Computing the Petersson scalar product $\langle f^0, f_0 \rangle$.

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Prépublication de l'Institut Fourier n°654 (2004)
http://www-fourier.ujf-grenoble.fr/prepublications.html

1 Introduction

Let $f = \sum_{n \geqslant 1} a(n) \mathfrak{q}^n$ be a primitive cusp form of weight k, conductor C and character ψ (see Miyake [5] page 164).

Let p be a prime not dividing C. We define modular forms f_0 and f^0 (as in Dabrowski and Delbourgo's paper [2]) by

$$f_0(z) := f(z) - \alpha'_p f(pz) \tag{1.1}$$

where

$$X^2 - a(q)X + \psi(q)q^{k-1} = (X - \alpha_q)(X - \alpha'_q)$$

is the Hecke polynomial of f at prime q ; and

$$f^0 := f_0^\rho \big| W(4Cp) \tag{1.2}$$

where

$$f^\rho(z) := \overline{f(-\overline{z})}. \tag{1.3}$$

The main purpose of this paper is to show the following equality.

Proposition 1. *With the above notations the following equality holds*

$$\langle f^0, f_0 \rangle_{4Cp} = \overline{\gamma} \beta_p \left(4^{1-k/2} a(4) - \psi(2) \right) \langle f, f \rangle_C \tag{1.4}$$

where γ is an algebraic number of modulus 1 uniquely determined by f and $\beta_p = \alpha'_p p^{-k/2}(p+1) \left(\frac{a(p)}{\alpha'_p} \frac{p}{p+1} - 2 + \frac{a(p)}{\alpha_p} \frac{1}{p+1} \right)$.

Remark : The main motivations of (1.4) are due to the properties : $f_0 \big| U_p = \alpha_p f_0$ et $f^0 \big| U_p^* = \overline{\alpha_p} f^0$, where $(\sum b_n \mathfrak{q}^n) \big| U_p = \sum b_{pn} \mathfrak{q}^n$ is the Atkin's operator, U_p^* its adjoint on modular forms of level Np.

In particular, it is important to have a control on the behavior of both parts of (1.4) in a p-adic family $\{f_k\}$ of modular forms (Coleman [1], Panchishkin [7]),

Keywords : modular forms, Hecke operators.
2000MSC : 11F11, 11F25.

because these quantities come up in special values of $L_{f_k}(j,\chi)$ $(1 \leqslant j \leqslant k-1)$, χ is a Dirichlet character.

The authors want to thank professor Alexeï Panchishkin for his help and many advices.

2 Notations

Let us fix an embedding $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

We let every element $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2^+(\mathbb{R})$ act on the upper half-plane $\mathfrak{h} := \left\{ z \in \mathbb{C} \middle| \ \mathrm{Im(z)} > 0 \right\}$ by the rule

$$\alpha \cdot z := \frac{az+b}{cz+d}, \quad \text{for all } z \in \mathfrak{h}.$$

The group $GL_2^+(\mathbb{R})$ acts on functions $g : \mathfrak{h} \to \mathbb{C}$ by the rule

$$\left(g \Big|_k \gamma \right)(z) := (\det(\gamma))^{k/2}(cz+d)^{-k} g(\gamma z). \tag{2.1}$$

Let $G_k(N,\psi)$ denote the space of modular forms of weight k, level N and character ψ. An element f of $G_k(N,\psi)$ satisfies

$$f \Big|_k \gamma = \psi(d) f, \quad \text{for all } \gamma \in \Gamma_0(N),$$

where

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \middle| \ c \equiv 0 \pmod N \right\}. \tag{2.2}$$

Let define the operators $V(m)$, $W(N)$ and U_p on functions of complex variable z by

$$\begin{aligned} g\big|V(m)(z) &:= g(mz) \\ &= m^{-k/2} g \Big|_k \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}(z). \end{aligned} \tag{2.3}$$

$$\begin{aligned} g\big|W(N)(z) &:= (\sqrt{N}z)^{-k} g\left(-\frac{1}{Nz}\right) \\ &= g \Big|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}(z). \end{aligned} \tag{2.4}$$

$$g\big|U_p(z) := p^{k/2-1} \sum_{u \bmod p} g \Big|_k \begin{pmatrix} 1 & u \\ 0 & p \end{pmatrix}. \tag{2.5}$$

$V(m)$ takes $G_k(N,\psi)$ to $G_k(mN,\tilde{\psi})$, where $\tilde{\psi}$ is the character *modulo* mN induced by ψ (Lemma 4.6.1 page 153 of Miyake [5]), and $W(N)$ takes $G_k(N,\psi)$ to $G_k(N,\overline{\psi})$ (Lemma 4.3.2 page 115 of Miyake [5]).

Proposition 2. *These operators satisfy the following relations :*

$$(f|W(N))|W(N) = (-1)^k f; \tag{2.6}$$

$$f|W(Nm) = m^{k/2}(f|W(N))|V(m) = f|W(N)\Big|_k \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}; \tag{2.7}$$

$$f|W(N) = m^{k/2}\left(f|V(m)\right)|W(mN) = f\Big|_k \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}\Big|W(mN). \tag{2.8}$$

If g and h are both modular forms in $G_k(N,\psi)$ such that one of them is a cusp form, then we define the Petersson inner product $\langle g,h\rangle_N$ by

$$\langle g,h\rangle_N := \int_{\Gamma_0(N)\backslash\mathfrak{h}} \overline{g}hy^k \frac{dxdy}{y^2}.$$

Let us view g and h as two modular forms of level M where $N|M$, then we have the identity

$$\langle g,h\rangle_M = [\Gamma_0(N):\Gamma_0(M)]\,\langle g,h\rangle_N\,, \tag{2.9}$$

because of $-I_2$ belongs to $\Gamma_0(N)$ (see Li [4]).

Finally, for g an element of $G_k(N,\psi)$ and q a prime number one put

$$g|T_q = g|U_q + \psi(q)q^{k-1}g|V_q. \tag{2.10}$$

As proved by Hecke, T_q is a linear operator on $G_k(N,\psi)$ (see Theorem 2.8.1 page 74 and pages 134-135 of Miyake [5]). We also note T_q^* the adjoint of T_q (for the above scalar product) on modular forms of level N.

The cusp form $f = \sum_{n\geqslant 1} a(n)\mathfrak{q}^n$ being primitive of weight k, conductor C and character ψ, we have the identities

$$f|T_q = a(q)f, \quad \text{if } q\nmid C; \tag{2.11}$$

$$f|U_q = a(q)f, \quad \text{if } q|C; \tag{2.12}$$

where q denotes an arbitrary prime number and the Hecke operators T_q and U_q are defined on the space $G_k(C,\psi)$.

3 The Trace Operator

Let us consider a modular form g in $G_k(N,\psi)$. We view g as an element in $G_k(M,\psi)$ where M is a multiple of N. Following Serre's paper [9] we define the trace operator Tr_N^M by

$$g|Tr_N^M := \sum_{\gamma\in\mathcal{R}} \overline{\psi}(\gamma)g\Big|_k\gamma, \tag{3.1}$$

where we put $\psi(\gamma) := \psi(d)$ when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and where $\mathcal{R}$ is a complete representative set of $\Gamma_0(M)\diagdown\Gamma_0(N)$.

Proposition 3. *Tr_N^M sends $G_k(M,\psi)$ into $G_k(N,\psi)$.*

Proof : First, let us check that Tr_N^M is well defined. Let γ and γ' be in the same coset. Then there exists an element $\alpha \in \Gamma_0(M)$ such that $\gamma' = \alpha \cdot \gamma$. We find

$$\begin{aligned} \overline{\psi}(\gamma')g\big|_k\gamma' &= \overline{\psi}(\alpha\cdot\gamma)g\Big|_k \alpha\cdot\gamma \\ &= \overline{\psi}(\alpha)\overline{\psi}(\gamma)\left(\psi(\alpha)g\big|_k\gamma\right) \\ &= \overline{\psi}(\gamma)g\big|_k\gamma. \end{aligned}$$

Next consider α an element of $\Gamma_0(N)$.

$$\begin{aligned} g\big|Tr_N^M\big|_k\alpha &= \sum_{\gamma\in\mathcal{R}} \overline{\psi(\gamma)g}\Big|_k \gamma\cdot\alpha \\ &= \psi(\alpha)\sum_{\gamma\in\mathcal{R}} \overline{\psi}(\gamma\cdot\alpha)g\Big|_k\gamma\cdot\alpha. \end{aligned}$$

When γ runs through $\mathcal{R}$, the matrix $\gamma\cdot\alpha$ runs through another complete set of representative of $\Gamma_0(M)\diagdown\Gamma_0(N)$. Therefore,

$$\sum_{\gamma\in\mathcal{R}} \overline{\psi}(\gamma\cdot\alpha)g\Big|_k\gamma\cdot\alpha = \sum_{\gamma\in\mathcal{R}}\overline{\psi}(\gamma)g\Big|_k\alpha,$$

and we find

$$g\big|Tr_N^M\big|_k\alpha = \psi(\alpha)g\big|Tr_N^M, \quad \text{for all } \alpha\in\Gamma_0(N).$$

Finally we look at the behavior at cusps. Let γ_0 be an element of $SL_2(\mathbb{Z})$. Since g is an element of $G_k(M,\tilde{\psi})$, we have $g\big|_k\gamma_0 = \sum_{n\geqslant 0} a_n(\gamma_0)q^{n/M}$. Thus we find

$$\begin{aligned} g\big|Tr_N^M\gamma_0 &= \sum_{\gamma\in\Gamma_0(M)\diagdown\Gamma_0(N)} \overline{\psi}(\gamma)g\big|_k\gamma\gamma_0 \\ &= \sum_{\gamma\in\Gamma_0(M)\diagdown\Gamma_0(N)} \overline{\psi}(\gamma)\sum_{n\geqslant 0} a_n(\gamma\gamma_0)q^{n/M} \\ &= \sum_{n\geqslant 0}\left(\sum_{\gamma\in\Gamma_0(M)\diagdown\Gamma_0(N)}\overline{\psi}(\gamma)a_n(\gamma\gamma_0)\right)q^{n/M}. \end{aligned}$$

This shows that $g\big|Tr_N^M$ is holomorphic at each cusp. □

We now give the link between the trace operator and the Petersson inner product

Proposition 4. *Let g be as above and let h be a modular form in $G_k(N,\psi)$, then*

$$\langle g,h\rangle_M = \left\langle g\big|Tr_N^M, h\right\rangle_N. \tag{3.2}$$

Proof : Indeed, for all $\gamma\in\Gamma_0(N)$

$$\begin{aligned} \langle g,h\rangle_M &= \left\langle g\big|_k\gamma, h\big|_k\gamma\right\rangle_M \\ &= \psi(\gamma)\left\langle g\big|_k\gamma, h\right\rangle_M \end{aligned}$$

because h is an element of $G_k(N,\psi)$.

Therefore,

$$\begin{aligned}\langle g|Tr_N^M, h\rangle_M &= \sum_{\gamma\in\mathcal{R}} \psi(\gamma)\,\langle g|_k\gamma, h\rangle_M \\ &= \sum_{\gamma\in\mathcal{R}} \langle g, h\rangle_M\,.\end{aligned}$$

To conclude, notice that the cardinality of $\mathcal{R}$ is precisely $[\Gamma_0(N):\Gamma_0(M)]$. □

4 A general result

Let N be a positive integer and let p be a prime integer not dividing N.

Proposition 5. *Let g be a modular form in $G_k(N,\overline{\psi})$ and let h be a modular form in $G_k(N,\psi)$. Then we have*

$$\left\langle g|W(N)\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, h\right\rangle_{Np} = p^{1-k/2}\,\langle g|T_p|W(N), h\rangle_N\,. \tag{4.1}$$

Proof : Since p and N are relatively prime, the Bezout's identity gives two integers α and β such that $\alpha p - \beta N = 1$. So we obtain a complete set of representatives of $\Gamma_0(Np)\diagdown\Gamma_0(N)$,

$$\mathcal{R} := \bigcup_{u \bmod p} \left\{\begin{pmatrix} 1 & 0 \\ Nu & 1 \end{pmatrix}\right\} \bigcup \left\{\begin{pmatrix} \alpha & \beta \\ N & p \end{pmatrix}\right\}. \tag{4.2}$$

Using the definition (3.1), we get

$$\begin{aligned} & g|W(N)\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\Big|Tr_N^{Np} \\ &= \sum_{u \bmod p} g|W(N)\Big|_k \begin{pmatrix} p & 0 \\ Nu & 1 \end{pmatrix} + \overline{\psi(p)}g|W(N)\Big|_k \begin{pmatrix} p\alpha & p\beta \\ N & p \end{pmatrix}. \end{aligned} \tag{4.3}$$

Now $\begin{pmatrix} p\alpha & p\beta \\ N & p \end{pmatrix} = \begin{pmatrix} p\alpha & \beta \\ N & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $\begin{pmatrix} p\alpha & \beta \\ N & 1 \end{pmatrix} \in \Gamma_0(N)$, so it comes that

$$g|W(N)\Big|_k \begin{pmatrix} p\alpha & p\beta \\ N & p \end{pmatrix} = \psi(1)g|W(N)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}. \tag{4.4}$$

Furthermore,

$$W(N)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} W(N)$$

and, for each $u \bmod p$, we have

$$\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ Nu & 1 \end{pmatrix} = \begin{pmatrix} 1 & -u \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

so (4.3) becomes

$$\begin{aligned}
&g\big|W(N)\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Big|Tr_N^{Np} \\
&= \sum_{u \bmod p} g\Big|_k \begin{pmatrix} 1 & -u \\ 0 & p \end{pmatrix} \Big|W(N) + \overline{\psi(p)} g\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Big|W(N) \\
&= \left(\sum_{u \bmod p} g\Big|_k \begin{pmatrix} 1 & -u \\ 0 & p \end{pmatrix} + \overline{\psi(p)} g\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) \Big|W(N). \qquad (4.5)
\end{aligned}$$

Let us use the identity (4.2.26) page 142 of Miyake [5]

$$p^{1-k/2} g\big|T_p = \overline{\psi(p)} g\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{v \bmod p} g\Big|_k \begin{pmatrix} 1 & v \\ 0 & p \end{pmatrix}.$$

The last equality and (4.5) implies that

$$g\big|W(N)\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Big|Tr_N^{Np} = p^{1-k/2} g\big|T_p\big|W(N). \qquad (4.6)$$

Using (3.2) and (4.6) and assuming $M = Np$, we have Proposition 5 . □

Corollary 1. *With the same assumptions as in Proposition 5, we have*

$$\left\langle g\big|W(N), h\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_{Np} = p^{1-k/2} \left\langle g\big|T_p^*\big|W(N), h \right\rangle_N. \qquad (4.7)$$

Proof : Indeed, let us write

$$\begin{aligned}
&\left\langle g\big|W(N), h\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_{Np} \\
&= \left\langle g\big|W(N), h\big|W(N)^{-1} W(N)\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_{Np} \\
&= \overline{\left\langle h\big|W(N)^{-1} W(N)\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, g\big|W(N) \right\rangle_{Np}}.
\end{aligned}$$

Observing that $g\big|W(N) \in G_k(N, \psi)$ and $h\big|W(N)^{-1} \in G_k(N, \overline{\psi})$, we see that the assumptions of Proposition 5 are satified and

$$\begin{aligned}
p^{k/2-1} \left\langle g\big|W(N), h\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_{Np} &= \overline{\left\langle h\big|W(N)^{-1}\big|T_p\big|W(N), g\big|W(N) \right\rangle_N} \\
&= \left\langle g\big|W(N), h\big|W(N)^{-1}\big|T_p\big|W(N) \right\rangle_N \\
&= \left\langle g, h\big|W(N)^{-1}\big|T_p \right\rangle_N \\
&= \left\langle g\big|T_p^*, h\big|W(N)^{-1} \right\rangle_N \\
&= \left\langle g\big|T_p^*\big|W(N), h \right\rangle_N.
\end{aligned}$$

□

5 Proof of Proposition 1

Let f be a primitive modular form of weight k, conductor C and character ψ. Put $N = 4C$.

Proposition 6. *There exists an algebraic number γ of modulus 1 such that*

$$f^\rho \big| W(C) = \gamma f. \tag{5.1}$$

Proof : According to Theorem 4.6.15 in Miyake [5], there exists an algebraic number γ such that equality (5.1) is verified. We just need to check γ is of modulus 1.

Let us recall that $f^\rho(z) = \overline{f(-\overline{z})}$. Applying $W(C)$ to (5.1), by using (2.7) we get

$$\begin{aligned}
(-1)^k f^\rho &= \gamma f \big| W(C) \\
(-1)^k f^\rho &= \gamma(\sqrt{C}z)^{-k} f\left(-\frac{1}{Cz}\right) \\
(-1)^k f &= \overline{\gamma(\sqrt{C}(-\overline{z}))^{-k} f\left(\frac{1}{C\overline{z}}\right)} \\
(-1)^k f &= \overline{\gamma}(\sqrt{C}(-z))^{-k} f^\rho\left(-\frac{1}{Cz}\right) \\
f &= \overline{\gamma}(\sqrt{C}z)^{-k} f^\rho\left(-\frac{1}{Cz}\right) \\
f &= \overline{\gamma} f^\rho \big| W(C) \\
f &= \overline{\gamma}\gamma f.
\end{aligned}$$

Since f is not the zero function, the last equality proves Proposition 6 . □

The proof of Proposition 1 splits into two steps. First, we express $\langle f^0, f_0 \rangle_{Np}$ in terms of $\langle f^\rho | W(N), f \rangle_N$ (Lemma 1) , next $\langle f^\rho | W(N), f \rangle_N$ in terms of $\langle f, f \rangle_C$ (Lemma 2). Using both lemmas we will obtain (1.4) .

Lemma 1.

$$\langle f^0, f_0 \rangle_{Np} = \beta_p \langle f^\rho | W(N), f \rangle_N \tag{5.2}$$

where $\beta_p := \alpha'_p p^{-k/2}(p+1)\left(\frac{a(p)}{\alpha'_p}\frac{p}{p+1} - 2 + \frac{a(p)}{\alpha_p}\frac{1}{p+1}\right)$.

Proof : According to Definition (1.1) and to equality (2.3), $f_0 = f - \alpha'_p f \big| V(p)$. According to definition (1.2) and to equality (2.8)

$$\begin{aligned}
f^0 &= \left(f^\rho - \overline{\alpha'_p} p^{-k/2} f^\rho \Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) \Big| W(Np) \\
&= f^\rho \big| W(Np) - \overline{\alpha'_p} p^{-k/2} f^\rho \big| W(N).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left\langle f^0, f_0\right\rangle_{Np} \\
&= \left\langle f^\rho\middle|W(Np) - \overline{\alpha'_p}p^{-k/2}f^\rho\middle|W(N), f - \alpha'_p p^{-k/2} f\middle|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right\rangle_{Np} \\
&= \left\langle f^\rho\middle|W(Np), f\right\rangle_{Np} - p^{-k/2}\alpha'_p \left\langle f^\rho\middle|W(Np), f\middle|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right\rangle_{Np} \\
&\quad -\alpha'_p p^{-k/2}\left\langle f^\rho\middle|W(N), f\right\rangle_{Np} + (\alpha'_p)^2 p^{-k}\left\langle f^\rho\middle|W(N), f\middle|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right\rangle_{Np}.
\end{aligned}$$

Let us use the equality $W(Np) = W(N)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ (see (2.7)).

Finally we obtain

$$\begin{aligned}
&\left\langle f^0, f_0\right\rangle_{Np} \\
&= \left\langle f^\rho\middle|W(N)\middle|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, f\right\rangle_{Np} - 2\alpha'_p p^{-k/2}\left\langle f^\rho\middle|W(N), f\right\rangle_{Np} \\
&\quad +(\alpha'_p)^2 p^{-k}\left\langle f^\rho\middle|W(N), f\middle|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right\rangle_{Np}.
\end{aligned} \tag{5.3}$$

By Proposition 5, we have

$$\begin{aligned}
\left\langle f^\rho\middle|W(N)\middle|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, f\right\rangle_{Np} &= p^{1-k/2}\left\langle f^\rho\middle|T_p\middle|W(N), f\right\rangle_N \\
&= a(p)p^{1-k/2}\left\langle f^\rho\middle|W(N), f\right\rangle_N.
\end{aligned}$$

In order to evaluate the last term in (5.3) let us use Corollary 1, giving

$$\begin{aligned}
\left\langle f^\rho\middle|W(N), f\middle|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right\rangle_{Np} &= p^{1-k/2}\left\langle f^\rho\middle|T_p^*\middle|W(N), f\right\rangle_N \\
&= \overline{a(p)}p^{1-k/2}\left\langle f^\rho\middle|W(N), f\right\rangle_N.
\end{aligned}$$

Using (5.3) we find

$$\left\langle f^0, f_0\right\rangle_{Np} = \beta_p\left\langle f^\rho\middle|W(N), f\right\rangle_N \tag{5.4}$$

where $\beta_p = a(p)p^{1-k/2} - 2\alpha'_p p^{-k/2}(p+1) + (\alpha'_p)^2\overline{a(p)}p^{-k}p^{1-k/2}$. Note that $\alpha_p \neq 0$ because $p \nmid C$, thus we have

$$\begin{aligned}
\beta_p &= \alpha'_p p^{-k/2}(p+1)\left(\frac{a(p)}{\alpha'_p}\frac{p}{p+1} - 2 + \alpha'_p\frac{p^{1-k}}{p+1}\overline{a(p)}\right) \\
&= \alpha'_p p^{-k/2}(p+1)\left(\frac{a(p)}{\alpha'_p}\frac{p}{p+1} - 2 + \frac{a(p)}{\alpha_p}\frac{1}{p+1}\right)
\end{aligned}$$

proving Lemma 1. □

Notice next that

$$\frac{a(p)}{\alpha'_p}\frac{p}{p+1} - 2 + \frac{a(p)}{\alpha_p}\frac{1}{p+1} = \frac{\alpha_p}{\alpha'_p}\frac{p}{p+1} - 1 + \frac{\alpha'_p}{\alpha_p}\frac{1}{p+1}.$$

We find the following factorization properties

$$\begin{aligned}\frac{\alpha_p}{\alpha'_p}\frac{p}{p+1}-1+\frac{\alpha'_p}{\alpha_p}\frac{1}{p+1} &= \frac{\alpha_p}{\alpha'_p}\frac{p}{p+1}-\frac{\alpha'_p}{\alpha_p}\frac{p}{p+1}-1+\frac{\alpha'_p}{\alpha_p}\\ &= \left(1-\frac{\alpha'_p}{\alpha_p}\right)\left(\frac{a(p)}{\alpha'_p}\frac{p}{p+1}-1\right) \qquad (5.5)\end{aligned}$$

and

$$\begin{aligned}\frac{\alpha_p}{\alpha'_p}\frac{p}{p+1}-1+\frac{\alpha'_p}{\alpha_p}\frac{1}{p+1} &= \frac{\alpha_p}{\alpha'_p}-1+\frac{\alpha'_p}{\alpha_p}\frac{1}{p+1}-\frac{\alpha_p}{\alpha'_p}\frac{1}{p+1}\\ &= \left(1-\frac{\alpha_p}{\alpha'_p}\right)\left(\frac{a(p)}{\alpha_p}\frac{1}{p+1}-1\right). \qquad (5.6)\end{aligned}$$

Corollary 2. *The algebraic number β_p is not zero if and only if $\alpha_p \neq \alpha'_p$.*

Proof : Use factorization (5.6) together with the Petersson-Deligne's evaluation $|a(p)|^2 \leqslant 4p^{k-1}$ (giving $|a(p)/\alpha_p| \leqslant 2$). □

Moreover we can use (5.5) to simplify the formulas inside the section 4.4 of J. Puydt's thesis [8] (see Corollary 3 below).

Before proving Lemma 2, we give a more general result. Let consider S a finite set of primes not dividing N. We put $M_S := \prod_{q\in S} q$ and we define

$$f_{0,S} := f\Big|_k \prod_{l\in S}\left(Id-\alpha'_l V(l)\right), \qquad (5.7)$$

and

$$f_S^0 := f_{0,S}^\rho\big|_k W(NM_S). \qquad (5.8)$$

Let us notice that

$$\begin{aligned} f_{0,\{p\}} &= f_0,\\ f_{\{p\}}^0 &= f^0.\end{aligned}$$

Corollary 3. *(see [8], chapter 4) We have*

$$\left\langle f_S^0, f_{0,S}\right\rangle_{NM_S} = \left(\prod_{l\in S}\beta_l\right)\left\langle f^\rho\middle|W(N), f\right\rangle_N. \qquad (5.9)$$

Proof : We make an induction on the number of elements of S. If S is the void set, the result is clear and if $S=\{p\}$, it is Lemma 1.

Let $\Sigma = S\cup\{p\}$ where p is a prime not in S. We have

$$f_{0,\Sigma} = f_{0,S}\big|_k\left(Id-\alpha'_p V(p)\right), \qquad (5.10)$$

and

$$f_\Sigma^0 = p^{k/2} f_S^0 - \overline{\alpha'_p} p^{-k/2} f_S^0. \qquad (5.11)$$

Hence

$$
\begin{aligned}
&\left\langle f^0_\Sigma, f_{0,\Sigma}\right\rangle_{NM_\Sigma} \\
&= \left\langle p^{k/2} f^0_S - \overline{\alpha'_p} p^{-k/2} f^0_S, f_{0,S}\big|_k \left(Id - \alpha'_p V(p)\right)\right\rangle_{NM_\Sigma} \\
&= \left\langle f^\rho_{0,S}\Big|_k W(NM_S)\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, f_{0,S}\right\rangle_{NM_\Sigma} \\
&\quad -2\alpha'_p p^{-k/2}\left\langle f^0_S, f_{0,S}\right\rangle_{NM_\Sigma} \\
&\quad + \left(\alpha'_p\right)^2 p^{-k}\left\langle f^\rho_{0,S}\big|_k W(NM_S), f_{0,S}\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right\rangle_{NM_\Sigma} \\
&= \beta_p \left\langle f^0_S, f_{0,S}\right\rangle_{NM_S}.
\end{aligned}
$$

as in the proof of Lemma 1. Hence the corollary is proved, by induction. □

Let us now prove the second step.

Lemma 2.

$$
\left\langle f^\rho\big|W(N), f\right\rangle_N = \overline{\gamma}\left(4^{1-k/2}a(4) - \psi(2)\right)\langle f, f\rangle_C . \tag{5.12}
$$

Proof : We have to consider two cases :

i) Suppose $2|C$. In that case, the action of the Hecke operator $U_4 = U_2U_2$ on f is given by

$$
\begin{aligned}
f\big|U_4 &= 4^{k/2-1}\sum_{\nu \bmod 4} f\Big|_k \begin{pmatrix} 1 & \nu \\ 0 & 4 \end{pmatrix} \\
&= 4^{k/2-1}\sum_{\nu \bmod 4} f\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

As a consequence, by noticing that $\begin{pmatrix} 1 & -\nu \\ 0 & 1 \end{pmatrix} \in \Gamma_0(C)$,

$$
\begin{aligned}
&\left\langle f^\rho\big|W(C), f\big|U_4\right\rangle_N \\
&= 4^{k/2-1}\sum_{\nu \bmod 4}\left\langle f^\rho\big|W(C), f\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}\right\rangle_N \\
&= 4^{k/2-1}\sum_{\nu \bmod 4}\left\langle f^\rho\big|W(C)\Big|_k \begin{pmatrix} 1 & -\nu \\ 0 & 1 \end{pmatrix}, f\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\right\rangle_N \\
&= 4^{k/2-1}\sum_{\nu \bmod 4}\left\langle f^\rho\big|W(C), f\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\right\rangle_N \\
&= 4^{k/2}\left\langle f^\rho\big|W(C), f\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\right\rangle_N .
\end{aligned}
$$

We deduce from the last equality

$$\begin{aligned}
\left\langle f^\rho \middle| W(N), f\right\rangle_N &= \left\langle f^\rho \middle| W(C)\middle|_k \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, f \right\rangle_N \\
&= \left\langle f^\rho \middle| W(C), f\middle|_k \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \right\rangle_N \\
&= 4^{-k/2} \left\langle f^\rho \middle| W(C), f \middle| U_4 \right\rangle_N \\
&= \overline{\gamma} 4^{-k/2} a(4) \left\langle f, f\right\rangle_N \qquad \text{(by equality (5.1))} \\
&= \overline{\gamma} 4^{1-k/2} a(4) \left\langle f, f\right\rangle_C .
\end{aligned}$$

ii) Suppose $2 \nmid C$. A complete system of representative $\mathcal{S}$ for $\Gamma_0(4C)\backslash\Gamma_0(2C)$ is given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ et $\begin{pmatrix} 1 & 0 \\ 2C & 1 \end{pmatrix}$. So we have for any $g \in G_k(C,\psi)$

$$\begin{aligned}
g\middle|W(4C)\middle|Tr_{2C}^{4C} &= \sum_{u \bmod 2} g\middle|_k \begin{pmatrix} 0 & -1 \\ 4C & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u2C & 1 \end{pmatrix} \\
&= \sum_{u \bmod 2} g\middle|_k \begin{pmatrix} 1 & -u \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 2C & 0 \end{pmatrix} \\
&= \left[2^{1-k/2} g\middle|T_2 - \psi(2) g\middle|_k \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right] \middle| W(2C).
\end{aligned}$$

It follows that

$$\begin{aligned}
f^\rho\middle|W(4C)\middle|Tr_{2C}^{4C} &= 2^{1-k/2} f^\rho\middle|T_2 W(2C) - \overline{\psi(2)} f^\rho\middle|_k \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} W(2C) \\
&= 2^{1-k/2}\overline{a(2)} f^\rho\middle|W(C)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \overline{\psi(2)} f^\rho\middle|W(C).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\left\langle f^\rho\middle|W(N), f\right\rangle_{4C} \\
&= \left\langle f^\rho\middle|W(4C)Tr_{2C}^{4C}, f\right\rangle_{2C} \\
&= 2^{1-k/2} a(2) \left\langle f^\rho\middle|W(C)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, f\right\rangle_{2C} - \psi(2)\left\langle f^\rho\middle|W(C), f\right\rangle_{2C} \\
&= 2^{1-k/2} a(2) 2^{1-k/2} \left\langle f^\rho\middle|T_2 W(C), f\right\rangle_C - 3\psi(2)\left\langle f^\rho\middle|W(C), f\right\rangle_C \\
&= \left[4^{1-k/2}(a(2))^2 - 3\psi(2)\right]\left\langle f^\rho\middle|W(C), f\right\rangle_C .
\end{aligned}$$

Let us now use Lemma 4.5.7 (2) of Miyake [5],

$$T_2T_2 = T_4 + 2\cdot 4^{k/2-1}\psi(2)\mathrm{Id}$$

implying

$$(a(2))^2 = a(4) + 2\cdot 4^{k/2-1}\psi(2). \tag{5.13}$$

It follows immediately that

$$4^{1-k/2}(a(2))^2 - 3\psi(2) = 4^{1-k/2}a(4) - \psi(2) \tag{5.14}$$

so

$$\left\langle f^\rho\middle|W(N), f\right\rangle_{4C} = \overline{\gamma}\left[4^{1-k/2}a(4) - \psi(2)\right]\left\langle f, f\right\rangle_C ,$$

proving Lemma 2. □

References

[1] Robert F. Coleman, *P-adic Banach Spaces and Families of Modular Forms*, Inventiones mathematicae **127** (1997), 417–479.

[2] Andrzej Dabrowski and Daniel Delbourgo, *S-Adic L-Functions Attached to the Symmetric Square of a Newform*, Proc. London Math. Soc. (1991), 559–611.

[3] Neal Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Graduate Texts in Mathematics, vol. 97, Springer-Verlag, 1993.

[4] Wen-Ch'ing Winnie Li, *Newforms and functional equations*, Math. Annalen **212** (1975), 285–315.

[5] Toshitsune Miyake, *Modular Forms*, Springer-Verlag, 1989.

[6] Alexeï A. Panchishkin, *Admissible Measures for Standard L-Functions and Nearly Holomorphic Siegel Modular Forms*, Preprint MPI **42** (2002), 1–65.

[7] ______, *Two Variables p-adic L-Functions Attached to Eigenfamilies of Positive Slope*, Inventiones mathematicae **154** (2003), 551–615.

[8] Julien Puydt, *Valeurs Spéciales de Fonctions L de Formes Modulaires Adéliques* , Ph.D. thesis, Institut Fourier, 2003.

[9] Jean-Pierre Serre, *Formes Modulaires et Fonctions Zêta p-adiques*, Lectures Notes in Mathematics, vol. 350, Springer-Verlag, 1972.

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