On Minkowski's bound for lattice-packings

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Abstract: We give a new proof of the Minkowski-Hlawka bound on the existence of dense lattices. The proof is based on an elementary method for constructing dense lattices which is almost effective. [1]

1 Introduction

Let $\mu \geq 2$ be a strictly positive integer. A $\mu-$*sequence* is a sequence $s_0 = 1, s_1, s_2, \ldots$ of strictly positive integers such that the $n-$dimensional lattice

$$\Lambda_n = \{(z_0, z_1, \ldots, z_n) \in \mathbf{Z}^{n+1} \mid \sum_{k=0}^{n} s_k z_k = 0\} = (s_0, \ldots, s_n) \cap \mathbf{Z}^{n+1}$$

has minimum $\geq \mu$ for all $n \geq 1$. Since $\det(\Lambda_n) = \sum_{k=0}^{n} s_k^2$ we get a lower bound for the center-density

$$\delta(\Lambda_n) = \sqrt{\frac{(\min \Lambda_n)^n}{4^n \det \Lambda_n}} \geq \sqrt{\frac{\mu^n}{4^n \sum_{k=0}^{n} s_k^2}}$$

(or for the density $\Delta(\Lambda_n) = \delta(\Lambda_n)\pi^{n/2}/(n/2)!$) of the $n-$dimensional lattice Λ_n associated to a $\mu-$sequence.

Theorem 1.1 *Given an integer $\mu \geq 2$ as above there exists a $\mu-$sequence $s_0 = 1, s_1, \ldots$ satisfying for all $n \geq 1$*

$$s_n \leq 1 + \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^{\,n} \frac{\sqrt{\pi}^n}{(n/2)!} \leq \sqrt{\mu}\sqrt{\mu + n/4}^{\,n} \frac{\sqrt{\pi}^n}{(n/2)!} .$$

Remark 1.2 *(i) The condition $s_0 = 1$ is of no real importance and can be omitted after minor modifications. It is of course also possible (but not very useful) to consider sequences with coefficients in $\mathbf{Z}$.*

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(ii) Any subsequence $s_{i_0} = s_0, s_{i_1}, s_{i_2}, \ldots$ *of a* $\mu-$*sequence is again a* $\mu-$*sequence. Reordering the terms of a* $\mu-$*sequence (in increasing order) yields of course again a* $\mu-$*sequence.*

(iii) The lattices associated to a $\mu-$*sequence are generally neither perfect nor eutactic (cf. [4] for definitions) and one can thus generally improve their densities by suitable deformations.*

The proof of Theorem 1.1 is very elementary and consists of an analysis of the "greedy algorithm" which constructs the first $\mu-$sequence with respect to the lexicographic order on sequences. An easy analysis shows that the lexicographically first sequence satisfies the first inequalities of Theorem 1.1. The greedy algorithm, although very simple, is however quite useless for applications because of astronomical memory requirements (which can be lowered at the price of an astronomical amount of computations).

$\mu-$sequences satisfying the inequalities of Theorem 1.1 yield rather dense lattices as shown by the next result.

Corollary 1.3 *For any* $\mu \geq 2$*, there exists a* $\mu-$*sequence* $(s_0, s_1, \ldots, s_n) \in \mathbf{Z}^{n+1}$ *such that the density of the associated lattice* $\Lambda_n = (s_0, \ldots, s_n)^\perp \cap \mathbf{Z}^{n+1}$ *satisfies*

$$\Delta(\Lambda_n) \geq \frac{(1 + n/(4\mu))^{-n/2}}{2^n \sqrt{(n+1)\mu}} \ .$$

Remark 1.4 *Taking* $\mu \sim n^2/4$ *we get the existence of lattices in dimension* n *(for large* n*) with density* Δ *roughly at least equal to*

$$\frac{1}{2^{n-1}\ n\ \sqrt{(n+1)\ e}} \ .$$

This is already close to the Minkowski-Hlawka bound (which shows the existence of lattices with density at least $\zeta(n)\ 2^{1-n}$*, cf. formula (14) in [2], Chapter 1. The best known lower bound concerning densities of lattice packings (together with a very nice proof) seems to be due to Ball and asserts the existence of* $n-$*dimensional lattices with density at least* $2(n-1)2^{-n}\zeta(n)$*, see [1].*

A more careful analysis of $\mu-$sequences yields the following result.

Theorem 1.5 *For every* $\epsilon > 0$*, there exist* $n-$*dimensional lattices with density*

$$\Delta \geq \frac{1-\epsilon}{2^n\ \sum_{k=1}^{\infty} e^{-k^2\pi}} \sim (1-\epsilon)\ 23.1388\ 2^{-n}$$

for all n *large enough.*

2 Definitions

For the convenience of the reader this section contains all needed facts concerning lattices. Reference for lattices and lattice-packings are [2] and [4].

An $n-$*dimensional lattice* is a discret-cocompact subgroup Λ of the $n-$dimensional Euclidean vector space $\mathbf{E}^n$ (with scalar product denoted by $\langle\ ,\ \rangle$). The *determinant* of a lattice Λ is the square of the volume of a fundamental domain $\mathbf{E}^n/\Lambda$ and equals $\det(\langle b_i, b_j\rangle)$ where $b_1, \dots, b_n$ denotes a $\mathbf{Z}-$base of Λ. The *norm* of a lattice element $\lambda \in \Lambda$ is defined as $\langle \lambda, \lambda\rangle$ (and is thus the squared Euclidean norm of λ). A lattice Λ is *integral* if the scalar product takes only integral values on $\Lambda \times \Lambda$. The *minimum*

$$\min \Lambda\ = \min_{\lambda\in\Lambda\setminus\{0\}} \langle \lambda, \lambda\rangle$$

of a lattice Λ is the norm of a shortest non-zero vector in Λ. The *density* $\Delta(\Lambda)$ and the *center-density* $\delta(\Lambda)$ of an $n-$dimensional lattice Λ are defined as

$$\Delta(\Lambda) = \sqrt{\frac{(\min\ \Lambda)^n}{4^n\ \det\ \Lambda}}\, V_n \qquad \text{and} \qquad \delta(\Lambda) = \sqrt{\frac{(\min\ \Lambda)^n}{4^n\ \det\ \Lambda}}$$

where $V_n = \pi^{n/2}/(n/2)!$ denotes the volume of the $n-$dimensional unit-ball in $\mathbf{E}^n$. These two densities are proportional (for a given dimension n) and measure the quality of the sphere-packing associated to the lattice Λ obtained by packing $n-$dimensional Euclidean balls of radius $\sqrt{\min \Lambda/4}$ centered at all points of Λ.

Given an $n-$dimensional lattice $\Lambda \subset \mathbf{E}^n$ the subset

$$\Lambda^\sharp = \{x \in \mathbf{E}^n \mid \langle x, \lambda\rangle \in \mathbf{Z} \quad \forall \lambda \in \Lambda\}$$

is also a lattice called the *dual lattice* of Λ. The lattice Λ is integral if and only if $\Lambda \subset \Lambda^\sharp$. For an integral lattice the *determinant group* $\Lambda^\sharp/\Lambda$ is a finite abelian group consisting of $(\det\ \Lambda)$ elements.

A sublattice $M \subset \Lambda$ is *saturated* if Λ/M is a free group (or equivalently if $M = (M \otimes_{\mathbf{Z}} \mathbf{R}) \cap \Lambda$).

We leave the proof of the following well-known result to the reader.

Proposition 2.1 *(cf. Chapter I, Proposition 9.8 in [4]) Let Λ be an integral lattice of determinant* 1. *Let $M, N \subset \Lambda$ be two sublattices of Λ such that*

$$M = \Lambda \cap N^\perp \text{ and } N = \Lambda \cap M^\perp$$

where $X^\perp \subset \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ denotes the subspace of all vectors orthogonal to X in the Euclidean vector-space $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ (i.e. M and N are saturated sublattices, orthogonal to each other and $M \oplus N$ is of finite index in Λ).

Then the two determinant groups

$$M^\sharp/M \text{ and } N^\sharp/N$$

are isomorphic. In particular, the determinants of the lattices M and N are equal.

Given a $\mathbf{Z}-$basis $b_1, \ldots, b_n \in \Lambda$ of an $n-$dimensional lattice Λ, the symmetric positive definite matrix G with coefficients

$$G_{i,j} = \langle b_i, b_j \rangle$$

is called a *Gram matrix* of Λ. Its determinant $\det(G)$ is independent of the choice of the basis and equals the determinant of Λ.

Two lattices Λ and M are *similar*, if there exists a bijection $\Lambda \longrightarrow M$ which extends to an Euclidean similarity between $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ and $M \otimes_{\mathbf{Z}} \mathbf{R}$. The set of similarity classes of lattices is endowed with a natural topology: a neighbourhood of a lattice Λ is given by all lattices having a Gram matrix in $\mathbf{R}_{>0}\ V(G)$ where $V(G)$ is a neighbourhood of a fixed Gram matrix G of Λ.

Similar lattices have identical densities and the density function $\Lambda \longmapsto \Delta(\Lambda)$ is continuous with respect to the natural topology on similarity classes.

Consider the set $\mathcal{L}_n$ of all $n-$dimensional lattices of the form

$$\Lambda = \{z \in \mathbf{Z}^{n+1} \mid \langle z, s \rangle = 0\}$$

for $s \in \mathbf{N}^{n+1} \setminus \{0\}$.

Proposition 2.2 *The set $\mathcal{L}_n$ is dense in the set of similarity classes of $n-$dimensional Euclidean lattices.*

The upper bound for densities of lattices in $\mathcal{L}_n$ is thus equal to the maximum for densities of all $n-$dimensional lattices.

Proof of Proposition 2.2 Given a Gram matrix $G = \langle b_i, b_j \rangle$ with respect to a $\mathbf{Z}-$basis $b_1, \ldots, b_n \in \Lambda$ of an $n-$dimensional lattice Λ, Gram-Schmidt orthogonalization shows that

$$G = L\ L^t$$

where $L = (l_{i,j})_{1 \leq i,j \leq n}$ is lower triangular (and invertible).

Choose $\kappa > 0$ large and consider the integral lower triangular matrix $\tilde{L}(\kappa)$ whose coefficients $\tilde{l}_{i,j} \in \mathbf{Z}$ satisfy

$$|\tilde{l}_{i,j} - \kappa l_{i,j}| \leq 1/2$$

and are obtained by rounding off the coefficients of κL to the nearest integers.

Define an integral matrix

$$B(\kappa) = \begin{pmatrix} \tilde{l}_{1,1} & 1 & 0 & 0 & \\ \tilde{l}_{2,1} & \tilde{l}_{2,2} & 1 & 0 & 0 \\ \vdots & & & \ddots & \end{pmatrix}$$

of size $n \times (n+1)$ with coefficients

$$b_{i,j} = \begin{cases} \tilde{l}_{i,j} & \text{if } j \leq i \\ 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} . \end{cases}$$

It is easy to see that the rows of $B(\kappa)$ span a saturated integral sublattice $\tilde{\Lambda}(\kappa)$ of dimension n in $\mathbf{Z}^{n+1}$. The special form of $B(\kappa)$ shows that there exists an integral row-vector

$$v(\kappa) = \begin{pmatrix} 1 \\ -\tilde{l}_{1,1} \\ \tilde{l}_{1,1}\tilde{l}_{2,2} - \tilde{l}_{2,1} \\ \vdots \end{pmatrix} \in \mathbf{Z}^{n+1}$$

such that $B(\kappa)v(\kappa) = 0$. We have thus

$$\tilde{\Lambda}(\kappa) = v(\kappa)^{\perp} \cap \mathbf{Z}^{n+1} \subset \mathbf{E}^{n+1} .$$

Since $\lim_{\kappa\to\infty} \frac{1}{\kappa^2}\ B(\kappa)$ is given by the matrix L with an extra row of zeros appended, we have

$$\lim_{\kappa\to\infty} \frac{1}{\kappa^2}\ B(\kappa)(B(\kappa))^t = G$$

and the lattice $\frac{1}{\kappa}\tilde{\Lambda}(\kappa)$ converges thus to the lattice Λ for $\kappa \to \infty$. Considering the integral vector $s = (s_0, s_1, \ldots)$ defined by $s_i = |v(\kappa)_{i+1}|$ for $i = 0, \ldots, n$, we get an integral lattice

$$\{z = (z_0, \ldots, z_n) \in \mathbf{Z}^{n+1} \mid \langle z, s\rangle = 0\}$$

which is isometric to $\tilde{\Lambda}(\kappa)$. □

3 Proof of Theorem 1.1

Lemma 3.1 *The standard Euclidean lattice* $\mathbf{Z}^n$ *contains at most*

$$2\sqrt{\mu + n/4}^{\,n} \frac{\pi^{n/2}}{(n/2)!}$$

vectors of (squared Euclidean) norm $\leq \mu$.

Proof We denote by

$$B_{\leq\sqrt{\rho}}(x) = \{z \in \mathbf{E}^n \mid \langle z - x, z - x\rangle \leq \rho\}$$

the closed Euclidean ball with radius $\sqrt{\rho} \geq 0$ and center $x \in \mathbf{E}^n$. Given $0 \leq \sqrt{\mu}$, $\sqrt{\rho}$ and $x \in B_{\leq\sqrt{\mu}}(0)$, the closed half-ball

$$\{z \in \mathbf{E}^n \mid \langle z, x\rangle \leq \langle x, x\rangle\} \cap B_{\leq\sqrt{\rho}}(x)$$

(obtained by intersecting the closed halfspace $H_x = \{z \in \mathbf{E}^n \mid \langle z, x\rangle \leq \langle x, x\rangle\}$ with the Euclidean ball $B_{\leq\sqrt{\rho}}(x)$ centered at $x \in \partial H_x$) is contained in $B_{\leq\sqrt{\mu+\rho}}(0)$.

Since the regular standard cube

$$C = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbf{E}^n$$

of volume 1 is contained in a ball of radius $\sqrt{n/4}$ centered at the origin, the intersection

$$(z + C) \cap \{x \in \mathbf{E}^n \mid \langle x, x\rangle \leq \mu + n/4\} = (z + C) \cap B_{\leq\sqrt{\mu+n/4}}(0)$$

is of volume at least $1/2$ for any element $z \in \mathbf{E}^n$ of norm $\langle z, z\rangle \leq \mu$.

Since integral translates of C tile $\mathbf{E}^n$, we have

$$\frac{1}{2}\sharp\{z \in \mathbf{Z}^n \mid \langle z, z\rangle \leq \mu\} \leq \mathrm{Vol}\ \{x \in \mathbf{E}^n \mid \langle x, x\rangle \leq \mu + n/4\}\ .$$

Using the fact that the unit ball in Euclidean $n-$space has volume $\pi^{n/2}/(n/2)!$ (cf. Chapter 1, formula 17 in [2]) we get the result. □

Proof of Theorem 1.1 For $n = 0$, the first inequality boils down to $s_0 = 1 \leq 1 + \sqrt{\mu - 2}$ and holds for $\mu \geq 2$. Consider now for $n \geq 1$ a $\mu-$sequence $(s_0, \ldots, s_{n-1}) \in \mathbf{N}^n$.

Introduce the set

$$\mathcal{F} = \{(a, k) \in \mathbf{N}^2 \mid \exists\ z = (z_0, \ldots, z_{n-1}) \in \mathbf{Z}^n \setminus \{0\} \text{ such that}$$
$$ak = |\langle z, (s_0, \ldots, s_{n-1})\rangle| \text{ and } \langle z, z\rangle + k^2 < \mu\}\ .$$

Since Λ_{n-1} has minimum $\geq \mu$, the equality $\langle z, (s_0, \ldots, s_{n-1})\rangle = 0$ implies $\langle z, z\rangle \geq \mu$ for $z \in \mathbf{Z}^n \setminus \{0\}$. This shows that we have $a, k > 0$ for $(a, k) \in \mathcal{F}$.

Since for a given pair of opposite non-zero vectors $\pm z \in \mathbf{Z}^n$ with norm $0 < \langle z, z\rangle < \mu$ there are at most $\sqrt{\mu - 1 - \langle z, z\rangle} \leq \sqrt{\mu - 2}$ strictly positive integers k such that $\langle z, z\rangle + k^2 < \mu$, such a pair $\pm z$ of vectors contributes at most $\sqrt{\mu - 2}$ distinct elements to $\mathcal{F}$. The cardinality $f = \sharp(\mathcal{F})$ of $\mathcal{F}$ is thus bounded by

$$f \leq \sqrt{\mu - 2}\frac{\sharp\{z \in \mathbf{Z}^n \mid 0 < \langle z, z\rangle \leq \mu - 1\}}{2} \leq \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^{\,n}\ \frac{\pi^{n/2}}{(n/2)!}$$

where the last inequality follows from Lemma 3.1. There exists thus a strictly positive integer

$$s_n \leq f + 1 \leq 1 + \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^{\,n}\ \frac{\pi^{n/2}}{(n/2)!}$$

such that $(s_n, k) \notin \mathcal{F}$ for all $k \in \mathbf{N}$. The strictly positive integer s_n satisfies the first inequality of the Theorem and it is straightforward to check that the $n-$dimensional lattice

$$\Lambda_n = \{z \in \mathbf{Z}^{n+1} \mid \sum_{i=0}^{n} s_i z_i = 0\}$$

has minimum $\geq \mu$. This shows the first inequality.

The second inequality

$$1 + \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} \leq \sqrt{\mu}\sqrt{\mu + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!}$$

of Theorem 1.1 boils down to

$$1 \leq \sqrt{2}\sqrt{2 + n/4}\frac{\sqrt{\pi}^n}{(n/2)!}$$

for $\mu = 2$. This inequality is clearly true since the $n-$dimensional Euclideean ball of radius $\sqrt{2 + n/4}$ has volume $\sqrt{2 + n/4}\frac{\sqrt{\pi}^n}{(n/2)!}$ and contains the regular cube $[-\frac{1}{2}, \frac{1}{2}]^n$ of volume 1.

For $\mu \geq 3$ we have to establish the inequality $\Phi(1) - \Phi(0) \geq 1$ where

$$\Phi(t) = \sqrt{\mu - 2 + 2t}\sqrt{\mu - 1 + t + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} \ .$$

We get thus

$$\begin{aligned} \Phi(1) - \Phi(0) &\geq \inf_{\xi \in (0,1)} \Phi'(\xi) \\ &\geq \frac{1}{\sqrt{\mu}}\sqrt{\mu - 1 + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} + \frac{n}{2}\sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^{n-2} \frac{\sqrt{\pi}^n}{(n/2)!} \ . \end{aligned}$$

For $n = 1$ and $\mu \geq 2$ we have

$$\Phi(1) - \Phi(0) \geq \sqrt{1 - \frac{1}{\mu}}\frac{\sqrt{\pi}}{\sqrt{\pi}/2} \geq \frac{2}{\sqrt{2}} > 1 \ .$$

For $n \geq 2$ and $\mu \geq 3$ we get

$$\Phi(1) - \Phi(0) \geq \sqrt{2 + n/4}^{n-2} \frac{\sqrt{\pi}^{n-2}}{((n-2)/2)!} \pi$$

and the right-hand side equals $\pi > 1$ for $n = 2$. For $n > 2$, the right hand side equals π times the volume of the $(n-2)-$dimensional ball of radius $\sqrt{2 + n/4}$ containing the regular cube $[-\frac{1}{2}, \frac{1}{2}]^{n-2}$ of volume 1. The second inequality follows. □

Proof of Corollary 1.3 Theorem 1.1 shows the existence of a $\mu-$sequence $(s_0 = 1, \ldots, s_n)$ satisfying

$$s_0, \ldots, s_n \leq \sqrt{\mu}\sqrt{\mu + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} \ .$$

This shows for the lattice $\Lambda_n = (s_0, \ldots, s_n)^\perp \cap \mathbf{Z}^{n+1}$ the inequality

$$\det \Lambda_n = \sum_{i=0}^{n} s_i^2 \leq (n+1)\mu(\mu + n/4)^n \frac{\pi^n}{((n/2)!)^2} = (n+1)\mu(\mu+n/4)^n V_n^2$$

and implies

$$\Delta(\Lambda_n) \geq \sqrt{\frac{\mu^n}{4^n (n+1)\mu(\mu+n/4)^n V_n^2}} V_n \ .$$

This proves the Corollary. □

4 Proof of Theorem 1.5

The main idea for proving Theorem 1.5 is to get rid of a factor $\sqrt{\mu}$ when computing an upper bound f for the size of the finite set $\mathcal{F}$ considered above during the proof of Theorem 1.1. This is possible since the volume of the $n-$dimensional unit-ball centered at the origin is concentrated along hyperplanes for large n. For the sake of simplicity, we consider sequences in the $\mu \to \infty$ limit. This allows us to neglect boundary effects when replacing counting arguments by volume-computations.

In the sequel we write

$$g(x) \sim_{x\to\alpha} h(x) \ , \ \text{respectively} \ g(x) \leq_{x\to\alpha} h(x) \ ,$$

for

$$\lim_{x\to\alpha} \frac{g(x)}{h(x)} = 1 \ , \ \text{respectively} \ \text{limsup}_{x\to\alpha} \frac{g(x)}{h(x)} \leq 1 \ ,$$

where $g(x), \ h(x) > 0$.

The following easy Lemma will be useful.

Lemma 4.1 *We have*

$$\sqrt{n}\ \frac{V_n}{V_{n-1}} = \sqrt{2\pi}(1 - \frac{1}{4n} + O(\frac{1}{n^2})) \ .$$

Proof. Using the definition $V_n = \frac{\pi^{n/2}}{(n/2)!}$ and Stirlings formula $n! \sim \sqrt{2\pi n}\ n^n\ e^{-n}(1 + \frac{1}{12n} + O(\frac{1}{n^2}))$, we have

$$\begin{aligned}
\sqrt{n}\ \frac{V_n}{V_{n-1}} &= \sqrt{n}\frac{\pi^{n/2}}{\pi^{(n-1)/2}}\ \frac{((n-1)/2)!}{(n/2)!} \\
&= \sqrt{n\ \pi}\frac{\sqrt{2\pi(n-1)/2}}{\sqrt{2\pi n/2}}\ \frac{(n-1)^{(n-1)/2}}{2^{(n-1)/2}}\ \frac{2^{n/2}}{n^{n/2}}\ \frac{e^{n/2}}{e^{(n-1)/2}} \\
&\quad \left(\frac{(1+\frac{1}{6(n-1)}+O(\frac{1}{(n-1)^2}))}{(1+\frac{1}{6n}+O(\frac{1}{n^2}))}\right) \\
&= \sqrt{2\pi e}(1-\tfrac{1}{n})^{n/2}(1+O(\tfrac{1}{n^2})) \\
&= \sqrt{2\pi e}\ e^{-\frac{n}{2}(\frac{1}{n}+\frac{1}{2n^2}+O(\frac{1}{n^3}))}(1+O(\tfrac{1}{n^2})) \\
&= \sqrt{2\pi}(1-\tfrac{1}{4n}+O(\tfrac{1}{n^2}))
\end{aligned}$$

which ends the proof. □

Proof of Theorem 1.5 Let $(s_0,\ldots,s_{n-1})$ be a finite $\mu-$sequence. For $\epsilon>0$ fixed and suitable $\sigma_n>0$, we show the existence of a $\mu-$sequence $(s_0,\ldots,s_{n-1},s_n)$ with $s_n\in I\cap\mathbf{N}$ where $I=[\sigma_n\mu^{n/2}V_n,(1+\epsilon)\sigma_n\mu^{n/2}V_n]$.

For $k=1,2,\ldots,\in\mathbf{N}$ we define finite subsets

$$I_k=\{s\in I\cap\mathbf{N}\mid\ \sum_{i=0}^{n-1}s_ix_i=ks \text{ for some } (x_0,\ldots,x_{n-1})\in B_{<\sqrt{\mu-k^2}}\cap\mathbf{Z}^n\}$$

of natural integers in $I\cap\mathbf{N}$ where $B_{<\sqrt{\mu-k^2}}\cap\mathbf{Z}^n$ denotes the set of all integral vectors $(x_0,\ldots,x_{n-1})\in\mathbf{Z}^n$ having (squared Euclidean) norm strictly smaller than $\mu-k^2$. A sequence $(s_0,\ldots,s_{n-1},s_n)$ with $s_n\in I$ is a $\mu-$sequence if and only if $s_n\notin I_k$ for $k=1,2,\ldots,\lfloor\sqrt{\mu}\rfloor$. Introducing the sets

$$X_k=\{(x_0,\ldots,x_{n-1})\in\mathbf{Z}^n\mid\ \frac{1}{k}\sum_{i=0}^{n-1}s_ix_i\in I\cap\mathbf{N},\ \sum_{i=0}^{n-1}x_i^2<\mu-k^2\}$$

we have

$$\sharp(\cup_{k=0}^{\lfloor\sqrt{\mu}\rfloor}I_k)\leq\sum_{k=0}^{\lfloor\sqrt{\mu}\rfloor}\sharp(I_k)\leq\sum_{k=0}^{\lfloor\sqrt{\mu}\rfloor}\sharp(X_k)\ .$$

For μ large enough this ensures the existence of a $\mu-$sequence $(s_0,\ldots,s_{n-1},s_n)$ with $s_n\leq(1+\epsilon)\sigma_n\mu^{n/2}V_n$ if

$$(1+\epsilon')\sum_{k=0}^{\lfloor\sqrt{\mu}\rfloor}\sharp(X_k)\leq_{\mu\rightarrow\infty}\epsilon\sigma_n\mu^{n/2}V_n \tag{1}$$

for some $0<\epsilon'$.

Set

$$X_k(*)=\{(x_0,\ldots,x_{n-1})\in\mathbf{Z}^n\mid\ \frac{1}{k}\sum_{i=0}^{n-1}s_ix_i\in I,\ \sum_{i=0}^{n-1}x_i^2<\mu-k^2\}$$

and consider the partition $X_k(*)=X_k(0)\cup X_k(1)\cup\ldots\cup X_k(k-1)$ defined by the disjoint subsets

$$X_k(a)=\{\sharp\{(x_0,\ldots,x_{n-1})\in X_k(*)\mid\ \sum_{i=0}^{n-1}s_ix_i\equiv a\pmod{k}\}\subset X_k(*)\ .$$

Since $s_0=1$ we have (for $\epsilon>0$ fixed) the asymptotic equalities

$$\sharp(X_k(j))\sim_{\mu\rightarrow\infty}\frac{1}{k}\sharp(X_k(*))$$

for $j=0,\ldots,k-1$.

The obvious identity $X_k = X_k(0)$ yields thus

$$\begin{aligned}\sharp(X_k) \quad &\sim_{\mu\to\infty} \quad \frac{1}{k}\sharp\{(x_0,\dots,x_{n-1})\in \mathbf{Z}^n \mid \frac{1}{k}\sum_{i=0}^{n-1} s_i x_i \in I,\ \sum_{i=0}^{n-1} x_i^2 < \mu - k^2\} \\ &\sim_{\mu\to\infty} \quad \frac{1}{k}\mathrm{Vol}\{(t_0,\dots,t_{n-1})\in \mathbf{E}^n \mid \sum_{i=0}^{n-1} t_i^2 \le \mu - k^2,\ \frac{1}{k}\sum_{i=0}^{n-1} s_i t_i \in I\}\ .\end{aligned}$$

Setting $\tilde{s}_{n-1} = \sqrt{\sum_{i=0}^{n-1} s_i^2}$ and $\tilde{\sigma}_{n-1} = \frac{\tilde{s}_{n-1}}{\sqrt{\mu}^{n-1} V_{n-1}}$ we have

$$\sharp(X_k) \sim_{\mu\to\infty} \frac{1}{k}\int_{\alpha}^{(1+\epsilon)\alpha} \sqrt{\mu - t^2}^{\,n-1} dt V_{n-1} \le \frac{\epsilon\alpha}{k}\sqrt{\mu-\alpha^2}^{\,n-1} V_{n-1}$$

where

$$\alpha = k\frac{\sigma_n \mu^{n/2} V_n}{\tilde{s}_{n-1}} = k\sqrt{\mu}\frac{\sigma_n V_n}{\tilde{\sigma}_{n-1} V_{n-1}}\ .$$

We have thus

$$\sharp(X_k) \le_{\mu\to\infty} \epsilon\sqrt{\mu}^n \frac{\sigma_n V_n}{\tilde{\sigma}_{n-1}}\sqrt{1 - k^2\left(\frac{\sigma_n V_n}{\tilde{\sigma}_{n-1}V_{n-1}}\right)^2}^{\,n-1}$$

implying

$$\sum_{k=1} \sharp(X_k) \le_{\mu\to\infty} \epsilon\sqrt{\mu}^n \sigma_n V_n\ \frac{1}{\tilde{\sigma}_{n-1}}\sum_{k=1}^{\infty}\sqrt{1 - k^2\left(\frac{\sigma_n V_n}{\tilde{\sigma}_{n-1}V_{n-1}}\right)^2}^{\,n-1}$$

and showing that the asymptotic inequality (1) above is satisfied for all $\epsilon > 0$ if

$$\sum_{k=1}^{\infty}\sqrt{1 - k^2\left(\frac{\sigma_n V_n}{\tilde{\sigma}_{n-1}V_{n-1}}\right)^2}^{\,n-1} \le \frac{\tilde{\sigma}_{n-1}}{1+\epsilon'}\ . \tag{2}$$

Using Lemma 4.1 we get the asymptotics

$$\begin{aligned}\sqrt{1 - k^2\left(\frac{\sigma_n V_n}{\tilde{\sigma}_{n-1}V_{n-1}}\right)^2}^{\,n-1} &= \sqrt{1 - k^2\left(\frac{\sigma_n}{\tilde{\sigma}_{n-1}}\right)^2\frac{2\pi}{n}(1-\frac{1}{2n}+O(\frac{1}{n^2}))}^{\,n-1} \\ &= e^{-k^2\left(\frac{\sigma_n}{\tilde{\sigma}_{n-1}}\right)^2\pi}\left(1 + k^2\left(\frac{\sigma_n}{\tilde{\sigma}_{n-1}}\right)^2\pi(\frac{3}{2} - k^2\left(\frac{\sigma_n}{\tilde{\sigma}_{n-1}}\right)^2\pi)\frac{1}{n} + O(\frac{1}{n^2})\right) \\ &< e^{-k^2\left(\frac{\sigma_n}{\tilde{\sigma}_{n-1}}\right)^2\pi}\end{aligned}$$

for $\sigma_n \ge \frac{\tilde{\sigma}_{n-1}}{\sqrt{2}} > \frac{\tilde{\sigma}_{n-1}}{k}\sqrt{\frac{3}{2\pi}}$ and n large enough.

Notice that $\frac{1}{2^{n-1}\tilde{\sigma}_{n-1}}$ is a lower bound for the density of the $(n-1)-$dimensional integral lattice

$$\{(x_0,\dots,x_{n-1})\in \mathbf{Z}^n \mid \sum_{i=0}^{n-1} s_i x_i = 0\} = (s_0,\dots,s_{n-1})^{\perp}\cap \mathbf{Z}^n$$

associated to the $\mu-$sequence $(s_0, \dots, s_{n-1})$.

For σ_n of order $O(1)$ we have

$$s_n \sim_{\mu\to\infty} \tilde{s}_n = \sqrt{\sum_{i=0}^{n} s_i^2}\ . \tag{3}$$

Supposing $\tilde{\sigma}_{n-1} = \sum_{k=1} e^{-k^2\pi}$ the choice $\sigma_n = \tilde{\sigma}_{n-1}(1+\tilde{\epsilon})$ implies thus the asymptotic inequality (1) for n large enough and all $\tilde{\epsilon} > 0$. The asymptotic equality (3) implies now easily the result and the argument can be iterated.

In the case $\tilde{\sigma}_{n-1} > \sum_{k=1} e^{-k^2\pi}$ choose ϵ' small enough such that inequation (2) holds for some $\sigma_n < \tilde{\sigma}_{n-1}$. This implies that the asymptotic inequality (1) is valid and a closer inspection shows that we can iterate this construction using a decreasing sequence $\tilde{\sigma}_{n-1} > \sigma_n \geq \sigma_{n+1} \geq \dots$ with limit $\sum_{k=1} e^{-k^2\pi}$. This proves the result in this case.

The remaining case $\tilde{\sigma}_{n-1} < \sum_{k=1} e^{-k^2\pi}$ can for instance be treated by replacing the $\mu-$sequence $(s_0, \dots, s_{n-1})$ with a $\mu-$sequence of smaller density. □

Remark 4.2 *(i) Theorem 1.5 can be slightly sharpened in a standard way which yields the $\zeta(n)$ factor in the best known bounds for the density of the densest lattice packing.*

(ii) The main error during the proof of Theorem 1.5 occurs during the majoration

$$\sharp(I_k) \leq \sharp(X_k)$$

which is very crude.

(iii) Instead of working with sublattices of $\mathbf{Z}^{n+1}$ orthogonal to a given vector $(s_0, \dots, s_n) \in \mathbf{Z}^{n+1}$, it is possible to consider sublattices $\mathbf{Z}^{n+a}$ which are orthogonal to a set of $a \geq 2$ linearly independent vectors in $\mathbf{Z}^{n+a}$. One might also replace the standard lattice $\mathbf{Z}^{n+1}$ by other lattices, e.g. sublattices of finite index in $\mathbf{Z}^{n+1}$.

(iv) Let us conclude by mentioning that extending finite $\mu-$sequences in an optimal way into longer $\mu-$sequences amounts geometrically to the familiar process of lamination for lattices (see for instance [2] or [4]). The existence of an integer $s \in I \setminus I_1$ implies indeed the existence of a point $P \in \mathbf{E}^{n-1}$ which is far away from any lattice point of the affine lattice $\{(x_0, \dots, x_{n-1}) \mid \sum x_i s_i = s\} \subset \mathbf{Z}^n$ and corresponds thus to a "hole" of the lattice.

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