On generating series of complementary planar trees

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Prépublication de l'Institut Fourier n° 649 (2004),
http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract[1]*:* We generalize and reprove an identity of Parker and Loday. It states that certain pairs of generating series associated to pairs of labelled rooted planar trees are mutually inverse under composition.

1 Introduction

In [1] Carlitz, Scoville and Vaughan consider finite words (in a finite alphabet $\mathcal{A}$) such that all pairs of consecutive letters belong to a fixed subset $L \subset \mathcal{A} \times \mathcal{A}$. They show (Theorems 6.8 and 7.3 of [1]) that suitably defined pairs of signed generating series counting such words associated to $L \subset \mathcal{A} \times \mathcal{A}$ and to its complementary set $\overline{L} = \mathcal{A} \times \mathcal{A} \setminus L$ are each others inverse. Their result was generalized in the first part of Parker's thesis [5] who showed an analogous result for suitable classes of finite trees having labelled vertices. Loday in [3], motivated by questions concerning combinatorial realisations of operads, rediscovered Parkers result toghether with a different proof, based on homological arguments.

This paper presents a further generalisation of Parkers and Lodays result.

A typical example of our identity can be described as follows: Associate to two complex matrices

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ M_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

the following two systems of algebraic equations

$$\left\{ \begin{array}{lcl} g_1 & = & (-X + ag_1 + bg_2)(-X + \alpha g_1 + \beta g_2) \\ g_2 & = & (-X + cg_1 + dg_2)(-X + \gamma g_1 + \delta g_2) \\ g & = & -X + g_1 + g_2 \end{array} \right.$$

and

$$\left\{ \begin{array}{lcl} \tilde{g}_1 & = & (X - (1-a)\tilde{g}_1 - (1-b)\tilde{g}_2)(X - (1-\alpha)\tilde{g}_1 - (1-\beta)\tilde{g}_2) \\ \tilde{g}_2 & = & (X - (1-c)\tilde{g}_1 - (1-d)\tilde{g}_2)(X - (1-\gamma)\tilde{g}_1 - (1-\delta)\tilde{g}_2) \\ \tilde{g} & = & -X + \tilde{g}_1 + \tilde{g}_2 \end{array} \right. .$$

[1]Math. class: 05A15, 05C05, 06A10. Keywords: Integer sequence, generating function, inversion of power series, planar tree, spin model

Choosing continuous determinations satisfying $g = g_1 = g_2 = \tilde{g} = \tilde{g}_1 = \tilde{g}_2 = 0$ at $X = 0$ we get holomorphic functions $g = g(X)$, $\tilde{g} = \tilde{g}(X)$ for X in an open neighbourhood of $0 \in \mathbb{C}$. We have now

$$g(\tilde{g}(X)) = X$$

for all X in a small open disc centered at 0.

This result holds of course formally for the corresponding generating series and can be verified by computing for instance a minimal polynomial $P(u,v) = \sum_{i,j} p_{i,j} u^i v^j$ for g (i.e. satisfying in particular $P(g(X), X) = 0$) and checking that we have $P(X, \tilde{g}(X)) = 0$. Since this identity is algebraic, the field $\mathbb{C}$ can be replaced by an arbitrary commutative ring.

The sequel of this paper is organized as follows: The next section states the main result in purely algebraic terms over a not necessarily commutative ring. Parkers and Lodays result corresponds to the special case of matrices with coefficients in $\{0, 1\}$. Our main result removes the restriction on the coefficients. It is also somewhat easier to state (at least in a commutative setting) since it avoids combinatorial descriptions. It follows from our formulation that all involved generating functions are algebraic in a commutative setting and over a finite alphabet. Section 3 fixes notations concerning trees. Section 4 describes spin models on trees and recasts our main result using partition functions of spin models. Section 5 proves the main result using a spin model on grafted trees (called "graftings" in [3]). The proof avoids homological arguments and is thus in some sense more elementary (although perhaps not simpler) than the proof of [3]. Section 6 describes briefly a further generalisation involving arbitrary (not necessarily regular) finite trees which appears already in Parkers work. Section 7 is a digression generalizing the notion of grafted trees. Section 8 contains the computations for example (i) of [3]. We display the defining polynomial of the relevant (algebraic) generating function and discuss briefly its asymptotics.

2 Main result

Consider a (not necessarily finite) alphabet $\mathcal{A}$ and a (not necessarily commutative) associative ring R having a unit 1. We denote by $Y_{\mathcal{A}}$ a set of (non-commutative) variables indexed by elements $\alpha \in \mathcal{A}$ and by X a supplementary (non-commutative) variable. We denote by $\tilde{R}[[X]] = \mathbf{Z}[[R, Y_{\mathcal{A}}, R]] \otimes_{\mathbf{Z}} \mathbf{Z}$ the ring of formal power series. An element of $\tilde{R}[[X]]$ is a (generally infinite) sum of monomials of the form

$$\pm r_1 Z_1 r_2 Z_2 \cdots r_l Z_l r_{l+1}$$

with $r_i \in R$ and $Z_i \in \{X\} \cup Y_{\mathcal{A}}$. Let $k \geq 2$ be a natural integer and let $M_1, \dots, M_k$ be a set of matrices with rows and columns indexed by $\mathcal{A}$ and coefficients $M_j(\alpha,\beta) \in R$ for $\alpha,\beta \in \mathcal{A}$. For $\alpha \in \mathcal{A}$, let $g_\alpha \in \tilde{R}[[X]]$ be the power series $g_\alpha = Y_\alpha X^k +$ (terms of higher order in X) which satisfies

$$g_\alpha = Y_\alpha(X - (M_1\ V)_\alpha)(X - (M_2\ V)_\alpha)\cdots(X - (M_k\ V)_\alpha)$$

where V is the column vector with coordinates $g_\beta,\ \beta \in \mathcal{A}$ and where $(M_j\ V)_\alpha = \sum_{\beta\in\mathcal{A}} M_j(\alpha,\beta)g_\beta$ denotes the $\alpha-$th coordinate of the matrix-product $M_j\ V$. Set $g = -X + \sum_{\alpha\in\mathcal{A}} g_\alpha$.

Remark that our notation is slightly misleading: $g, g_1, \ldots, g_l$ are power series in X defining "functions" of X. The letter X stands of course for the power series at $X = 0$ of the identity function $X \longmapsto X$. Remark also that the requirements $k \geq 2$ and $g_\alpha = Y_\alpha X^k +$ terms of higher order in X ensure that g_α is well-defined: Its $n-$th coefficient involving X n times depends only on coefficients with indices $\leq n-k+1$ of $g_\beta,\ \beta \in {}_mathcalA$ involving at most $n-k+1$ occurences of X.

Define "complementary" matrices $\tilde{M}_1, \ldots, \tilde{M}_k$ with coefficients $\tilde{M}_j(\alpha,\beta) = 1 - M_j(\alpha,\beta)$ by setting $\tilde{M}_j = J - M_j$ where J is the all 1 matrix with rows and columns indexed by $\mathcal{A}$. For $\alpha \in \mathcal{A}$ we introduce "complementary" functions $\tilde{g}_\alpha = Y_\alpha(-X)^k +$ terms of higher order in $X \in \tilde{R}[[X]]$ satisfying

$$\tilde{g}_\alpha = Y_\alpha(-X + (\tilde{M}_1\ \tilde{V})_\alpha)\cdots(-X + (\tilde{M}_k\ \tilde{V})_\alpha)$$

where $\tilde{V}$ is the column vector with coordinates $\tilde{g}_\beta,\ \beta \in \mathcal{A}$ and where $(\tilde{M}_j\ \tilde{V})_\alpha = \sum_{\beta\in\mathcal{A}} \tilde{M}_j(\alpha,\beta)\tilde{g}_\beta$. We define now $\tilde{g} = -X + \sum_{j=1}^{l} \tilde{g}_j$.

Given two formal power series $f, h \in \tilde{R}[[X]]$ such that every monomial of h is at least of degree 1 in X, the composition $f \circ_X h$ of f with h is defined as the formal power series obtained by replacing every occurence of $X^k,\ k = 1, 2, 3, \ldots$ in every monomial of f by the series h^k.

The main result of this paper can now be stated as follows:

Theorem 2.1 *We have (formally)*

$$g \circ_X \tilde{g} = \tilde{g} \circ_X g = X\ .$$

Remark 2.2 *(i) In a commutative setting with $R = \mathbf{C}$ the field of complex numbers, a finite alphabet $\mathcal{A}$, and $Y_\alpha \in \mathbf{C}$ for $\alpha \in \mathcal{A}$, the functions $g_\alpha, \tilde{g}_\alpha$ and thus also g and $\tilde{g}$ are holomorphic determinations of algebraic functions in an open neighbourhood of* 0.

(ii) The above definitions of g_α and $\tilde{g}_\alpha$ are well-suited for iterative computations of the power-series $g_\alpha, \tilde{g}_\alpha$ by "bootstrapping" (see e.g. section 5.4 of [4]). Indeed, given $g_{\alpha,n},\ \alpha \in \mathcal{A}$ such that $g_\alpha - g_{\alpha,n} = O(X^{1+(n+1)(k-1)})$ we have

$$g_\alpha - Y_\alpha(t - (M_1\ V_n)_\alpha)\cdots(t - (M_k\ V_n)_\alpha) = O(X^{1+(n+2)(k-1)})$$

where V_n is the column vector with coordinates $g_{\alpha,n},\ \alpha \in \mathcal{A}$. An analogous result holds of course for $\tilde{g}_1, \ldots, \tilde{g}_l$.

(iii) Changing the grading and considering also all variables Y_α, $\alpha \in \mathcal{A}$ as beeing of degree 1, Theorem 2.1 holds also for $k = 1$ (as a special case, one obtains the main result of [1]). In this case, $g \circ_X \tilde{g}$ boils down to a simple product in $\tilde{R}$ since g and $\tilde{g}$ are of the form cX with $c \in R[[(Y_\alpha)_{\alpha\in\mathcal{A}}]]$.

(iv) The choice of an integer $k \geq 2$ (or $k = 1$) corresponds to the case of $k-$regular trees. This restriction can be removed: Section 6 describes a further generalization (already contained in [5]), corresponding to arbitrary finite trees.

3 Trees

A *tree* is a connected graph without cycles. A *rooted* tree contains a marked vertex r, called the *root*. In particular, a rooted tree is non-empty. A rooted tree $T = \{r\}$ reduced to its root is *trivial*. The edges of a rooted tree are canonically oriented by requiring the root to be the unique source of the directed tree obtained by orienting all edges away from the root. We write $e = (\alpha(e), \omega(e))$ for the edge e oriented from $\alpha(e)$ to $\omega(e)$. Given an edge $e = (\alpha(e), \omega(e))$ we call $\omega(e)$ a *son* of $\alpha(e)$ and $\alpha(e)$ the *father* of $\omega(e)$. Each vertex $v \neq r \in T$ except the root has a unique father. The set of vertices sharing a common father is a *brotherhood*. A *leaf* is a vertex without sons. A vertex having at least one son is *interior*. The *level* of a vertex is its (combinatorial) distance to the root. A rooted tree is *planar* if every brotherhood is totally ordered. Every brotherhood of a *locally finite* rooted tree is finite . A rooted tree is thus locally finite if and only if the set of vertices of given level N is finite for all $N \in \mathbf{N}$. A rooted tree is *$k-$regular* if every non-empty brotherhood of strictly positive level contains exactly k vertices. A rooted tree is finite if and only if the set of vertices of level N is finite for all N and empty for N large enough. Given a vertex $v \in T$, we denote by $T(v)$ the subtree of T rooted at v which is defined by considering all vertices $w \in T$ such that v belongs to the unique geodesic path joining the root $r \in T$ to w (the vertices of $T(v)$ are thus v and its descendants). We call such a subtree *maximal*. A *principal* subtree of a non-trivial tree T is a maximal subtree $T(s_i)$ rooted at a son s_i of the root $r \in T$. A non-trivial $k-$regular tree has thus exactly k principal subtrees.

Unless otherwise stated, a ($k-$regular) tree will henceforth always denote a ($k-$regular) finite rooted planar tree.

Every tree T can be represented by a plane tree of the halfplane $\mathbf{R} \times \mathbf{R}_{\geq 0}$ such that a totally ordered brotherhood $s_1 < s_2 < \ldots$ of level $n \geq 0$ corresponds to vertices $(x_1, n), (x_2, n), \ldots,\ x_1 < x_2 < \ldots$.

A vertex $v \in T$ of level n has a unique recursively defined *address*

$$a(v) = a_0\ a_1\ \ldots\ a_{n-1} \in \{1, 2, \ldots\}^n$$

where a_{n-1} is the number of elements $\leq v$ in the brotherhood of v and where

$a_0 \ \ldots \ a_{n-2}$ is the address of the (unique) father of v. The address of the root $r \in T$ is empty. The lexicographic order on addresses orders the set of vertices of a planar rooted tree completely.

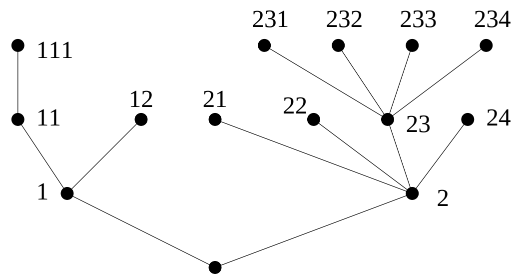

Figure 1: A (finite rooted planar) tree with addressed vertices.

Figure 1 illustrates the notions of this section. It displays a tree with 14 vertices

$$\emptyset < 1 < 11 < 111 < 12 < 2 < 21 < 22 < 23 < 231 < 232 < 233 < 234 < 24 \, ,$$

completely ordered by their addresses. The root corresponds to the lowest vertex having an empty address. The oriented edge e with vertices 23 and 232 starts at $\alpha(e) = (23)$ and ends at the leave $\omega(e) = 232$. The vertex 23 is thus the father of 232. The brotherhood of 232 are the four vertices 231, 232, 233 and 234 at level 3 having $\alpha(e) = 23$ as their common father. This tree is of course not regular since the interior vertices 1 and 2 (for instance) have respectively 2 and 4 sons. The tree of Figure 1 has two principal subtrees. Two vertices with a non-empty adress belong to a common principal subtree if and only if the first letter of their adresses coincides.

4 Non-commutative spin models on rooted planar trees

Let $T = (E, V)$ be a (rooted planar) tree with edges E and vertices V. We denote by V° the set of interior vertices (having at least one son) of T and by $E' \subset E$ the subset of leafless edges (i.e. $e \in E'$ if $\omega(e) \in V^\circ$). We define a *spin-model* (T, w) on T by considering a (not necessarily) finite set or alphabet $\mathcal{A}$ of *spins* and a *weight-function*

$$w : E' \times \mathcal{A} \times \mathcal{A} \longrightarrow R$$

with values in a (not necessarily commutative) ring R containing 1. Setting $w_e(\alpha, \beta) = w(e, \alpha, \beta)$, a weight function can be identified with an application

$$E' \longrightarrow M_{\mathcal{A}}(R)$$

where $M_{\mathcal{A}}(R)$ denotes the set of matrices with coefficients in R and rows and columns indexed by $\mathcal{A}$. We call $w_e \in M_{\mathcal{A}}(R)$ the *weight-matrix* of $e \in E'$.

We introduce furthermore (non-commutative) variables Y_α for $\alpha \in \mathcal{A}$ and a supplementary (non-commutative) variable X.

Given a spin-model (T, w), its *complementary spin-model* is defined as $(T, \tilde{w})$ where $\tilde{w}_e(\alpha, \beta) = 1 - w_e(\alpha, \beta)$ for $e \in E'$ and $\alpha, \beta \in \mathcal{A}$.

A colouring

$$\varphi : V^\circ \longrightarrow \mathcal{A}$$

of all interior vertices in T by elements of $\mathcal{A}$ is a *state*. Its *energy* $f_T(\varphi) \in R[X, (Y_\alpha)_{\alpha\in\mathcal{A}}]$ is recursively defined as follows: If $T = \{r\}$ is trivial then $f_T(\varphi) = X$. Otherwise, we set

$$f_T(\varphi) = Y_{\varphi(r)} \left(w(1,\varphi) f_{T_1}(\varphi_1)\right) \left(w(2,\varphi) f_{T_2}(\varphi_2)\right) \cdots$$

where $T_1, T_2, \dots$ are the principal subtrees associated to the linearly ordered sons $s_1 < s_2 < \dots$ of the root $r \in T$, where

$$w(i, \varphi) = \begin{cases} w((r, s_i), \varphi(r), \varphi(s_i)) & \text{if } s_i \in V^\circ \\ 1 & \text{otherwise} \end{cases}$$

and where $f_{T_i}(\varphi)$ is the energy of the spinmodel on the principal subtree T_i defined by the $i-$th son s_i of the root $r \in T$ with weights and state obtained by restriction.

In a commutative setting, this boils down to

$$f(\varphi) = X^{\sharp(V \setminus V^\circ)} \prod_{v \in V^\circ} Y_{\varphi(v)} \prod_{e \in E'} w_e(\varphi(\alpha(e)), \varphi(\omega(e)))$$

which is the familiar definition used in statistical physics.

The total sum

$$Z = Z(T) = \sum_{\varphi \in \mathcal{A}^{V^\circ}} f(\varphi)$$

of energies over all states is the *partition function*. We denote by

$$Z_\alpha = Z_\alpha(T) = \sum_{\varphi \in \mathcal{A}^{V^\circ},\ \varphi(r) = \alpha} f(\varphi)$$

the *restricted* partition function obtained by computing the total energy of all states with prescribed colour $\varphi(r) = \alpha$ on the root. We have obviously

$$Z = \sum_{\alpha \in \mathcal{A}} Z_\alpha$$

if T is non-trivial. For $T = \{r\}$ trivial, we have $Z = X$ and $Z_\alpha = 0$ for $\alpha \in \mathcal{A}$.

The partition function of a tree T can be computed as follows: We have $Z(T) = X$ for $T = \{r\}$ the trivial tree. Otherwise, consider the principal subtrees $T_1, T_2, \dots$ associated to the k linearly ordered sons $s_1 <$

$s_2 < \ldots < s_k$ of r. For $\alpha \in \mathcal{A}$ denote by $Z_\alpha(T_j)$ the obvious restricted partition functions of the principal subtree T_j with colour α on its root s_j. We denote by $e_j = \{r = \alpha(e_j), s_j = \omega(e_j)\}$ the oriented edge joining the root r of T to its $j-$th son $s_j = \omega(e_j)$.

The definition of the restricted partition function implies then easily the following result:

Proposition 4.1 *If $T \neq \{r\}$ is non-trivial, we have*

$$Z_\alpha(T) = Y_\alpha C_1(\alpha) C_2(\alpha) \cdots C_k(\alpha)$$

where $T_1, \ldots, T_k$ are the principal subtrees associated to the k linearly ordered sons $s_1 < s_2 < \ldots < s_k$ of the root $r \in T$ and where

$$C_k(\alpha) = \begin{cases} \sum_{\beta \in \mathcal{A}} w((r, s_i), \alpha, \beta) Z_\beta(T_i) & \text{if } T_i \neq \{s_i\} \\ X & \text{otherwise .} \end{cases}$$

An edge $e \in E$ of a $k-$regular tree is of type j if the extremity $\omega(e)$ (having level $m+1$) of e has an address

$$a_0\ a_1\ \ldots\ a_m = a_0\ a_1\ \ldots a_{m-1}\ j$$

ending with $1 \leq j = a_m \leq k$. The endpoint $\omega(e)$ of a type j edge is thus the $j-$th element in its (totally ordered) brotherhood.

Given a (finite) $k-$regular tree T and k weight-matrices $M_1, \ldots, M_k$ (with indices in $\mathcal{A} \times \mathcal{A}$) we consider the spin model with spins $\mathcal{A}$ and weight function $w_e(\alpha, \beta) = M_j(\alpha, \beta)$ on leafless edges of type j.

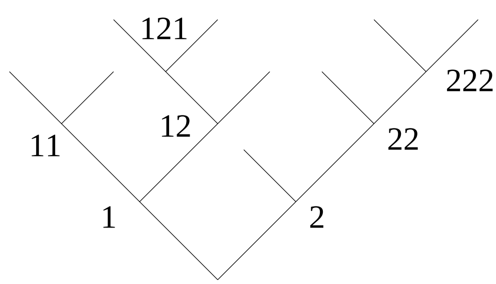

Figure 2: A $2-$regular tree.

Example. We consider the commutative spin model on the $2-$regular tree T of Figure 2 with $\mathcal{A} = \{1, 2\}$ and

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \; M_2 = (M_1)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} .$$

For the sake of simplicity, we set $X = Y_1 = Y_2 = 1$. Given an interior vertex v of T, we denote by $Z_*(v)$ the vector $\begin{pmatrix} Z_1(T(v)) \\ Z_2(T(v)) \end{pmatrix}$ where $T(v)$ denotes the maximal subtree of T with root v (obtained by considering the connected component of v in $T \setminus a(v)$ where $a(v)$ is the father of $v \neq r$). Since all sons of $11,\ 121,\ 222$ are leaves, we have

$$Z_*(11) = Z_*(121) = Z_*(222) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where we denote a vertex by its address. We get now

$$Z_*(12) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} Z_*(121) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$Z_*(22) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(222) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$Z_*(2) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(22) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Denoting by $U * W$ the Hadamard product ("student's vector product") $\begin{pmatrix} \alpha\alpha' \\ \beta\beta' \end{pmatrix}$ of two vectors $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$ we have then

$$\begin{aligned} Z_*(1) &= \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} Z_*(11) \right) * \left(\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(12) \right) \\ &= \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) * \left(\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Z_*(\emptyset) &= \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} Z_*(1) \right) * \left(\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} Z_*(2) \right) \\ &= \left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) * \left(\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 5 \\ 8 \end{pmatrix} * \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 25 \\ -24 \end{pmatrix}. \end{aligned}$$

This yields the partition function

$$Z(T) = Z_1(T) + Z_2(T) = 25 - 24 = 1$$

for the tree of Figure 2 with respect to the weights $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $M_2 = M_1^{-1}$ associated to leafless edges indicating NW (type 1), respectively NE (type 2).

Given a spinset (alphabet) $\mathcal{A}$ and pairs of complementary matrices $M_1, \tilde{M}_1 = J - M_1, \ldots, M_k, \tilde{M}_k = J - M_k$ (where J denotes the all 1 matrix with row and column-indices in $\mathcal{A}$), we consider the associated complementary spin models on $k-$regular trees with partition function $Z(T)$, $\tilde{Z}(T)$ as above.

Denoting by $\mathcal{T}_k$ the set of all (finite, rooted, planar) $k-$regular trees, we introduce the signed formal generating series

$$Z(\mathcal{T}_k) = -\sum_{T\in\mathcal{T}_k} (-1)^{d^\circ(T)}\ Z(T)$$

and

$$\tilde{Z}(\mathcal{T}_k) = \sum_{T\in\mathcal{T}_k} (-1)^{d(T)} \tilde{Z}(T)$$

where $d^\circ(T) = \sharp(V^\circ)$ denotes the number of interior vertices and $d(T) = \sharp(V \setminus V^\circ)$ the number of leaves of a $k-$regular tree $T \in \mathcal{T}_k$.

Let us moreover consider the restricted generating series

$$Z_\alpha(\mathcal{T}'_k) = -\sum_{T\in\mathcal{T}'_k} (-1)^{d^\circ(T)}\ Z_\alpha(T)$$

and

$$\tilde{Z}_\alpha(\mathcal{T}'_k) = \sum_{T\in\mathcal{T}'_k} (-1)^{d(T)}\ \tilde{Z}_\alpha(T)$$

where $\mathcal{T}'_k = \mathcal{T}_k \setminus \{r\}$ denotes the set of all $k-$regular trees which are non-trivial. We have obviously

$$Z(\mathcal{T}_k) = -X + \sum_{\alpha\in\mathcal{A}} Z_\alpha(\mathcal{T}'_k) \qquad \text{and} \qquad \tilde{Z}(\mathcal{T}_k) = -X + \sum_{\alpha\in\mathcal{A}} \tilde{Z}_\alpha(\mathcal{T}'_k)\ .$$

Proposition 4.2 *We have*

$$Z_\alpha(\mathcal{T}'_k) = Y_\alpha\,(X - (M_1\ W)_\alpha) \cdots (X - (M_k\ W)_\alpha)$$

and

$$\tilde{Z}_\alpha(\mathcal{T}'_k) = Y_\alpha\left(-X + (\tilde{M}_1\ \tilde{W})_\alpha\right) \cdots \left(-X + (\tilde{M}_k\ \tilde{W})_\alpha\right)$$

where W (respectively $\tilde{W}$) is the column vector with coordinates $Z_\alpha(\mathcal{T}'_k)$ (respectively $\tilde{Z}_\alpha(\mathcal{T}'_k)$) indexed by $\alpha \in \mathcal{A}$.

For a fixed natural integer $k \geq 1$ and k (weight-)matrices $M_1, \ldots, M_k$ we have now:

Corollary 4.3 *We have $g_\alpha = Z_\alpha(\mathcal{T}'_k)$, $\tilde{g}_\alpha = \tilde{Z}_\alpha(\mathcal{T}'_k)$ for all $\alpha \in \mathcal{A}$ and $g = Z(\mathcal{T}_k)$, $\tilde{g} = \tilde{Z}(\mathcal{T}_k)$ where g_α, $\tilde{g}_\alpha$, g, $\tilde{g}$ are as in Theorem 2.1 and where $Z_\alpha(\mathcal{T}'_k)$, $\tilde{Z}_\alpha(\mathcal{T}'_k)$, $Z(\mathcal{T}_k)$, $\tilde{Z}(\mathcal{T}_k)$ are as above.*

Proof of Proposition 4.2 Given a non-trivial $k-$regular tree $T \neq \{r\}$ with root r, we denote by $T_1, \ldots, T_k$ the principal subtrees associated to the k linearly ordered sons $s_1 < s_2 \ldots < s_k$ of r.

The generating function

$$Z_\alpha(\mathcal{T}'_k) = -\sum_{T\in\mathcal{T}'_k} (-1)^{d^\circ(T)}\ Z_\alpha(T) = \sum_{T\in\mathcal{T}'_k} (-1)^{\sum_j d^\circ(T_j)}\ Z_\alpha(T)$$

(with the last equality following from $d^\circ(T) = 1 + \sum_{j=1}^k d^\circ(T_j)$) factorizes now as

$$\begin{aligned} Z_\alpha(\mathcal{T}'_k) &= Y_\alpha\Big(X + \textstyle\sum_{\beta\in\mathcal{A}} M_1(\alpha,\beta)\sum_{T\in\mathcal{T}'_k}(-1)^{d^\circ(T)} Z_\beta(T)\Big)\cdot \\ &\quad\cdot\Big(X + \textstyle\sum_{\beta\in\mathcal{A}} M_2(\alpha,\beta)\sum_{T\in\mathcal{T}'_k}(-1)^{d^\circ(T)} Z_\beta(T)\Big)\cdots \\ &\quad\cdots\ \Big(X + \textstyle\sum_{\beta\in\mathcal{A}} M_k(\alpha,\beta)\sum_{T\in\mathcal{T}'_k}(-1)^{d^\circ(T)} Z_\beta(T)\Big) \\ &= Y_\alpha\Big(X - \textstyle\sum_{\beta\in\mathcal{A}} M_1(\alpha,\beta)\ Z_\beta(\mathcal{T}'_k)\Big)\cdots\Big(X - \textstyle\sum_{\beta\in\mathcal{A}} M_k(\alpha,\beta)\ Z_\beta(\mathcal{T}'_k)\Big)\ . \end{aligned}$$

Indeed, neglecting the trivial term Y_α, the partition function $Z_\alpha(T)$ of a tree $T \in \mathcal{T}'_k$ with prescribed spin $\varphi(r) = \alpha$ on its root decomposes into k obvious factors corresponding to the k principal subtrees of T: A leafless edge $e \in E'$ with $\omega(e) \in T_j$ yields a contribution to the $j-$th factor of $Z_\alpha(\mathcal{T}'_k)$. Summing over all $k-$regular trees $T \in \mathcal{T}'_k$ yields the above expression. The term X in the $j-$th factor corresponds to trees $T \in \mathcal{T}'_k$ whose $j-$th principal subtree T_j is trivial. This proves the first equality of Proposition 4.2.

For

$$\tilde{Z}_\alpha(\mathcal{T}'_k) = \sum_{T\in\mathcal{T}'_k} (-t)^{d(T)}\ \tilde{Z}_\alpha(T) = \sum_{T\in\mathcal{T}'_k} (-t)^{\sum_j d(T_j)}\ \tilde{Z}_\alpha(T)$$

we get the analogous factorization

$$\begin{aligned} \tilde{Z}_\alpha(\mathcal{T}'_k) &= Y_\alpha\Big(-X + \textstyle\sum_{\beta\in\mathcal{A}} \tilde{M}_1(\alpha,\beta)\sum_{T\in\mathcal{T}'}(-1)^{d(T)} \tilde{Z}_\beta(T)\Big)\cdot \\ &\quad\cdot\Big(-X + \textstyle\sum_{\beta\in\mathcal{A}} \tilde{M}_2(\alpha,\beta)\sum_{T\in\mathcal{T}'}(-1)^{d(T)} \tilde{Z}_\beta(T)\Big)\cdots \\ &\quad\cdots\Big(-X + \textstyle\sum_{\beta\in\mathcal{A}} \tilde{M}_k(\alpha,\beta)\sum_{T\in\mathcal{T}'}(-1)^{d(T)} \tilde{Z}_\beta(T)\Big) \\ &= Y_\alpha\Big(-X + \textstyle\sum_{\beta\in\mathcal{A}} \tilde{M}_1(\alpha,\beta)\ \tilde{Z}_\beta(\mathcal{T}'_k)\Big)\cdots\Big(-X + \textstyle\sum_{\beta\in\mathcal{A}} \tilde{M}_k(\alpha,\beta)\ \tilde{Z}_\beta(\mathcal{T}_k)\Big) \end{aligned}$$

which proves the second equality of Proposition 4.2. □

The proof of Corollary 4.3 is immediate: corresponding series are recursively defined by the same formulae and initial data.

5 Proof of Theorem 2.1

By Corollary 4.3, the formal power series $g = g(X) = -X + \sum_{\alpha\in\mathcal{A}} g_\alpha(X)$ and $\tilde{g} = \tilde{g}(X) = -X + \sum_{\alpha\in\mathcal{A}} \tilde{g}_\alpha(X)$ involved in Theorem 2.1 are suitably signed generating series for the partition functions of complementary spin-models defined on $k-$regular trees. In order to prove Theorem 2.1 we define spin-models on a set of combinatorial objects which we call grafted trees. A suitably signed generating series of the partition functions for these spin-models equals $g \circ_X \tilde{g}$ and a direct computation establishes Theorem 2.1.

5.1 A spin model on grafted trees

A *grafted tree* is given by

$$(A; B_1, B_2, \ldots, B_{d(A)})$$

where A is a tree with $d(A)$ leaves and where $B_1, \ldots, B_{d(A)}$ is a sequence of $d(A)$ trees. A grafted tree is *k-regular* if it involves only $k-$regular trees.

The *skeleton* of a grafted tree $(A; B_1, \ldots, B_{d(A)})$ is the tree obtained by grafting (gluing) the root of B_1 to the smallest (leftmost) leaf of A, by grafting the root of B_2 to the second-smallest leaf of A etc. The skeleton of $(A; B_1, \ldots, B_{d(A)})$ has thus $\sum_{j=1}^{d(A)} d(B_j)$ leaves.

The data of a k-regular grafted tree $(A; B_1, \ldots, B_{d(A)})$ and k weight-matrices $M_1, \ldots, M_k$ indexed by $\mathcal{A} \times \mathcal{A}$ defines a "compositional" spin model as follows: Compute first the ordinary partition function $Z(A)$ as defined previously. Compute also the partition functions $\tilde{Z}(B_1), \ldots, \tilde{Z}(B_{d(A)})$ with respect to the complementary weight matrices $\tilde{M}_1 = J - M_1, \ldots, \tilde{M}_k = J - M_k$ (where we denote by J the all 1 matrix indexed by $\mathcal{A} \times \mathcal{A}$). Replace now the $i-$th (occurence of the) letter X in every monomial of $Z(A)$ by $\tilde{Z}(B_i)$. The resulting formal power series (or non-commutative polynomial for a finite alphabet $\mathcal{A}$) is by definition $Z(A; B_1, \ldots, B_{d(A)})$. It is of degree $\sum_{i=1}^{d(A)} d(B_i)$ in X.

In particular, we have

$$Z(A; B_1, \ldots, B_{d(A)}) = X^{-d(V_A)} Z(A) \prod_{j=1}^{d(A)} \tilde{Z}(B_j)$$

(with $d(V_A)$ denoting the number of leaves in A) in a commutative setting.

Let $g(X) = -X + \sum_{\alpha \in \mathcal{A}} g_\alpha(X)$, $\tilde{g}(X) = -X + \sum_{\alpha \in \mathcal{A}} \tilde{g}_\alpha(X)$ be the generating series involved in Theorem 2.1 associated to k matrices $M_1, \ldots, M_k$.

Proposition 5.1 *We have*

$$g \circ_X \tilde{g}(X) = - \sum_{(A; B_1, \ldots, B_{d(A)}) \in \mathcal{G}_k} (-1)^{d^\circ(A) + \sum_{j=1}^{d(A)} d(B_j)} \; Z(A; B_1, \ldots, B_{d(A)})$$

where $\mathcal{G}_k$ denotes the set of all $k-$regular grafted trees.

Proof. This follows at once from Corollary 4.3 and the definition of spin-models on grafted trees. □

Given a $k-$regular tree $T \in \mathcal{T}_k$ we denote by $\mathcal{S}(T)$ the set of all $k-$regular grafted trees with skeleton T. Elements of $\mathcal{S}(T)$ are in bijection with $k-$regular rooted subtrees of T containing the root $r \in T$: The subtree $A \subset T$ of a $k-$regular grafted tree $(A; B_1, \ldots, B_{d(A)}) \in \mathcal{S}(T)$ with skeleton T clearly contains r. Since the subtree $B_j \subset T$ is the maximal subtree of T rooted in the $j-$th leaf v_j of A, the pair of trees $A \subset T$ defines the grafted tree $(A; B_1, \ldots, B_{d(A)})$ completely.

We denote by $\mathcal{S}'(T) = \mathcal{S} \setminus \{(\{r\}; T)\}$ the set of all $k-$regular grafted trees $(A; B_1, \ldots, B_{d(A)})$ with skeleton T and non-trivial $A \neq \{r\}$.

For $\alpha \in \mathcal{A}$ and $(A; B_1, \ldots, B_{d(A)}) \in \mathcal{S}'(T)$ we define $Z_\alpha(A; B_1, \ldots, B_{d(A)})$ in the obvious way by replacing the $i-$th occurence of the letter X in $Z_\alpha(A)$ with $\tilde{Z}(B_i)$ for $i = 1, \ldots, d(A)$.

Proposition 5.2 *We have for a non-trivial $k-$regular tree $T \in \mathcal{T}'_k$ and for all $\alpha \in \mathcal{A}$*

$$\sum_{(A;B_1,\ldots,B_{d(A)}) \in \mathcal{S}'(T)} (-1)^{d^\circ(A)} Z_\alpha(A; B_1, \ldots, B_{d(A)}) = -\tilde{Z}_\alpha(T) \ .$$

Proof. The proof is by induction on the number $d(T)$ of leaves in T. If all k principal subtrees associated to the linearly ordered sons $s_1 < \ldots < s_k$ of the root $r \in T$ are trivial, we have $\mathcal{S}'(T) = (T; \{s_1\}, \ldots, \{s_k\})$ and

$$\begin{aligned} & \textstyle\sum_{(A;B_1,\ldots,B_{d(A)}) \in \mathcal{S}'(T)} (-1)^{d^\circ(A)} Z_\alpha(A; B_1, \ldots, B_{d(A)}) \\ = \quad & (-1)^{d^\circ(T)} Z_\alpha(T; \{s_1\}, \ldots, \{s_k\}) = -\, Y_\alpha \ . \end{aligned}$$

Since we have $\tilde{Z}_\alpha(T) = Y_\alpha$, Proposition 5.2 follows in this case.

Otherwise, we have

$$\sum_{(A;B_1,\ldots,B_{d(A)}) \in \mathcal{S}'(T)} (-1)^{d^\circ(A)} Z_\alpha(A; B_1, \ldots, B_{d(A)}) = -Y_\alpha C_1 C_2 \cdots C_k$$

where $C_i = X$ if the principal subtree $T_i = \{s_i\}$ is trivial and

$$C_i = \tilde{Z}(T_i) + \sum_{\beta \in \mathcal{A}} M_i(\alpha, \beta) \sum_{(A;B_1,\ldots,B_{d(A)}) \in \mathcal{S}'(T_i)} Z_\beta(A; B_1, \ldots, B_{d(A)})$$

if $T_i \neq \{s_i\}$. The contribution of a non-trivial principal subtree $T_i \neq \{s_i\}$ is thus by induction

$$\begin{aligned} C_i &= \tilde{Z}(T_i) - \textstyle\sum_{\beta \in \mathcal{A}} M_i(\alpha, \beta) \tilde{Z}_\beta(T_i) \\ &= \textstyle\sum_{\beta \in \mathcal{A}} \left(\tilde{Z}_\beta(T_i) - M_i(\alpha, \beta) \tilde{Z}_\beta(T_i) \right) \\ &= \textstyle\sum_{\beta \in \mathcal{A}} (1 - M_i(\alpha, \beta))\, \tilde{Z}_\beta(T_i) \\ &= \textstyle\sum_{\beta \in \mathcal{A}} \tilde{M}_i(\alpha, \beta) \tilde{Z}_\beta(T_i) \end{aligned}$$

and

$$-Y_\alpha C_1 C_2 \cdots C_k = -\tilde{Z}_\alpha(T) \ . \qquad \square$$

Corollary 5.3 *We have for a non-trivial $k-$regular tree $T \in \mathcal{T}'_k$*

$$\sum_{(A;B_1,\dots,B_{d(A)})\in\mathcal{S}(T)} (-1)^{d^\circ(A)} Z(A; B_1, \dots, B_{d(A)}) = 0 \ .$$

Proof. Sum the equality of Proposition 5.2 over $\alpha \in \mathcal{A}$ and use

$$Z(\{r\}; T) = \tilde{Z}(T) = \sum_{\alpha\in\mathcal{A}} \tilde{Z}_\alpha(T) \ . \qquad \square$$

Proof of Theorem 2.1. By Proposition 5.1 we have

$$g \circ_X \tilde{g} = -\sum_{T\in\mathcal{T}_k} (-1)^{d(T)} \sum_{(A;B_1,\dots,B_{d(A)})\in\mathcal{S}(T)} (-1)^{d^\circ(A)} Z(A; B_1, \dots, B_{d(A)}) \ .$$

It follows from Corollary 5.3 that only the trivial tree $T = \{r\}$ contributes to the right hand side. The contribution of $T = \{r\}$ amounts to

$$-(-1)^{d(\{r\})} Z(\{r\}; \{r\}) = -(-1) \cdot X = X$$

which proves Theorem 2.1. $\square$

6 Further generalizations

We give first a generalization of Theorem 2.1 associated to not necessarily regular trees. The special case of this generalization where all weight-matrices have coefficients in $\{0, 1\}$ appears already in [5].

We state then an even more general version involving trees with vertices labelled by a set $\mathcal{T}$ of types and formal power-series in $R[[(X_\tau)_{\tau\in\mathcal{T}}, (Y_\alpha)_{\alpha\in\mathcal{A}}]]$.

6.1 Trees with vertices of arbitrary degrees

Consider a subset $\mathcal{K} \subset \mathbb{N}_{>0}$ of strictly positive integers and a partition $\mathcal{A} = \cup_{k\in\mathcal{K}} \mathcal{A}_k$ of the alphabet $\mathcal{A}$ into non-empty parts indexed by $\mathcal{K}$. For each $k \in \mathcal{K}$ choose k matrices $M^k_1,\ M^k_2, \dots, M^k_k$ with rows indexed by $\mathcal{A}_k$ and columns indexed by $\mathcal{A}$. For $i = 1, \dots, k$, set $\tilde{M}^k_i = J - M^k_i$ where J is the all 1 matrix with indices in $\mathcal{A}_k \times \mathcal{A}$.

For $\alpha \in \mathcal{A}_k$ consider the uniquely defined (non-commutative) formal power series $g_\alpha,\ \tilde{g}_\alpha$ consisting only of monomials of degree at least 1 in X, involving at least one variable of $(Y_\beta)_{\beta\in\mathcal{A}}$ and such that

$$g_\alpha = Y_\alpha(X - (M^k_1\ V)_\alpha)(X - (M^k_2\ V)_\alpha) \cdots (X - (M^k_k\ V)_\alpha)$$

and

$$\tilde{g}_\alpha = Y_\alpha(-X + (\tilde{M}_1^k\ \tilde{V})_\alpha)\cdots(-X + (\tilde{M}_k^k\ \tilde{V})_\alpha)$$

where V, respectively $\tilde{V}$, is the column vector with coordinates g_β, respectively $\tilde{g}_\beta$, for $\beta \in \mathcal{A}$.

Set $g = -X + \sum_{\alpha\in\mathcal{A}} g_\alpha$ and $\tilde{g} = -X + \sum_{\alpha\in\mathcal{A}} \tilde{g}_\alpha$.

Theorem 6.1 *We have*

$$g \circ_X \tilde{g} = \tilde{g} \circ_X g = X \ .$$

The proof is an adaption of the proof of Theorem 2.1: Consider trees whose internal vertices have degrees in $\mathcal{K}$ and colour internal vertices of degree $k \in \mathcal{K}$ by elements in $\mathcal{A}_k$.

Example (Inversion of power series). In this example, we work over a commutative ring R and set $Y_\alpha = 1$. Choose an integer $l \geq 1$ and set $\mathcal{K} = \{2, 3, 4, \ldots\}$ (this ensures existence of all relevant power-series). We consider trees with internal vertices of degree ≥ 2 having spins in a finite alphabet $\mathcal{A}$ containing l elements.

For $1 \leq j \leq k,\ 2 \leq k$ choose elements $\beta_{k,j} \in R$ and consider the diagonal weight function

$$w_e(\varphi(\alpha(e)), \varphi(\omega(e)) = \beta_{k,j}\ \delta_{\varphi(\alpha(e)),\varphi(\omega(e))} = \begin{cases} \beta_{k,j} & \text{if } \varphi(\alpha(e)) = \varphi(\omega(e)) \\ 0 & \text{otherwise} \end{cases}$$

where the leafless edge e joins a vertex $\alpha(e)$ of degree k to its $j-$th son $\omega(e)$.

Introduce the formal power series $g_*,\ \tilde{g}_* \in R[[X]]$ without constant term which satisfy

$$\begin{aligned} g_* &= \sum_{k=2}^\infty \prod_{j=1}^k (X - \beta_{k,j}\ g_*) \\ \tilde{g}_* &= \sum_{k=2}^\infty \prod_{j=1}^k (-X + (l - \beta_{k,j})\ \tilde{g}_*) \ . \end{aligned}$$

Set $g(X) = -X + l\ g_*$ and $\tilde{g}(X) = -X + l\ \tilde{g}_*$. Theorem 6.1, applied to the present situation shows that we have

$$g(\tilde{g}(X)) = \tilde{g}(g(X)) = X \ .$$

Since this formula holds for any $l \in \mathbb{N}$, it extends to an arbitrary value of $l \in R$ which can thus be considered as a parameter.

This formula provides a mean (other than the celebrated Lagrange inversion formula) for computing the compositional inverse of a formal power series $h(X) = \sum_{k=1}^\infty \gamma_k\ X^k$ with $\gamma_1 \in R^*$ invertible: Set $f = h(-X/\gamma_1) = -X + \sum_{k=2}^\infty \beta_k\ t^k$. If $\beta_2 \neq 0$ set $l = \beta_2$ and choose constants $\beta_{k,j}$ such that $g(X) = f(X)$ with $g(X)$ defined as above.

Each such choice can be used to compute the compositional inverse $\tilde{g} = f^{-1}$ of f. The compositional inverse of the initial series $h(X)$ is then given by $-\gamma_1\ \tilde{g}(X)$. The case $\beta_2 = 0$ can be handled similarly by allowing the trees to have internal vertices of degree 1.

6.2 Vertices of different types

This still more general version of Theorem 2.1 is perhaps easier to formulate in a combinatorial way:

Consider a set $\mathcal{T}$ of *vertex-types* indexing the non-empty parts of a partition $\mathcal{A} = \cup_{\tau\in\mathcal{T}}\mathcal{A}_\tau$. Consider also a set $\mathcal{E}$ of *edge-types* together with weight-functions

$$w_\epsilon : \mathcal{A}\times\mathcal{A} \longrightarrow R$$

indexed by edge-types $\epsilon \in \mathcal{E}$.

A *labelled* tree is a finite rooted planar tree L with leaves labelled by elements of $\mathcal{T}$, internal vertices labelled by pairs $(\tau, \alpha \in \mathcal{A}_\tau) \in \mathcal{T}\times\mathcal{A}$ and edges labelled by elements of $\mathcal{E}$. A vertex $v \in L$ is of type $\tau \in \mathcal{T}$ if it is either a leaf labelled τ or an internal vertex labelled $(\tau, \alpha \in \mathcal{A}_\tau)$. A labelled tree L is of type $\tau \in \mathcal{T}$ if its root vertex is of type τ.

The *energy* $f(L) \in R[(X_\tau)_{\tau\in\mathcal{T}}, (Y_\alpha)_{\alpha\in\mathcal{A}}]$ of a labelled tree L of type $\tau \in \mathcal{T}$ is defined in the obvious way: $f(L) = X_\tau$ if L is trivial (reduced to its labelled root) and

$$f(L) = Y_\alpha C_1 \cdots C_k$$

otherwise where $(\tau, \alpha \in \mathcal{A}\tau)$ is the label of the root $r \in L$ and where $C_i \in R[(X_\tau)_{\tau\in\mathcal{T}}, (Y_\alpha)_{\alpha\in\mathcal{A}}]$ is associated to the $i-$th labelled principal subtree $L(s_i)$ of L as follows: $C_i = X_{\tau_i}$ if $L(s_i)$ is trivial of type τ_i and

$$C_i = w_{\epsilon_i}(\alpha, \alpha_i) f(L_i)$$

where ϵ_i is the label of the edge joining the root r to its $i-$th son s_i which is labelled $(\tau_i, \alpha_i \in \mathcal{A}_{\tau_i})$.

Define complementary weight-functions by $\tilde{w}_\epsilon(\alpha, \beta) = 1 - w_\epsilon(\alpha, \beta)$ and compute the *complementary energy* $\tilde{f}(L)$ of L as above using the complementary weight-functions.

Call two labelled trees L_1, L_2 *edge-equivalent* if they differ only on their edge-labels (but share the same underlying tree-structure and vertex-labels).

Let $\mathcal{L}$ be a subset of the set of all labelled (finite rooted planar) trees. We denote by $\mathcal{L}_\tau \subset \mathcal{L}$ the subset of all labelled trees of type τ in $\mathcal{L}$ and assume that the set $\mathcal{L}$ satisfies the following three conditions:

(i) If $L \in \mathcal{L}$ and v is a leaf of $\mathcal{L}$, then $L(v) \in \mathcal{L}$ where $L(v)$ denotes the labelled subtree defined in the obvious way by considering the maximal subtree of L rooted at v.

(ii) If $L' \in \mathcal{L}_\tau$ and v is a leaf of type τ in $L \in \mathcal{L}$ then the labelled tree obtained in the obvious way by gluing L' onto the leaf $v \in L$ is again in $\mathcal{L}$.

(iii) All equivalence classes of the edge-equivalence relation are finite.

Otherwise stated, all maximal labelled subtrees of an element in $\mathcal{L}$ are in $\mathcal{L}$ by (i) and $\mathcal{L}$ is "closed under composition" by (ii). Condition (iii) is a finiteness condition (which can perhaps be slightly weakened or replaced by a similar statement) ensuring the existence of the generating series $Z_\mathcal{T}$ and $\tilde{Z}_\mathcal{T}$ defined below.

For such a set $\mathcal{L}$ we define

$$Z_\mathcal{T} = -\sum_{L\in\mathcal{L}} (-1)^{d^\circ(L)} f(L)$$

and

$$\tilde{Z}_\mathcal{T} = \sum_{L\in\mathcal{L}} (-1)^{d(L)} \tilde{f}(L)$$

where $d^\circ(L)$ denotes the number of internal vertices of a labelled tree L and where $d(L)$ denotes the number of leaves in L.

Theorem 6.2 *We have*

$$Z_\mathcal{T} \circ_{X_\mathcal{L}} (\tilde{Z}_\sigma)_{\sigma\in\mathcal{T}} = \tilde{Z}_\mathcal{T} \circ_{X_\mathcal{L}} (Z_\sigma)_{\sigma\in\mathcal{T}} = X_\mathcal{T}$$

where the notation $Z_\mathcal{T} \circ_{X_\mathcal{L}} (\tilde{Z}_\sigma)_{\sigma\in\mathcal{T}}$ *means that every occurence of* $X_\mathcal{T}$ *in* $Z_\mathcal{T}$ *is replaced by the generating series* $\tilde{Z}_\mathcal{T}$*.*

This result can be used for the formal inversion of power-series in several variables.

Sketch of Proof. Define grafted labelled trees in the obvious way and check that Proposition 5.2 remains valid in the present context.

7 Morphisms of rooted trees into posets

This section is a digression discussing a generalization of grafted trees.

The vertex set V of a rooted (not necessarily finite or planar) tree T with oriented edges E can be considered as a poset (partially ordered set) by considering the order relation induced by $\omega(e) > \alpha(e)$ for $e \in E$.

Remark 7.1 *Considering a rooted tree as a poset, one might wonder how many unordered pairs of comparable vertices are contained in (k−regular) rooted planar trees having n vertices. Let* a_n *(respectively* $a_{n,k}$*) denote this number. We have then*

$$\sum_{n=1}^{\infty} a_n\, t^n = \frac{\partial y}{\partial u}(t,1)$$

where $y(t,u) = t+($ *terms of higher order in* t $)$ *satisfies the functional equation*

$$y(t,u) = t + t\frac{y(tu,u)}{1-y(tu,u)} \ .$$

The corresponding sequence $a_2, a_3, a_4, \ldots$ *starts as*

$$1, 5, 22, 93, 386, 1586, 6476, 26333, 106762, 431910, \ldots \quad \textit{cf. A346 of [2].}$$

Similarly, for the numbers $a_{n,k}$ associated to $k-$regular trees, we have

$$\sum_{n=1}^{\infty} a_{n,k}\ t^n = \frac{\partial y}{\partial u}(t,1)$$

with $y(t,u) = t +$ (terms of higher order in t) satisfying

$$y(t,u) = t + t\ (y(tu,u))^k\ .$$

For $k = 2$ we get the sequence $\frac{1}{2}(a_{3,2}, a_{5,2}, a_{7,2}, \ldots)$ starting as

$$1, 6, 29, 130, 562, 2380, 9949, 41226, 169766, \ldots, \quad \textit{cf. A8549 of [2]}$$

and for $k = 3$, the sequence $\frac{1}{3}(a_{4,3}, a_{7,3}, a_{10,3}, \ldots)$ starting as

$$1, 9, 69, 502, 3564, 24960, 173325, 1196748, \ldots, \quad \textit{cf. A75045 of [2].}$$

Given a second poset P, a *morphism* from T to P is an application $\mu : V \longrightarrow P$ such that $\mu(\omega(e)) \geq \mu(\alpha(e))$ for every edge e of T.

Denote by $\{1, 2\}$ (with $1 < 2$) the obvious totally ordered poset. Call a morphism $\mu : T \longrightarrow \{1, 2\}$ *restricted* if $\mu^{-1}(2)$ contains all leaves of T.

Proposition 7.2 *If T is a rooted (planar) tree, then the set of grafted trees $(A; B_1, \ldots, B_{d(A)})$ with skeleton T corresponds bijectively to the set of restricted morphisms $\mu : T \longrightarrow \{1, 2\}$.*

Proof. Given a restricted morphism $\mu : T \longrightarrow \{1, 2\}$ we set $A = \{\mu^{-1}(1) \cup$ sons of $\mu^{-1}(1)\}$ where the root $r \in T$ is by convention the son of the empty set. The rooted trees $\{B_1, \ldots, B_d(A)\}$ are the connected components of the maximal subforest with vertices $\mu^{-1}(2)$. It is obvious that $(A; B_1, \ldots, B_{d(A)})$ is a grafted tree with skeleton T. Reciprocally, given a grafted tree $(A; B_1, \ldots, B_{d(A)})$ with skeleton T, we get a morphism $\mu : T \longrightarrow \{1, 2\}$ by setting $\mu(v) = 2$ if $v \in B_j$ for some $j = 1, \ldots, d(A)$ and $\mu(v) = 1$ otherwise. Since the forest $\cup_{j=1}^{d(A)} B_j$ contains all leaves of T, the morphism μ is restricted. □

Proposition 7.2 suggests to generalize the notion of grafted trees by considering sets of (suitable) morphisms from rooted trees into posets. We consider now a few special cases. Henceforth all rooted trees and posets will be finite. The number of morphisms of a rooted tree T having $\sharp(V)$ vertices into a poset P with $\sharp(P)$ elements is bounded by $\sharp(P)^{\sharp(V)}$.

7.1 Grafted trees

In this subsection we indicate how to count the number of grafted trees with given skeleton T. A slight modification counts all morphisms from T into

$\{1,2\}$. Further generalizations consist in counting all (suitable) morphisms from T into the totally ordered set $\{1,\ldots,n\}$.

Given a (finite rooted planar) tree T we denote by $\gamma(T)$ the number of grafted trees with skeleton T. We denote by $\tilde{\gamma}(T) \geq \gamma(T)$ the number of all morphisms $T \longrightarrow \{1,2\}$. We have $\gamma(\{r\}) = 1,\ \tilde{\gamma}(\{r\}) = 2$ if $T = \{r\}$ is the trivial tree reduced to its root. Remark that $\tilde{\gamma}(\tilde{T}) = \gamma(T)$ where $\tilde{T}$ is obtained by removing all leaves from the rooted tree γ (where $\tilde{\gamma}(\emptyset) = 1$ by convention). This remark generalizes easily to the analogous numbers enumerating (restricted) morphismes $T \longrightarrow \{1,\ldots,m\}$.

Proposition 7.3 *For T a non-trivial (finite rooted planar) tree we have*

$$\gamma(T) = 1 + \prod_{j=1}^{k} \gamma(T_i)$$

and

$$\tilde{\gamma}(T) = 1 + \prod_{j=1}^{k} \tilde{\gamma}(T_i)$$

where $T_1,\ldots,T_k$ are the principal subtrees defined by the k sons $s_1,\ldots,s_k$ of the root $r \in T$.

Proof. We count all (restricted) morphisms μ from T into $\{1,2\}$.

Consider a morphism $\mu : V \longrightarrow \{1,2\}$. If $\mu(r) = 2$ we have $\mu(v) = 2$ for all $v \in V$ and there is exactly one such morphism yielding a contribution of 1 to $\gamma(T)$ and $\tilde{\gamma}(T)$. If $\mu(r) = 1$, the restriction μ_i of μ to a principal subtrees T_i is an arbitrary (restricted) morphism from T_i into $\{1,2\}$ and the restrictions $\mu_1,\ldots,\mu_k$ can be arbitrary thus proving the formula. □

Proposition 7.3 leads to a fast algorithm for computing $\gamma(T)$ illustrated by the following example.

Example. For the tree T of Figure 2 we have

$$\begin{array}{l}
\gamma(v_1) = \gamma(v_3) = \gamma(v_8) = 2 \\
\gamma(v_4) = 1 + \gamma(v_3)\cdot 1 = 3,\ \gamma(v_7) = 1 + 1\cdot\gamma(v_8) = 3 \\
\gamma(v_2) = 1 + \gamma(v_1)\cdot\gamma(v_4) = 7,\ \gamma(v_6) = 1 + 1\cdot\gamma(v_7) = 4 \\
\gamma(v_5) = 1 + \gamma(v_2)\cdot\gamma(v_6) = 29
\end{array}$$

and

$$\begin{array}{l}
\tilde{\gamma}(v_1) = \tilde{\gamma}(v_3) = \tilde{\gamma}(v_8) = 1 + 2\cdot 2 = 5 \\
\gamma(v_4) = 1 + \gamma(v_3)\cdot 2 = 11,\ \gamma(v_7) = 1 + 2\cdot\gamma(v_8) = 11 \\
\gamma(v_2) = 1 + \gamma(v_1)\cdot\gamma(v_4) = 56,\ \gamma(v_6) = 1 + 2\cdot\gamma(v_7) = 23 \\
\gamma(v_5) = 1 + \gamma(v_2)\cdot\gamma(v_6) = 1289
\end{array}$$

which shows that the tree T of Figure 2 is the skeleton of $\gamma(T) = \gamma(v_5) = 29$ different grafted trees (or equivalently, that T contains 29 different $2-$regular subtrees sharing the root r with T) and has 1289 distinct morphisms into the totally ordered poset $\{1,2\}$.

Proposition 7.3 shows also that the generating function

$$y(t)=\sum_{T\in\mathcal{T}_2}\gamma(T)\ t^{d(T)}$$

counting the number $\gamma(k)=[t^k]y$ of $2-$regular grafted trees whose skelettons have k leaves satisfies

$$\begin{aligned} y &= \tfrac{1-\sqrt{1-4t}}{2}+y^2=\tfrac{1-\sqrt{-1+2\sqrt{1-4t}}}{2} \\ &= t+2t^2+6t^3+21t^4+80t^5+322t^6+1348t^7+5814t^8+\dots\ . \end{aligned}$$

The similar generating function $\tilde{y}(t)=\sum_{T\in\mathcal{T}_2}\tilde{\gamma}(T)\ t^{d(T)}$ (which counts all morphisms of binary rooted trees into $\{1,2\}$) is given by

$$\begin{aligned} \tilde{y} &= t+\tfrac{1-\sqrt{1-4t}}{2}+y^2=\tfrac{1-\sqrt{-1-4t+2\sqrt{1-4t}}}{2} \\ &= 2t+5t^2+22t^3+118t^4+706t^5+4530t^6+\dots \end{aligned}\ .$$

More generally, consider a fixed natural integer $k\geq 2$. We call a morphism $\mu:T\longrightarrow\{1,\dots,k\}$ *restricted* if μ^{-1} contains all leaves of the rooted tree T. We denote by $y_m(t)$ the generating function counting the number of (non-isomorphic) restricted morphisms from a $k-$regular tree having n leaves into the completely ordered finite set $\{1,\dots,m\}$. Similarly, we denote by $\tilde{y}_m(t)$ the analogous generating function counting all morphisms without restriction on the leaves.

Proposition 7.4 *We have*

$$y_1(t)=\tilde{y}_1(t)=t+(y_1(t))^k$$

$$y_m(t)=t+\sum_{h=1}^{m}(y_h(t))^k$$

$$\tilde{y}_m(t)=mt+\sum_{h=1}^{m}(\tilde{y}_h(t))^k$$

Sketch of proof. The trivial tree T yields a contribution of t to y_k and of mt to $\tilde{y}(t)$. For a morphism μ from a nontrivial tree into $\{1,\dots,m,\}$ we consider the image $\mu(r)$ of its root and the remaining possibilities of the induced restrictions $\mu_1,\dots,\mu_k$ on the principal subtrees $T_1,\dots,T_k$ of T. $\Box$

For arbitrary finite planar non-empty trees we consider the generating function counting the number of different restricted morphisms μ from a tree having n vertices into the completely ordered finite set $\{1,\dots,m\}$. The generating function $\tilde{y}_m(t)$ counts all such morphisms into $\{1,\dots,m\}$. One can then prove the following result.

Proposition 7.5 *We have*

$$y_1(t) = \tilde{y}_1(t) = t + \frac{ty_1}{1 - y_1}$$

$$y_m(t) = t + t \sum_{h=1}^{m} \frac{y_h}{1 - y_h}$$

$$\tilde{y}_m(t) = mt + t \sum_{h=1}^{m} \frac{\tilde{y}_h}{1 - \tilde{y}_h}$$

The first instances are $y_1 = \tilde{y}_1 = \frac{1-\sqrt{1-4t}}{2}$ (defining the Catalan numbers, cf. A108 of [2])

$$\begin{aligned} y_2 &= \frac{3 - 2t - \sqrt{1-4t} - \sqrt{2 - 16t + 4t^2 + (2+4t)\sqrt{1-4t}}}{4} \\ &= t + 2t^2 + 5t^3 + 15t^4 + 50t^5 + 178t^6 + 663t^7 + 2553t^8 + \dots \end{aligned}$$

which is sequence A7853 of [2] and

$$\begin{aligned} \tilde{y}_2 &= \frac{3 - \sqrt{1-4t} - \sqrt{2 - 20t + 2\sqrt{1-4t}}}{4} \\ &= 2t + 3t^2 + 9t^3 + 34t^4 + 145t^5 + 667t^6 + 3231t^7 + 16247t^8 + \dots \end{aligned}$$

Finally, let us mention the well-known fact that the function $p(m)$ counting all morphisms from a fixed finite poset E into the totally ordered poset $\{1, \dots, m\}$ is a polynomial of degree $\sharp(E)$. Indeed

$$p(m) = \sum_{k=1} \alpha_k \binom{m}{k}$$

where α_k denotes the number of surjective morphismes of E into $\{1, \dots, k\}$. One can thus also consider generating functions associated to such polynomials. The number of restricted morphisms from a fixed rooted tree into $\{1, \dots, m\}$ is of course also a polynomial function. Its degree in m is the number of interior leaves in T.

7.2 Surjective morphisms

Given a finite rooted tree T having n vertices, the number $\sigma(T)$ of surjective morphisms from T into $\{1, \dots, n\}$ can be recursively computed by remarking that

$$\sigma(T) = (n-1)! \prod_{j=1}^{k} \frac{\sigma(T_j)}{n_j!}$$

where $T_1, \ldots, T_k$ are the principal (rooted) subtrees of T having $n_1, \ldots, n_k$ vertices. Denoting by $\binom{n-1}{n_1,\ldots,n_k} = \frac{(n-1)!}{n_1!\cdots n_k!}$ the obvious multinomial coefficient (where $n-1 = \sum n_i$) we get for our favorite (binary) tree T of Figure 2 the following numbers: $\sigma(v) = 1$ if v is a leaf and

$$\sigma(v_1) = \sigma(v_3) = \sigma(v_8) = \binom{2}{1,1} = 2$$

$$\sigma(v_4) = \sigma(v_7) = \binom{4}{1,3} 2 = 8$$

$$\sigma(v_2) = \binom{8}{3,5} 2 \cdot 8 = 896, \ \sigma(v_6) = \binom{6}{1,5} 1 \cdot 8 = 48$$

$$\sigma(T) = \sigma(v_5) = \binom{16}{9,7} 896 \cdot 48 = 492011520$$

The generating function

$$y(t) = \sum_{k=1}^{\infty} \alpha_k t^k$$

encoding the number α_n of surjective morphisms into $\{1, \ldots, n\}$ from all rooted binary planar trees on n vertices satisfies

$$\alpha_n = (n-1)! \sum_{k=1}^{n-2} \frac{\alpha_k \ \alpha_{n-1-k}}{k! \ (n-1-k)!}$$

(cf. sequence A182 of [2]). Otherwise stated, the exponential generating function

$$z(t) = \sum_{k=1}^{\infty} \alpha_k \ \frac{t^k}{k!}$$

satisfies $z' = z^2 - 1$ thus proving that $z(t) = \tanh(t)$.

Similarly, considering the exponential generating function

$$z(t) = \sum_{k=1}^{\infty} \beta_k \ \frac{t^k}{k!}$$

enumerating the number β_n of all surjective homomorphisms from rooted planar trees having n vertices into $\{1, \ldots, n\}$ we have

$$z' = \frac{z}{1-z} + 1 = \frac{1}{1-z}$$

which implies $z(t) = \frac{1-\sqrt{1-4t}}{2}$ (cf. sequence A108 of [2]) and shows that $\beta_n = n!\binom{2(n-1)}{n-1}/n = \frac{(2n-2)!}{(n-1)!}$.

8 Loday's example (i)

The aim of this section is a partial analysis of example (i) in [3]. The framework is somewhat simpler as in the previous sections: We work over the commutative ground field $\mathbf{C}$ of complex numbers. The alphabet $\mathcal{A}$ consists of nine elements and equals

$$\mathcal{A} = \{\circ, N, NW, W, SW, S, SE, E, NE\}$$

suggesting the graphical notations of [3]. We set $X = t$ in order to stick to [3].

Loday's example (i) corresponds to $k = 2$. The two 9×9 matrices M_1, M_2 are given by

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0\\ 1&0&1&1&0&0&0&0&0\\ 1&1&1&1&1&0&0&0&0\\ 1&1&0&0&0&0&0&0&0\\ 1&1&1&1&0&1&0&0&0\\ 1&1&0&1&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&1&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&1&1&1&1&0&0&0\\ 0&0&0&0&1&0&1&1&1\\ 0&0&1&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&1&1 \end{pmatrix}$$

Writing $V = (g_\circ, g_N, g_{NW}, g_W, g_{SW}, g_S, g_{SE}, g_E, g_{NE})^t$ we get the equations

$$\begin{array}{lcl}
g_\circ &=& (-t+g_\circ)(-t+g_N+g_{NW}+g_W)\\
g_N &=& (-t+g_\circ+g_N+g_{NW})(-t+g_W)\\
g_{NW} &=& (-t+g_\circ+g_N+g_{NW}+g_W)(-t)\\
g_W &=& (-t+g_\circ+g_{NW}+g_W)(-t+g_N)\\
g_{SW} &=& (-t+g_\circ+g_N+g_{NW}+g_W+g_{SW})(-t+g_S)\\
g_S &=& (-t+g_\circ+g_N)(-t+g_{NW}+g_W+g_{SW}+g_S)\\
g_{SE} &=& (-t+g_\circ+g_N+g_{NW}+g_W+g_S)\\
&& \quad (-t+g_{SW}+g_{SE}+g_E+g_{NE})\\
g_E &=& (-t+g_\circ+g_N+g_W)(-t+g_{NW}+g_E+g_{NE})\\
g_{NE} &=& (-t+g_\circ+g_N+g_{NW}+g_W)(-t+g_E+g_{NE})
\end{array}$$

One sees easily that we have

$$g_o = g_N = g_W \ .$$

Eliminating g_N, g_W we get the simpler equations:

$$\begin{array}{lcl}
g_\circ &=& (-t+g_\circ)(-t+2g_\circ+g_{NW})\\
g_{NW} &=& (-t+3g_\circ+g_{NW})(-t)\\
\hline
g_{SW} &=& (-t+3g_\circ+g_{NW}+g_{SW})(-t+g_S)\\
g_S &=& (-t+2g_\circ)(-t+g_\circ+g_{NW}+g_S+g_{SW})\\
\hline
g_E &=& (-t+3g_\circ)(-t+g_{NW}+g_E+g_{NE})\\
g_{NE} &=& (-t+3g_\circ+g_{NW})(-t+g_E+g_{NE})\\
\hline
g_{SE} &=& (-t+3g_\circ+g_{NW}+g_S)(-t+g_{SW}+g_{SE}+g_E+g_{NE})
\end{array}$$

where functions above a horizontal line are independent of functions below the line.

Computations (done with Maple) using Groebner-bases show that

$$y = g_\circ + g_N + g_{NW} + g_W + g_{SW} + g_S + g_{SE} + g_E + g_{NE}$$

satisfies the algebraic equation $P(y(t), t) = 0$ where

$$P(y,t) = c_0 + c_1\ y + c_2\ y^2 + c_3\ y^3 + c_4\ y^4$$

with

$$\begin{aligned} c_0 \;=\; & t\ \big(288\,t^{31} - 1008\,t^{30} - 17696\,t^{29} + 35124\,t^{28} + 513042\,t^{27} \\ & -352654\,t^{26} - 8834409\,t^{25} - 2315100\,t^{24} + 94293622\,t^{23} \\ & +92841847\,t^{22} - 608228325\,t^{21} - 1031578684\,t^{20} \\ & +2072381165\,t^{19} + 5859780674\,t^{18} - 1127775119\,t^{17} \\ & -16287829166\,t^{16} - 15833938922\,t^{15} + 9251292427\,t^{14} \\ & +38652814035\,t^{13} + 44572754075\,t^{12} + 10866248029\,t^{11} \\ & -40129564125\,t^{10} - 59007425756\,t^{9} - 36829453004\,t^{8} \\ & -10216139916\,t^{7} - 63849664\,t^{6} + 693364800\,t^{5} + 187804368\,t^{4} \\ & +24111840\,t^{3} + 1694752\,t^{2} + 63488\,t + 1024\big) \end{aligned}$$

$$\begin{aligned} c_1 \;=\; & -768\,t^{31} + 1488\,t^{30} + 44636\,t^{29} - 49538\,t^{28} - 1198111\,t^{27} \\ & +359773\,t^{26} + 19127286\,t^{25} + 7602856\,t^{24} - 192894854\,t^{23} \\ & -193322898\,t^{22} + 1208988418\,t^{21} + 1968967542\,t^{20} \\ & -4212884427\,t^{19} - 10893520130\,t^{18} + 4099837581\,t^{17} \\ & +31343794098\,t^{16} + 22508418249\,t^{15} - 27437733598\,t^{14} \\ & -67449042813\,t^{13} - 62529644946\,t^{12} - 3629552721\,t^{11} \\ & +84243589625\,t^{10} + 130637976096\,t^{9} + 104165077688\,t^{8} \\ & +50704667612\,t^{7} + 15902199040\,t^{6} + 3290858704\,t^{5} \\ & +451630576\,t^{4} + 40498432\,t^{3} + 2274080\,t^{2} + 72704\,t + 1024 \end{aligned}$$

$$\begin{aligned} c_2 \;=\; & t\ \big(680\,t^{29} - 932\,t^{28} - 39270\,t^{27} + 33926\,t^{26} + 1020385\,t^{25} \\ & -335552\,t^{24} - 15580483\,t^{23} - 2999586\,t^{22} + 151572589\,t^{21} \\ & +104425399\,t^{20} - 945694543\,t^{19} - 1131146667\,t^{18} \\ & +3558106593\,t^{17} + 6544185368\,t^{16} - 6226627151\,t^{15} \\ & -20759382401\,t^{14} - 4197042728\,t^{13} + 29133893401\,t^{12} \\ & +33005986439\,t^{11} + 5921959164\,t^{10} - 22398414511\,t^{9} \\ & -37477032816\,t^{8} - 35648192872\,t^{7} - 21631096056\,t^{6} \\ & -8298733544\,t^{5} - 1992896768\,t^{4} - 295849440\,t^{3} \\ & -26179392\,t^{2} - 1255424\,t - 24832\big) \end{aligned}$$

$$
\begin{aligned}
c_3 \;=\; & -t^2\ (t-2)\,(t+1)\,\big(2\,t^4+6\,t^3-11\,t^2-30\,t-4\big) \\
& \big(124\,t^{21}-398\,t^{20}-5146\,t^{19}+14694\,t^{18}+92616\,t^{17}-213234\,t^{16} \\
& -966327\,t^{15}+1518831\,t^{14}+6391763\,t^{13}-5003278\,t^{12} \\
& -26227554\,t^{11}+1248286\,t^{10}+58532080\,t^{9}+36103178\,t^{8} \\
& -41699603\,t^{7}-64544195\,t^{6}-41818519\,t^{5}-21472740\,t^{4} \\
& -8578026\,t^{3}-1961960\,t^{2}-218928\,t-9296\big)
\end{aligned}
$$

and

$$
\begin{aligned}
c_4 \;=\; & (t^2-2\,t-2)(t^2-2\,t-7)(t-2)^2(t+1)^2 \\
& (2\,t^4+6\,t^3-11\,t^2-30\,t-4)^2 t^5 \\
& (8\,t^7-10\,t^6-171\,t^5+209\,t^4+948\,t^3-721\,t^2-1892\,t-249)
\end{aligned}
$$

The first coefficients of the series $y(t)$ are

$$
-t+9\,t^2-49\,t^3+284\,t^4-1735\,t^5+10955\,t^6-70695\,t^7+463087\,t^8
$$

$$
-3066450\,t^9+20471641\,t^{10}-137540539\,t^{11}+928791019\,t^{12}\mp\ldots
$$

The "complementary" function

$$
\tilde{y}=\tilde{g}_\circ+\tilde{g}_N+\tilde{g}_{NW}+\tilde{g}_W+\tilde{g}_{SW}+\tilde{g}_S+\tilde{g}_{SE}+\tilde{g}_E+\tilde{g}_{NE}
$$

is associated to the matrice $J-M_1$ and $J-M_2$ (where J is the square matrix of order 9 with all entries equal to 1) satisfies of course the same polynomial equation, after transposition of y and t.

Remark. For computations of huge terms in the series expansion of an algebraic function, one can use the following well-known trick: Any algebraic function $y(t)=\sum a_n t^n$ of degree d satisfies a linear differential equation

$$
\sum_{k=0}^{d} q_k(t)y^{(k)}(t)=0
$$

with polynomial coefficients $q_0,\ldots,q_t\in\mathbf{C}[t]$. This allows a recursive computations of a_n with time and memory requirements linear in n. In our case, we get

$$
q_0(t)y+q_1(t)y'+q_2(t)y''+q_3(t)y^{(3)}+q_4(t)y^{(4)}=0
$$

where $q_0,\ldots,q_4\in\mathbf{Z}[t]$ are polynomials of degrees respectively 150, 151, 155, 156 and 157.

8.1 Asymptotics

The asymptotic growth rate of the coefficients of $y(t)$ is governed by the distance of the origin to the first ramification point of the corresponding sheet, see [4]. Ramifications are above the zeros of the discriminant $D(t)$ of $P(y,t)$ with respect to y. This discriminant is given by

$$D(t) = r_1^2\, r_2\, r_3^2\, r_4^2\, r_\infty$$

where

$$\begin{aligned}
r_1 &= t^4 - 4\,t^3 + 6\,t^2 + 8\,t + 1\\
r_2 &= t^{13} - 3\,t^{12} - 16\,t^{11} + 100\,t^{10} - 86\,t^9 - 222\,t^8 + 312\,t^7 - 544\,t^6\\
&\quad +4845\,t^5 + 10665\,t^4 + 9536\,t^3 + 4084\,t^2 + 528\,t + 16\\
r_3 &= 20\,t^{16} + 50\,t^{15} - 849\,t^{14} - 937\,t^{13} + 11563\,t^{12} + 7833\,t^{11}\\
&\quad -64177\,t^{10} - 58882\,t^9 + 141152\,t^8 + 259280\,t^7 + 82253\,t^6\\
&\quad -366913\,t^5 - 698955\,t^4 - 468324\,t^3 - 122700\,t^2 - 13720\,t - 552\\
r_4 &= 5760\,t^{53} + 8864\,t^{52} - 425056\,t^{51} + \ldots - 7200309248\,t - 76152832
\end{aligned}$$

(r_4 is not involved in coarse asymptotics of $y(t)$). The roots of the polynomial

$$r_\infty = t^6\ (t-2)^2\ (t+1)^3\ (2\,t^4 + 6\,t^3 - 11\,t^2 - 30\,t - 4)^2$$

are critical points for the critical value ∞.

Since the coefficients of $y(t)$ have alternating signs, the "smallest" singularity of $y(t)$ is on the negative real halfline. The following table resumes the relevant data for its computation. More precisely, the algebraic function defined by P has a ramification of order 3 with $y = \infty$ above $t = 0$. The remaining sheet is unramified above $t = 0$ and defines the generating function $y(t)$ under consideration.

The table contains the following informations: The first column displays the argument t considered in the corresponding row. The second row indicates the factor of the discriminant $D(t)$ if t is a root of $D(t)$. The remaining row displays information about the inverse images of t defined by the algebraic equation of $y(t)$.

We have

$t_0 = 0$	r_∞	$y_1 = y_2 = y_3 = \infty,\ y_4 = 0$
$t_0 > t > t_1$		$y_1 < y_2 < y_3 < 0 < y_4$
$t_1 \sim -0.04355$	r_2	$y_1 \sim -2922 < y_2 = y_3 \sim -1.083 < 0 < y_4 \sim 0.066$
$t_1 > t > t_2$		$y_1 < 0 < y_4,\ y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_2 \sim -0.1118$	r_3	$y_1 \sim -82.3 < y_2 = y_3 \sim 0.2194 < y_4 \sim 0.4499$
$t_2 > t > t_3$		$y_1 < 0 < y_4,\ y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_3 \sim -0.14047$		$0 < y_4 \sim 4.113,\ y_1 = \infty,\ y_2 = \overline{y_3} \sim -2.98 \pm 8.481i$
$t_3 > t > t_4$		$0 < y_4 < y_1,\ y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_4 \sim -0.14118$	r_∞	$0 < y_4 \sim 8.692 < y_1 \sim 28.07,\ y_2 = y_3 = \infty$
$t_4 > t > t_5$		$0 < y_4 < y_1,\ y_2 = \overline{y_3} \in \mathbf{C} \setminus \mathbf{R}$
$t_5 \sim -0.14127$	r_2	$0 < y_4 = y_1 \sim 14.89,\ y_2 = \overline{y_3} \sim 23.95 \pm 59.72i$

The convergency radius of the series for $y(t)$ is of course given by $|t_5| = -t_5$ and the asymptotic growth of the coefficients of $y(t)$ is roughly exponential with argument

$$1/t_5 \sim -7.0785745851241030382064125273753858681 6317182$$

A slightly more precise asymptotical behaviour of the coefficients of $y(t)$ can be computed as follows:

At the root

$$\rho = t_5 \sim -.14127137998962933757540882196178714222253950575630$$

of r_2, the ramified sheet corresponds to the double root

$$y_\rho \sim 14.88738808602894055277970788094544394$$

of $P(y,\rho) \in \mathbf{C}[y]$. (Caution: when computing y_ρ as a root of $P(y,\rho)$ one loses roughly half the digits since an error of order ϵ on ρ induces an error of order $\sqrt{\epsilon}$ on the corresponding two roots approximating the double root y_ρ of $P(y,\rho)$. A better strategy is of course to compute y_ρ as a (simple) root of the derivative $\frac{d}{dy}P(y,\rho)$ of $P(y,\rho)$).

In a open neighbourhood of ρ we get now a Puiseux series expansion

$$y(t) = h(t) + \sqrt{\rho - t}\; g(t)$$

with $h(t),\ g(t)$ holomorphic in an open disc of radius

$$.1414780159629839137794 03501 > \rho \sim .14127137998962933757540882 1$$

containing no other singularities of $y(t)$. The asymptotics of the generating function $y(t)$ are thus roughly given by

$$\begin{aligned} & \gamma_\rho\sqrt{\rho}\sqrt{1-t/\rho} \\ = \; & \gamma_\rho\sqrt{\rho}\textstyle\sum_{n=0}^\infty \binom{1/2}{n}\left(\frac{-t}{\rho}\right)^n \\ = \; & \gamma_\rho\sqrt{\rho}\left(1-\textstyle\sum_{n=1}^\infty \frac{1}{2}\,\frac{1/2\cdot 3/2\cdots(n-3/2)}{n!}\left(\frac{t}{\rho}\right)^n\right) \\ = \; & \gamma_\rho\sqrt{\rho}\left(1-\textstyle\sum_{n=1}^\infty \frac{1\cdot 3\cdots(2n-3)}{2^n\;n!}\left(\frac{t}{\rho}\right)^n\right) \\ = \; & \gamma_\rho\sqrt{\rho}\left(1-\frac{1}{2}\textstyle\sum_{n=1}^\infty \frac{(2n-2)!}{4^{n-1}\;n!\;(n-1)!\;\rho^n}\,t^n\right) \end{aligned}$$

where $\gamma(\rho) = g(\rho)$. Since

$$\frac{(2n-2)!}{n!\;(n-1)!} \sim \frac{1}{n}\,\frac{\sqrt{4\pi(n-1)}\;4^{n-1}\;(n-1)^{2n-2}\;e^{-2n+2}}{2\pi\;(n-1)\;(n-1)^{2(n-1)}\;e^{-2(n-1)}} \sim \frac{4^{n-1}}{\sqrt{\pi}\;n^{3/2}}$$

we get the asymptotics

$$a_n \sim \frac{\gamma_\rho}{2\;\sqrt{\pi}\;n^{3/2}\;\rho^{n-1/2}}$$

The constant γ_ρ can be computed by remarking that

$$\begin{aligned} 0 &= P(h(t)+\sqrt{\rho-t}\ g(t),t) \\ &= P(y_\rho+\gamma_\rho\sqrt{\rho-t}+O((\rho-t)),t) \\ &= \frac{\partial^2 P}{\partial y^2}|_{(y_\rho,\rho)}\frac{\gamma_\rho^2\ (\rho-t)}{2}+\frac{\partial P}{\partial t}|_{(y_\rho,\rho)}(t-\rho)+O((\rho-t)^{3/2})) \end{aligned}$$

yielding

$$\gamma_\rho\sqrt{\rho}=\sqrt{2\rho\frac{\frac{\partial P}{\partial t}|_{(y_\rho,\rho)}}{\frac{\partial^2 P}{\partial y^2}|_{(y_\rho,\rho)}}}\sim\ 337.171657540870\ .$$

We have thus asymptotically

$$a_n\sim 95.114368526045118940688836\ \rho^{-n}\ n^{-3/2}$$

with $\rho\sim -.14127137998962933757540882196178714222253950575630064\,18$. Unfortunately, the right hand side is a fairly correct approximation of a_n only for very huge values of n (concretely, $n\sim 10^4$ yields a only very few decimals). Indeed, the function $y(t)$ ramifies again for

$$t\sim -.14147801596298391377\,94$$

which is extremely close to ρ.

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