HAUSDORFF DIMENSION OF CANTOR SETS AND POLYNOMIAL HULLS

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Abstract

We give examples of Cantor sets in $\mathbb{C}^n$ of Hausdorff dimension 1 whose polynomial hulls have non-empty interior.

In the 60'ies W. Rudin [R] posed the following problem which arose in connection with Banach algebras and polynomial approximation. How small can the dimension of a compact subset K of $\mathbb{C}^n$ be, provided its polynomial hull $\widehat{K}$ has non-empty interior. Asking about topological dimension, Vitushkin [V] and Henkin [H] constructed Cantor sets E in $\mathbb{C}^2$ with the latter property. Note that Cantor sets have topological dimension zero. However, the set in Vitushkin's example has Hausdorff dimension 2 and in Henkin's example the Hausdorff dimension was even bigger. The known results gave rise to the conjecture (see also [V]) that the Hausdorff dimension of a set $K \subset \mathbb{C}^n$ must be at least n if $\widehat{K}$ has non-empty interior.

In the present note we show that this is not the case. However, it would be still interesting to give a reasonable sense to the notion of dimension which approves the corresponding conjecture as true.

The main result of this paper is the following

THEOREM . — *For any natural n there exists a Cantor set E in $\mathbb{C}^n$ of Hausdorff dimension 1 whose polynomial hull contains the unit polydisc.*

The estimate of the Hausdorff dimension is optimal.

LEMMA 1. — *If K is a compact subset of $\mathbb{C}^n$ of zero linear measure (in particular, if the Hausdorff dimension of K is strictly less than one) then K is polynomially convex.*

For convenience of the reader we include a proof of the lemma.

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Proof. — The lemma is true for $n = 1$. Indeed, take an arbitrary point $z \notin K$. After a translation we may assume that $z = 0$. The radial projection of K to the unit circle has vanishing length, hence there is a ray $re^{i\theta}$, $r > 0$, which does not meet K. By Runge's theorem this means that $0 \notin \widehat{K}$.

Assume the lemma is true for n. Prove it for $n+1$. Let $K \subset \mathbb{C}^{n+1}$ have zero length. Take an arbitrary point $z \notin K$. After a translation we may assume that $z = 0$. Then there is a complex line through 0 which avoids K. Indeed, put $A_{n+1} = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_{n+1}| = \max_{j=1,\dots,n+1} |z_j|\}$. If a complex line through 0 intersects $A_{n+1} \setminus \{0\}$, it is contained in A_{n+1}. Since $0 \notin K$, there is a neighbourhood of the set $K \cap A_{n+1}$ which is covered diffeomorphically by the mapping

$$(z', \zeta) \overset{\text{def}}{=} (z_1, \dots, z_n, \zeta) \longrightarrow (\zeta z', \zeta)$$

where z' runs over a neighbourhood of $\overline{\mathbb{D}}^n$ and ζ is in a suitable open subset of $\mathbb{C}$. Here $\mathbb{D}$ denotes the open disc in $\mathbb{C}$ and $\overline{\mathbb{D}}$ its closure.

The linear measure of $K \cap A_{n+1}$ in coordinates (z', ζ) is also zero, hence so is the linear measure of its projection parallel to the ζ-direction. Hence, for some $z' \in \overline{\mathbb{D}}^n$, the line $\zeta \to (\zeta z', \zeta)$ does not meet $K \cap A_{n+1}$, hence it does not meet K.

Denote by π the orthogonal projection in $\mathbb{C}^{n+1}$ onto the orthogonal complement $L \cong \mathbb{C}^n$ of the above line. Then $\pi(K)$ has zero linear measure and does not contain the origin. By hypothesis its polynomial hull in L, $\widehat{\pi(K)}_L$, does not contain the origin. But then $0 \notin \pi(\widehat{K})$. (Consider a polynomial p on L for which $p(0) = 1$ and $\max_{\pi(K)} |p| < 1$ and extend p to $\mathbb{C}^{n+1}$ not depending on the direction orthogonal to L.) Hence $0 \notin \widehat{K}$. $\square$

We prove the theorem first for the case $n = 2$. The building block for the proof will be the following lemma, which is a refinement of the main lemma in [J]. By a complex affine mapping (opposed to a complex linear mapping) we mean a mapping of the form $z \to b + Az$, $z \in \mathbb{C}^k$, $b \in \mathbb{C}^m$, A a constant $k \times m$ matrix, k and m natural numbers. In the same way we will distinguish complex affine and complex linear subspaces of $\mathbb{C}^n$. By an affine quasicircle $C \subset \mathbb{C}^n$ surrounding a point $p \in \mathbb{C}^n$ we mean the following: There exists a complex affine line in $\mathbb{C}^n$ which contains p and C and a smooth quasiconformal mapping of $\mathbb{C}$ onto this line which maps the origin to p and the unit circle to C. Denote by $\mathbb{B}^n$ the open unit ball in $\mathbb{C}^n$.

Lemma 2. — *Let f_j, $j = 1, \dots, N$, be N complex linear functions in $\mathbb{C}^2$ which are transversal to each other and have gradient ∇f_j of length 1. Let σ be any positive number. Consider for each j an affine quasicircle C_j on $\{f_j = 0\}$ which surrounds the origin and is contained in $\mathbb{C}^2 \setminus \sigma\overline{\mathbb{B}^2}$. Denote by $T_j(\varepsilon)$ the closed ε-neighbourhood of C_j.*

There exists a positive constant a depending only on the f_j but not on σ and C_j such that for each sufficiently small $\varepsilon > 0$

$$(a\sigma)^{\frac{N-1}{N}} \varepsilon^{\frac{1}{N}} \overline{\mathbb{B}^2} \subset \widehat{\bigcup_j T_j(\varepsilon)}. \tag{1}$$

(The set on the left hand side of (1) is the ball of radius $(a\sigma)^{\frac{N-1}{N}} \varepsilon^{\frac{1}{N}}$ and center 0.)

Proof. — Assume $N \geqslant 2$. (The assertion for $N = 1$ is trivial.) Replacing z by $\frac{z}{\sigma}$ and ε by $\frac{\varepsilon}{\sigma}$ we may reduce the general case to the case $\sigma = 1$. Let now $\sigma = 1$.

If $\varepsilon > 0$ is small (smaller than the distance of C_j to $\overline{\mathbb{B}^2}$) then for each j

$$\widehat{T_j(\varepsilon)} \supset \{|f_j| \leqslant \varepsilon\} \cap \partial\mathbb{B}^2. \tag{2}$$

By assumption all sets $\{f_j = 0\} \cap \partial\mathbb{B}^2$ are disjoint, hence there is a positive number a such that the sets $\{|f_j| \leqslant a\} \cap \partial\mathbb{B}^2$ are disjoint. We will prove that

$$\Big\{|f_1 \cdots f_N| \leqslant a^{N-1}\varepsilon\Big\} \cap \partial\mathbb{B}^2 \subset \bigcup_{j=1}^{N} \Big\{|f_j| \leqslant \varepsilon\Big\} \cap \partial\mathbb{B}^2. \tag{3}$$

Indeed, let z be a point in the left hand side of (3). Since $z \in \partial\mathbb{B}^2$, all except, maybe, one of the factors $|f_j(z)|$ exceed a, say all but, maybe, $|f_{j_0}(z)|$. Then

$$|f_{j_0}(z)|\, a^{N-1} < |(f_1 \cdots f_N)(z)| \leqslant a^{N-1}\varepsilon.$$

Hence, $|f_{j_0}(z)| \leqslant \varepsilon$ and z is in the right hand side of (3).

Taking the polynomial hull in (3) and taking into account (2) we obtain

$$\widehat{\bigcup_{j=1}^{N} T_j(\varepsilon)} \supset \Big\{|f_1 \cdots f_N| \leqslant a^{N-1}\varepsilon\Big\} \cap \overline{\mathbb{B}^2} \supset a^{\frac{N-1}{N}}\,\varepsilon^{\frac{1}{N}}\,\overline{\mathbb{B}^2}$$

since $f_j(0) = 0$, $|\nabla f_j| = 1$. □

The proof of the theorem for $n = 2$ will be based on the following lemma, which is a consequence of lemma 2.

LEMMA 3. — *Let $C \subset \mathbb{C}^2$ be an affine quasicircle and let σ be a small enough positive number. Denote by $T(3\sigma)$ the 3σ-neighbourhood of C. For any natural number $N \geqslant 5$ there exists a constant c, depending only on N and on the torus $T(3\sigma)$, and for each small enough $\varepsilon > 0$ there exist closed disjoint tori $\widetilde{T}(3\varepsilon)$ around affine quasicircles with the following properties.*

The number $q(\varepsilon)$ of tori satisfies the inequality

$$q(\varepsilon) \leqslant c\varepsilon^{-\frac{4}{N}}, \tag{4}$$

the affine quasicircles have length not exceeding $10\pi\sigma$, the tori $\widetilde{T}(3\varepsilon)$ are contained in $T(3\sigma) \setminus T(2\sigma)$ and

$$\widehat{\bigcup \widetilde{T}(\varepsilon)} \supset T(\sigma). \tag{5}$$

Proof. — Let g be a complex affine function such that $|\nabla g| = 1$ and $C \subset \{g = 0\}$. With the number N as in the statement consider complex linear functions $f_1, \ldots, f_N$, all transversal to each other and such that $|\nabla f_j| = 1$ and the Hermitian scalar product $\langle \nabla f_j, \nabla g\rangle$ is small enough.

Consider a point $p \in T(\sigma)$, an index j and a number $b(p)$ strictly between 2 and 3. Denote by $C_{p,j}$ the intersection of $\partial T(b\sigma)$ with the complex line $\mathscr{L}_j(p)$ through p which is parallel to $\{f_j = 0\}$. Then $C_{p,j}$ is an affine quasicircle surrounding p. Indeed, if F is complex affine with $|\nabla F| = 1$ and ∇F orthogonal to ∇g then $\partial T(b\sigma) \cap \{F = F(p)\}$ bounds a disc in a complex line, the disc containing p. $C_{p,j}$ are small diffeomorphic perturbations of the circle.

Let $\widetilde{T}_{p,j}(\varepsilon)$ be the ε-neighbourhood of $C_{p,j}$. If $\varepsilon > 0$ is small enough then for fixed p the 3ε-neighbourhoods $\widetilde{T}_{p,j}(3\varepsilon)$, $j = 1, ..., N$, are pairwise disjoint and contained in

$$\overline{T(b\sigma + 3\varepsilon) \setminus T(b\sigma - 3\varepsilon)} \subset T(3\sigma) \setminus T(2\sigma), \quad j = 1, \ldots, N.$$

Lemma 2 (applied after a translation) gives

$$\widehat{\bigcup_{j=1}^{N} \widetilde{T}_{p,j}(\varepsilon)} \supset p + (a\sigma)^{\frac{N-1}{N}} \varepsilon^{\frac{1}{N}} \overline{\mathbb{B}}^2 \tag{6}$$

for a constant $a > 0$ which depends on N (precisely, on the choice of the f_j for given g and N), but not on p nor on ε.

Let now p run through a suitable $(a\sigma)^{\frac{N-1}{N}} \varepsilon^{\frac{1}{N}}$ net for $T(\sigma)$. One can choose the latter set so that it contains not more than $c' \cdot \varepsilon^{-\frac{4}{N}}$ points p_k, $k = 1, 2, \ldots$. (c' is a constant depending only on N and on the torus $T(3\sigma)$.) Choose $b(p_k) \cdot \sigma = 3\sigma - 5(2k-1)\varepsilon$. If

$$10c' \cdot \varepsilon^{-\frac{4}{N}} \cdot \varepsilon < \sigma, \tag{7}$$

then all $b(p_k) \cdot \sigma$ are in $[2\sigma + 5\varepsilon, 3\sigma - 5\varepsilon]$. Consider for each k the N affine quasicircles $C_{p_k,j}$ and the tori $\widetilde{T}_{p_k,j}(3\varepsilon)$ associated with p_k, $b(p_k)$ and j as described above.

If (7) holds all tori $\widetilde{T}(3\varepsilon)$ (corresponding to all k and j) are pairwise disjoint and contained in $T(3\sigma) \setminus T(2\sigma)$. Their number $q(\varepsilon)$ does not exceed $c \cdot \varepsilon^{-\frac{4}{N}}$ with $c = N \cdot c'$ and the lengths of the affine quasicircles $C_{p_k,j}$ do not exceed $10\pi \cdot \sigma$ if $\sigma > 0$ is small and the angle between $\{g = 0\}$ and $\{f_j = 0\}$ is close to the right angle. Moreover, by (6)

$$\widehat{\bigcup_{k,j} \widetilde{T}_{p_k,j}(\varepsilon)} \supset T(\sigma),$$

since for p running over the p_k the balls on the right of (6) cover $T(\sigma)$. □

Proof of the theorem for $n = 2$. — Let $C^{(0)}$ be the circle $\{z_1 = 0, |z_2| = 10\}$, let $\sigma = 1$ and $T^{(0)}(3)$ be the closed 3-neighbourhood of $C^{(0)}$. $T^{(0)}(3)$ is a closed solid torus and $\widehat{T^{(0)}(1)} \supset \overline{\mathbb{D}}^2$. Put $E_0 = T^{(0)}(3)$.

Choose a sequence of numbers $N_k \geqslant 5$, $k = 1, 2, \ldots, N_k \to \infty$, for $k \to \infty$. Construct inductively a sequence of closed sets E_k, $k = 1, 2, \ldots$, $E_{k+1} \subset E_k$ for $k = 0, 1, \ldots$. Suppose the set E_k is obtained and has the following properties. E_k is the finite union of disjoint closed tori $T^{(k)}(3\varepsilon_k)$ around affine quasicircles (tori in the k-th generation), and

$$\widehat{\bigcup T^{(k)}(\varepsilon_k)} \supset \overline{\mathbb{D}}^2. \tag{8}$$

The construction of the set E_{k+1} goes as follows.

Put $N = N_{k+1}$. Choose for each torus $T^{(k)}(\varepsilon_k)$ functions $f_1, \dots, f_N$ according to lemma 3. Let $\varepsilon > 0$ be so small that the inequalities (7) are satisfied for each of the tori $T^{(k)}(\varepsilon_k)$. Apply lemma 3 to each of the tori $T^{(k)}(\varepsilon_k)$ and obtain in each of them disjoint closed tori of width 3ε in the $(k+1)^{\text{st}}$ generation.

Denote all tori in the $(k+1)^{\text{st}}$ generation by $T^{(k+1)}(3\varepsilon)$ (omitting indices labeling them). Their total number $q_{k+1}(\varepsilon)$ does not exceed $c_{k+1} \cdot \varepsilon^{-\frac{4}{N_{k+1}}}$, where the constant c_{k+1} depends on N_{k+1} and on all tori $T^{(k)}(3\varepsilon_k)$ of generation k, in particular on the number of those tori.

Further, the $T^{(k+1)}(3\varepsilon)$ are 3ε-neighbourhoods of affine quasicircles of length $\leqslant 10\pi\varepsilon_k$ and the $T^{(k+1)}(3\varepsilon)$ are contained in $\bigcup T^{(k)}(3\varepsilon_k)$. Moreover, by lemma 3 (see (5) for each torus of generation k) we obtain

$$\widehat{\bigcup T^{(k+1)}}(\varepsilon) \supset \bigcup T^{(k)}(\varepsilon_k)\,, \tag{9}$$

hence, by (8)

$$\widehat{\bigcup T^{(k+1)}}(\varepsilon) \supset \overline{\mathbb{D}}^2. \tag{10}$$

The set $\bigcup T^{(k+1)}(\varepsilon)$ can be covered by not more than s_{k+1} balls of radius ε, where

$$\begin{aligned} s_{k+1} &= \text{const} \cdot q_{k+1}(\varepsilon) \cdot 10\pi\varepsilon_k \cdot \varepsilon^{-1} \\ &\leqslant \text{const}\, c_{k+1} \cdot 10\pi\varepsilon_k \cdot \varepsilon^{-1-\frac{4}{N_{k+1}}}. \end{aligned} \tag{11}$$

Choose now for ε a number ε_{k+1} so that (7) is satisfied for all tori of generation k,

$$\varepsilon_{k+1} \leqslant (c_{k+1} \cdot 10\pi\varepsilon_k)^{-(k+1)} \tag{12}$$

and $\varepsilon_{k+1} \to 0$ for $k \to \infty$.

Put $E_{k+1} = \bigcup T^{(k+1)}(3\varepsilon_{k+1})$. Then $E_{k+1} \subset E_k$ and $\widehat{E_{k+1}} \supset \overline{\mathbb{D}}^2$. The set $E \stackrel{\text{def}}{=} \bigcap_{k=0}^{\infty} E_k$ is a Cantor set with $\hat{E} \supset \overline{\mathbb{D}^2}$.

For each k the inclusion $E \subset E_{k+1}$ holds and E_{k+1} can be covered by not more than s_{k+1} balls of radius ε_{k+1} and for any positive α (11) and (12) imply

$$s_{k+1} \cdot \varepsilon_{k+1}^{1+\alpha} \leqslant \text{const} \cdot 10\pi \cdot (\varepsilon_{k+1})^{\alpha - \frac{1}{k+1} - \frac{4}{N_{k+1}}} \underset{k\to\infty}{\longrightarrow} 0. \tag{13}$$

(13) shows that the Hausdorff measure of E of dimension $1+\alpha$ is zero for any positive number α. Hence, the Hausdorff dimension of E equals 1. (It cannot be less than 1 by lemma 1.) The theorem is proved for $\mathbb{C}^2$. □

The proof for $n > 2$ goes along the same lines with lemma 2 replaced by the following lemma 4. We will prove lemma 4, but skip the details of the proof of the theorem in higher dimension.

Let $n \geqslant 2$ and let f_j, $j = 1, \dots, N$, be complex linear functions in $\mathbb{C}^n$. We say that f_j are in general position if for each natural number $k \leqslant n$ the zero sets of any k of them intersect along an $(n-k)$-dimensional linear subspace of $\mathbb{C}^n$. In particular, the gradient of each of them is different from zero and the intersection of the zero set of any n of them is equal to the origin in $\mathbb{C}^n$.

LEMMA 4. — *Let $n \geqslant 2$ and let $f_j,\ j = 1, \dots, N$, be complex linear functions in $\mathbb{C}^n$ in general position, $|\nabla f_j| = 1$. Suppose $N \geqslant n$. For each subset $\mathscr{J} = \{j_1, \dots, j_{n-1}\}$ containing exactly $n-1$ distinct elements of $\{1, 2, \dots, N\}$ denote by $C_{\mathscr{J}}$ an affine quasicircle contained in the complex line $\mathscr{L}_{\mathscr{J}} = \left\{ f_{j_1} = \cdots = f_{j_{n-1}} = 0 \right\}$ and surrounding the origin. Suppose that for some constant $\sigma > 0$ each quasicircle $C_{\mathscr{J}}$ is contained in $\mathbb{C}^n \setminus \sigma\overline{\mathbb{B}}^n$. For small enough $\varepsilon > 0$ denote by $T_{\mathscr{J}}(\varepsilon)$ the ε-neighbourhood of $C_{\mathscr{J}}$.*

There exists a positive constant a depending only on f_j but not on σ nor on the $C_{\mathscr{J}}$ such that for each sufficiently small $\varepsilon > 0$

$$(a\sigma)^{1-\beta_n(N)} \varepsilon^{\beta_n(N)} \overline{\mathbb{B}^n} \subset \widehat{\bigcup_{\mathscr{J}} T_{\mathscr{J}}(\varepsilon)}. \tag{14}$$

Here $\beta_n(N) \overset{\text{def}}{=} \frac{n(n-1)}{2(N-n+2)}$. Note that $\beta_n(N) \to 0$ for n fixed and $N \to \infty$. The union on the right hand side is taken over all subsets $\mathscr{J}$ of $\{1, \dots, N\}$ containing $n-1$ distinct elements.

Proof. — The case of general $\sigma > 0$ can be reduced to the case $\sigma = 1$ by replacing z by $\frac{z}{\sigma}$ and ε by $\frac{\varepsilon}{\sigma}$. We may therefore assume that $\sigma = 1$. Lemma 2 implies lemma 4 for $n = 2$. Indeed, $\beta_2(N) = \frac{1}{N}$. Assume, lemma 4 is true for $n-1$ ($n-1 \geqslant 2$). Prove it for n.

Note first that there exists a constant $A > 0$ depending only on n and on the functions f_j such that for each subset $\mathscr{J} = \{j_0, \dots, j_{n-2}\}$ of $\{1, \dots, N\}$ containing $n-1$ distinct elements the set

$$\{|f_{j_0}| \leqslant A\varepsilon, \dots, |f_{j_{n-2}}| \leqslant A\varepsilon\}$$

is contained in the ε-neighbourhood of the line $\mathscr{L}_{\mathscr{J}} = \{f_{j_0} = \cdots = f_{j_{n-2}} = 0\}$. Moreover, writing $\mathscr{J} = \{j_0\} \cup \mathscr{J}',\ \mathscr{J}' = \{j_1, \dots, j_{n-2}\}$, we find for each point $\zeta \in \mathbb{C}$, $|\zeta| \leqslant A\varepsilon$, a subset $\tau_{j_0,\zeta,\mathscr{J}'}(A\varepsilon)$ of

$$\{f_{j_0} = \zeta\} \cap \left\{|f_{j_1}| \leqslant A\varepsilon, \dots, |f_{j_{n-2}}| \leqslant A\varepsilon\right\}$$

with the following properties.

Identify the set $\{f_{j_0} = \zeta\}$ with $\mathbb{C}^{n-1}$ by the affine mapping of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ which preserves length and maps $0 \in \mathbb{C}^{n-1}$ to the orthogonal projection of $0 \in \mathbb{C}^n$ to $\{f_{j_0} = \zeta\}$. With this identification we choose $\tau_{j_0,\zeta,\mathscr{J}'}(A\varepsilon)$ as the $A\varepsilon$-neighbourhood in $\{f_{j_0} = \zeta\}$ of an affine quasicircle, the latter being close to $C_{\mathscr{J}}$ and contained in $\{f_{j_0} = \zeta\} \cap \left\{f_{j_1} = \cdots = f_{j_{n-2}} = 0\right\}$. If ε is small the sets $\tau_{j_0,\zeta,\mathscr{J}'}(A\varepsilon)$ are outside the unit ball of $\{f_{j_0} = \zeta\}$. Moreover, the choice can be done in such a way that

$$\tau_{j_0,\zeta,\mathscr{J}'}(A\varepsilon) \subset T_{\mathscr{J}}(\varepsilon). \tag{15}$$

By the lemma for $n-1$ (applied to the $N-1$ functions $f_j,\ j \neq j_0$) for each $j_0 \in \{1, \dots, N\}$ and each $|\zeta| \leqslant A\varepsilon$

$$\widehat{\bigcup_{\mathscr{J}'} \tau_{j_0,\zeta,\mathscr{J}'}}(A\varepsilon) \supset a_{n-1}(A\varepsilon)^{\beta_{n-1}(N-1)} \overline{\mathbb{B}_{\zeta}^{n-1}}, \tag{16}$$

where $\mathcal{J}'$ runs over subsets of $\{1,\dots,N\} \setminus \{j_0\}$ containing $n-2$ different elements, a_{n-1} is a positive constant depending on the f_j, and $\mathbb{B}^{n-1}_\zeta$ is the unit ball in $\{f_{j_0} = \zeta\}$. Take in (16) the union over $|\zeta| \leqslant A\varepsilon$, we obtain by (15)

$$\widehat{\bigcup_{\mathcal{J}=\{j_0\}\cup\mathcal{J}'} T_{\mathcal{J}}(\varepsilon)} \supset \{|f_{j_0}| \leqslant A\varepsilon\} \cap a'\varepsilon^{\beta_{n-1}(N-1)}\overline{\mathbb{B}^n}, \tag{17}$$

for another positive constant a'. □

It remains to prove the following

Claim. — *For small $\varepsilon > 0$ the polynomial hull of*

$$\bigcup_{j_0=1}^{N} \{|f_{j_0}| \leqslant A\varepsilon\} \cap a'\varepsilon^{\beta_{n-1}(N-1)}\partial\mathbb{B}^n \tag{18}$$

contains the ball $a^{1-\beta_n(N)}\varepsilon^{\beta_n(N)}\overline{\mathbb{B}^n}$ for a positive constant a depending only on the f_{j_0}.

Proof. — Denote $\alpha = a'\varepsilon^{\beta_{n-1}(N-1)}$ and change variables, $\tilde{z} = \frac{z}{\alpha}$. We have to consider the polynomial hull of

$$\bigcup_{j=1}^{N} \left\{|f_j(\tilde{z})| \leqslant \frac{A\varepsilon}{\alpha}\right\} \cap \partial\mathbb{B}^n. \tag{19}$$

There exists a constant $\tilde{a} > 0$ such that on $\partial\mathbb{B}^n$ at most $n-1$ of the $|f_j|$ do not exceed $\tilde{a}$. This follows from the genericity assumption for the f_j. It implies that the set

$$\left\{\tilde{z} \in \partial\mathbb{B}^n : \prod_{j=1}^{N} |f_j(\tilde{z})| \leqslant \tilde{a}^{N-(n-1)} \left(\frac{A\varepsilon}{\alpha}\right)^{n-1}\right\} \tag{20}$$

is contained in (19). Indeed, if $\tilde{z}$ is contained in (20) then for some set $\mathcal{J}$ containing $n-1$ elements

$$\prod_{j\in\mathcal{J}} |f_j(\tilde{z})| \leqslant \left(\frac{A\varepsilon}{\alpha}\right)^{n-1},$$

hence, at least one of the $|f_j(\tilde{z})|$ does not exceed $\frac{A\varepsilon}{\alpha}$.

The obtained inclusion implies that the polynomial hull of (19) contains

$$\left\{\prod_{j=1}^{N} |f_j(\tilde{z})| \leqslant \tilde{a}^{N-(n-1)} \left(\frac{A\varepsilon}{\alpha}\right)^{n-1}\right\} \cap \overline{\mathbb{B}^n} \supset \tilde{a}^{\frac{N-(n-1)}{N}} \left(\frac{A\varepsilon}{\alpha}\right)^{\frac{n-1}{N}} \overline{\mathbb{B}^n}.$$

Here we used that $f_j(0) = 0$ and $|\nabla f_j| = 1$.

Rescaling, $z = \alpha\tilde{z}$, we conclude that the polynomial hull of (18) contains the ball with center 0 and radius

$$\tilde{a}^{1-\frac{N-1}{N}} \cdot (A\varepsilon\alpha^{-1})^{\frac{n-1}{N}} \cdot \alpha. \tag{21}$$

By definition of α and $\beta_{n-1}(N-1)$ the power of ε in the last expression is

$$\frac{n-1}{N}+\left(1-\frac{n-1}{N}\right)\frac{(n-1)(n-2)}{2(N-n+2)}<\frac{n-1}{N-n+2}+\frac{(n-1)(n-2)}{2(N-n+2)}$$
$$=\frac{n(n-1)}{2(N-n+2)}=\beta_n(N).$$

Note that increasing the power of ε will decrease the expression (21) provided $\varepsilon<1$. Denote the absolute constant in (21) by $a^{1-\beta_n(N)}$. We obtained that the polynomial hull in (17) contains the ball of radius $a^{1-\beta_n(N)}\cdot\varepsilon^{\beta_n(N)}$ with center 0. □

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