^*(\mathcal{T}_1^*(\omega_{S^2})-\mathcal{T}_0^*(\omega_{S^2}))$$

where according to Lemma 4.1,

$$(\theta(c)\times\mathrm{Id})^*(\mathcal{T}_1^*(\omega_{S^2})-\mathcal{T}_0^*(\omega_{S^2}))=-\frac{1}{4\pi}d(v_1d\theta(c)).$$

Therefore, according to the Stokes theorem, the variation of the right-hand side is

$$\int_{\partial S(n)}-\frac{1}{4\pi}v_1d\theta(c)=-\frac{\langle c,\partial S\rangle}{2},$$

and we are done. $\diamond$

4.2 Extensions of trivialisations on 3-manifolds

This section is useless for the proofs. It only justifies why I could not avoid the following subsections, and some of the difficulties they contain.

Let A be a compact oriented connected 3-manifold with boundary ∂A. Consider the $\mathbb{Z}/2\mathbb{Z}$-*Lagrangian* of A

$$\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}=\mathrm{Ker}(H_1(\partial A;\mathbb{Z}/2\mathbb{Z})\longrightarrow H_1(A;\mathbb{Z}/2\mathbb{Z}))$$

This is a Lagrangian subspace of $(H_1(\partial A;\mathbb{Z}/2\mathbb{Z});\langle .,.\rangle)$.

Let K be a framed knot in an oriented 3-manifold M, that is a knot equipped with a normal nonzero vector field $\vec{N}$, or equivalently with a parallel (up to homotopies). These data induce the direct trivialisation τ_K of $TM_{|K}$ (up to homotopy) such that $\tau_K^{-1}(e_1)=T\vec{K}$ and $\tau_K^{-1}(e_2)=\vec{N}$, where $T\vec{K}$ is a tangent vector of K that is equipped with an arbitrary orientation. The homotopy class of the trivialisation τ_K is well-defined and does not depend on the orientation of K.

Assume that K bounds a possibly non-oriented compact surface Σ that induces the given parallelisation of K. Then if τ is a trivialisation of the tangent space of M over Σ, the restriction of τ to K is not homotopic to τ_K. (This is clear when Σ is a disk, and the trivialisation must be homotopic in the other cases, since the tangent bundle of an oriented 3-manifold over a closed surface is trivialisable. Recall that $\pi_1(SO(3))=\mathbb{Z}/2\mathbb{Z}$.)

If K is a framed knot in an oriented 3-manifold M and if τ is a trivialisation of the restriction of TM to K, we shall say that K is τ-*bounding* if τ is not homotopic to τ_K.

Proposition 4.6 *Let ∂A be a connected oriented compact surface. Let τ be a trivialisation of $T(\partial A \times [-2,2])$. Then there exists a unique map*

$$\phi_\tau : H_1(\partial A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}}$$

such that

1. *when x is a connected curve of $\partial A = \partial A \times \{0\}$, $\phi_\tau(x) = 0$ if and only if x (equipped with its parallelisation induced by ∂A) is τ-bounding and,*
2. $$\phi_\tau(x+y) = \phi_\tau(x) + \phi_\tau(y) + \langle x, y \rangle.$$

Let c be curve of ∂A and let $\mathcal{T}_c$ denote the twist across c, then

$$\phi_{\mathcal{T}_c \circ \tau}(x) = \phi_\tau(x) + \langle x, c \rangle.$$

When A is a compact oriented connected 3-manifold with boundary ∂A, τ extends as a trivialisation over A if and only if $\phi_\tau(\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}) = \{0\}$.

Proof: Define ϕ_τ for the embedded possibly non-connected curves x in ∂A by $\phi_\tau(x) = 0$ if and only if x is τ-bounding, (that is such that τ would extend to a possibly non-oriented connected surface with framed boundary x).

With this definition that is consistent with the first part of the above definition, ϕ_τ is additive under disjoint union. This is easy to see if one of the considered embedded curve is τ-bounding, and this additivity property is preserved by a trivialisation change.

Then this definition only depends on the class of x in $H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$. Indeed let x be an embedded (possibly non-connected) curve in ∂A and let y be another such in $\partial A \times \{-1\}$ that is homologous to x mod 2. Then there exists a framed (possibly non-orientable) cobordism between x and y in $\partial A \times [-1,1]$, and it is easy to see that x is τ-bounding if and only if y is τ-bounding.

Let us check that ϕ_τ behaves as predicted under addition. Because we are dealing with elements of $H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$, we can consider representatives of x and y that are disjoint or that intersect once. The first case has already been treated. Note that both sides of the equality to be proved vary in the same way under trivialisation changes. When x and y intersect once, consider the punctured torus neighborhood of $x \cup y$, and a trivialisation τ that restricts to the punctured torus as the direct sum of a trivialisation of the torus and the normal vector to ∂A. Then $\phi_\tau(x+y) = \phi_\tau(x) = \phi_\tau(y) = 1$. The last two assertions are left to the reader. $\diamond$

Example 4.7 For any $\mathbb{Q}$-handlebody A, there exists a Lagrangian subspace $\mathcal{L}^{\mathbb{Z}}$ of $(H_1(\partial A;\mathbb{Z}); \langle .,. \rangle)$, such that $\mathcal{L}_A = \mathcal{L}^{\mathbb{Z}} \otimes \mathbb{Q}$. However, as the following example shows, $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}$ is not necessarily equal to $\mathcal{L}^{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z}$.

The $\mathbb{Q}$-handlebody A will be the exterior of a knot ∂M in $S^2 \times S^1$ described below. Consider a Moebius band M embedded in the interior of a solid torus $D^2 \times S^1$ so that the core of the solid torus is equal to the core of M. Embed $D^2 \times S^1$ into $S^2 \times S^1 = D^2 \times S^1 \cup_{\partial D^2 \times S^1} (-D^2 \times S^1)$ as the first copy. Let m be the meridian of the knot ∂M that is oriented so that ∂M pierces twice $S^2 \times 1$ positively, and let ℓ be the parallel of ∂M induced by M. Then A is a $\mathbb{Q}$-handlebody such that $\mathcal{L}_A^{\mathbb{Z}} = \mathbb{Z}[2m]$, $\mathcal{L}_A = \mathbb{Q}[m]$, and $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}[\ell]$.

4.3 Special admissible forms and their p_1.

In this subsection, we fix

- a rational homology sphere M,
- an embedding of a neighborhood $N(c) = [a,b]\times c\times[-1,1]$ of a framed link c in B_M that respects the framing of c,
- a trivialisation τ of $T(M \setminus (\infty \cup N(c))$ that is standard near ∞, and
- a trivialisation τ_b of $T(N(c))$ such that

$$\tau_b = \begin{cases} \tau & \text{over } \partial([a,b]\times c\times[-1,1]) \setminus (\{a\}\times c\times[-1,1]) \\ \mathcal{T}_c^{-1}\circ\tau & \text{over } \{a\}\times c\times[-1,1]. \end{cases}$$

With these data, we will associate:

- in Notation 4.9, an admissible 2-form $\omega(c;\tau,\tau_b)$ on $\partial C_2(M)$ that reads $p_M(\tau)^*(\omega_{S^2})$ over $ST(M\setminus(\infty\cup N(c))$, and then

$$z_n(c;\tau,\tau_b) = z_n(\omega(c;\tau,\tau_b)) = \sum_{\Gamma\in\mathcal{E}_n} I_\Gamma(M;\bigwedge_{i=1}^{3n} p_i^*(\omega(c;\tau,\tau_b)))[\Gamma]$$

  as in [L2, Theorem 1.9],
- in Notation 4.14, a trivialisation $\tau_{\mathbb{C}}(c;\tau,\tau_b)$ of $T(M\setminus\infty)\otimes\mathbb{C}$ that reads $\tau\otimes 1_{\mathbb{C}}$ over $M\setminus(\infty\cup N(c))$, and its relative Pontryagin class $p_1(c;\tau,\tau_b) = p_1(\tau_{\mathbb{C}}(c;\tau,\tau_b))$ that is defined like in the real case.

Then in Subsection 4.4, we shall prove

Proposition 4.8 *For any rational homology sphere M equipped with a framed link c in B_M, with a trivialisation τ of $T(M\setminus\infty)$ that is standard near ∞ outside $N(c)$, and with a trivialisation τ_b of $T(N(c))$ such that*

$$\tau_b = \begin{cases} \tau & \textit{over } \partial([a,b]\times c\times[-1,1]) \setminus (\{a\}\times c\times[-1,1]) \\ \mathcal{T}_c^{-1}\circ\tau & \textit{over } \{a\}\times c\times[-1,1]. \end{cases}$$

for any trivialisation τ_M of $M\setminus\infty$ that is standard near ∞,

$$z_n(c;\tau,\tau_b) - z_n(\tau_M) = \frac{p_1(\tau_M) - p_1(c;\tau,\tau_b)}{4}\delta_n.$$

In particular, $Z(M) = Z(M;\omega(c;\tau,\tau_b))\exp(\frac{p_1(\tau(c;\tau,\tau_b))}{4}\xi)$.

The forms $\omega(c;\tau,\tau_b)$ are called *special admissible forms.* We shall see that they satisfy Lemma 4.5 (see Lemma 4.12).

The two main ideas that make our constructions work are the following ones.

1. This neighborhood will be filled in by gadgets that factor through the projection $p_c : [a,b]\times c\times[-1,1] \longrightarrow [a,b]\times[-1,1]$.
2. Over $N(c)$, $\omega(c;\tau,\tau_b)$ will be the average of two forms coming from genuine trivialisations.

Notation 4.9 Let $\varepsilon > 0$ be a small positive number and let F be a smooth map such that

$$\begin{array}{llll} F: & [a,b]\times[-1,1] & \longrightarrow & SO(3) \\ & (t,u) & \mapsto & \left\{\begin{array}{ll} \text{Identity} & \text{if } |u| > 1-\varepsilon \\ R_{\theta(u)} & \text{if } t < a+\varepsilon \\ R_{-\theta(u)} & \text{if } t > b-\varepsilon \end{array}\right. \end{array}$$

where θ has been defined in Definition 4.2. The map F extends to $[a,b]\times[-1,1]$ because its restriction to the boundary is trivial in $\pi_1(SO(3))$.

Let $F(c,\tau_b)$ be defined on $ST(N(c)) \stackrel{\tau_b}{=} [a,b]\times c\times[-1,1]\times S^2$ as follows

$$\begin{array}{llll} F(c,\tau_b): & [a,b]\times c\times[-1,1]\times S^2 & \longrightarrow & S^2 \\ & (t,\sigma,u,v) & \mapsto & F(t,u)(v). \end{array}$$

Define the closed two-form $\omega(c,\tau_b)$ on $ST([a,b]\times c\times[-1,1])$ as

$$\omega(c,\tau_b) = \frac{\pi_{S^2}(\mathcal{T}_c\circ\tau_b)^*(\omega_{S^2}) + F(c,\tau_b)^*(\omega_{S^2})}{2}. \tag{4.10}$$

Set

$$\omega(c;\tau,\tau_b) = \left\{\begin{array}{ll} p_M(\tau)^*(\omega_{S^2}) & \text{on } ST(M\setminus(N(c)\cup\infty)) \\ \omega(c,\tau_b) & \text{on } ST(N(c)). \end{array}\right.$$

Extend $\omega(c,\tau_b)$ on $\partial C_2(M)\setminus ST(B_M)$ like in the case of fundamental forms.

Observe that $\omega(c,\tau_b)$ is the average of two forms corresponding to trivialisations. The definition of $\omega(c;\tau,\tau_b)$ is consistent because using Lemma 4.1, we see that:

$$\omega(c,\tau_b) = \left\{\begin{array}{ll} \pi_{S^2}(\tau_b=\tau)^*(\omega_{S^2}) & \text{on } ST(\{b\}\times c\times[-1,1]) \\ \pi_{S^2}(\tau_b=\tau)^*(\omega_{S^2}) & \text{on } ST\left([a,b]\times c\times\{-1,1\}\right) \\ \pi_{S^2}(\mathcal{T}_c\circ\tau_b=\tau)^*(\omega_{S^2}) & \text{on } ST(\{a\}\times c\times[-1,1]) \end{array}\right.$$

Of course, $\omega(c;\tau,\tau_b)$ depends on many choices. However, we shall see that $z_n(\omega(c;\tau,\tau_b))$ only depends on the isotopy class of the framed link c, and on the homotopy classes of τ and τ_b.

Bundle isomorphisms ϕ from $T(M\setminus\infty)$ to itself over a diffeomorphism $\overline{\phi}$ that is the identity on $(M\setminus(B_M\cup\infty))$ induce isomorphisms of $ST(M\setminus\infty)$ that trivially extend to $\partial C_2(M)$. The pull-back $\phi^*(\omega(c;\tau,\tau_b))$ of a special admissible form $\omega(c;\tau,\tau_b)$ under such an isomorphism is a special admissible form $\omega(\overline{\phi}^{-1}(c);\tau\circ\phi,\tau_b\circ\phi)$.

Lemma 4.11 *If ϕ_1 is a bundle isomorphism from $T(M\setminus\infty)$ to itself that is the identity over $(M\setminus(B_M\cup\infty))$ and that is isotopic to the identity among these, then $z_n(\omega(\overline{\phi_1}^{-1}(c);\tau\circ\phi_1,\tau_b\circ\phi_1)) = z_n(\omega(c;\tau,\tau_b))$.*
Furthermore, $z_n(\omega(c;\tau,\tau_b))$ is independent of the choices of F, c inside its homotopy class, τ_b and τ in their homotopy classes.

Proof: For the first assertion, consider the isotopy

$$\phi: [0,1]\times ST(M) \longrightarrow ST(M).$$

Then Proposition 2.27 in [L2] tells that

$$z_n(\phi_1^*(\omega(c;\tau,\tau_b))) - z_n(\omega(c;\tau,\tau_b)) = z_n([0,1]\times ST(B_M);\phi^*(\omega(c;\tau,\tau_b))).$$

The right-hand side vanishes thanks to Lemma 2.26 in [L2]. This proves that when τ and τ_b vary in their homotopy classes so that they always satisfy

$$\tau_b = \left\{\begin{array}{ll} \tau & \text{over } \partial([a,b]\times c\times[-1,1])\setminus(\{a\}\times c\times[-1,1]) \\ \mathcal{T}_c^{-1}\circ\tau & \text{over } \{a\}\times c\times[-1,1], \end{array}\right.$$

$z_n(c;\tau,\tau_b)$ remains the same.

Similarly, the choice of the tubular neighborhood of c in Σ does not matter because it can be realised by an isotopy of M that induces a bundle isomorphism isotopic to the identity on $ST(B_M)$.

Now, since $\pi_2(SO(3))$ is trivial, two maps $F=G_0$ and G_1 that satisfy the hypotheses of Notation 4.9 are homotopic by a homotopy

$$G:[0,1]\times[a,b]\times[-1,1]\longrightarrow SO(3).$$

Extend $\omega(G)$ that satisfies

$$\omega(G)=\begin{cases}\omega(G_1) & \text{on } \{1\}\times ST(B_M)\\ \omega(G_0) & \text{on } \{0\}\times ST(B_M)\\ \pi^*_{S^2}(\tau)(\omega_{S^2}) & \text{on } [0,1]\times ST(B_M\setminus N(c)),\end{cases}$$

by using Formula 4.10 on $[0,1]\times[a,b]\times c\times[-1,1]\times S^2$. Then according to Proposition 2.27 in [L2], $z_n(\omega(G_1))-z_n(\omega(G_0))=z_n([0,1]\times ST(B_M);\omega(G))$, and since $\omega(G)$ pulls-back under a bundle morphism onto $[0,1]\times[a,b]\times[-1,1]\times S^2$ over $[0,1]\times N(c)$, then

$$z_n([0,1]\times ST(B_M);\omega(G))=0,$$

thanks to Lemma 2.26 in [L2]. $\diamond$

Lemma 4.12 *Let S be an oriented surface of M whose boundary does not meet $N(c)$, then $\int_{S(n)}\omega(c;\tau,\tau_b)$ only depends on the topology of S and on the restriction of τ on ∂S, and if the positive normal n of S is the first vector of the trivialisation τ on ∂S, then*

$$\int_{S(n)}\omega(c;\tau,\tau_b)=\frac{1}{2}\chi(TS;\tau^{-1}(.,e_2)_{|\partial S}).$$

PROOF: First isotope S so that it meets c along meridian squares $D(x_i)=[a,b]\times\{x_i\}\times[-1,1]$ of c. Without changing the sides of the equality to be shown, perform a bundle isomorphism of $ST(M\setminus\infty)$ over 1_M that is isotopic to the identity and supported near $\partial D(x_i)$ so that the normal n to the square $D(x_i)$ becomes the first vector of τ around $\partial D(x_i)$. Let D_i be a disk inside $D(x_i)$ with a smooth boundary outside $[a+\varepsilon,b-\varepsilon]\times\{x_i\}\times[-1+\varepsilon,1-\varepsilon]$ such that $\tau(n)=e_1$ on ∂D_i. Set $\omega_M=\omega(c;\tau,\tau_b)$. Then we have the following sublemma.

Sublemma 4.13 $\int_{D_i(n)}\omega_M=\frac{1}{2}\chi(TS;\tau^{-1}(.,e_2)_{|\partial D_i})$.

PROOF: Indeed, on D_i, ω_M is the average of two forms corresponding to the trivialisations $\mathcal{T}_c\circ\tau_b$ and $F(c,\tau_b)\circ\tau_b$. For both of these, n is the first vector of the trivialisation on ∂D_i, and Lemma 4.5 tells us that $2\int_{D_i(n)}\omega_M$ is the average of the obstructions to extend the second vector of these trivialisations to D_i. These obstructions vary like the degrees of the induced maps from ∂D_i to the fiber S^1 of $S(TD_{i|\partial D_i})$ equipped with an arbitrary trivialisation. We conclude because the degree of the map induced by τ is the average of the degrees of the map induced by $\mathcal{T}_c\circ\tau_b$ and the map induced by $F(c,\tau_b)\circ\tau_b$. $\diamond$

Back to the proof of Lemma 4.12, use Lemma 4.5 and the additivity of both sides of the equality under gluing to treat the case when the positive normal n of S is the first vector of the trivialisation τ on ∂S. For the other case, let $[-1,0]\times\partial S$ be a collar of $\partial S=\partial S\times\{0\}$ inside S. After an arbitrary isotopy of τ around $[-1,0]\times\partial S$ supported away from ∂S, the positive normal n of S is the first vector of the trivialisation τ on $\partial S\times\{-1\}$. Then, since $\int_{S(n)}\omega_M=\int_{(S\setminus]-1,0]\times\partial S)(n)}\omega_M+\int_{([-1,0]\times\partial S)(n)}\omega_M$, according to the first case, $\int_{S(n)}\omega_M$ only depends on our arbitrary isotopy, on the topology of S and on the restriction of τ on ∂S. $\diamond$

Notation 4.14 Let F_U be a smooth map such that

$$\begin{array}{llll} F_U: & [a,b]\times[-1,1] & \longrightarrow & SU(3) \\ & (t,u) & \mapsto & \begin{cases} \text{Identity} & \text{if } |u|>1-\varepsilon \\ R_{\theta(u)} & \text{if } t<a+\varepsilon \\ \text{Identity} & \text{if } t>b-\varepsilon. \end{cases} \end{array}$$

F_U extends to $[a,b]\times[-1,1]$ because $\pi_1(SU(3))$ is trivial. Define the trivialisation $\tau_{\mathbb{C}}=\tau_{\mathbb{C}}(c;\tau,\tau_b)$ of $T(M\setminus\infty)\otimes\mathbb{C}$ as follows.

- On $T(M\setminus(\infty\cup N(c)))$, $\tau_{\mathbb{C}}=\tau\otimes 1_{\mathbb{C}}$,
- Over $[a,b]\times c\times[-1,1]$, $\tau_{\mathbb{C}}(t,\gamma,u;v)=F_U(t,u)(\tau_b\otimes 1_{\mathbb{C}})(t,\gamma,u;v)$.

(Here, as often τ_b that is valued in $N(c)\times\mathbb{R}^3$ is identified with $\pi_{\mathbb{R}^3}\circ\tau_b$, and $\tau_{\mathbb{C}}$ is identified with $\pi_{\mathbb{C}^3}\circ\tau_{\mathbb{C}}$.) Since $\pi_2(SU(3))$ is trivial, the homotopy class of $\tau_{\mathbb{C}}$ is well-defined. Since $\tau_{\mathbb{C}}=\tau_M\otimes 1_{\mathbb{C}}$ outside B_M, the definition of p_1 for trivialisations that are standard near ∞ extends to this kind of trivialisations, hence $\tau_{\mathbb{C}}$ has a well-defined integral p_1 defined as in Subsection 1.5 in [L2]. Set

$$p_1(c;\tau,\tau_b)=p_1(\tau_{\mathbb{C}}(c;\tau,\tau_b)).$$

4.4 Proof of Proposition 4.8.

Of course it is enough to prove this proposition for some trivialisation τ_M of $M\setminus\infty$ that is standard near ∞, thanks to Propositions 1.8 and 2.11 in [L2]. The last sentence of the proposition follows from the previous one by the arguments given at the end of Subsection 2.3 in [L2].

Fixing the main notation for the proof.
Let A be a rational homology handlebody that meets $N(c)$ along $\{a\}\times c\times[-1,1]$. ($A$ can be the genuine handlebody obtained by first thickening $N(c)$ to an embedding of $[a-5,b]\times c\times[-2,2]$ and then construct A as a regular neighborhood of a connected graph whose loops are the connected components of $\{a-2\}\times c\times\{0\}$.) Let $[a,b]\times\partial A$ be a collar of ∂A in the closure of $B_M\setminus A$ such that the notation $[a,b]\times c\times[-1,1]$ is consistent.

Let D be a disk of ∂A that does not meet the tubular neighborhood of c. Let D^3 be a topological ball of M that contains $M\setminus B_M$, that intersects $[a,b]\times\partial A$ as $[a,b]\times D$, and A along $\{a\}\times D$. Choose a trivialisation τ_M of $M\setminus\infty$ that is standard near ∞ so that τ and τ_M coincide on the complement of $\{\infty\}$ in the ball D^3. Set $B_M^D=M\setminus\mathrm{Int}(D^3)$, $\Sigma=\partial A\setminus\mathrm{Int}(D)$, and

$$B=B_M^D\setminus(A\cup([a,b[\times\Sigma))$$

Then B_M^D is the union of three rational homology handlebodies A, B and $[a,b]\times\Sigma$.

There exists a curve c_B of Σ that is transverse to c, such that the restriction of τ to $\{b\}\times\Sigma$ is homotopic to $\mathcal{T}_{c_B}\circ\tau_{M|\{b\}\times\Sigma}$. Homotope τ_M so that

- $\tau=\mathcal{T}_{c_B}\circ\tau_M$ on $[a,b]\times(\Sigma\setminus c\times[-1,1])$, and
- $\tau_b=\mathcal{T}_{c_B}\circ\tau_M$ on $[a,b]\times c\times[-1,1]$.

Let c_A be the curve obtained from $c\cup c_B$ by replacing the neighborhoods $\times$ of the double points by $)($ according to the orientation. Then c_A represents $(c+c_B)$ in $H_1(\Sigma)$, and the restriction of $\tau_A=\tau_{|A}$ to $\{a\}\times\Sigma$ is homotopic to $\mathcal{T}_{c_A}\circ\tau_{M|\{a\}\times\Sigma}$.

Set $\omega_M=\omega(c;\tau,\tau_b)$.

Sketch of the proof of Proposition 4.8.
Consider the part

$$B_0 = \partial([0,1] \times B_M^D) \cup ([0,1] \times \{a,b\} \times \Sigma)$$

of the base $[0,1] \times B_M^D$ of the $S(\mathbb{R}^3)$-bundle $E = [0,1] \times ST(B_M^D)$. We shall fix a closed two-form ω over B_0 such that

$$\omega = \begin{cases} \omega_M & \text{on } \{0\} \times ST(B_M^D) \\ \omega(\tau_M) & \text{on } \{1\} \times ST(B_M^D) \end{cases}$$

and such that any extension ω of ω to E will satisfy

$$z_n(\tau_M) - z_n(\omega_M) = z_n(E;\omega).$$

Simultaneously, we shall fix a related trivialisation τ_0 of $T([0,1]\times B_M^D)\otimes\mathbb{C}$ over B_0, that coincides with the trivialisation associated to ω_M and τ_M on $\{0,1\} \times B_M^D$. Now, for each closure C of a connected component of $([0,1] \times B_M^D) \setminus B_0$, that is for $C = [0,1] \times A$, $[0,1] \times B$ or $[0,1] \times [a,b] \times \Sigma$, we are going to prove that:
ω extends as a closed form ω over C, and $z_n(E_{|C};\omega) = -\frac{p_1(C;\tau_{0|\partial C})}{4}\delta_n$ for each C. Of course, once these goals are achieved (in Lemmas 4.17, 4.19 and 4.20), the proof will be finished.

Definition of ω over B_0.
Define the bundle isomorphisms over the Identity of $[0,1] \times \{d\} \times \Sigma$

$$\begin{array}{llll} \mathcal{Q}: & ST([0,1] \times \{d\} \times \Sigma) & \longrightarrow & [0,1] \times ST(M)_{|\{d\}\times\Sigma} \\ & T((t,d) \times 1_\Sigma)(v \in T_\sigma\Sigma) & \mapsto & (t; T(\{d\} \times 1_\Sigma)(v)) \\ & \frac{\partial}{\partial u}(t+u,d,\sigma)_{u=0} & \mapsto & (t; \frac{\partial}{\partial u}(d+u,\sigma)_{u=0}). \end{array}$$

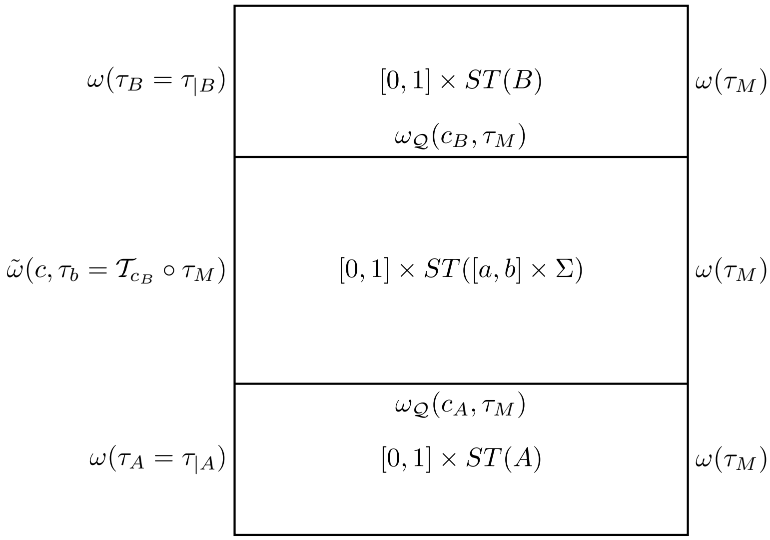
Define $\omega_{\mathcal{Q}}(c_B,\tau_M)$ on $[0,1] \times ST(M)_{|\{b\}\times\Sigma}$ and $\omega_{\mathcal{Q}}(c_A,\tau_M)$ on $[0,1] \times ST(M)_{|\{a\}\times\Sigma}$ by:

$$\omega_{\mathcal{Q}}(c_B,\tau_M) = \begin{cases} \left(\mathcal{Q}^{-1}\right)^*(\omega(c_B,\tau_M \circ \mathcal{Q})) & \text{on } [0,1] \times ST(M)_{|\{b\}\times c_B\times[-1,1]} \\ \omega(\tau_M) & \text{on } [0,1] \times ST(M)_{|\{b\}\times(\Sigma\setminus c_B\times[-1,1])}, \end{cases}$$

$$\omega_{\mathcal{Q}}(c_A,\tau_M) = \begin{cases} \left(\mathcal{Q}^{-1}\right)^*(\omega(c_A,\tau_M \circ \mathcal{Q})) & \text{on } [0,1] \times ST(M)_{|\{a\}\times c_A\times[-1,1]} \\ \omega(\tau_M) & \text{on } [0,1] \times ST(M)_{|\{a\}\times(\Sigma\setminus c_A\times[-1,1])}, \end{cases}$$

using the notation of Equation 4.10. Set

$$\omega = \begin{cases} \omega_M & \text{on } \{0\} \times ST(M \setminus \infty) \\ \omega(\tau_M) & \text{on } \{1\} \times ST(M \setminus \infty) \\ \omega_{\mathcal{Q}}(c_B,\tau_M) & \text{on } [0,1] \times ST(M)_{|\{b\}\times\Sigma} \\ \omega_{\mathcal{Q}}(c_A,\tau_M) & \text{on } [0,1] \times ST(M)_{|\{a\}\times\Sigma}. \end{cases}$$



Here, $\tilde{\omega}(c, \tau_b)$ is the following slight modification of $\omega(c; \tau, \tau_b)$ (allowed by Lemma 4.11).

$$\omega = \begin{cases} \omega(c; \tau, \tau_b) & \text{on } ST([a+\varepsilon/2, b] \times \Sigma) \\ \pi_{S^2}(H \circ \mathcal{T}_{c_A} \circ \tau_M)^*(\omega_{S^2}) & \text{on } ST([a, a+\varepsilon/2] \times \Sigma) \end{cases}$$

where

$$H : [a, a+\varepsilon/2] \times \Sigma \longrightarrow SO(2) \subset SO(3)$$

is a homotopy supported near the intersection points of c and c_B, valued in the subgroup $SO(2)$ of $SO(3)$ that fixes e_1, such that $H(\{a\} \times \Sigma) = \{\text{Identity}\}$, and $H_{a+\varepsilon/2} \circ \mathcal{T}_{c_A} = \mathcal{T}_c \mathcal{T}_{c_B}$.

Definition of the trivialisation τ_0 of $T([0,1] \times B_M^D) \otimes \mathbb{C}$ over B_0.
Wherever ω is associated to a trivialisation τ of $T(B_M^D)$, τ_0 is its natural stabilisation $S(\tau)$ obtained by mapping the unit tangent vector to $T([0,1] \times \{x\})$ to the first basis vector E_1 of $\mathbb{C}^4$, and the i^{th} basis vector of $\mathbb{R}^3$ to the $(i+1)^{\text{th}}$ basis vector of $\mathbb{C}^4$. On the remaining parts of B_0 that are of the form $c \times D^2$, with a trivialisation τ_b of $T(B_M)$ over $c \times D^2$ involved (possibly via $\mathcal{Q}$),

$$\tau_0(\gamma \in c, d \in D^2; v) = \tilde{F}_U(d)(S(\tau_b)(\gamma, d; v))$$

where $\tilde{F}_U$ is a fixed map from D^2 to the stabilizer $SU(3)$ of E_1 in $SU(4)$.

Lemma 4.15 *The form ω extends as a closed two-form on $[0,1] \times ST(A)$, on $[0,1] \times ST(B)$ and on $[0,1] \times ST([a,b] \times \Sigma)$.*

PROOF: Let us first treat the case of $[0,1] \times ST(A)$. We know ω on $\partial([0,1] \times A) \times S^2$, and it is sufficient to prove that the integral of ω vanishes on the kernel of the map induced by the inclusion $H_2(\partial([0,1] \times A) \times S^2) \longrightarrow H_2([0,1] \times A \times S^2)$. This kernel is generated by the $\partial([0,1] \times S(a_i)(n))$, for $i = 1, \dots, g(\Sigma)$, where the a_i are simple curves that generate $\mathcal{L}_A$, $S(a_i)$ is a surface whose boundary is made of k_i copies of a_i and $S(a_i)(n)$ is the section of $ST(A)_{|S(a_i)}$ given by its positive normal that belongs to $T(\{x\} \times \partial A)$ on a neighborhood $[a-1, a+1] \times \partial A$ of ∂A.

$$\int_{\partial([0,1] \times S(a_i)(n))} \omega = \int_{S(a_i)(n)} \omega(\tau_M) - \int_{0 \times S(a_i)(n) \cup [0,1] \times \partial S(a_i)(n)} \omega.$$

Lemma 4.12 tells that the first integral is a well-determined function of the restriction of τ_M to $\partial S(a_i)(n)$ and of the topology of $S(a_i)(n)$.

Now, $\tilde{A} = (\{0\} \times A) \cup_{\partial A} ([0,1] \times \partial A)$ is equipped with a smooth structure in a standard way so that the S^2-bundle over it $ST(\{x\} \times M)$ is identified with its unit tangent bundle with the help of the bundle isomorphism $\mathcal{Q}$ over $[0,1] \times \partial A$. The smoothing makes

$$\tilde{S}(a_i)(n) = 0 \times S(a_i)(n) \cup [0,1] \times \partial S(a_i)(n)$$

smooth and leaves the normal to the surface unchanged. On $\tilde{A}$, the form ω is a special admissible form with respect to a trivialisation $\tilde{\tau}$ of $T(\tilde{A} \setminus N(c_A))$ and a trivialisation $\tilde{\tau}_b$ of $T(N(c_A))$. Now, $(\tilde{A}, \tilde{S}(a_i)(n), \tilde{\tau}, \tilde{\tau}_b)$ is isomorphic to some $(A, S(a_i)(n), \tau', \tau'_b)$ such that the restrictions of τ' and τ_M to $\partial S(a_i)(n)$ coincide. Therefore, Lemma 4.12 applies to identify $\int_{\tilde{S}(a_i)(n)} \omega$ to $\int_{S(a_i)(n)} \omega(\tau_M)$. Thus, ω extends as a closed 2-form on $[0,1] \times ST(A)$.

Similarly, for any surface $S(b_i)$ of B whose boundary is in $\{b\} \times \Sigma$, Lemma 4.12 applies to identify $\int_{\{0\} \times S(b_i)(n)} \omega$ to $\int_{-[0,1] \times \partial S(b_i)(n) \cup \{1\} \times S(b_i)(n)} \omega$, and allows us to prove that ω extends as a closed 2-form on $[0,1] \times ST(B)$.

Now, for any curve γ of Σ, Lemma 4.12 also applies to prove

$$\int_{\{0\} \times [a,b] \times \gamma \cup [0,1] \times \{b\} \times \gamma} \omega = \int_{[0,1] \times \{a\} \times \gamma \cup \{1\} \times [a,b] \times \gamma} \omega,$$

and we easily deduce from this fact that ω extends as a closed 2-form on $[0,1] \times ST([a,b] \times \Sigma)$. $\diamond$

Lemma 4.16 *Let K be a rational homology handlebody, and let τ be a trivialisation of TK over $\partial([0,1] \times K)$. Let ω denote the associated two-form on $\partial([0,1] \times ST(K))$ and let $\tau_{\mathbb{C}}$ be the associated trivialisation of $T([0,1] \times K) \otimes \mathbb{C}$. Then ω extends to $[0,1] \times ST(K)$ and for any closed extension of ω to $[0,1] \times ST(K)$,*

$$z_n([0,1] \times ST(K); \omega) = -\frac{1}{4} p_1([0,1] \times K; \tau_{\mathbb{C}|\partial([0,1] \times K)}) \delta_n.$$

PROOF: The existence of ω is shown as in the above proof of Lemma 4.15. By Lemma 2.25 in [L2], since the restriction injects $H^2([0,1] \times K \times S^2)$ into $H^2(\partial([0,1] \times K) \times S^2)$ and maps $H^1([0,1] \times K \times S^2)$ onto $H^1(\partial([0,1] \times K) \times S^2)$, $z_n([0,1] \times ST(K); \omega)$ only depends on the values of ω on the boundary of $[0,1] \times ST(K)$.

Use the restriction τ_1 of τ to $\{1\} \times K$ to identify $ST(K)$ to $K \times S^2$. Then $\tau = G \circ \tau_1$ for some map $G : \partial([0,1] \times K) \longrightarrow SO(3)$ that maps $\{1\} \times K$ to 1. Then both sides only depend on the homotopy class of G among these maps. In particular, we may assume that G maps $[0,1] \times \partial K$ to 1. But in this case, it is enough to embed K into a rational homology sphere M where τ_1 extends and to apply Propositions 1.8, 2.11, and 2.27 in [L2]. $\diamond$

Lemma 4.17

$$z_n([0,1] \times ST(A); \omega) = -\frac{1}{4} p_1([0,1] \times A; \tau_{0|\partial[0,1] \times A}) \delta_n.$$

PROOF: There exists a bundle isomorphism $\psi : ST(A) \longrightarrow ST(A)$ over the identity map of A such that $\tau_A = \tau_M \circ \psi$.

Let ψ still denote $\psi : \begin{array}{ccc} [-1,0] \times ST(A) & \longrightarrow & [0,1] \times ST(A) \\ (t,y) & \mapsto & (t+1, \psi(y)). \end{array}$

Extend ω on $[-1,1] \times ST(A)$ by $\psi^*(\omega_{|[0,1] \times ST(A)})$ on $[-1,0] \times ST(A)$. Then according to Lemma 2.26 in [L2],

$$z_n([-1,0] \times ST(A); \omega) = z_n([0,1] \times ST(A); \omega) = \frac{1}{2} z_n([-1,1] \times ST(A); \omega).$$

Again, $z_n([-1,1]\times ST(A);\omega)$ only depends on the values of ω on the boundary of $[-1,1]\times ST(A)$ that are

$$\omega = \begin{cases} \omega(\tau_M \circ \psi^2) & \text{on } \{-1\}\times ST(A) \\ \omega(\tau_M) & \text{on } \{1\}\times ST(A) \\ \omega_{\mathcal{Q}}(c_A, \tau_M) & \text{on } [0,1]\times ST(A)_{|\partial A} \stackrel{\mathcal{Q}}{\cong} ST([0,1]\times\partial A) \\ \omega_{\mathcal{Q}}(c_A, \tau_M\circ\psi) & \text{on } [-1,0]\times ST(A)_{|\partial A} \stackrel{\mathcal{Q}}{\cong} ST([-1,0]\times\partial A). \end{cases}$$

Furthermore, with the data on $\partial([-1,1]\times A)$, we again naturally associate a trivialisation τ_0 of $T([-1,1]\times A)\otimes\mathbb{C}$ over $\partial([-1,1]\times A)$ such that

$$p_1([-1,1]\times A;\tau_{0|\partial([-1,1]\times A)}) = 2p_1([0,1]\times A;\tau_{0|\partial([0,1]\times A)}).$$

Therefore, we are left with the proof of the following equality.

$$z_n([-1,1]\times ST(A);\omega) = -\frac{p_1([-1,1]\times A;\tau_{0|\partial([-1,1]\times A)})}{4}\delta_n. \tag{4.18}$$

There exists a smooth map G for a small positive number $\varepsilon>0$ such that

$$\begin{array}{rccl} G: & [-1,1]\times[-1,1] & \longrightarrow & SO(3) \\ & (t,u) & \mapsto & \begin{cases} \text{Identity} & \text{if } |u|>1-\varepsilon \\ R_{2\theta(u)} & \text{if } t<-1+\varepsilon \\ \text{Identity} & \text{if } t>1-\varepsilon. \end{cases} \end{array}$$

Define $\tilde{G}$ as the Identity map over $[-1,1]\times(\partial A\setminus c_A\times]-1,1[)\times\mathbb{R}^3$ and by

$$\tilde{G}((t,\sigma,u;v)\in[-1,1]\times c_A\times]-1,1[\times\mathbb{R}^3) = (t,\sigma,u;G(t,u)(v)).$$

Thanks to Lemma 4.16, in order to prove Equality 4.18, it is enough to prove that changing the value of ω on $[-1,1]\times ST(\partial A)$ into $\omega(\tilde{G}\circ\tau_M)$ does not change the left-hand side of the equality and that changing the trivialisation τ_0 of $T([-1,1]\times A)\otimes\mathbb{C}$ into $\tilde{G}\circ\tau_M\otimes 1_{\mathbb{C}}$ over $[-1,1]\times\partial A$ does not change its right-hand side.

To do this, first assume without loss that over a collar $[-1,1]\times[a-1,a]\times\partial A$ everything behaves as a product by $[a-1,a]$, and use this to extend both ω as $\pi^*_{[-1,1]\times\partial A\times S^2}(\omega)$ on this collar, and the trivialisation τ_0. In particular, ω and τ_0 are associated to the trivialisation τ_M over $[-1,1]\times[a-1,a]\times(\partial A\setminus(c_A\times[-1,1]))$, and over $\{1\}\times A$. Over $[-1,1]\times[a-1,a]\times c_A\times[-1,1]$, $\pi^*_{[-1,1]\times\partial A\times S^2}(\omega)$ factors through $[-1,1]\times[-1,1]\times S^2$. We are going to modify ω on $]-1,1[\times[a-1,a]\times c_A\times[-1,1]\times S^2$ by a form that still factors through the bundle projection onto $]-1,1[\times[a-1,a]\times[-1,1]\times S^2$, and therefore without changing $z_n([-1,1]\times A;\omega)$ (thanks to Lemma 2.26 in [L2]) so that $\omega=\omega(\tilde{G}\circ\tau_M)$ over $[-1,1]\times\{a\}\times\partial A$. Then the associated trivialisation will read $\psi(K)\circ\tau_M$ on $\partial([-1,1]\times[a-1,a]\times[-1,1])$ for some fixed

$$K:\partial([-1,1]\times[a-1,a]\times[-1,1])\longrightarrow SO(3)$$

Since $\pi_2(SU(4))=0$, K will extend to $SU(4)$, and such an operation will not change $p_1([-1,1]\times A;\tau_0(\omega)_{\partial([-1,1]\times A)})$ either.

Change ω over $[-1,1]\times\{a\}\ (\times c_A)\times[-1,1]$ into $\omega(\tilde{G}\circ\tau_M)$. In order to achieve our goal, it is enough to see that ω that is defined on the boundary of $[-1,1]\times[a-1,a]\times[-1,1]\times S^2$ extends to a closed form on the whole space. To do that, it is enough to prove that

$$\int_{\partial([-1,1]\times[a-1,a]\times[-1,1])\times v}\omega = 0$$

for some $v\in S^2$. This integral equals

$$\int_{\partial([-1,1]\times[a-1,a])\times[-1,1]\times v}\omega.$$

It depends neither on c_A nor on τ_M. Assume that there exists a closed curve d that intersects $c_A\times[-1,1]$ along $\{\sigma\}\times[-1,1]$. Then

$$\int_{\partial([-1,1]\times[a-1,a])\times[-1,1]\times v}\omega=\int_{\partial([-1,1]\times[a-1,a])\times d\times v}\omega.$$

and this integral vanishes as in the proof of Lemma 4.15.

Therefore, Lemma 4.16 (together with Lemma 2.26 in [L2]) applies to conclude the proof. $\diamond$

Similarly, we get the following lemma.

Lemma 4.19

$$z_n([0,1]\times ST(B);\omega)=-\frac{1}{4}p_1([0,1]\times B;\tau_{0|\partial[0,1]\times B})\delta_n$$

Proof: We can also deduce this lemma from the previous one by gluing a rational homology handlebody A' along $\{b\}\times\Sigma$ to B so that both τ_B and $\mathcal{T}_{c_B}\circ\tau_B$ extend to A', and $A'\cup B$ is rational homology ball. $\diamond$

Unfortunately, the same methods do not apply to prove the following similar lemma over $[0,1]\times[a,b]\times\Sigma$.

Lemma 4.20 *Let Σ be an oriented compact surface with one boundary component. Let τ_M be a trivialisation of $ST([a,b]\times\Sigma)$, let c, c_A, c_B be three non-necessarily connected curves of Σ, let*

$$\omega=\begin{cases}\tilde{\omega}(c,\mathcal{T}_{c_B}\circ\tau_M) & \text{on } \{0\}\times ST([a,b]\times\Sigma)\\ \omega(\tau_M) & \text{on } \{1\}\times ST([a,b]\times\Sigma)\\ \omega_{\mathcal{Q}}(c_B,\tau_M) & \text{on } [0,1]\times ST([a,b]\times\Sigma)_{|\{b\}\times\Sigma}\\ \omega_{\mathcal{Q}}(c_A,\tau_M) & \text{on } [0,1]\times ST([a,b]\times\Sigma)_{|\{a\}\times\Sigma}.\end{cases}$$

$$z_n([0,1]\times ST([a,b]\times\Sigma);\omega)=-\frac{1}{4}p_1([0,1]\times[a,b]\times\Sigma;\tau_{0|\partial([0,1]\times[a,b]\times\Sigma)})\delta_n.$$

Remark 4.21 Here, we have three genuine trivialisations, namely τ_M, $\mathcal{T}_{c_A}\circ\tau_M$ and $\mathcal{T}_{c_B}\circ\tau_M$, and it may be possible that the three of them cannot simultaneously extend to a rational homology handlebody whose boundary contains Σ. (Indeed, according to Proposition 4.6, if these three trivialisations extend to a $\mathbb{Q}$-handlebody K, $\phi_{\tau_M}(\mathcal{L}_K^{\mathbb{Z}/2\mathbb{Z}})=\phi_{\mathcal{T}_{c_A}\circ\tau_M}(\mathcal{L}_K^{\mathbb{Z}/2\mathbb{Z}})=\{0\}$, this implies that $\langle c_A,.\rangle$ vanishes on $\mathcal{L}_K^{\mathbb{Z}/2\mathbb{Z}}$ and hence that c_A belongs to $\mathcal{L}_K^{\mathbb{Z}/2\mathbb{Z}}$. Similarly c_B must be in $\mathcal{L}_K^{\mathbb{Z}/2\mathbb{Z}}$. Thus, if the intersection of c_A and c_B mod 2 does not vanish, such a K cannot exist.) Therefore, we have to give further arguments to prove Lemma 4.20 that will conclude the proof of Proposition 4.8.

Proof of Lemma 4.20: Fix an arbitrary trivialisation τ_M of $T([a,b]\times\Sigma)$. Any other trivialisation is obtained from τ_M by a bundle isomorphism. By Lemma 2.26 in [L2], the left-hand side does not depend on τ_M. The right-hand side does not depend on τ_M either for the same reason. Hence, if Lemma 4.20 is true for τ_M, it is true for any other trivialisation of $T([a,b]\times\Sigma)$.

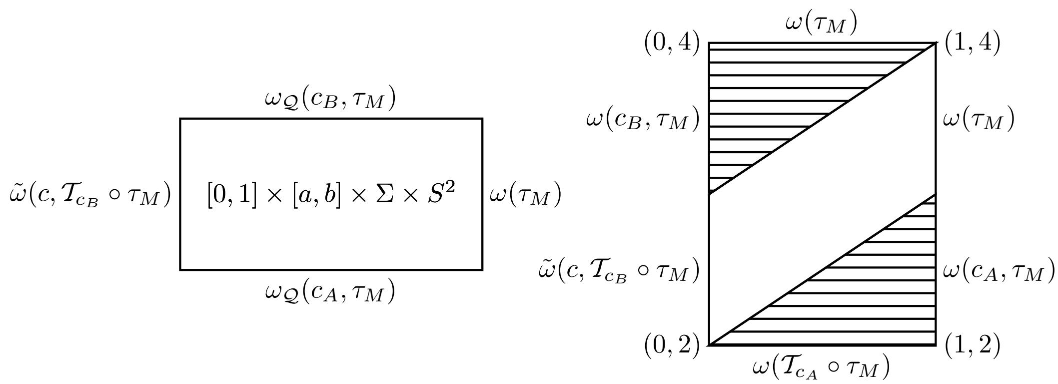
Then for simplicity, we shall assume that $\tau_M(\frac{\partial}{\partial u}((u,\sigma)\in[a,b]\times\Sigma))=e_1$ and that $\tau_M\circ T(\Sigma\hookrightarrow\{c\}\times\Sigma)$ is independent of c. In particular, τ_M makes unambiguous sense on all $[\alpha,\beta]\times\Sigma$.

Lemma 4.22 *Under the hypotheses above and the assumptions of Lemma 4.20, equip $[0,1]\times([2,4]\times\Sigma)$ with the trivialisation τ_M of $[2,4]\times\Sigma$. Let ω_1 be a closed form on $[0,1]\times ST([2,4]\times\Sigma)$ such that*

$$\omega_1=\begin{cases}\omega(\tau_M) & \text{on } [0,1]\times ST([2,4]\times\Sigma)_{|\{4\}\times\Sigma}\\ \omega(\mathcal{T}_{c_A}\circ\tau_M) & \text{on } [0,1]\times ST([2,4]\times\Sigma)_{|\{2\}\times\Sigma}\\ \omega(\tau_M) & \text{on } \{1\}\times ST([3,4]\times\Sigma)\\ \tilde{\omega}(c,\mathcal{T}_{c_B}\circ\tau_M) & \text{on } \{0\}\times ST([2,3]\times\Sigma)\\ \omega(c_B,\tau_M) & \text{on } \{0\}\times ST([3,4]\times\Sigma)\\ \omega(c_A,\tau_M) & \text{on } \{1\}\times ST([2,3]\times\Sigma).\end{cases}$$

Then $z_n([0,1]\times ST([2,4]\times\Sigma);\omega_1) = z_n([0,1]\times ST([a,b]\times\Sigma);\omega)$. *Let* $W=[0,1]\times[2,4]\times\Sigma$. *Let* τ_1 *be the trivialisation of* $TW\otimes\mathbb{C}$ *over* ∂W *corresponding to the given trivialisations. Then*

$$p_1(W;\tau_{1|\partial W}) = p_1([0,1]\times[a,b]\times\Sigma;\tau_{0|\partial([0,1]\times[a,b]\times\Sigma)})\delta_n.$$



PROOF: Let us first treat the p_1 case, there is a map G_U from ∂W to $SU(3)$ such that τ_1 is the stabilisation of $G_U\circ\tau_M$ over ∂W,

$$G_U = \begin{array}{ll} 1 & \text{on } [0,1]\times\{4\}\times\Sigma \\ 1 & \text{on } [0,1]\times\{2\}\times(\Sigma\setminus c_A\times[-1,1]) \\ 1 & \text{on } \{1\}\times[3,4]\times\Sigma \\ 1 & \text{on } \{1\}\times[2,4]\times(\Sigma\setminus c_A\times[-1,1]) \\ 1 & \text{on } \{0\}\times[3,4]\times(\Sigma\setminus c_B\times[-1,1]) \\ 1 & \text{on } \{0\}\times[2,4]\times(\Sigma\setminus(c\cup c_B\times[-1,1])) \\ \mathcal{T}_{c_A} & \text{on } [0,1]\times\{2\}\times c_A\times[-1,1] \\ \mathcal{T}_{c_B} & \text{on } \{0\}\times\{3\}\times c_B\times[-1,1] \\ \mathcal{T}_{c_B} & \text{on } \{0\}\times[2,3]\times(c_B\times[-1,1]\setminus c\times[-1,1]) \end{array}$$

Furthermore, on $\{1\}\times[2,3]\times c_A\times[-1,1]$, G_U factors through the natural projection onto $[2,3]\times[-1,1]$, on $\{0\}\times[3,4]\times c_B\times[-1,1]$, G_U factors through the natural projection onto $[3,4]\times[-1,1]$, and on $\{0\}\times[2,3]\times(c\times[-1,1]\setminus c_B\times[-1,1])$, G_U factors through the natural projection onto $[2,3]\times[-1,1]$.

$p_1(W;\tau_{1|\partial W})$ only depends on the homotopy class of G_U with respect to $\{(1,4)\}\times\Sigma$. Since $p_1([0,1]\times[a,b]\times\Sigma;\tau_{0|\partial([0,1]\times[a,b]\times\Sigma)})$ is defined by a homotopic map. The two p_1 coincide.

The coincidence of the integrals can be seen on the picture where the projection of $ST([0,1]\times[a,b]\times\Sigma) \stackrel{\tau_M}{=} [0,1]\times[a,b]\times\Sigma\times S^2$ on $[0,1]\times[a,b]$ is represented on the left-handside, and the projection of $ST([0,1]\times[2,4]\times\Sigma) \stackrel{\tau_M}{=} [0,1]\times[2,4]\times\Sigma\times S^2$ onto $[0,1]\times[2,4]$ is represented on the right-hand side. Next ω_1 can be defined on the (products by $\Sigma\times S^2$ of the) two hatched triangles by pulling-back $\omega(c_B,\tau_M)$ and $\omega(c_A,\tau_M)$ using the pictured horizontal projections. On the remaining part, ω_1 can be filled in by the pull-back of ω under the obvious diffeomorphism from this remaining part to $[0,1]\times[a,b]\times\Sigma\times S^2$. Now, the z_n corresponding to the hatched triangles will vanish while the z_n corresponding to the remaining part equals $z_n([0,1]\times ST([a,b]\times\Sigma);\omega)$ thanks to Lemma 2.26 in [L2]. $\diamond$

Lemma 4.23 *Let* c'_A *be a curve with the same class as* c_A *in* $H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$. *Let*

$$H:[0,1]\times\{2\}\times\Sigma\longrightarrow SO(3)$$

satisfy

$$\begin{array}{l} H([0,1]\times\{2\}\times\partial\Sigma)=\{1\}\\ H=\mathcal{T}_{c_A} \quad on \ \ \{(0,2)\}\times\Sigma,\\ H=\mathcal{T}_{c'_A} \quad on \ \ \{(1,2)\}\times\Sigma.\end{array}$$

Let ω_3 be a closed form on $ST([0,1]\times[2,3]\times\Sigma)$ such that

$$\omega_3=\left\{\begin{array}{ll} \omega(\tau_M) & on\ [0,1]\times ST([2,3]\times\Sigma)_{|\{3\}\times\Sigma}\\ \omega(H\circ\tau_M) & on\ [0,1]\times ST([2,3]\times\Sigma)_{|\{2\}\times\Sigma}\\ \omega(c_A,\tau_M) & on\ \{0\}\times ST([2,3]\times\Sigma)\\ \omega(c'_A,\tau_M) & on\ \{1\}\times ST([2,3]\times\Sigma).\end{array}\right.$$

Let τ_3 be the trivialisation of $T([0,1]\times[2,3]\times\Sigma)\otimes\mathbb{C}$ corresponding to the trivialisations used to define ω_3. Then

$$z_n([0,1]\times ST([2,3]\times\Sigma);\omega_3)=-\frac{p_1([0,1]\times[2,3]\times\Sigma;\tau_{3|\partial([0,1]\times[2,3]\times\Sigma)})}{4}\delta_n.$$

Proof: By an obvious modification of Lemma 4.22 above, we would not change either side of the equality to be shown by setting rather

$$\omega_3=\left\{\begin{array}{ll} \omega_{\mathcal{Q}}(c_A,\tau_M) & \text{on } [0,1]\times ST([2,3]\times\Sigma)_{|\{3\}\times\Sigma}\\ \omega_{\mathcal{Q}}(c'_A,\tau_M) & \text{on } [0,1]\times ST([2,3]\times\Sigma)_{|\{2\}\times\Sigma}\\ \omega(H\circ\tau_M) & \text{on } \{0\}\times ST([2,3]\times\Sigma)\\ \omega(\tau_M) & \text{on } \{1\}\times ST([2,3]\times\Sigma).\end{array}\right.$$

Now, both ω_3 and τ_3 trivially extend over $[0,1]\times[2,3]\times\partial A$ and can be glued along $[0,1]\times\{2\}$ with the implicit picture of Lemma 4.17 with c'_A instead of c_A, and therefore ω' instead of ω and τ'_0 instead of τ_0. Then Lemma 4.17 applied to $A\cup_{\partial A}[2,3]\times\partial A$ instead of A tells us that

$$4z_n(\omega_3)+4z_n([0,1]\times A;\omega')=-p_1([0,1]\times[2,3]\times\partial A;\tau_3)\delta_n-p_1([0,1]\times A;\tau'_0)\delta_n,$$

and that

$$4z_n([0,1]\times A;\omega')=-p_1([0,1]\times A;\tau'_0)\delta_n.$$

$\diamond$

This lemma has the following corollary whose proof is similar and therefore left to the reader.

Lemma 4.24 *If there exist curves c', c'_A and c'_B that are homologous modulo 2 to c, c_A and c_B, respectively such that Lemma 4.20 is true when replacing (c,c_A,c_B) by (c',c'_A,c'_B), then Lemma 4.20 is true for (c,c_A,c_B).*

Lemma 4.25 *Lemma 4.20 is true when c and c_B do not intersect.*

Proof: According to Lemma 4.22, it is enough to prove that in this case,

$$z_n([0,1]\times ST([2,4]\times\Sigma);\omega_1)=-\frac{1}{4}p_1(W;\tau_{1|\partial W})\delta_n.$$

We may assume that

$$c_A\times[-1,1]=c\times[-1,1]\coprod c_B\times[-1,1].$$

Use an isotopy of $\{0\}\times[2,4]\times\Sigma$ supported away from $\{0\}\times[2,4]\times c\times[-1,1]$ to lower lower $[3+\varepsilon,4-\varepsilon]\times c_B\times[-1,1]$ to $[2+\varepsilon,3-\varepsilon]\times c_B\times[-1,1]$ without changing either side of the equality. After this isotopy, the trivialisation and the form over $\{0\}\times[2,4]\times\Sigma$ coincide with the trivialisation and the form over $\{1\}\times[2,4]\times\Sigma$. Therefore both sides of the equality to be shown vanish. $\diamond$

As a direct corollary of these two lemmas, we get the following lemma.

Lemma 4.26 *Lemma 4.20 is true when the algebraic intersection $\langle c, c_B\rangle$ of c and c_B in Σ is even.*

$\diamond$

Lemma 4.27 *There exists an integer p^+ and an element $I_n \in \mathcal{A}_n(\emptyset)$ such that if, under the assumptions of Lemma 4.20, c and c_B are transverse and the sign of all their intersection points is positive, then*
$z_n([0,1] \times ST([a,b] \times \Sigma); \omega) = \langle c, c_B\rangle I_n$ *and*
$p_1([0,1] \times [a,b] \times \Sigma; \tau_{0|\partial([0,1]\times[a,b]\times\Sigma)}) = \langle c, c_B\rangle p^+$.

Once this lemma is proved, using Lemma 4.26 for two curves c and c_B as in Lemma 4.27 such that $\langle c, c_B\rangle = 2$ shows that $I_n = -\frac{p^+}{4}\delta_n$. Therefore, Lemma 4.20 is true for any two curves that only intersect positively. Now, since it is easy to change c_B without changing its class in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ so that c and c_B only intersect positively, Lemma 4.20 will be proved right after Lemma 4.27 is proved.
Proof of Lemma 4.27: First isolate the intersection points of c and c_B inside boxes $[-2,2] \times [-2,2]$ that $c \times [-1,1]$ intersects as $[-2,2] \times [-1,1]$ and $c_B \times [-1,1]$ as $-[-1,1] \times [-2,2]$. Then lower $[3+\varepsilon, 4-\varepsilon] \times c_B \times [-1,1]$ to $[2+\varepsilon, 3-\varepsilon] \times c_B \times [-1,1]$ in $\{0\} \times [2,4] \times \Sigma$ except on the cubes $[2,4] \times [-2,2]^2$ by an isotopy of the framed link $c \cup c_B$ supported away from $[2,4] \times [-1,1]^2$. This does not change either side of the equality and after this isotopy, we may assume that

$$\omega_1 = \pi^*_{\{1\}\times ST([2,4]\times\Sigma)}\left(\omega_{1|\{1\}\times ST([2,4]\times\Sigma)}\right)$$

except over the cubes $[0,1] \times [2,4] \times [-2,2]^2$. But on the boundaries of the products by S^2 of these cubes, the value of ω is always the same and since $H^2(\partial([0,1] \times [2,4] \times [-2,2]^2 \times S^2)) = H^2([0,1] \times [2,4] \times [-2,2]^2 \times S^2)$, ω extends as a closed form there, and we may choose the same extension for all the cubes that are the only ones to produce nonzero integrals according to Lemma 2.26 in [L2] and that all produce the same integral I_n. Similarly, p^+ is the obstruction to extend the trivialisation associated to ω_1 on the boundary of such a cube. $\diamond$

5 Simultaneous normalization of the fundamental forms

This section is devoted to the proof of Proposition 3.3. We use real coefficients for homology and cohomology.

5.1 Sketch

First note that the closed 2-forms $\omega(M_I)$ on $\partial C_2(M_I)$ defined in the beginning of Section 3 (after Remark 3.1) are antisymmetric and extend as closed antisymmetric 2-forms on $C_2(M_I)$ because of Lemma 2.4 in [L2].

Now, we wish to arrange the forms $\omega_{M_I} = \omega(M_I)$ as in Proposition 3.3. To do that, we shall first show how to make ω_M explicit in some part of $C_2(M)$.

Recall that $[-4,4] \times \partial A^i$ denotes a regular neighborhood of ∂A^i embedded in M, that intersects A^i as $[-4,0] \times \partial A^i$. All the neighborhoods $[-4,4] \times \partial A^i$ are disjoint from each other. Throughout this paragraph, we shall use the corresponding coordinates on the image of this implicit embedding.

For $t \in [-4,4]$, set

$$A^i_t = \left\{ \begin{array}{ll} A^i \cup ([0,t] \times \partial A^i) & \text{if } t \geq 0 \\ A^i \setminus (]t,0] \times \partial A^i) & \text{if } t \leq 0 \end{array} \right.$$

$$\partial A^i_t = \{t\} \times \partial A^i.$$

Choose curves $(b^i_j)_{j=1,\dots,g_i}$, and $(y^i_j)_{j=1,\dots,g_i}$ of ∂A^i such that

- the homology classes of the $(b^i_j)_{j=1,\dots,g_i}$ form a basis of $\mathcal{L}(M \setminus \mathrm{Int}(A^i))$,
- and $\langle y^i_j, [b^i_k] \rangle_{\partial A^i} = \delta_{jk}$ (thus, the homology classes of the (y^i_j) form a basis of $H_1(M \setminus A^i)$).

Choose a basepoint p^i in ∂A^i outside the neighborhoods $a^i_j \times [-1,1]$ of the a^i_j and outside neighborhoods $b^i_j \times [-1,1]$ the b^i_j. Fix a path $[p^i,(0,0,1)]$ from p^i to $(0,0,1)$ in

$$B_M(1) \setminus \left(\mathrm{Int}(A^i) \cup_{k,k\neq i} A^k_4\right),$$

that is extended into a path $[p^i,\infty(v)]$ by the vertical line $(0,0) \times [1,\infty[\cup \{\infty(v)\}$, where $\infty(v)$ is the intersection with $\partial C_1(M)$ of the closure in $C_1(M)$ of the vertical half-line $(0,0) \times [1,\infty[$. Fix a closed two-form $\omega(p^i)$ on $(M \setminus \mathrm{Int}(A^i))$ such that

- the integral of $\omega(p^i)$ along a closed surface of $(M \setminus \mathrm{Int}(A^i))$ is its algebraic intersection with $[p^i,\infty(v)]$,
- the support of $\omega(p^i)$ intersects $(B_M \setminus \mathrm{Int}(A^i))$ inside a tubular neighborhood of $[p^i,\infty(v)]$ disjoint from
$$\left(\cup_{k,k\neq i} A^k_4\right) \cup \left([-4,4] \times \left(\cup_{j=1}^{g_i}((a^i_j \times [-1,1]) \cup (b^i_j \times [-1,1]))\right)\right).$$
- $\omega(p^i)$ restricts as the usual volume form on $\partial C_1(M) = S^2$.

Here, a *two-chain* is a linear rational combination of smooth compact oriented surfaces with boundaries. The *integral* of a 2-form along such a chain is the corresponding linear rational combination of its integral along the surfaces. The *support* of such a 2-chain is the union of the involved surfaces. A *2-cycle* is a two-chain with empty (or null) boundary.

For any $i \in N$, and for any $j \in \{1,2,\dots,g_i\}$, extend $\eta(a^i_j)$ on $[-4,4] \times \partial A^i$ into a closed one-form $\eta(a^i_j)$ supported on $[-4,4] \times a^i_j \times [-1,1]$ where $\eta(a^i_j)$ is again given by the formula.

$$\eta(a^i_j) = \pi^*_{[-1,1]}(\eta_{[-1,1]}).$$

Let $S(a^i_j)$ be a 2-chain in A^i_4 with boundary $4\times a^i_j\times 0$ that intersects $[-4,4]\times\partial A^i$ along $[-4,4]\times a^i_j\times 0$.

Let $i\in N$, and let $j\in\{1,2,\dots,g_i\}$. Choose a 2-chain $S(b^i_j)$ in $(B_M\setminus \mathrm{Int}(A^i))$ that is bounded by b^i_j whose support is disjoint from all the supports of the $\omega(p^k)$, and that intersects A^k_4 as a combination of $S(a^k_\ell)$ for all $k\in N$ such that $k\neq i$. Define a closed one-form $\eta(b^i_j)$ on $(M\setminus\mathrm{Int}(A^i))$ such that

- the integral of $\eta(b^i_j)$ along a closed curve of $(M\setminus\mathrm{Int}(A^i))$ is its algebraic intersection with $S(b^i_j)$,
- the support of $\eta(b^i_j)$ is in a neighborhood of $S(b^i_j)$ disjoint from the support of the $\omega(p^k)$, for all $k\neq i$.
- for all $k\neq i$, the restriction of $\eta(b^i_j)$ to A^k_4, $k\neq i$ is a rational combination of the $\eta(a^k_\ell)$, $\ell=1,\dots,g_k$ (the coefficients are linking numbers determined by the first condition).

In Subsection 5.2, we shall prove that these forms can be used as follows to make ω_M explicit in some parts of $C_2(M)$.

Proposition 5.1 *With the above notations, we can choose ω_M so that:*

1. *for every $i\in N$, the restriction of ω_M to*
$$A^i\times(C_1(M)\setminus A^i_3)\subset C_2(M)$$
equals
$$\sum_{(j,k)\in\{1,\dots,g_i\}^2}\ell(z^i_j,(4\times y^i_k))p_1^*(\eta(a^i_j))\wedge p_2^*(\eta(b^i_k))+p_2^*(\omega(p^i))$$
where p_1 and p_2 denote the first and the second projection of $C_2(M)$ onto $C_1(M)$, respectively;
2. *for every i, for any $j\in\{1,2,\dots,g_i\}$,*
$$\int_{S(a^i_j)\times p^i}\omega_M=0;$$
3. *ω_M is fundamental with respect to τ_M.*

Assume that Proposition 5.1 is proved. This is the goal of Subsection 5.2. When changing some A^i into some B^i with the same Lagrangian, it is easy to change the restrictions of ω_M inside the parts mentioned in the first paragraph of the statement of Proposition 5.1 (and inside their symmetric parts under ι that are also determined by the statement). Indeed, all the forms $\eta(a^i_j)$, $\eta(b^i_j)$ and $\omega(p^i)$ can be defined on the parts of the M_I where they are needed so that these forms coincide with each other whenever it makes sense, and so that they have the properties that were required for M. (Recall that the $\eta(a^i_j)$ are defined both in A^i and B^i and that they are identical near ∂A^i and ∂B^i while $\omega(p^i)$ is supported in $(M\setminus(\cup_{k\in N}\mathrm{Int}(A_k)))$ and while the $\eta(b^i_j)$ restrict to the A^k as a (fixed by the clover data) combination of $\eta(a^k_\ell)$. Define $\omega_0(M_I)$ on
$D(\omega_0(M_I))=$
$$\left(C_2(M_I)\setminus\left(\cup_{i\in I}p_{12}^{-1}\left((B^i_{-1}\times B^i_3)\cup(B^i_3\times B^i_{-1})\right)\right)\right)\cup p_{12}^{-1}(\mathrm{diag}(M\setminus\{\infty\}))$$
so that

1. $\omega_0(M_I)=\omega_M$ on $C_2\left(M\setminus(\cup_{i\in I}B^i_{-1})\right)$,

2.

$$\omega_0(M_I) = \sum_{(j,k)\in\{1,\dots g_i\}^2} \ell(z^i_j, (4\times y^i_k))p_1^*(\eta(a^i_j)) \wedge p_2^*(\eta(b^i_k)) + p_2^*(\omega(p^i))$$

on $p_{12}^{-1}(B^i \times (M \setminus B^i_3))$ when $i \in I$,

3. $\omega_0(M_I) = -\iota^*(\omega_0(M_I))$ on $p_{12}^{-1}((M \setminus B^i_3) \times B^i)$ when $i \in I$,

4. $\omega_0(M_I) = \omega(M_I)$ on $\partial C_2(M_I)$. (See the definition after Remark 3.1.)

Note that this definition is consistent.

Lemma 5.2 *With the above notation, for any $i \in N$, $\omega_0(M_i = M_{\{i\}})$ vanishes on the kernel of the map induced by the inclusion*

$$H_2\left(D(\omega_0(M_i)\right) \longrightarrow H_2(C_2(M_i)).$$

This lemma is surprisingly difficult to prove for me. It will be proved in Subsection 5.3. Assume it for the moment. Then (the cohomology class of) $\omega_0(M_i)$ is in the image of the natural map

$$H^2(C_2(M_i)) \longrightarrow H^2(D(\omega_0(M_i))).$$

Therefore $\omega_0(M_i)$ extends to a closed form $\omega_1(M_i)$, and

$$\omega(M_i) = \frac{\omega_1(M_i) - \iota^*(\omega_1(M_i))}{2}$$

is an admissible form for M_i .

Now, for any $I \subset N$, we may define

$$\omega(M_I) = \begin{cases} \omega_0(M_I) & \text{on } C_2(M_I) \setminus \left(\cup_{i\in I} p_{12}^{-1}\left((B^i_{-1} \times B^i_4) \cup (B^i_4 \times B^i_{-1})\right)\right) \\ \omega(M_i) & \text{on } C_2(B^i_4) \text{ for } i \in I \end{cases}$$

since the $C_2(B^i_4)$ do not intersect. These forms $\omega(M_I)$ satisfy the conclusions of Proposition 3.3 that will be proved once Proposition 5.1 and Lemma 5.2 are proved. Their proofs will occupy the next two subsections.

5.2 Proof of Proposition 5.1

The homology classes of the $(z^i_j \times (4 \times y^i_k))_{(j,k)\in\{1,\dots,g_i\}^2}$ and $(p^i \times \partial C_1(M))$ form a basis of

$$H_2\left(A^i \times (C_1(M) \setminus A^i_3)\right) = (H_1(A^i) \otimes H_1(M \setminus A^i)) \oplus H_2(C_1(M) \setminus A^i).$$

The evaluation of L_M (defined after Lemma 2.1 in [L2]) along these classes is $\ell(z^i_j, (4 \times y^i_k))$ for the first ones and 1 for the last one. In particular the form of the statement integrates correctly on this basis.

Let us first prove Proposition 5.1 when $N = \{1\}$. Set $A^1 = A$, and forget about the superfluous superscripts 1. Let ω_0 be a 2-form fundamental with respect to τ_M given by Lemma 2.4 in [L2], and let ω be the closed 2-form defined on $(A_1 \times (C_1(M) \setminus \mathrm{Int}(A_2)))$ by the statement (naturally extended). Since this form ω integrates correctly on $H_2\left(A_1 \times (C_1(M) \setminus \mathrm{Int}(A_2))\right)$, there exists a one-form η on $(A_1 \times (C_1(M) \setminus \mathrm{Int}(A_2)))$ such that $\omega = \omega_0 + d\eta$.

This form η is closed on $A_1 \times \partial C_1(M)$. Since $H^1\left(A_1 \times (C_1(M) \setminus \mathrm{Int}(A_2))\right)$ maps surjectively to $H^1(A_1 \times \partial C_1(M))$, we may extend η to a closed one-form $\tilde{\eta}$ on $(A_1 \times (C_1(M) \setminus \mathrm{Int}(A_2)))$. Changing η into $(\eta - \tilde{\eta})$, turns η into a primitive of $(\omega - \omega_0)$ that vanishes on $A_1 \times \partial C_1(M)$.

Let χ be a smooth function on $C_2(M)$ supported in $(A_1 \times (C_1(M) \setminus \mathrm{Int}(A_2)))$, and constant with the value 1 on $(A \times C_1(M) \setminus A_3)$.

Set

$$\omega_a = \omega_0 + d\chi\eta.$$

Then ω_a is a closed form that has the required form on $(A \times (C_1(M) \setminus A_3))$. Furthermore, the restrictions of ω_a and ω_0 agree on $\partial C_2(M)$ since $d\chi\eta$ vanishes there (because η vanishes on $A_1 \times \partial C_1(M)$).

Adding to η a combination η_c of the closed forms $p_2^*(\eta(b_j))$ that vanish on $A_1 \times \partial C_1(M)$ does not change the above properties, but adds

$$\int_{p\times([2,3]\times a_j)} d(\chi\eta_c) = \int_{p\times(3\times a_j)} \eta_c$$

to $\int_{p\times S(a_j)} \omega_a$. Therefore since the $p_2^*(\eta(b_j))$ generate the dual of $\mathcal{L}_A$, we may choose η_c so that all the $\int_{p\times S(a_j)} \omega_a$ vanish. After this step, ω_a is a closed form that takes the prescribed values on

$$PS_a = \partial C_2(M) \cup (A \times (C_1(M) \setminus A_3))$$

and such that all the $\int_{p\times S(a_j)} \omega_a$ vanish. In order to make ω_a antisymmetric with respect to ι^*, we apply similar modifications to ω_a on the symmetric part $(C_1(M) \setminus \mathrm{Int}(A_2)) \times A_1$. The support of these modifications is disjoint from the support of the previous ones. Thus, they do not interfer and transform ω_a into a closed form ω_b with the additional properties:

- ω_b has the prescribed form on $(C_1(M)\setminus A_3)\times A$, (It is prescribed there because of the prescribed form on $(A \times (C_1(M) \setminus A_3))$ and because of the prescribed antisymmetry with respect to ι^*.)
- $\int_{S(a_j)\times p} \omega_b = 0$, for all $j = 1, \ldots g_1$.

Now, the form $\omega_M = \frac{\omega_b - \iota^*(\omega_b)}{2}$ has all the required properties, and the proposition is proved for $N = \{1\}$.

We now proceed by induction on $\sharp N = i$. We start with a 2-form ω_0 that satisfies all the hypotheses on $\{1, \ldots, i-1\}$ instead of $N = \{1, \ldots, i\}$, and by the first step, we also assume that we have a 2-form ω that satisfies all the hypotheses on $\{i\}$ instead of $N = \{1, \ldots, i\}$, with A_1^i replacing A^i.

Now, we proceed similarly. There exists a one-form η on $C_2(M)$ such that $\omega = \omega_0 + d\eta$. The exact sequence

$$0 = H^1(C_2(M)) \longrightarrow H^1(\partial C_2(M)) \longrightarrow H^2(C_2(M), \partial C_2(M)) \cong H_4(C_2(M)) = 0$$

shows that $H^1(\partial C_2(M))$ is trivial. Therefore, η is exact on $\partial C_2(M)$, we can assume that η vanishes on $\partial C_2(M)$, and we do assume so.
Let χ be a smooth function on $C_2(M)$ supported in $\left(A_1^i \times (C_1(M) \setminus \mathrm{Int}(A_2^i))\right)$, and constant with the value 1 on $\left(A^i \times (C_1(M) \setminus A_3^i)\right)$. Again, we are going to modify η by some closed forms so that

$$\omega_a = \omega_0 + d\chi\eta$$

has the prescribed value on

$$PS_a = \partial C_2(M) \cup \left(\cup_{k\in N} \left(A^k \times (C_1(M) \setminus A_3^k)\right)\right) \cup \left(\cup_{k\in N\setminus\{i\}} \left((C_1(M) \setminus A_3^k) \times A^k\right)\right).$$

Our form ω_a is as required anywhere except possibly in

$$\left(A_1^i \times (C_1(M) \setminus \mathrm{Int}(A_2^i))\right) \setminus \left(A^i \times (C_1(M) \setminus A_3^i)\right)$$

and in particular in the intersection of this domain with the domains where it was previously normalized, that are included in

$$\left(A_1^i \times (\partial C_1(M) \cup (\cup_{k=1}^{i-1} A^k))\right).$$

Recall that η vanishes on $A_1^i \times \partial C_1(M)$. Our assumptions also imply that η is closed on $A_1^i \times A^k$. Let us prove that they imply that η is exact on $A_1^i \times A^k$ for any $k < i$. To do that it suffices to check that:

1. For any $j = 1, \dots, g_i$, $\int_{b_j^i \times p^k} \eta = 0$.
2. For any $j = 1, \dots, g_k$, $\int_{p^i \times b_j^k} \eta = 0$.

Let us prove the first assertion. Since $\int_{b_j^i \times \infty(v)} \eta = 0$,

$$\int_{b_j^i \times p^k} \eta = \int_{\partial(b_j^i \times [p^k, \infty(v)])} \eta = \int_{b_j^i \times [p^k, \infty(v)]} (\omega - \omega_0).$$

where $\int_{b_j^i \times [p^k, \infty(v)]} \omega = 0$ because the supports of the $\eta(b_\ell^i)$ do not intersect $[p^k, \infty(v)]$. Now,

$$\int_{b_j^i \times [p^k, \infty(v)]} \omega_0 = -\int_{S(b_j^i) \times \partial[p^k, \infty(v)]} \omega_0 = \int_{S(b_j^i) \times \{p^k\}} \omega_0.$$

The latter integral vanishes because

1. $S(b_j^i)$ intersects A_4^k as copies of $S(a_\ell^k)$,
2. $\int_{S(a_\ell^k) \times p^k} \omega_0 = 0$ (that is the second condition of Proposition 5.1), and,
3. the integral of ω_0 also vanishes on the remaining part of $S(b_j^i) \times p^k$ because ω_0 is determined on $((C_1(M) \setminus A_4^k) \times A^k)$ and because the support of $\omega(p^k)$ is disjoint from $S(b_j^i)$.

Let us prove the second assertion. Again, since η vanishes on $\partial C_2(M)$, $\int_{\infty(v) \times b_j^k} \eta = 0$ and therefore

$$\int_{p^i \times b_j^k} \eta = -\int_{[p^i, \infty(v)] \times b_j^k} (\omega - \omega_0).$$

$\int_{[p^i, \infty(v)] \times b_j^k} \omega_0 = 0$ because of the form of ω_0 on $(C_1(M) \setminus A_4^k) \times A^k$.

$$\int_{[p^i, \infty(v)] \times b_j^k} \omega = \int_{\partial[p^i, \infty(v)] \times S(b_j^k)} \omega = -\int_{\{p^i\} \times S(b_j^k)} \omega.$$

Again, we know that this integral is zero along the intersection of $\{p^i\} \times S(b_j^k)$ with $A^i \times (C_1(M) \setminus A_4^i)$ because $S(b_j^k)$ does not meet the support of $\omega(p^i)$, and we conclude because $\int_{\{p^i\} \times S(a_\ell^i)} \omega = 0$ and because $S(b_j^k)$ intersects A_4^i along copies of $S(a_\ell^i)$.

Since η is exact on the annoying parts, we can assume that it identically vanishes there.

Thus, ω_a takes the prescribed values on $A^i \times (C_1(M) \setminus A_4^i)$, ω_a coincides with ω_0 where ω_0 was prescribed and ω_a integrates correctly along the $S(a_\ell^k) \times p^k$ and their symmetric with respect to ι, for $k \neq i$. Let us now modify η by adding a linear combination of $p_2^*(\eta(b_j^i))$ that vanishes on the $A_1^i \times A^k$, for $k < i$, and thus without changing the above properties so that the integrals of ω_a along the $\{p^i\} \times S(a^i_\ell)$ vanish, for $\ell = 1, \dots, g_i$, too. Let $f : H_1(M \setminus \mathrm{Int}(A^i)) \longrightarrow \mathbb{R}$ be the linear map defined by

$$f(a^i_\ell) = -\int_{\{p^i\}\times S(a^i_\ell)} \omega_a.$$

There exists a combination η_c of $p_2^*(\eta(b^i_j))$ such that for any $x \in \mathcal{L}_A$, $f(x) = \int_{p^i \times x} \eta_c$.

Observe that

$$\int_{\{p^i\}\times S(b^k_j)} \omega_a = \int_{\{\infty(v)\}\times S(b^k_j)} \omega_a - \int_{[p^i,\infty(v)]\times b^k_j} \omega_a = 0.$$

This implies that $f(\mathrm{Im}(H_1(A^k) \longrightarrow H_1(M \setminus \mathrm{Int}(A^i)))) = 0$. Thus, η_c vanishes on $A^i_1 \times A_k$. Changing η into $(\eta+\eta_c)$ does not change ω_a on the prescribed set but adds $\int_{\{p^i\}\times(4\times a^i_\ell)} \eta_c = f(a^i_\ell)$ to $\int_{\{p^i\}\times S(a^i_\ell)} \omega_a$ that becomes 0.

After this step, ω_a is a closed form that takes the prescribed values on PS_a such that the integrals of ω_a along the $(\{p^i\} \times S(a^i_\ell))$ vanish, for $\ell = 1, \dots, g_i$. In order to make ω_a antisymmetric with respect to ι^*, we apply similar modifications to ω_a on the symmetric part $(C_1(M) \setminus \mathrm{Int}(A^i_2)) \times A^i_1$. Again, the support of these modifications is disjoint from the support of the previous ones. Thus, they do not interfer and transform ω_a into a closed form ω_b with the additional properties:

- ω_b has the prescribed form on $(C_1(M) \setminus A^i_3) \times A^i$,
- $\int_{S(a^i_j)\times p^i} \omega_b = 0$, for all $j = 1, \dots g_i$.

Now, the form $\omega_M = \frac{\omega_b - \iota^*(\omega_b)}{2}$ has all the required properties, and Proposition 5.1 is proved.

$\diamond$

5.3 Proof of Lemma 5.2

We need some more notation before stating the key proposition that will lead to the proof of Lemma 5.2.

We assume that our trivialisation τ_M (fixed since the beginning of Section 3) maps the unit tangent vector of $(\{s\} \times x \times [0,1])$ at (s,x,t) to the first basis vector $e_1 \in \mathbb{R}^3$ for any $(s,x,t) \in [-4,0] \times a^i_j \times [0,1]$. When X is a unit vector field on the image of some chain $F(P)$ of $M \setminus \infty$, then $\mathrm{diag}(X)(F(P))$ denotes the chain of the blow-up of the diagonal that is the image of P under the map $(p \mapsto (F(p), X(F(p))))$.

Fix a basepoint $p(a^i_j)$ on every curve a^i_j.

We shall construct several 2-chains. When chains are presented as products, the orientation is the product orientation with respect to the order of the factors from left to right. Similarly, when chains are described with coordinates varying inside oriented manifolds, they are oriented as the image of the product of the oriented manifolds ordered by the order used to write the coordinates. The field $\mathbb{R}$ is given its standard orientation. A minus sign reverses the orientation.

Let a denote a curve $a^i_j = 0 \times a^i_j \times 0$ that lives inside the fixed neighborhood $([-4,4] \times (a \times [-1,1] \subset \partial A^i))$ in M. Fix $I \subset N$. We are about to explicitly construct a 2-dimensional cycle $F(a)$ in $C_2(M_I)$ of the form

$$F(a) = C(a) \cup e(S_0(a))(S^2_{\mathrm{diag}} = ST(M)_{|p(a)})$$
$$-(S_0(a) \times (4 \times p(a))) \cup -((4 \times p(a)) \times S_0(a)) \cup \mathrm{diag}(n)(S_0(a))$$

for a rational two-chain $S_0(a)$ in C^i_I $(= A^i$ or $B^i)$ whose boundary is $(0 \times a \times 0)$, equipped with a vector field n and with a rational number $e(S_0(a))$, and for a two-chain $C(a)$ in $C_2([0,4] \times \partial A^i)$ described below.

Let us first describe $S_0(a)$, n and $e(S_0(a))$.

There exists a minimal positive nonzero integer k such that $ka = 0$ in $H_1(C_I^i;\mathbb{Z})$. Then there exists a connected surface $S_{2,-4}$ embedded in $C^i_{I,-4}$ whose boundary is

$$\partial S_{2,-4} = (\{-4\} \times a \times \{0, \frac{2}{2k-1}, \frac{4}{2k-1}, \dots, \frac{2k-2}{2k-1}\})$$

and whose normal is $\tau_M^{-1}(.,e_1)$ on $\partial S_{2,-4}$. It can be easily extended to an immersed connected surface S_2 with boundary k copies of $\{0\} \times a \times \{0\}$ such that $S_2 \cap ([-2,0] \times \partial A)$ is made of k copies of $[-2,0] \times a \times \{0\}$, and such that S_2 is obtained from $S_{2,-4}$ by gluing k embedded annuli transverse to the vector field $\tau_M^{-1}(.,e_1)$. Then $S_0(a) = \frac{1}{k}S_2$, n is the normal vector to S_2, (that is homotopic to $\tau_M^{-1}(.,e_1)$ on $S_2 \setminus S_{2,-4}$,) and

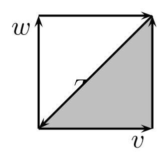
$$e(S_0(a)) = \frac{g(S_2)+k-1}{k} = \frac{1}{2} - \frac{\chi(S_2)}{2k}.$$

Let us now describe the wanted two-chain $C(a)$ in $C_2([0,4] \times \partial A^i)$ with boundary

$$\partial C(a) = (\{0\} \times a \times \{0\} \times (4 \times p(a)))$$

$$\cup ((4 \times p(a)) \times \{0\} \times a \times \{0\})) \cup -\mathrm{diag}(e_1)(\{0\} \times a \times \{0\}).$$

For two given based parametrized closed curves $\{x(v); v \in [0,1]/(0 \sim 1)\}$ and $\{y(v); v \in [0,1]/(0 \sim 1)\}$ with respective basepoints $x(0)$ and $y(0)$, we fix a cobordism $T(x,y)$ in the torus $x \times y$ between the diagonal $\{(x(v),y(v)); v \in [0,1]\}$ and $(x(1) \times y) \cup (x \times y(0))$. (The notation $(x(1) \times y)$ stands for $(\{x(1)\} \times y)$ for lightness). The fixed cobordism $T(x,y)$ is the image of the triangle $\{(v,w) \in (0,1)^2; v \geq w\}$ by the map $((v,w) \mapsto (x(v),y(w))$ in the torus $x \times y$.



Let t and u be such that $0 \leq t < u < 1$. Let $s \in [-4,4]$. Define

$$A(t,u;s) = \overline{\{((s,x,t),(s,x,t+\lambda(u-t))); \lambda \in]0,1], x \in a\}} \subset C_2(M)$$

$$= \{((s,x,t),(s,x,t+\lambda(u-t))); \lambda \in]0,1], x \in a\} \cup \mathrm{diag}(e_1)(s \times a \times t).$$

Let $C(a)$ denote the sum of the following 2-chains in $C_2(M)$:

1. $T((0 \times a \times 0),(0 \times a \times 1))$
2. $A(0,1;0)$
3. $(0 \times a \times 0) \times [-([0,4] \times p(a) \times 1) \cup (4 \times p(a) \times [0,1])]$
4. $((4 \times p(a) \times 0) \times (0 \times a \times [0,1])) \cup (([0,4] \times p(a) \times 0) \times (0 \times a \times 1))$

Now, $F(a)$ is completely defined as a 2-dimensional cycle in $C_2(M_I)$, for all $I \subset N$.

We postpone the proof of the most difficult lemma to the end of this subsection.

Lemma 5.3 *With the above definition, $F(a) = F(a^i_j)$ is null-homologous in $C_2(M)$.*

Assuming this lemma, the proof of Lemma 5.2 goes as follows.

We first prove:

Lemma 5.4 *For any $i \in N$, the homology classes of the $F(a^i_j)$ for $j = 1, \dots g_i$ generate the kernel of*

$$H_2(D(\omega_0(M_i)) \longrightarrow H_2(C_2(M_i)).$$

PROOF: Since the inclusion from $D(\omega_0(M_i)$ to $(C_2(M_i)\setminus C_2(B^i_{-1}))\cup ST(B^i)$ is a homotopy equivalence, it is the same to prove that the $F(a^i_j)$ generate the kernel of

$$H_2((C_2(M_i) \setminus C_2(B^i_{-1})) \cup ST(B^i)) \longrightarrow H_2(C_2(M_i)).$$

It is also the same to prove that the $F(a^i_j)$ generate the kernel of

$$H_2((C_2(M) \setminus C_2(A^i_{-1})) \cup ST(A^i)) \longrightarrow H_2(C_2(M)).$$

Set $C = C_1(M)$ and $A = A^i$. Then $C_2(M) \setminus C_2(A)$ has the homotopy type of

$$\check{C}_2(C) \setminus \check{C}_2(A) \stackrel{\text{def}}{=} \left(C^2 \setminus A^2\right) \setminus \text{diag}(C \setminus A).$$

Let us compute the real homology of $C_2(M) \setminus C_2(A)$ in degrees 1 and 2. Recall that C has the homology of a point, that $H_2(C \setminus A) = \mathbb{R}[\partial C]$, and that $H_1(C \setminus A)$ is isomorphic to $\mathcal{L}_A$. Therefore, in the Mayer-Vietoris sequence

$$H_i((C \setminus A)^2) \xrightarrow{\alpha_i} H_i(C \times (C \setminus A)) \oplus H_i((C \setminus A) \times C)$$
$$\longrightarrow H_i\left((C \times (C \setminus A)) \cup_{(C\setminus A)^2} ((C \setminus A) \times C)\right) \xrightarrow{\partial_i} H_{i-1}((C \setminus A)^2),$$

the map α_i is onto for $i \geq 1$, and the map ∂_i is an injection into the kernel of α_{i-1} that is an injection, when $i = 1$ and 2. This shows that

$$H_i\left(C^2 \setminus A^2 = (C \times (C \setminus A)) \cup ((C \setminus A) \times C)\right) = \{0\} \text{ for } i \in \{1, 2\}.$$

Let us now compute the effect of removing $\text{diag}(C \setminus A)$, by using the long exact sequence:

$$\longrightarrow H_{i+1}(C^2 \setminus A^2, \check{C}_2(C) \setminus \check{C}_2(A)) \longrightarrow H_i(\check{C}_2(C) \setminus \check{C}_2(A)) \longrightarrow H_i(C^2 \setminus A^2) \longrightarrow ...$$

By excision, $H_{i+1}(C^2 \setminus A^2, \check{C}_2(C) \setminus \check{C}_2(A))$ is isomorphic to

$$H_{i+1}\left((C \setminus A) \times \mathbb{R}^3, (C \setminus A) \times (\mathbb{R}^3 \setminus \{0\})\right) \cong H_{i-2}(C \setminus A) \otimes H_2(S^2).$$

This shows that $H_1(C_2(M) \setminus C_2(A)) = H_1(\check{C}_2(C) \setminus \check{C}_2(A))$ is trivial, and that $H_2(C_2(M) \setminus C_2(A))$ is generated by the homology class of a fiber of the unit tangent bundle of $(C \setminus A)$. Since this fiber is not null homologous in $C_2(M)$, we conclude that $H_2(C_2(M) \setminus C_2(A)) = \mathbb{R}[S^2_{\text{diag}}]$.

We end the computation of $H_2((C_2(M) \setminus C_2(A)) \cup ST(A))$ by gluing $(ST(C) \cong C \times S^2)$ to $(C_2(M) \setminus C_2(A))$ along $(ST(C \setminus A) \cong (C \setminus A) \times S^2)$, and by using the Mayer-Vietoris sequence that yields the exact sequence

$$0 \longrightarrow \mathbb{R}[S^2_{\text{diag}}] \longrightarrow H_2((C_2(M) \setminus C_2(A_{-1})) \cup ST(A))$$
$$\xrightarrow{\partial_{MV}} H_1((C \setminus A_{-1}) \times S^2) \xrightarrow{\gamma_1} H_1(C \times S^2).$$

The kernel of γ_1 is freely generated by the curves $\text{diag}(e_1)(a_j)$ for $j = 1, \dots, g_i$, and the assumptions on the $F(a^i_j)$ ensure that ∂_{MV} maps $F(a^i_j)$ to $\pm\text{diag}(e_1)(a_j)$. Therefore,

$$H_2((C_2(M) \setminus C_2(A_{-1})) \cup ST(A)) = \oplus_{j=1}^{g_i}[F(a^i_j)] \oplus \mathbb{R}[S^2_{\text{diag}}].$$

and Lemma 5.3 ensure that the kernel of

$$H_2(C_2(M) \setminus C_2(A_{-1}) \cup ST(A)) \longrightarrow H_2(C_2(M))$$

is generated by the $[F(a^i_j)]$. $\diamond$

Thus, in order to prove Lemma 5.2, it is enough to prove that, for any $j = 1, \dots, g_i$,

$$\int_{F(a^i_j)} \omega_0(M_i) = 0.$$

Set $a = a^i_j$. Since $C(a)$ only depends on $\mathcal{L}_{A^i}$ and since $C(a)$ lives inside the part $C_2\left([0,4] \times \partial A^i\right)$ of $C_2(M)$ or $C_2(M_i)$ where ω_M and $\omega_0(M_i)$ coincide,

$$\int_{C(a)} \omega_M = \int_{C(a)} \omega_0(M_i).$$

The normalizations also imply that the integrals of the forms vanish on

$$-\left(S_0(a) \times (4 \times p(a))\right) \cup -\left((4 \times p(a)) \times S_0(a)\right).$$

Now, since $F(a)$ is null-homologous in $C_2(M)$, $\int_{F(a)} \omega_M = 0$, and we are left with the proof that

$$\int_{\mathrm{diag}(n)(S_0(a))} \omega_M + e(S_0(a)) = \int_{\mathrm{diag}(n)(S_{0,i}(a))} \omega_0(M_i) + e(S_{0,i}(a)).$$

$$\int_{\mathrm{diag}(n)(S_0(a))} \omega_M = \frac{1}{k}\int_{\mathrm{diag}(n)(S_2)} \omega_M = \frac{1}{2k}\chi(TS_2; \tau_M^{-1}(.,e_2)_{|\partial S_2})$$

according to Lemma 4.5. Then Lemma 4.4 implies that

$$\int_{\mathrm{diag}(n)(S_0(a))} \omega_M = \frac{kd + \chi(S_2)}{2k}$$

where d is the degree of the map $\tau_M^{-1}(.,e_2)$ from a to $S(\mathbb{R}\vec{N}(a) \oplus \mathbb{R}\vec{T}(a))$. Thus,

$$\int_{\mathrm{diag}(n)(S_0(a))} \omega_M + e(S_0(a)) = \frac{d+1}{2}.$$

Since similar equalities hold when $(M_i, \omega_0(M_i), S_{0,i}(a))$ replaces $(M, \omega_M, S_0(a))$ according to Lemma 4.12,

$$\int_{\mathrm{diag}(n)(S_{0,i}(a))} \omega_0(M_i) + e(S_{0,i}(a)) = \frac{d+1}{2},$$

and this concludes the proof of Lemma 5.2 up to Lemma 5.3. $\diamond$

Proof of Lemma 5.3 We shall first replace $F(a)$ by an integral cycle.

Let $S = S_{2,-4} \cup ([-4,0] \times a \times \{0, \frac{2}{2k-1}, \frac{4}{2k-1}, \dots, \frac{2k-2}{2k-1}\})$ be the connected surface embedded in A whose boundary is

$$\partial S = \{0\} \times a \times \{0, \frac{2}{2k-1}, \frac{4}{2k-1}, \dots, \frac{2k-2}{2k-1}\},$$

such that

$$S \cap ([-4,0] \times \partial A) = [-4,0] \times a \times \{0, \frac{2}{2k-1}, \frac{4}{2k-1}, \dots, \frac{2k-2}{2k-1}\}.$$

Choose a tubular neighborhood $S \times [-\frac{1}{2k}, \frac{1}{2k-1}]$ of S such that, when $v = (t, (x, \frac{2(\ell-1)}{2k-1}))$ belongs to the part $([-4,0] \times a \times \{0, \frac{2}{2k-1}, \frac{4}{2k-1}, \dots, \frac{2k-2}{2k-1}\})$ of S, the element (v,u) of this neighborhood reads $(t, x, \frac{2(\ell-1)}{2k-1} + u)$ in $[-4,0] \times (a \times [-1,1])$.

We use S to find an integral cycle that is homologous to $kF(a)$.

Define two basepoints on $(a \times [0,1] \subset \partial A \subset M)$, one left one and one right one:

$$p_\ell = (0, p(a), 0) \text{ and } p_r = (0, p(a), 1)$$

Sublemma 5.5 *$kF(a)$ is homologous to the integral cycle $G(a)$, where $G(a)$ is the sum of the seven following 2-chains.*

1. $\cup_{\ell=1}^{k} T((0\times a\times\{\frac{2(\ell-1)}{2k-1}\})\times(0\times a\times\{\frac{2\ell-1}{2k-1}\}))$
2. $A(\partial S)=\cup_{\ell=1}^{k}A(\frac{2(\ell-1)}{2k-1},\frac{2\ell-1}{2k-1};0)$
3. $A_r(0)=-\cup_{\ell=1}^{k}(0\times p(a)\times[0,\frac{2(\ell-1)}{2k-1}])\times(0\times a\times\{\frac{2\ell-1}{2k-1}\})$
4. $A_\ell(0)=-\cup_{\ell=1}^{k}(0\times a\times\{\frac{2(\ell-1)}{2k-1}\})\times(0\times p(a)\times[\frac{2\ell-1}{2k-1},1])$
5. $-S\times p_r$
6. $-p_\ell\times(S\times\{\frac{1}{2k-1}\})$
7. $diag(n)(S)+(g(S)+k-1)S^2_{diag}$

Proof: The cycle $\left(k\mathrm{diag}(n)S_0(a)-\left(\mathrm{diag}(n)(S)\cup_{\ell=1}^{k}\mathrm{diag}(e_1)(a\times[0,\frac{2(\ell-1)}{2k-1}])\right)\right)$ is homologous to

$$\mathrm{diag}(e_1)\left((S_2\setminus S_{2,-4})-\left(\cup_{\ell=1}^{k}([-4,0]\times a\times\{\frac{2(\ell-1)}{2k-1}\}\cup a\times[0,\frac{2(\ell-1)}{2k-1}])\right)\right)$$

and is therefore null-homologous because $H_2(A;\mathbb{Q})=0$.

Therefore, in the definition of $kF(a)$, we may change $k\mathrm{diag}(n)(S_0(a))$ into

$$\mathrm{diag}(n)(S)\cup\cup_{\ell=1}^{k}\mathrm{diag}(e_1)\left(a\times[0,\frac{2(\ell-1)}{2k-1}]\right)$$

and stay in the same homology class.

We may also change

$$-k\,(S_0(a)\times(4\times p(a)))\cup$$
$$\cup k(0\times a\times 0)\times(-([0,4]\times p(a)\times 1)\cup(4\times p(a)\times[0,1]))$$

into

$$-\left(S\cup_{\ell=1}^{k}(a\times[0,\frac{2(\ell-1)}{2k-1}])\right)\times p_r.$$

Indeed these chains have the same boundary and they live inside

$$(A\setminus(0\times a\times 1))\times((-[0,4]\times p(a)\times 1)\cup(4\times p(a)\times[0,1]))$$

that has the homotopy type of A, and that therefore has a trivial H_2.

Similarly, we may change

$$-k\,((4\times p(a))\times S_0(a))\cup$$
$$\cup k\,((4\times p(a)\times 0)\times(0\times a\times[0,1]))\cup k\,(([0,4]\times p(a)\times 0)\times(0\times a\times 1))$$

into

$$-p_\ell\times\left((S\times\frac{1}{2k-1})\cup(0\times -a\times(\cup_{\ell=1}^{k}([\frac{2\ell-1}{2k-1},1]))\right)$$

inside

$$(([0,4]\times p(a)\times[0,1])\times A)\setminus(\mathrm{diag}(0\times p(a)\times[0,1]))$$

that has the homotopy type of A.

Afterwards, it is enough to check that, for any $\ell = 1, \dots k$,

$$T((0 \times a \times 0) \times (0 \times a \times 1)) \cup A(0,1;0)$$

may be replaced by the chain $C(a)(\frac{2(\ell-1)}{2k-1}, \frac{2\ell-1}{2k-1})$ with boundary

$$p_\ell \times (0 \times a \times 1) \cup (0 \times a \times 0) \times p_r \cup -\mathrm{diag}(e_1)(0 \times a \times 0)$$

where, for any $(t,u) \in [0,1]^2$ such that $0 \le t < u \le 1$, $C(a)(t,u)$ is the sum of the following chains:

1. $T((0 \times a \times \{t\}), (0 \times a \times \{u\}))$
2. $A(t,u;0)$
3. $-(0 \times p(a) \times [0,t]) \times (0 \times a \times \{u\})$
4. $-(0 \times a \times \{t\}) \times (0 \times p(a) \times [u,1])$
5. $-\mathrm{diag}(e_1)\,(a \times [0,t])$
6. $(a \times [0,t]) \times p_r$
7. $-p_\ell \times (0 \times a \times [u,1])$.

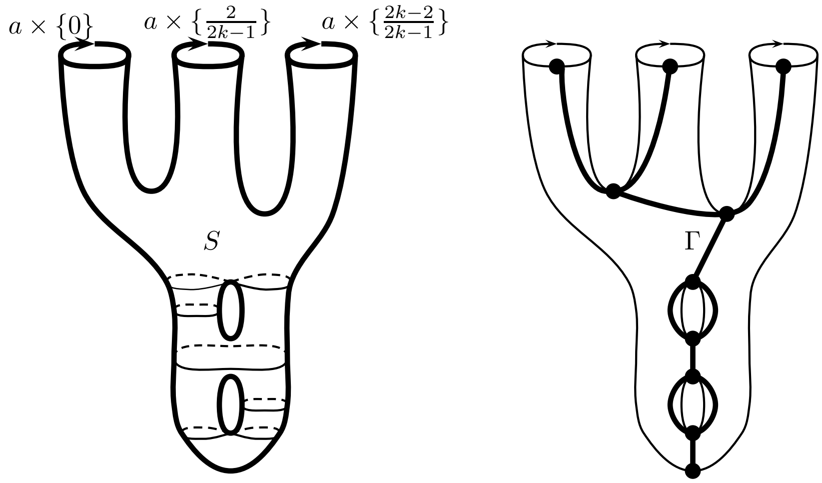
Since $C(a)(0,1)$ is homologous to $T((0 \times a \times 0) \times (0 \times a \times 1)) \cup A(0,1;0)$ and since the $C(a)(t,u)$ form a continuous family of chains with the same boundary indexed by a connected set, we are done.

$\diamond$

Then to prove Lemma 5.3, it is sufficient (and necessary) to prove that $G(a)$ represents 0 in $(H_2(C_2(M);\mathbb{R}) = \mathbb{R}[S^2_{\mathrm{diag}}])$. To do that, we shall describe some homotopies explicitly, and we need some more notation. All our homotopies will take place inside $C_2\left((S \times [0, \frac{1}{2k-1}]) \cup (\{0\} \times a \times [0,1])\right)$. We shall use the implicit coordinates there.

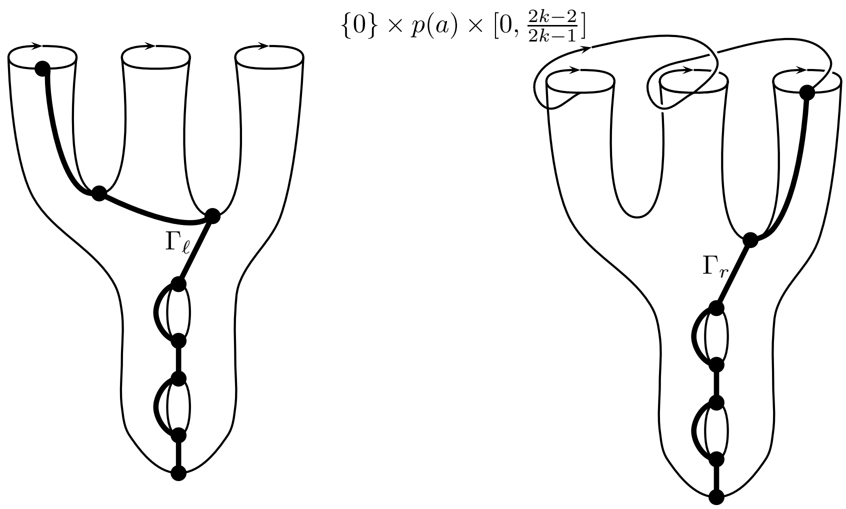
Let $g(S)$ be the genus of S. We define a Morse function h_S from S to $[-6-3g(S),0]$ that is the height function of S with respect to an embedding such as in the following picture of S, and such that h_S coincides with the projection on $[-4,0]$ on $S \cap ([-4,0] \times \partial A)$. In particular, h_S is maximal and constant on the boundary of S, h_S has a unique minimum at the height $(-6-3g(S))$, h_S has $(2g(S)+(k-1))$ index one critical points, and h_S has no other critical points. Furthermore, $h_S^{-1}(-6-3j)$ is a circle for $j = 0, \dots, g(S)-1$ and there are two critical points between $h_S^{-1}(-6-3j)$ and $h_S^{-1}(-6-3(j+1))$ for $j = 0, \dots, g(S)-1$.

We construct a connected (compact) graph Γ on S that intersects every connected component of every height level of S exactly once, and that intersects $S \cap ([-4,0] \times \partial A)$ as $\cup_{i=0}^{k-1}\left([-4,0] \times p(a) \times \{\frac{2i}{2k-1}\}\right)$.



We also extract two connected subgraphs Γ_ℓ and Γ_r of Γ that intersect every height level of S exactly once, and such that:

- $\Gamma_\ell \cap h_S^{-1}([-6-3g(S),-6]) = \Gamma_r \cap h_S^{-1}([-6-3g(S),-6])$,
- $\Gamma_\ell \cap ([-4,0] \times \partial A) = [-4,0] \times p(a) \times \{0\}$, and,
- $\Gamma_r \cap ([-4,0] \times \partial A) = [-4,0] \times p(a) \times \{\frac{2k-2}{2k-1}\}$.



Then, define three associated projections p, p_ℓ and p_r from S to Γ such that for any element s of S,

- $p(s)$ is the intersection point of Γ and the connected component of s in $h_S^{-1}(h_S(s))$,
- $\{p_\ell(s)\} = \Gamma_\ell \cap h_S^{-1}(h_S(s))$,
- $\{p_r(s)\} = \Gamma_r \cap h_S^{-1}(h_S(s))$.

The maps p_ℓ and p_r factor through h_S, and the quotient maps will still be denoted by p_ℓ and p_r.

Our Morse function h_S is such that Γ_ℓ contains all the critical points of h_S, and such that the shortest path in Γ from $p_\ell(-6)$ to $0 \times p(a) \times \{\frac{2i}{2k-1}\}$

- is $p_\ell[-6,0]$, if $i = 0$,

- is $p_r[-6,0]$, if $i = k-1$,
- contains exactly $k-i$ critical points for $0 < i < k$.

For any $p \in \Gamma \setminus \Gamma_\ell$, define the path $[p, p_\ell(h_S(p))]$ as the injective path from p to $p_\ell(h_S(p))$ in

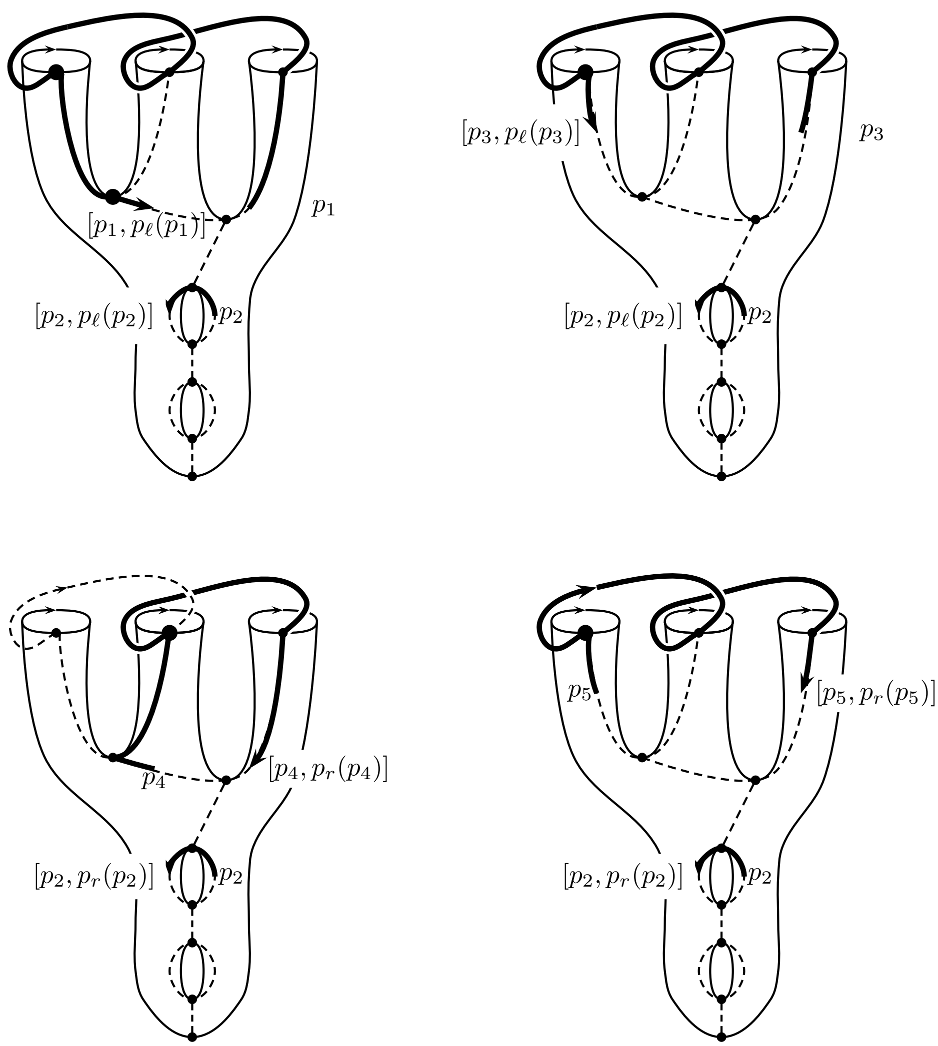
$$\left(\Gamma \cap h_S^{-1}([h_S(p), 0])\right) \cup (\{0\} \times p(a) \times [0, \frac{2k-2}{2k-1}])$$

that is in Γ_ℓ on its way down (that is when h_S is decreasing). Define $[p, p_\ell(h_S(p))]$ as the constant path for $p \in \Gamma_\ell$.

For any $p \in \Gamma \setminus \Gamma_r$, define the path $[p, p_r(h_S(p))]$ as the injective path from p to $p_r(h_S(p))$ in

$$\left(\Gamma \cap h_S^{-1}([h_S(p), 0])\right) \cup (\{0\} \times p(a) \times [0, \frac{2k-2}{2k-1}])$$

whose image intersects $(\{0\} \times p(a) \times [0, \frac{2k-2}{2k-1}])$ as little as possible, and define $[p, p_r(h_S(p))]$ as the constant path for $p \in \Gamma_r$.



For $t \in [-6-3g(S), 0]$, set

$$S_t = h_S^{-1}([-6-3g(S), t]) \subseteq S.$$

$$S_\ell(t) = \{(x, 0, p(x), \frac{1}{2k-1}); x \in S_t\},$$

$$A_\ell(t) = -\cup_{c \text{ component of } \partial S_t}\, c \times [p(c), p_r(t)] \times \{\frac{1}{2k-1}\},$$

and define the cycle

$$C_\ell(t) = -\left(S_t \times \{p_r(t)\} \times \{\frac{1}{2k-1}\}\right) \cup S_\ell(t) \cup A_\ell(t).$$

In a symmetric way, set

$$S_r(t) = \{(p(x), 0, x \times \{\frac{1}{2k-1}\}); x \in S_t\},$$

$$A_r(t) = \cup_{c \text{ component of } \partial S_t} [p(c), p_\ell(t)] \times c \times \{\frac{1}{2k-1}\},$$

and define the cycle

$$C_r(t) = -\left(p_\ell(t) \times S_t \times \{\frac{1}{2k-1}\}\right) \cup S_r(t) \cup A_r(t).$$

Let n denote the unit normal positive vector field of S. Set

$$C(0) = -S_\ell(0) \cup -S_r(0) \cup \mathrm{diag}(n)(S) \cup A(\partial S)$$

$$\cup \cup_{c \text{ component of } \partial S}\, T(c, c \times \frac{1}{2k-1}).$$

Note that the homology class of $G(a)$ that has been defined in Sublemma 5.5 is

$$[G(a)] = [C(0)] + (g(S) + k - 1)[S^2_{\mathrm{diag}}] + [C_\ell(0)] + [C_r(0)].$$

Therefore, the proof of Lemma 5.3 reduces to the proof of the two following lemmas.

Lemma 5.6 $[C(0)] = 0$ *in* $H_2(C_2(M))$.

Lemma 5.7 $[C_\ell(0)] + [C_r(0)] = -(g(S) + k - 1)[S^2_{diag}]$.

Proof of Lemma 5.6: Set

$$\mathrm{diag}(\frac{1}{2k-1})(S_t) = \{(x, 0, x, \frac{1}{2k-1}); x \in S_t\}$$

and

$$\tilde{E}(t) = \cup_{c \text{ component of } \partial S_t} T(c, c \times \frac{1}{2k-1}).$$

Then $C(0)$ is homologous to $\tilde{C}(0)$ with

$$\tilde{C}(t) = -S_\ell(t) \cup -S_r(t) \cup \mathrm{diag}(\frac{1}{2k-1})(S_t) \cup \tilde{E}(t).$$

We shall now replace $\tilde{E}(t)$ by a chain $E(t)$ such that

1. $\partial E(t) = \partial \tilde{E}(t) = \partial S_\ell(t) \cup \partial S_r(t) \cup -\partial \mathrm{diag}(\frac{1}{2k-1})(S_t)$

2. $(E(0) - \tilde{E}(0))$ is null-homologous in $C_2(M)$,

3. $E(t)$ is defined continuously enough so that if

$$D(t) = -S_\ell(t) \cup -S_r(t) \cup \mathrm{diag}(\frac{1}{2k-1})(S_t) \cup E(t),$$

then $\cup_{t\in[-6-3g,0]} D(t)$ may be considered as a cobordism from $D(0)$ to 0, and used to show that $D(0)$ is null-homologous.

Since the first two conditions imply that $D(0)$ is homologous to $C(0)$, this will be enough to finish proving the lemma.

In order to define $E(t)$, we use our graph Γ on S. This graph equips each component c of ∂S_t with the basepoint $p(c)$. We furthermore equip the set of connected components of ∂S_t with the total order from left to right in the picture.

Then we set

$$E(t) = \tilde{E}(t) \cup \bigcup_{\{(c,c');c,c' \text{ components of } \partial S_t; c'<c\}} c \times \{0\} \times c' \times \{\frac{1}{2k-1}\}$$

Since the linking number of a and a curve parallel to a on ∂A is zero, the homology classes of the tori $a \times \{\frac{2i}{2k-1}\} \times a \times \{\frac{2j+1}{2k-1}\}$ are null in $C_2(M)$. Therefore, $(E(0) - \tilde{E}(0))$ is null-homologous in $C_2(M)$. It is also clear that adding tori has not changed the boundary of $\tilde{E}(t)$.

Since $D(-6-3g)$ is supported in a point, it is null-homologous. To conclude, we prove that for any subinterval $[h_1, h_2]$ of $[-6-3g, 0]$, $D(h_1)$ is homologous to $D(h_2)$. This is clear when $[h_1, h_2]$ does not contain any critical value of h_S. Let us prove that this is still true when $[h_1, h_2]$ contains exactly one critical value h_c in its interior.

There are two cases. Either the number of components of $h_S^{-1}(]h_c, h_2])$ is greater than the number of components of $h_S^{-1}([h_1, h_c[)$, or it is smaller. In the first case, the corresponding index one critical point will be called a *positive saddle point*, in the second case, the critical point will be called a *negative saddle point*. Let us treat the case of a positive saddle point.

Let r be the number of components of $D(h_2)$, we are going to use a continuous map

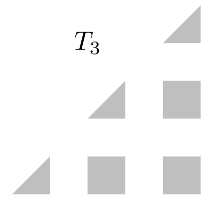
$$f : \sqcup_{i=0}^{r-1} [2i, 2i+1] \times [h_1, h_2] \longrightarrow S,$$

to parametrise $h_S^{-1}([h_1, h_2])$. Our parametrisation f has the following properties:

1. $h_S \circ f(x,t) = t$ on $\sqcup_{i=0}^{r-1} [2i, 2i+1] \times [h_1, h_2]$,
2. $f(.,t)$ provides a homeomorphism from $\frac{[2i,2i+1]}{2i\sim 2i+1}$ to the $(i+1)^{\text{th}}$ component from left to right of ∂S_t, for any $(i,t) \in \{0, 1, \dots, r-1\} \times]h_c, h_2]$,
3. $f(2i,t) = f(2i+1,t) \in \Gamma$ for any $(i,t) \in \{2, 3, \dots, r-1\} \times [h_1, h_2]$, and for any $(i,t) \in \{0, 1, \dots, r-1\} \times [h_c, h_2]$,
4. $f(0,t) = f(3,t) \in \Gamma$ and $f(1,t) = f(2,t)$ for any $t \in [h_1, h_c]$,
5. $f(.,t)$ provides a homeomorphism from the circle $\frac{[0,1]\coprod[2,3]}{0\sim 3, 1\sim 2}$ to the first component of ∂S_t for any $t \in [h_1, h_c[$, $f(.,t)$ provides a homeomorphism from $\frac{[2i,2i+1]}{2i\sim 2i+1}$ to the i^{th} component from left to right of ∂S_t for any $t \in [h_1, h_c]$, for any $i \in \{2, 3, \dots, r-1\}$.

Let T_r be the following part of $\mathbb{R}^2$:

$$T_r = \{(v,w) \in (\sqcup_{i=0}^{r-1} [2i, 2i+1])^2; v \geq w\}.$$

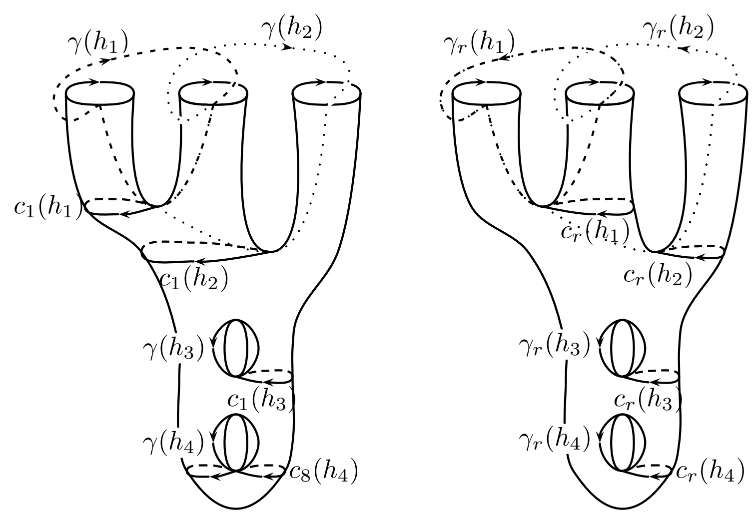


Then $\cup_{t\in[h_1,h_2]}E(t)$ is the image of the continuous map

$$\begin{array}{rccc} F: & T_r \times [h_1,h_2] & \longrightarrow & S\times\{0\}\times S\times\{\frac{1}{2k-1}\} \\ & (v,w,t) & \mapsto & (f(v,t),0,f(w,t),\frac{1}{2k-1}) \end{array}$$

that may be extended to provide a cobordism between $D(h_1)$ and $D(h_2)$. The case of a negative saddle point is symmetric. $\diamond$

Proof of Lemma 5.7: The graph Γ is as in the figure. We observe that $\cup_{t\in[-6-3g(S),0]}C_\ell(t)$ may not be considered as a cobordism between $C_\ell(0)$ and $C_\ell(-6-3g(S))$ (that is a point) because the chains $A_\ell(t)$ are not continuously defined with respect to t. More precisely, the jumps occur exactly at the heights of the positive saddle points, because the paths $[p(c),p_r(t)]$ are not continuously defined near a positive saddle point. Let h_1, h_2, $\dots$, h_{g+k-1} be the heights of the critical values ordered from top to bottom (in decreasing order) corresponding to the positive saddle points $p_\ell(h_1)$, $p_\ell(h_2)$, $\dots$, $p_\ell(h_{g+k-1})$. The figure 8 horizontal curve $c_8(h_i)$ that contains $p_\ell(h_i)$ is the union of two topological circles $c_1(h_i)$ and $c_2(h_i)$. Let $c_1(h_i)$ be the part of $c_8(h_i)$ where $[p(c),p_r(t)]$ changes. When approaching $c_1(h_i)$ from above h_i, this path approaches some path $[p^+(c_1(h_i)),p_r(h_i)]$ homotopic to the path composition of a loop $\gamma(h_i)$ based at $p_\ell(h_i)$ and the path $[p_\ell(h_i),p_r(h_i)]$. The loops $c_1(h_i)$ and $\gamma(h_i)$ are shown in the following picture.



Define $C^+_\ell(h_i)$ from $C_\ell(h_i)$ by replacing $c_8(h_i)\times[p(c_8(h_i)),p_r(h_i)]\times\{\frac{1}{2k-1}\}$ by the union

$$\left(c_1(h_i)\times[p^+(c_1(h_i)),p_r(h_i)]\times\{\frac{1}{2k-1}\}\right)\cup$$

$$\cup\left(c_2(h_i)\times[p(c_8(h_i)),p_r(h_i)]\times\{\frac{1}{2k-1}\}\right)$$

Set $h_0=0$, then for any $i=1,\dots,g+k-1$, $C^+_\ell(h_i)$ is homologous to $C_\ell(h_{i-1})$. Furthermore,

$$[C^+_\ell(h_i)]=[C_\ell(h_i)]-[c_1(h_i)\times\left(\gamma(h_i)\times\{\frac{1}{2k-1}\}\right)].$$

This shows that

$$[C_\ell(0)] = -\sum_{i=1}^{g(S)+k-1} [c_1(h_i) \times \left(\gamma(h_i) \times \{\frac{1}{2k-1}\}\right)].$$

Similarly, define $c_r(h_i)$ as the part of $c_8(h_i)$ where $[p(c), p_\ell(t)]$ changes. This part $c_r(h_i)$ is the right-hand side of $c_8(h_i)$ in the figure, it coincides with $c_1(h_i)$ when $i \geq k$. Define $\gamma_r(h_i)$ as the loop based at $p_\ell(h_i)$ that is the limit of $[p(c), p_\ell(t)]$ when c approaches c_r from above. Again, $\gamma_r(h_i) = \gamma(h_i)$ when $i \geq k$. Then, as above, we get that

$$[C_r(0)] = \sum_{i=1}^{g+k-1} [\gamma_r(h_i) \times c_r(h_i) \times \{\frac{1}{2k-1}\}].$$

Thus, in $C_2(M)$, we have

$$[C_\ell(0)] + [C_r(0)] =$$

$$\sum_{i=1}^{g+k-1} \left(\ell(\gamma_r(h_i), c_r(h_i) \times \{\frac{1}{2k-1}\}) - \ell(c_1(h_i), \gamma(h_i) \times \{\frac{1}{2k-1}\})\right) [S^2_{\mathrm{diag}}].$$

When $i \geq k$,

$$\ell(\gamma_r(h_i), c_r(h_i) \times \{\frac{1}{2k-1}\}) - \ell(c_1(h_i), \gamma(h_i) \times \{\frac{1}{2k-1}\})$$

$$= \ell(\gamma_r(h_i), c_r(h_i) \times \{\frac{1}{2k-1}\}) - \ell(\gamma_r(h_i), c_r(h_i) \times \{-\frac{1}{2k}\})$$

$$= -\langle \gamma_r(h_i), c_r(h_i) \rangle_S = -1.$$

When $i < k$,

$$\ell(\gamma_r(h_i), c_r(h_i) \times \{\frac{1}{2k-1}\})$$

$$= \ell(c_r(h_i), \gamma_r(h_i) \times \{-\frac{1}{2k}\})$$

where $c_r(h_i)$ is homologous to $a \times \{\frac{2i}{2k-1}\}$ inside a subsurface of S that does not intersect $(\gamma_r(h_i) \times \{-\frac{1}{2k}\})$. Therefore, this expression equals

$$\ell(1 \times a, \gamma_r(h_i) \times \{-\frac{1}{2k}\})$$

$$= \frac{1}{k}\ell(1 \times \partial S, \gamma_r(h_i) \times \{-\frac{1}{2k}\})$$

$$= \frac{1}{k}\langle S \cup ([0,1] \times \partial S), \gamma_r(h_i) \times \{-\frac{1}{2k}\}\rangle_M$$

$$= \frac{1}{k}\langle \partial S, \gamma_r(h_i) \times \{-\frac{1}{2k}\}\rangle_{\partial A} = -\frac{i}{k}$$

since $\gamma_r(h_i) \times \{-\frac{1}{2k}\}$ meets S only along its boundary.

When $i < k$, $c_1(h_i)$ is homologous to $\sqcup_{j=0}^{i-1} a \times \{\frac{2j}{2k-1}\}$ inside a subsurface of S that does not intersect $(\gamma(h_i) \times \{\frac{1}{2k-1}\})$. Therefore,

$$\ell(c_1(h_i), \gamma(h_i) \times \{\frac{1}{2k-1}\})$$

$$= i\ell(1 \times a, \gamma(h_i) \times \{\frac{1}{2k-1}\})$$

$$= \frac{i}{k}\langle S \cup ([0,1] \times \partial S), \gamma(h_i) \times \{\frac{1}{2k-1}\}\rangle_M$$

$$= \frac{i}{k}\langle \partial S, \gamma(h_i) \times \{\frac{1}{2k-1}\}\rangle_{\partial A} = \frac{i}{k}.$$

Therefore,

$$\sum_{i=1}^{k-1}\left(\ell(\gamma_r(h_i), c_r(h_i) \times \{\frac{1}{2k-1}\}) - \ell(c_1(h_i), \gamma(h_i) \times \{\frac{1}{2k-1}\})\right)$$

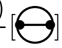
$$= -2\sum_{i=1}^{k-1}\frac{i}{k} = 1 - k.$$

$$[C_\ell(0)] + [C_r(0)] = -(g(S) + k - 1)[S^2_{\mathrm{diag}}].$$

$\diamond$

6 Comparison with the Walker invariant

This section is devoted to proving Theorem 2.6 as an application of the splitting formula for Z_1. Recall that Kevin Walker has extended the Casson invariant to rational homology spheres in [W]. Let λ_W denote his invariant normalized as in [W]. Here, we use the normalisation $\lambda = \frac{\lambda_W}{2}$ so that λ coincides with the Casson invariant normalised as in [AM, GM] for integral homology spheres. Thus, this section is devoted to proving that for any rational homology sphere M,

$$Z_1(M) = \frac{\lambda(M)}{2}[\quad].$$


6.1 Sketch of the proof of Theorem 2.6

Since [$\theta =$ ] freely generates $\mathcal{A}_1(\emptyset)$, there exists a rational invariant ν of rational homology spheres such that $Z_1 = \frac{\nu}{2}[\theta]$.

Recall from Proposition 1.10 in [L2] and Theorem 2.4 that ν satisfies:

- For any rational homology sphere, $\nu(-M) = -\nu(M)$.
- For any rational generalised 2-clover, $\nu(D) = \frac{\ell(D;\theta)}{6}$.

Then Theorem 2.6 is the direct consequence of the two following propositions.

Proposition 6.1 • *For any rational homology sphere, $\lambda(-M) = -\lambda(M)$.*

- *For any rational generalised 2-clover, $\lambda(D) = \frac{\ell(D;\theta)}{6}$.*

Proposition 6.2 *Let κ be a rational-valued invariant of rational homology spheres that satisfies the two following properties:*

- *For any rational homology sphere, $\kappa(-M) = -\kappa(M)$.*
- *For any rational generalised 2-clover, $\kappa(D) = 0$.*

Then κ vanishes identically.

Remark 6.3 For integral homology spheres, Proposition 6.2 is a consequence of the computation of the Goussarov-Habiro filtration of the space of integral homology spheres, that implies that the only rational invariants of integral homology spheres that vanish on integral generalised degree 2 clovers are constant, and therefore vanish provided that they vanish at the sphere S^3. For rational homology spheres, the invariant $\mathrm{Log}(|H_1|)$ is a non-constant invariant that vanishes at S^3 and on rational generalised degree two clovers, and we need the assumption on the behaviour under orientation reversing, and a proof that is given in Subsection 6.4.

Proof of Proposition 6.1. The first property of λ is well-known [W, Lemma 3.1]. Let $D = (M; 2; (A, B_0); (A', B'_0))$ be a generalised clover. Let $B = M \setminus \mathrm{Int}(A)$. Let B' be obtained from B by replacing B_0 by B'_0. Let $([0,1] \times \partial A)$ denote a collar of ∂A, disjoint from B_0, in B. Let $B_1 = B \setminus (\partial A \times [0,1[)$ and $B'_1 = B' \setminus (\partial A \times [0,1[)$. Then $D_1 = (M; 2; (A, B_1); (A', B'_1))$ is a rational generalised clover such that

$$\lambda(D) = \lambda(A \cup B) - \lambda(A' \cup B) - \lambda(A \cup B') + \lambda(A' \cup B') = \lambda(D_1)$$

is computed in [L1, Theorem 1.3].

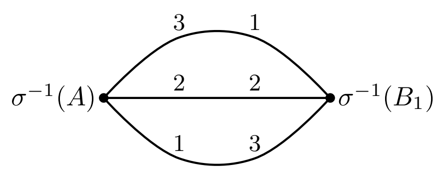
Similarly, $\nu(D_1) = \nu(D)$, and therefore $\ell(D_1;\theta) = \ell(D;\theta)$. Thus, it is enough to prove that $\lambda(D) = \ell(D_1;\theta)/6$. Let us compute $\ell(D_1;\theta)$. Let $(\alpha_1,\dots,\alpha_g)$ and $(\beta_1,\dots,\beta_g)$ be two bases for $\mathcal{L}_A$ and $\mathcal{L}_B$, respectively, such that $<\alpha_i,\beta_j>_{\partial A}$ is equal to the Kronecker symbol δ_{ij} for any i,j in $\{1,\dots,g\}$. Then $\mathcal{I}(A,A')$ reads

$$\sum_{f:\{1,2,3\}\to\{1,2,\dots,g\}} \mathcal{I}(A,A')(\alpha_{f(1)}\wedge\alpha_{f(2)}\wedge\alpha_{f(3)})\beta^*_{f(1)}\otimes\beta^*_{f(2)}\otimes\beta^*_{f(3)}$$

where $\beta^*_{f(i)} =<.,\beta_{f(i)}>_{\partial A}$, and $\mathcal{I}(B_1,B_1') = \mathcal{I}(B,B')$ reads

$$\sum_{h:\{1,2,3\}\to\{1,2,\dots,g\}} \mathcal{I}(B,B')(\beta_{h(1)}\wedge\beta_{h(2)}\wedge\beta_{h(3)})\alpha^*_{h(1)}\otimes\alpha^*_{h(2)}\otimes\alpha^*_{h(3)}$$

where $\alpha^*_{h(i)} =<.,\alpha_{h(i)}>_{\partial B_1}=<\alpha_{h(i)},.>_{\partial A}$. Let $\sigma : V(\theta)\to\{A,B_1\}$ be a bijection.



Then $\ell(D_1;\theta;\sigma)$ is the contraction of the tensor $\mathcal{I}(A,A')\otimes\mathcal{I}(B,B')$ with respect to the linking number and the edge data. This contraction maps

$$<.,\beta_{f(i)}>_{\partial A}\otimes<.,\alpha_{h(4-i)}>_{\partial B_1} \quad\text{to}\quad \ell(\beta_{f(i)},\alpha_{h(4-i)}\times\{1\}) = \delta_{f(i)h(4-i)}.$$

Therefore, $\ell = \ell(D_1;\theta;\sigma)$ equals

$$\sum_{f:\{1,2,3\}\to\{1,2,\dots,g\}} \mathcal{I}(A,A')(\alpha_{f(1)}\wedge\alpha_{f(2)}\wedge\alpha_{f(3)})\mathcal{I}(B,B')(\beta_{f(3)}\wedge\beta_{f(2)}\wedge\beta_{f(1)})$$

$$= -6\sum_{\{i,j,k\}\subset\{1,2,\dots,g\}} \mathcal{I}(A,A')(\alpha_i\wedge\alpha_j\wedge\alpha_k)\mathcal{I}(B,B')(\beta_i\wedge\beta_j\wedge\beta_k).$$

Since $\ell(D_1;\theta) = 2\ell(D_1;\theta;\sigma)$, $\ell(D_1;\theta)/6$ is equal to the right-hand side of the formula for $\lambda(D)$ in [L1, Theorem 1.3]. $\diamond$

6.2 Partial review of clover theory

In order to prove Proposition 6.2, we need to recall a few facts about *clovers.*

Consider the planar unoriented surface $N(Y)$ of Figure 1 that is a two-dimensional neighborhood of the bold *Y-graph* whose *leaves* are the three bold circles.

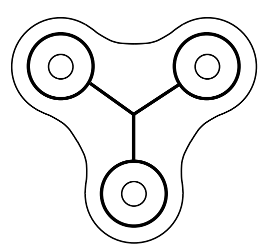


Figure 1: The planar unoriented surface $N(Y)$

A *degree k clover* is an embedding ϕ of a disjoint union of k copies $N(Y)^{(i)}$ of $N(Y)$, $i \in \{1, \dots, k\}$ in a rational homology sphere M. With such a data we associate the integral generalised clover

$$D = (M; k; (A^i)_{i\in\{1,\dots,k\}}, (B^i)_{i\in\{1,\dots,k\}})$$

where A^i is a regular neighborhood of $\phi(N(Y)^{(i)})$ and B^i is obtained from A^i by surgery on the six-component framed link in Figure 2 that inherits its framing from the embedding of $N(Y)$. The

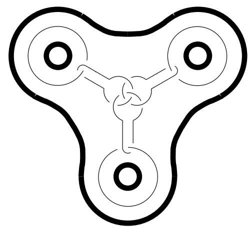


Figure 2: Surgery link associated to a framed Y-graph

surgery on D consists in replacing every A^i by the corresponding B^i. This surgery does not depend on the orientations of the $N(Y)^{(i)}$.

An orientation of the surface $N(Y)$ induces an orientation of the leaves and a cyclic order on them. When the pictured $N(Y)$ is given the orientation induced by the standard orientation of the plane, the induced orientation and order are the counterclockwise ones. Changing the orientation of $N(Y)$ changes both the orientation of the leaves and their cyclic order. A Y-graph equipped with such an orientation is said to be *oriented.*

A framed knot $J\sharp_b K$ is a *band sum* of two framed oriented knots J and K if there exists an embedding of a 2-hole disk that factors the three knot embeddings J, K and $J\sharp_b K$ by the embeddings of the three curves pictured in Figure 3 into the 2-hole disk, and that induces the three framings.

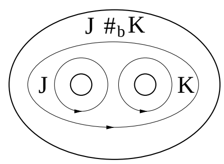


Figure 3: Band sum of two framed knots

We shall use the following lemma that allows one to cut a leaf of a clover.

Lemma 6.4 ([GGP, Theorem 3.1],[AL, Lemma 4.15]) *Let K_1, K_2, K_3 be three framed knots in an oriented 3-manifold M that are the leaves of an oriented framed Y-graph G in M. Assume that K_1 is a band sum of two framed knots K_1^2 and K_1^3. For $j = 2$ and 3, let K_j^2 and K_j^3 be two parallels of K_j equipped with the framing $\ell(K_j, K_j)$ of K_j, and such that $\ell(K_j^2, K_j^3) = \ell(K_j, K_j)$. Then there exist two oriented disjoint framed Y-graphs G^2 and G^3 in M whose framed leaves are K_1^2, K_2^2, K_3^2 and K_1^3, K_2^3, K_3^3, respectively such that the surgery along G is equivalent to the surgery along $G^2 \cup G^3$.*

Note that under the above assumptions, we have:

- for $i = 2$ or 3, and for $j, k \in \{2, 3\}$, $\ell(K_j^i, K_k^i) = \ell(K_j, K_k)$,

- for $j \in \{2,3\}$, $\ell(K_1, K_j) = \ell(K_1^2, K_j^2) + \ell(K_1^3, K_j^3)$,
- $\ell(K_1, K_1) = \ell(K_1^2, K_1^2) + \ell(K_1^3, K_1^3) + 2\ell(K_1^2, K_1^3)$.

6.3 Variation of κ under surgery along a Y-graph

In this subsection, we prove the following proposition.

Proposition 6.5 *A real-valued invariant κ of rational homology spheres that vanishes at rational generalised degree 2 clovers does not vary under a surgery along a Y-graph.*

Since $\kappa(D) = 0$ for any rational generalised 2-clover, the variation κ' of κ under surgery on an oriented framed Y-graph with leaves K_1, K_2, K_3 only depends on the lagrangian of the exterior of the Y-graph that is determined by the linking matrix $[\ell(K_i, K_j)]_{i,j\in\{1,2,3\}}$ of the framed leaves K_1, K_2 and K_3 of the framed Y-graph.

We are going to prove the following lemma:

Lemma 6.6 *Any symmetric matrix $[a_{ij}]_{i,j\in\{1,\dots,n\}}$ with rational coefficients is the linking matrix of some framed link in some rational homology sphere.*

In particular, any rational symmetric matrix $[a_{ij}]_{i,j\in\{1,2,3\}}$ is the linking matrix of some oriented Y-graph, and the variation of κ by surgery on that framed graph is denoted by $\kappa'([a_{ij}])$. We are going to study the properties of κ' to prove that it identically vanishes.

First note that by symmetry, for any cyclic permutation σ of $\{1,2,3\}$, $\kappa'([a_{ij}]) = \kappa'([a_{\sigma(i)\sigma(j)}])$.

As a consequence of Lemmas 6.4 and 6.6, for any rational numbers a_{**} and ℓ^{23} involved below, we have

$$\kappa'\begin{bmatrix} a_{11}^2 + a_{11}^3 + 2\ell^{23} & a_{12}^2 + a_{12}^3 & a_{13}^2 + a_{13}^3 \\ a_{12}^2 + a_{12}^3 & a_{22} & a_{23} \\ a_{13}^2 + a_{13}^3 & a_{23} & a_{33} \end{bmatrix} =$$

$$\kappa'\begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 \\ a_{12}^2 & a_{22} & a_{23} \\ a_{13}^2 & a_{23} & a_{33} \end{bmatrix} + \kappa'\begin{bmatrix} a_{11}^3 & a_{12}^3 & a_{13}^3 \\ a_{12}^3 & a_{22} & a_{23} \\ a_{13}^3 & a_{23} & a_{33} \end{bmatrix}.$$

Applying this when $a_{11}^3 = a_{12}^3 = a_{13}^3 = \ell^{23} = 0$ shows that, if $a_{11} = a_{12} = a_{13} = 0$, then $\kappa'([a_{ij}]) = 0$. Applying this property again when $a_{11}^3 = a_{12}^3 = a_{13}^3 = a_{11}^2 + 2\ell^{23} = 0$ shows that κ' does not depend on a_{11}. By symmetry, it does not depend on the other diagonal terms either. Therefore,

$$\kappa'([a_{ij}]) = \kappa'(a = a_{12}, b = a_{13}, c = a_{23})$$

where

$$\kappa'(a,b,c) = \kappa'(c,a,b),$$

$$\kappa'(0,0,c) = 0,$$

and

$$\kappa'(a + a', b + b', c) = \kappa'(a,b,c) + \kappa'(a',b',c).$$

In particular,

$$\begin{aligned} \kappa'(a,b,c) &= \kappa'(a,0,c) + \kappa'(0,b,c) = \kappa'(c,a,0) + \kappa'(b,c,0) \\ &= \kappa'(c,0,0) + \kappa'(0,a,0) + \kappa'(b,0,0) + \kappa'(0,c,0) = 0. \end{aligned}$$

This ends the proof of Proposition 6.5 assuming Lemma 6.6. $\diamond$

Proof of Lemma 6.6: Let $p > 0$ and q denote coprime integers. Recall that the H_1 of the lens space $L(p, -q)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, and that one of its generators $[\gamma_{p,q}]$ satisfies

$$\ell([\gamma_{p,q}], [\gamma_{p,q}]) = \frac{q}{p} \mod \mathbb{Z},$$

and therefore when $\gamma_{p,q}$ is a knot equipped with a suitable framing that represents $[\gamma_{p,q}]$,

$$\ell(\gamma_{p,q}, \gamma_{p,q}) = \frac{q}{p}.$$

Now, for $d > 0$, consider the following connected sum

$$M(d) = L(d, -1)\sharp L(d, 1)\sharp L(d, -1).$$

and three curves α, β, γ, one in each factor, such that

$$\ell(\alpha, \alpha) = -\ell(\beta, \beta) = \ell(\gamma, \gamma) = \frac{1}{d} \mod \mathbb{Z}.$$

Then, for any integer k we may choose curves $\delta(d, k), \varepsilon(d) \in M(d)$ that are homologous to $(k\alpha + k\beta)$ and to $(\gamma - \beta)$, respectively, such that

$$\ell(\delta(d, k), \delta(d, k)) = \ell(\varepsilon(d), \varepsilon(d)) = 0$$

and

$$\ell(\delta(d, k), \varepsilon(d)) = \frac{k}{d}.$$

Now, write the a_{ij} as irreducible fractions with positive denominators $a_{ii} = q_i/p_i$ and $a_{ij} = k_{ij}/d_{ij}$. Set

$$M = \sharp_{i\{1,2,\dots,n\}} L(p_i, -q_i)\sharp\sharp_{i,j\in\{1,2,\dots,n\},i<j} M(d_{ij}).$$

Choose knots K_i that are homologous to

$$\gamma_{p_i,q_i} + \sum_{j\in\{1,2,\dots,n\},i<j} \delta(d_{ij}, k_{ij}) + \sum_{j\in\{1,2,\dots,n\},j<i} \varepsilon(d_{ji}).$$

Then the linking matrix of $(K_1, \dots, K_n)$ is the wanted one modulo $\mathbb{Z}$. Next crossing changes and framing changes adjust it to the wanted one. $\diamond$

6.4 Proof of Proposition 6.2

Recall the following lemma.

Lemma 6.7 *Let N be a compact oriented 3-manifold with boundary. Then $H_2(N, \partial N; \mathbb{Z})$ is isomorphic to Hom$(H_1(N; \mathbb{Z}); \mathbb{Z})$ by the isomorphism that maps the homology class $[F]$ of a surface F with boundary in ∂N to the algebraic intersection with F.*

Proof: Use the Poincaré duality to identify $H_2(N, \partial N; \mathbb{Z})$ to $H^1(N; \mathbb{Z})$ and the universal coefficients theorem to identify $H^1(N; \mathbb{Z})$ to $\mathrm{Hom}(H_1(N; \mathbb{Z}); \mathbb{Z})$. $\diamond$

Lemma 6.7 yields the following characterisation of integral homology handlebodies.

Lemma 6.8 *Let H be a connected compact oriented 3-manifold with connected boundary. If the map from $H_1(\partial H; \mathbb{Z})$ to $H_1(H; \mathbb{Z})$ induced by the inclusion is a surjection, then H is an integral homology handlebody.*

PROOF: The long exact sequence associated to $(H, \partial H)$ yields the exact sequence with integral coefficients:

$$H_2(H) \longrightarrow H_2(H, \partial H) \longrightarrow H_1(\partial H) \longrightarrow H_1(H) \to 0$$

and shows that $H_1(H, \partial H; \mathbb{Z}) = 0$. Then its dual $H^1(H, \partial H; \mathbb{Z}) = H_2(H)$ is also trivial, and $H_2(H, \partial H) = \mathrm{Hom}(H_1(H; \mathbb{Z}); \mathbb{Z})$ is a free $\mathbb{Z}$-module whose rank is the genus of ∂H.

We are now left with the proof that $H_1(H; \mathbb{Z})$ has no torsion. If $H_1(H; \mathbb{Z})$ had torsion, there would exist a primitive element x of $H_1(\partial H; \mathbb{Z})$, and a $k > 0$ minimal such that $kx = 0$ in $H_1(H)$, and $k > 1$. Under these assumptions, kx would be in the image of the boundary map

$$H_2(H, \partial H) \longrightarrow H_1(\partial H)$$

and its preimage $[\Sigma(kx)]$ would be primitive. Therefore Lemma 6.7 implies that there would exist y in $H_1(H)$ such that $\langle [\Sigma(kx)], y \rangle_H = 1$. Now, this y could be thought of as an element of $H_1(\partial H; \mathbb{Z})$, and $\langle [\Sigma(kx)], y \rangle_H = \langle kx, y \rangle_{\partial H} = k \langle x, y \rangle_{\partial H} = 1$. Then k could not be larger than 1. $\diamond$

Recall that the *linking form* $\ell_{\mathbb{Q}/\mathbb{Z}}(M)$ of a rational homology sphere M is the bilinear form

$$\ell_{\mathbb{Q}/\mathbb{Z}}(M) : H_1(M)^2 \to \frac{\mathbb{Q}}{\mathbb{Z}}$$

that maps a pair of homology classes to the linking number of two of their representatives.

The form $\ell_{\mathbb{Q}/\mathbb{Z}}(M)$ is *non-degenerate* in the following sense. For any element x of $H_1(M)$ of order $k > 1$, there exists y such that $\ell_{\mathbb{Q}/\mathbb{Z}}(x, y) = \frac{1}{k}$. This can be seen by applying Lemma 6.7 to the exterior of a knot which represents x.

Proposition 6.9 (Massuyeau [Mas]) *A real-valued invariant κ of rational homology spheres that does not vary under surgery along Y-graphs only depends on $(H_1(.); \ell_{\mathbb{Q}/\mathbb{Z}}(.))$.*

PROOF: Let M and M' be two rational homology spheres. Assume that there exists an isomorphism $\phi : H_1(M') \longrightarrow H_1(M)$ such that

$$\ell_{\mathbb{Q}/\mathbb{Z}}(M') = \ell_{\mathbb{Q}/\mathbb{Z}}(M) \circ \phi^2.$$

Then $L' = (K'_1, \dots, K'_r)$ be a family of framed knots in M' whose homology classes generate $H_1(M')$. Then there exists a framed link $L = (K_1, \dots, K_r)$ in M such that the K_i are homologous to the $\phi([K'_i])$ with the same linking matrix as L'. (The hypothesis implies that it is true in $\mathbb{Q}/\mathbb{Z}$, perform crossing changes and framing changes so that it is true in $\mathbb{Q}$.) Glue a tree T to L along its endpoints so that each component of L has exactly one point identified to an endpoint of T. Then L is part of a connected graph $T \cup L$ whose H_1 is freely generated by the classes of the K_i. Then the regular neighborhood $N(T \cup L)$ of L in M is a handlebody whose complement is a homology handlebody H, thanks to Lemma 6.8, because the map induced by the inclusion maps $H_1(\partial H)$ onto $H_1(H)$.

There is a similar diffeomorphic handlebody $N(T' \cup L')$ whose complement is again a homology handlebody H'. Using the diffeomorphism mentioned above, we may write $M = N(T \cup L) \cup H$ and $M' = N(T \cup L) \cup H'$ where H and H' are two handlebodies with the same lagrangian determined by the linking matrices of L and L'. Recall the following lemma from [AL, Lemma 4.11].

Lemma 6.10 *For any two integral homology handlebodies A and B whose boundaries are identified so that their lagrangians coincide, there exists a degree k clover in A such that B is obtained from A by surgery on D.*

$\diamond$

According to this lemma, H and H' can be obtained from one another by surgeries on Y-graphs. Therefore M' can be obtained from M by surgeries on Y-graphs. Then $\kappa(M) = \kappa(M')$. $\diamond$

Now, we furthermore assume that $\kappa(-M) = -\kappa(M)$. This implies that $\kappa(S^3) = 0$. Then since κ vanishes at rational generalised 2-clover of genus 0, κ is additive under connected sum.

In order to conclude the proof of Proposition 6.2 and hence of Theorem 2.6, we are left with the proof of the following proposition.

Proposition 6.11 *A real-valued invariant κ of rational homology spheres*

- *that only depends on $(H_1(.); \ell_{\mathbb{Q}/\mathbb{Z}}(.))$*
- *that is additive under connected sum, and*
- *whose sign changes under orientation reversing,*

vanishes identically.

Since we now consider κ as an invariant of $(H_1(M); \ell_{\mathbb{Q}/\mathbb{Z}}(M))$, κ is additive under orthogonal sum. Since $\ell_{\mathbb{Q}/\mathbb{Z}}$ maps a pair of elements with coprime order to zero, $(H_1(M); \ell_{\mathbb{Q}/\mathbb{Z}}(M))$ is the orthogonal sum of its p-components for prime integers p (made of elements γ such that there exists $k \in \mathbb{N} \setminus \{0\}$ such that $p^k\gamma = 0$).

The classification of linking forms of rational homology spheres has been started by Wall [Wa, Theorem 4] and completed by Kawauchi and Kojima [KK]. Part of the results in [Wa] recalled below, together with the behaviour of κ under orientation reversing and connected sum, imply that κ vanishes. Details are given below.

Lemma 6.12 ([Wa, Theorem 4]) *Let p be a prime integer greater than 2. Assume that the order of any element of $H_1(M)$ is a power of p. Let $n(p)$ be the smallest integer in $\{2, \ldots, p-1\}$ that is not a square mod p. Then $(H_1(M); \ell_{\mathbb{Q}/\mathbb{Z}}(M))$ is an orthogonal sum of modules of the forms*

$$[p^k, s = n(p) \text{ or } 1] = (\mathbb{Z}/p^k\mathbb{Z}[x]; \ell_{\mathbb{Q}/\mathbb{Z}}(x,x) = \frac{s}{p^k}).$$

Furthermore the orthogonal sum of two copies of $[p^k, n(p)]$ is isomorphic to the orthogonal sum of two copies of $[p^k, 1]$.

Proof: Let k be the largest integer such that p^k is the order of an element of $H_1(M)$. Then there exists x such that $p^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x,x) \in \mathbb{Z}/p^k\mathbb{Z}$ is coprime with p. (Otherwise, if the order of x is p^k, let y be such that $p^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x,y) = 1 \bmod p^k$, then $p^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x+y, x+y) = 2 \bmod p$.) Now, the square from $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^*$ to itself is a group morphism. Its image $\left(\left(\mathbb{Z}/p^k\mathbb{Z}\right)^*\right)^2$ has index two and is made of the squares. It does not contain $n(p)$. Thus

$$\left(\frac{\mathbb{Z}}{p^k\mathbb{Z}}\right)^* = \left(\left(\frac{\mathbb{Z}}{p^k\mathbb{Z}}\right)^*\right)^2 \cup \left(\left(\frac{\mathbb{Z}}{p^k\mathbb{Z}}\right)^*\right)^2 n(p).$$

Therefore, there exists $x \in H_1(M)$ such that $p^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x,x)$ equals 1 or $n(p)$. This allows for a proof that $H_1(M)$ decomposes as in the statement by induction on the order of $H_1(M)$, because the orthogonal of the subspace generated by x is equipped with a non-degenerate linking form.

Now, since $(n(p) - 1) = a^2$ in $\mathbb{Z}/p^k\mathbb{Z}$, in the orthogonal sum $(\mathbb{Z}/p^k\mathbb{Z})e \oplus (\mathbb{Z}/p^k\mathbb{Z})f$ of two copies of $[p^k, 1]$, $p^k\ell_{\mathbb{Q}/\mathbb{Z}}(ae+f, ae+f) = n(p)$. This orthogonal sum is isomorphic to the orthogonal sum of the module $[p^k, n(p)]$ generated by $(ae+f)$ and of its orthogonal that must be isomorphic to $[p^k, s = 1 \text{ or } n(p)]$. Now, because of the effect of a change of generating system, $sn(p)$ must be a square, and therefore $s = n(p)$. ◇

Note that conversely, all the modules above are realised as the H_1 of connected sums of lens spaces. In particular $2\kappa([p^k,1]) = 2\kappa([p^k,n(p)])$. Then κ is the same for all rational homology spheres M with $H_1(M) = \mathbb{Z}/p^k\mathbb{Z}$. In particular, for such a manifold, $\kappa(M) = \kappa(-M) = -\kappa(M) = 0$. Then, using the behaviour of κ under orthogonal sum, κ vanishes at all the rational homology spheres without 2-torsion in their H_1.

Lemma 6.13 ([Wa, Theorem 4]) *For $n \in \{-3,-1,1,3\}$, and for $k \geq 1$, let $A^k(n)$ denote the cyclic group $\mathbb{Z}/2^k\mathbb{Z}[x]$ equipped with the pairing ℓ where $\ell(x,x) = n/2^k$ Assume that the order of any element of $H_1(M)$ is a power of 2.*

- *Then the orthogonal sum of $H_1(M)$ and of a finite number of copies of modules of the form $A^k(1)$, $k \geq 1$, is an orthogonal sum of modules of the form $A^k(n)$.*
- *For any $k \geq 1$, the orthogonal sum of 2 copies of $A^k(1)$ is isomorphic to the orthogonal sum of 2 copies of $A^k(-3)$.*
- *For any $k \geq 1$, the orthogonal sum of 4 copies of $A^k(1)$ is isomorphic to the orthogonal sum of $A^k(3)$ and three other modules of the form $A^k(n)$.*

PROOF: When $k \geq 3$, the square from $\left(\mathbb{Z}/2^k\mathbb{Z}\right)^*$ to itself is a group morphism whose kernel has the four elements, 1, -1, $1+2^{k-1}$ and $-1+2^{k-1}$. Therefore its image $\left(\left(\mathbb{Z}/2^k\mathbb{Z}\right)^*\right)^2$ has index 4, since all its elements must be congruent to 1 mod 8, it is easy to see that and $\left(\left(\mathbb{Z}/2^k\mathbb{Z}\right)^*\right)^2$ is exactly made of the numbers that are congruent to 1 mod 8. In particular, any element of $\left(\mathbb{Z}/2^k\mathbb{Z}\right)^*$ belongs to $\left(\left(\mathbb{Z}/2^k\mathbb{Z}\right)^*\right)^2 \{-3,-1,1,3\}$.

Again, we try to diagonalise the linking form by induction on the order of $H_1(M)$. Let k be the largest integer such that 2^k is the order of an element of $H_1(M)$. If there exists x such that $2^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x,x)$ is odd, then $\mathbb{Z}/2^k\mathbb{Z}[x]$ is of the form $A^k(n)$, and its orthogonal of is a module of lower order that satisfies the hypotheses. Otherwise, for any x of order 2^k, $2^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x,x) \in \mathbb{Z}/2\mathbb{Z}^k$ is even. However, for such an x, there exists y such that $2^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x,y) = 1$. Consider the orthogonal sum of $H_1(M)$ with $A^k(1) = \mathbb{Z}/2^k\mathbb{Z}[z]$ where $2^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(z,z) = 1$. Then $H_1(M) \oplus A^k(1)$ splits as the orthogonal sum of the module generated by the two orthogonal elements $(x+z)$ and $(y-z)$, that is an orthogonal sum of modules of the form $A^k(n)$, and of its orthogonal whose order is lower than the order of $H_1(M)$. The first assertion follows by induction on the order of $H_1(M)$.

The orthogonal sum

$$\frac{\mathbb{Z}}{2^k\mathbb{Z}}a \oplus \frac{\mathbb{Z}}{2^k\mathbb{Z}}b$$

where the mentioned generators have self-linking $\frac{1}{2^k}$, is isomorphic to the orthogonal sum $\frac{\mathbb{Z}}{2^k\mathbb{Z}}(a+2b) \oplus \frac{\mathbb{Z}}{2^k\mathbb{Z}}(b-2a)$ of two modules isomorphic to $A^k(-3)$.

The orthogonal sum

$$\frac{\mathbb{Z}}{2^k\mathbb{Z}}a \oplus \frac{\mathbb{Z}}{2^k\mathbb{Z}}b \oplus \frac{\mathbb{Z}}{2^k\mathbb{Z}}c$$

where the mentioned generators have self-linking $\frac{1}{2^k}$ splits as the direct orthogonal sum of $A^k(3)$ that is generated by $(a+b+c)$ and its orthogonal that is $\frac{\mathbb{Z}}{2^k\mathbb{Z}}x \oplus \frac{\mathbb{Z}}{2^k\mathbb{Z}}y$ equipped with a non-degenerate linking form where $2^k\ell_{\mathbb{Q}/\mathbb{Z}}(M)(x,x)$ is even. As above, the orthogonal sum of $A^k(1) = \mathbb{Z}/2^k\mathbb{Z}[z]$ and

$\left(\frac{\mathbb{Z}}{2^k\mathbb{Z}}(a+b+c)\right)^{\perp}$ is the orthogonal sum of the orthogonal sum of two modules of the form $A^k(n)$ and its orthogonal that is necessarily cyclic and equipped with a non degenerate linking form, and that is therefore of the form $A^k(n)$. $\diamond$

By the behaviour of κ under orientation reversing we deduce that $\kappa(A^k(1)) = -\kappa(A^k(-1))$ and that $\kappa(A^k(3)) = -\kappa(A^k(-3))$. By the second point of the above lemma, $2\kappa(A^k(1)) = 2\kappa(A^k(-3))$.

$$\kappa(A^k(1)) = \kappa(A^k(-3)) = -\kappa(A^k(-1)) = -\kappa(A^k(3)).$$

Then by the third point of the above lemma, $4\kappa(A^k(1)) = r\kappa(A^k(1))$, where r is an integral number less or equal than 2. Therefore $\kappa(A^k(1)) = 0$ and $\kappa(A^k(n)) = 0$, for all n in $\{-3,-1,1,3\}$. The first point of the lemma together with the additivity of κ under orthogonal sum allows us to conclude that κ vanishes at the manifolds M such that the order of $H_1(M)$ is a power of two, and next that $\kappa(M) = 0$ for any rational homology sphere M.

This ends the proof of Proposition 6.11 and hence the proof of Theorem 2.6.

6.5 Alternative definitions of the Casson-Walker invariant

I thank Misha Polyak and Oleg Viro for encouraging me to write this last subsection.

As a corollary to Theorem 2.6, we get

Corollary 6.14 *For any rational homology sphere M, and for any trivialisation τ_M of $T(M\setminus\infty)$ that is standard near ∞,*

$$\lambda(M) = \frac{\int_{C_2(M)}\omega(\tau_M)^3}{6} - \frac{p_1(\tau_M)}{24}.$$

PROOF: According to Theorem 2.6, $Z_1 = \lambda/2[\ ]$. Then by definition of Z_1 [L2, Theorem 1.9], and by the proof of Proposition 2.45 in [L2],

$$Z_1(M) = \frac{\int_{C_2(M)}\omega(\tau_M)^3}{12}[\ ] + \frac{p_1(\tau_M)}{4}\xi_1.$$

According to Proposition 2.45 in [L2], $12\xi_1 = -[\ ]$. $\diamond$

The integral $\int_{C_2(M)}\omega(\tau_M)^3$ can be rewritten in various different ways as follows. Recall that a trivialisation τ_M of $T(M\setminus\infty)$ that is standard near ∞ induces a map $p_M(\tau_M)$ from $\partial C_2(M)$ to S^2. See [L2, Subsection 1.2].

Lemma 6.15 *Let M be a rational homology sphere. Let τ_M be a trivialisation of $T(M\setminus\infty)$ that is standard near ∞. Let $\omega_a(S^2)$, $\omega_b(S^2)$ and $\omega_c(S^2)$ be 3 two-forms on S^2 whose integrals over S^2 equal 1. Let ω_a, ω_b and ω_c be 3 closed two-forms on $C_2(M)$ that coincide with $p_M(\tau_M)^*(\omega_a(S^2))$, $p_M(\tau_M)^*(\omega_b(S^2))$ and $p_M(\tau_M)^*(\omega_c(S^2))$ on $\partial C_2(M)$, respectively. Then*

$$\int_{C_2(M)}\omega(\tau_M)^3 = \int_{C_2(M)}\omega_a\wedge\omega_b\wedge\omega_c.$$

PROOF: The arguments are already in [L2]. However, since only antisymmetric forms were considered in [L2], and since the proof is far quicker in this case, we give it. It is enough to prove that the right-hand side does not depend on the choices of ω_a, ω_b and ω_c. By symmetry, it is enough to prove that it is independent of ω_a. By Lemma 2.15 in [L2], it is enough to prove that $\int_{C_2(M)}\omega_a\wedge\omega_b\wedge\omega_c$ does not change when $\mathrm{d}\eta$ is added to ω_a, for some one-form η on $C_2(M)$ that reads $p_M(\tau_M)^*(\eta_{S^2})$ on $\partial C_2(M)$

for some one-form η_{S^2} on S^2. By the Stokes theorem, the variation of $\int_{C_2(M)} \omega_a \wedge \omega_b \wedge \omega_c$ under the addition of $\mathrm{d}\eta$ reads

$$\int_{\partial C_2(M)} \eta \wedge \omega_b \wedge \omega_c = \int_{\partial C_2(M)} p_M(\tau_M)^* \left(\eta_{S^2} \wedge \omega_b(S^2) \wedge \omega_c(S^2)\right)$$

and vanishes since 5-forms vanish on S^2. $\diamond$

Let d be a point in S^2. Consider the codimension 2 submanifold $p_M(\tau_M)^{-1}(d)$ of $\partial C_2(M)$. Since $H_3(C_2(M);\mathbb{Q}) = 0$, $p_M(\tau_M)^{-1}(d)$ is the boundary of a rational 4-chain Σ_d. If M is a $\mathbb{Z}$-sphere, we may even assume that Σ_d is a codimension 2 submanifold of $C_2(M)$.

Lemma 6.16 *Let M be a rational homology sphere. Let τ_M be a trivialisation of $T(M \setminus \infty)$ that is standard near ∞. Let a, b and c be three distinct points in S^2. Let Σ_a, Σ_b and Σ_c be three rational 4-chains in $C_2(M)$ with respective boundaries $p_M(\tau_M)^{-1}(a)$, $p_M(\tau_M)^{-1}(b)$ and $p_M(\tau_M)^{-1}(c)$. Then $\int_{C_2(M)} \omega(\tau_M)^3$ is the algebraic intersection of Σ_a, Σ_b and Σ_c in $C_2(M)$.*

PROOF: First note that the algebraic intersection of Σ_a, Σ_b and Σ_c is well-defined and independent of the involved choices because the boundaries are three disjoint fixed submanifolds of the boundary of $C_2(M)$ and because $H_4(C_2(M);\mathbb{Q}) = 0$. Thus, we must now compare two topological invariants of $(M;\tau_M)$.

Without loss assume that Σ_a, Σ_b and Σ_c are in general position so that they intersect in a finite number of points that have a neighborhood of the form $\mathbb{R}^2_a \times \mathbb{R}^2_b \times \mathbb{R}^2_c$ (for three copies $\mathbb{R}^2_a$, $\mathbb{R}^2_b$, $\mathbb{R}^2_c$ of $\mathbb{R}^2$) that is intersected by the support of Σ_a along $0 \times \mathbb{R}^2_b \times \mathbb{R}^2_c$, by the support of Σ_b along $\mathbb{R}^2_a \times 0 \times \mathbb{R}^2_c$, and by the support of Σ_c along $\mathbb{R}^2_a \times \mathbb{R}^2_b \times 0$.

Now, use the preceding lemma with three forms $\omega_a(S^2)$, $\omega_b(S^2)$ and $\omega_c(S^2)$ with disjoint supports concentrated near a, b and c respectively, and with three forms ω_d, for $d \in \{a, b, c\}$, such that:

- ω_d is supported in a small neighborhood $N(\Sigma_d)$ of Σ_d,
- for any oriented dimension 2-manifold F whose boundary is disjoint from $N(\Sigma_d)$, $\int_F \omega_d$ is the algebraic intersection of F and Σ_d, and
- in an affine neighborhood of an intersection point as above, ω_d vanishes along any two-dimensional plane that is not transverse to the affine 4-dimensional support of Σ_d.

$\diamond$

Let M be a rational homology sphere. Let $p_1(M)$ denote the image of the map p_1 from the set of trivialisations of $T(M \setminus \infty)$ that are standard near ∞ to $\mathbb{Z}$. If $j \in p_1(M)$, let $\tau_j(M)$ be a trivialisation of $T(M \setminus \infty)$ that is standard near ∞ such that $p_1(\tau_j(M)) = j$.

If $0 \in p_1(M)$, (according to Proposition 1.8 in [L2], this is always the case when M is a $\mathbb{Z}$-sphere), then

$$\lambda(M) = \frac{\int_{C_2(M)} \omega(\tau_0(M))^3}{6}$$

and the previous lemma allows us to express $6\lambda(M)$ as the algebraic intersection of three rational 4-chains in $C_2(M)$, (or even of three codimension 2 submanifolds if M is a $\mathbb{Z}$-sphere).

Otherwise, Proposition 1.8 in [L2] ensures that $p_1(M)$ contains a subset of the form $\{i-4, i\}$. Then

$$\lambda(M) = \frac{i}{24} \int_{C_2(M)} \omega(\tau_{i-4}(M))^3 + \frac{4-i}{24} \int_{C_2(M)} \omega(\tau_i(M))^3.$$

In this case, the previous lemma allows us to express $24\lambda(M)$ as the sum of two rational algebraic intersection numbers in $C_2(M)$.

References

[AM] **S. Akbulut and J. McCarthy**, *Casson's invariant for oriented homology 3-spheres, an exposition.* Mathematical Notes **36**, Princeton University Press, Princeton, 1990

[AL] **E. Auclair and C. Lescop**, *Algebraic version of the clover calculus for homology 3-spheres.* math.GT/0401251

[GGP] **S. Garoufalidis, M. Goussarov, M. Polyak,** *Calculus of clovers and finite type invariants of 3-manifolds.* Geometry and Topology, Vol. 5 (2001), Paper no. 3, pages 75–108

[GM] **L. Guillou, A. Marin,** *À la recherche de la topologie perdue,* Progress in Mathematics, 62. Birkhäuser Boston, Inc., Boston, MA, 1986

[GM] **L. Guillou, A. Marin,** *Notes sur l'invariant de Casson des sphres d'homologie de dimension trois.* Enseign. Math. (2) 38 (1992), no. 3-4, 233–290

[Ha] **K. Habiro**, *Claspers and finite type invariants of links.* Geometry and Topology **4** (2000), 1–83

[KK] **A. Kawauchi and S. Kojima**, *Algebraic Classification of Linking Pairings on* 3*-manifolds.* Mathematische Annalen **253** (1980), 29–42

[Ko] **M. Kontsevich**, *Jacobi diagrams and low-dimensional topology*, First European Congress of Mathematics **II** 97–121, Birkhäuser Basel, 1994

[KT] **G. Kuperberg and D. P. Thurston,** *Perturbative 3–manifold invariants by cut-and-paste topology*, preprint, math.GT/9912167

[Le] **T. T. Q. Le,** *An invariant of integral homology* 3*-spheres which is universal for all finite type invariants.* Solitons, geometry, and topology: on the crossroad, 75–100, Amer. Math. Soc. Transl. Ser. 2, 179, Amer. Math. Soc., Providence, RI, 1997

[LMO] **T. T. Q. Le, J. Murakami and T. Ohtsuki,** *On a universal perturbative invariant of* 3*-manifolds,* Topology 37 no. 3 (1998) 539–574

[L1] **C. Lescop**, *A sum formula for the Casson-Walker invariant,* Invent. math. **133** (1998) 613-681

[L2] **C. Lescop**, *On the construction of the Kontsevich-Kuperberg-Thurston invariant for rational homology* 3*-spheres,* Prépublication de l'Institut Fourier 655 (2004), http://www-fourier.ujf-grenoble.fr/prepublications.html

[M] **A. Marin,** *Un nouvel invariant pour les sphères d'homologie de dimension 3 (d'après Casson),* Semin. Bourbaki, 40ème Année, Vol. 1987/88, Exp. No.693, Astérisque **161-162**, (1988), 151–164

[Mas] **G. Massuyeau,** *Cohomology rings, Rochlin function, linking pairing and the Goussarov-Habiro theory of three-manifolds,* Algebraic and Geometric Topology, Volume 3 (2003), paper no. 41, 1139–1166

[19] **J. W. Milnor,** *Topology from the differentiable viewpoint.* Based on notes by David W. Weaver. The University Press of Virginia, Charlottesville, 1965

[MS] **J. W. Milnor, J. D. Stasheff** *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press, 1974

[O] **T. Ohtsuki,** *Finite type invariants of integral homology* 3*-spheres.* J. Knot Theory Ramifications 5 (1996), no. 1, 101–115

[W] **K. Walker,** *An extension of Casson's Invariant,* Annals of Math. Studies 126, Princeton University Press, 1992

[Wa] **C. T. C. Wall,** *Quadratic forms on finite groups and related topics,* Topology (1964) 2, 281–298

[Wi] **E. Witten,** *Quantum field theory and the Jones polynomial.* Comm. Math. Phys. 121 (1989), no. 3, 351–399

Terminology

Notation