On lacunary Rényi β-expansions of 1 with $\beta > 1$ a real algebraic number, Perron numbers and a classification problem

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Abstract

We prove that for all algebraic number $\beta > 1$ the strings of zeros in the Rényi β-expansion $d_\beta(1)$ of 1 exhibit a lacunarity bounded above by $\log(s(P_\beta))/\log(\beta)$, where $s(P_\beta)$ is the size of the minimal polynomial of β. The conjecture about the specification of the β-shift, equivalently the uniform discreteness of the sets $\mathbb{Z}_\beta$ of β-integers, for β a Perron number is discussed. We propose a classification of algebraic numbers $\beta > 1$ according to the asymptotic "quotient of the gap" value of the Rényi β-expansion of 1 and examplify it, in a complementary classification of Blanchard's with the classes C_1 to C_5.

1. Introduction

Let $\beta > 1$ be a real number. In [Ren] Rényi introduced the numeration in base β, establishing a correspondence between $\mathbb{R}^+$ and the set of sequences $(x_i)_{i\geqslant 0}$ on the alphabet $\{0, 1, 2, \ldots, \lceil\beta - 1\rceil\}$ ($\lceil\beta - 1\rceil$ is by definition the smallest integer greater than or equal to $\beta - 1$) by the formula : $x = \sum_{i=-\infty}^{k} x_{-i+k}\beta^i$, with $\beta^k \leqslant x < \beta^{k+1}$, $x_0 \neq 0$, called Rényi β-expansion of x. Parry [Par] [Fr1] [Fr2] has shown that the knowledge of the Rényi β-expansion $d_\beta(1)$ of 1 is sufficient to make this correspondence bijective, once so-called Parry's inequalities are satisfied. More recently the integers in base β, or β-integers, defined as the real numbers equal to the integer part of their Rényi β-expansion, denoted by $\mathbb{Z}_\beta$, were introduced by Gazeau [Ga] and Burdik et al. [BFGK]. The sets $\mathbb{Z}_\beta$ are discrete. However, it is an open question whether they are uniformly discrete in general [GVG] (on mathematical quasicrystals and Delone sets see also [La] [Me] [BM]), and it is conjectured that this is true for all Perron numbers

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[Be1] [Be2] [GVG]. When $\beta = h \geqslant 2$ is an integer then $\mathbb{Z}_\beta = \mathbb{Z}$ is uniformly discrete and the conjecture holds. Thurston [Th] has shown that this conjecture is true in the case where β is a Pisot number in general.

On the other hand Schmeling (Theorem A in [Sc]) has shown that the class (class C_3) of real numbers $\beta > 1$ such that the Rényi β-expansion of 1 contains bounded strings of zeros, but is not eventually periodic, has Hausdorff dimension 1. For all β in this class the β-shift is specified (Blanchard [Bl] p. 138). The specification of the β-shift is equivalent to the fact that the set $\mathbb{Z}_\beta$ is uniformly discrete (see Section 2; Prop. 3.5 in [Sc], Prop. 4.5 in [Bl], [GVG]). In terms of the ergodic properties of the β-shift, the above conjecture was already stated as one of the basic questions about Rényi β-expansions of real numbers in Blanchard [Bl] §4 pp 137–139. Restated in this context, the conjecture is expressed as follows:

CONJECTURE 1. — *All Perron numbers $\beta > 1$ for which $d_\beta(1)$ is infinite are such that $d_\beta(1)$ contains bounded strings of zeros.*

From [Sc] we expect that this conjecture holds for a large class of algebraic numbers.

THEOREM 1.1. — *Let $\beta > 1$ be a algebraic number. Denote by $d_\beta(1) := 0.t_1 t_2 t_3 \ldots$, with $t_i \in A_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$ its Rényi β-expansion of 1. Assume that $d_\beta(1)$ is infinite and lacunary in the following sense: there exist two sequences $\{m_n\}_{n \geqslant 1}$ and $\{s_n\}_{n \geqslant 0}$ such that*

$$1 = s_0 \leqslant m_1 < s_1 \leqslant m_2 < s_2 \leqslant \ldots \leqslant m_n < s_n \leqslant m_{n+1} < s_{n+1} \leqslant \ldots$$

with $(s_n - m_n) \geqslant 2$, $t_{m_n} \neq 0, t_{s_n} \neq 0$ and $t_i = 0$ if $m_n < i < s_n$ for all $n \geqslant 1$. Then

$$(i) \quad \limsup_{n \to +\infty} \frac{s_n}{m_n} \leqslant \frac{\log(s(P_\beta))}{\log(\beta)}, \tag{1}$$

where $s(P_\beta)$ is the size of the minimal polynomial of β. Moreover, if $\liminf_{n \to +\infty}(m_{n+1} - m_n) = +\infty$, then

$$(ii) \quad \limsup_{n \to +\infty} \frac{s_{n+1} - s_n}{m_{n+1} - m_n} \leqslant \frac{\log(s(P_\beta))}{\log(\beta)}. \tag{2}$$

Note that the term "*lacunary*" has often other meanings in literature. Note also that we do not assume $t_j \neq 0$ for all j such that $s_n \leqslant j \leqslant m_{n+1}$ (for all $n \geqslant 1$). The quotient $s_n/m_n \geqslant 1$ is called "quotient of the gap" following [Os] [Os2].

We will say that the lacunarity of $d_\beta(1)$ is *linearly bounded* when there exists a constant $C, 1 \leqslant C < +\infty$, such that $\limsup s_n/m_n \leqslant C$. Each time lacunarity appears in $d_\beta(1)$ for β an algebraic number > 1, it is linearly bounded by Proposition 2.1 below and more accurately by Theorem 1.1. Consequently we may try a classification of algebraic numbers $\beta > 1$ as follows: first, we assume that $t_j \neq 0$ for all j such that $s_n \leqslant j \leqslant m_{n+1}$ in order to describe uniquely the zeros and the strings of zeros in $d_\beta(1)$. Then, referring to Blanchard's classification [Bl], either $d_\beta(1)$ is finite or, equivalently the β-shift is of finite type (class C_1); or it is eventually periodic if and only if the β-shift is sofic (class C_2); or, when $d_\beta(1)$ is infinite, lacunary and not eventually periodic, β belongs to one of the following classes Q_0, Q_1, or Q_2, where Q_0 is the union of $Q_0^{(1)}$, $Q_0^{(2)}$ and $Q_0^{(3)}$, and where the class C_3 is the union of $Q_0^{(1)}$ and $Q_0^{(2)}$.

What are the classes of algebraic numbers $\beta > 1$ such that

$$(\mathrm{Q}_0^{(1)}) \ : \qquad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n} \qquad \text{with } (m_{n+1} - m_n) \text{ bounded ?} \tag{3}$$

$$(\mathrm{Q}_0^{(2)}) \ : \qquad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n} \quad \text{with } (s_n - m_n) \text{ bounded and } \lim(m_{n+1} - m_n) = +\infty \text{ ?} \tag{4}$$

$$(\mathrm{Q}_0^{(3)} : \text{non } \mathrm{C}_3) \ : \qquad 1 = \lim_{n\to+\infty} \frac{s_n}{m_n} \qquad \text{with } \limsup_{n\to+\infty}(s_n - m_n) = +\infty \text{ ?} \tag{5}$$

$$(\mathrm{Q}_1) \ : \qquad 1 < \limsup_{n\to+\infty} \frac{s_n}{m_n} < \frac{\log(s(P_\beta))}{\log(\beta)} \text{ ?} \tag{6}$$

$$(\mathrm{Q}_2) \ : \qquad \limsup_{n\to+\infty} \frac{s_n}{m_n} = \frac{\log(s(P_\beta))}{\log(\beta)} \text{ ?} \tag{7}$$

In Section 3 the above conjecture is investigated numerically for large Perron numbers $\beta > 1$, as a function of the degree of β and of the maximal modulus of the conjugates of β. Examples of transcendental numbers in $\mathrm{Q}_0^{(1)}$: the Komornik-Loreti constant [AC] [KL] and Sturmian numbers [CK], are also given, together with the example of a Perron number which is not a beta-number ([So], page 483).

2. Proof of Theorem 1.1

Since we exclude algebraic numbers $\beta > 1$ for which the Rényi β-expansion $d_\beta(1)$ of 1 is finite, we can consider that β does not belong to $\mathbb{N}$. Indeed, if $\beta = h \in \mathbb{N}$, then $d_h(1) = 0.h$ is finite (Lothaire [Lo], Chap. 7). If $\beta \notin \mathbb{N}$, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$ and the alphabet A_β equals $\{0, 1, 2, \ldots, \lfloor \beta \rfloor\}$, where $\lfloor \beta \rfloor$ denotes the greatest integer smaller than or equal to β.

Recall that, if $P(X) = \sum_{i=0}^{m} a_i X^i$, with $a_m \neq 0$, is an arbitrary polynomial with complex coefficients, we denote by $s(P) = \sum_{i=0}^{m} |a_i|$ the *size* of $P(X)$. If $P_\beta(X) = \sum_{i=0}^{d} \alpha_i X^i \in \mathbb{Z}[X]$, with $\alpha_d \neq 0, \alpha_0 > 0, d \geqslant 1$, is the minimal polynomial of $\beta > 1$ and $P_\beta^*(X) = X^d P(1/X)$ its reciprocal polynomial, the size of $P_\beta(X)$ is an integer and : $s(P_\beta) = s(P_\beta^*) > 1$.

From the representation $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$ of 1, let us construct the "lacunary" power series $f(z) := \sum_{i=1}^{+\infty} t_i z^i$ associated with the β-shift (*lacunary* in the sense of Theorem 1.1). Since $d_\beta(1)$ is assumed infinite, its radius of convergence is 1. By definition, it satisfies

$$f(\beta^{-1}) = 1. \tag{1}$$

By this equality, we mean that the function value $f(\beta^{-1})$ is algebraic, equal to 1, at the real algebraic number β^{-1} located inside the open disk of convergence $D(0,1)$ of $f(z)$ in the complex plane. This fact is an intrinsic feature of the Rényi expansion process which leads to the following important consequence.

Proposition 2.1. —

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} < +\infty. \tag{2}$$

Proof. — This is a consequence of Theorem 1 in [Ma]. Indeed, if we assume that there exists a sequence of integers (n_i) which tends to infinity such that $\lim_{i\to+\infty} s_{n_i}/m_{n_i} = +\infty$, then $f(z)$ would be *admissible* in the sense of [Ma]. Since $f(z)$ is a power series with nonnegative coefficients, which is not a polynomial, the function value $f(\beta^{-1})$ should not be algebraic. But it equals 1, which is algebraic. Contradiction. □

Let us improve Proposition 2.1. Let us assume that

$$\limsup \frac{s_n}{m_n} > \frac{\log(s(P_\beta))}{\log(\beta)} \tag{3}$$

and show the contradiction for the assertion (i) (similarly for (ii)). Güting [Gü] has proved that the approximation of algebraic numbers by algebraic numbers is fairly difficult to realize by polynomials. In the present proof, we use approximation theorems obtained by Güting [Gü] on the values of the "polynomial tails" of the power series $f(z)$ at the algebraic number β^{-1}, to obtain the contradiction. Let us write

$$f(z) = \sum_{n=0}^{+\infty} Q_n(z) \tag{4}$$

with

$$Q_n(z) := \sum_{i=s_n}^{m_{n+1}} t_i z^i, \qquad n = 0, 1, 2, \ldots. \tag{5}$$

By construction the polynomials $Q_n(z)$, of degree m_{n+1}, are not identically zero and $Q_n(1) > 0$ is an integer for all $n \geqslant 0$.

Denote by $S_n(z) = -1 + \sum_{i=1}^{m_n} t_i z^i$ the m_nth-section polynomial of the power series $f(z) - 1$ for all $n \geqslant 1$. We have: $s(S_n) = 1 + \sum_{i=1}^{m_n} t_i = 1 + \sum_{j=0}^{n-1} Q_j(1)$. From Theorem 5 in [Gü], we deduce that only one of the following cases (i) or (ii) holds, for all $n \geqslant 1$:

$$(i) \qquad S_n(\beta^{-1}) = 0, \tag{6}$$

$$(ii) \qquad \left| S_n(\beta^{-1}) \right| \geqslant \frac{1}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \left(s(P_\beta^*)\right)^{m_n}}. \tag{7}$$

Case (i) is impossible for any n. Indeed, if there exists an integer $n_0 \geqslant 1$ such that (i) holds, then, since all the digits t_i are positive and that $\beta^{-1} > 0$, we would have $t_i = 0$ for all $i \geqslant s_{n_0}$. This would mean that the Rényi expansion of 1 in base β is finite, which is excluded by assumption. Contradiction. Therefore, the only possibility is (ii), which holds for all integers $n \geqslant 1$.

On the other hand, since $\left| S_n(\beta^{-1}) \right| = \sum_{i=s_n}^{+\infty} t_i \beta^{-i}$ for all integers $n \geqslant 1$, we deduce that

$$\left| S_n(\beta^{-1}) \right| \leqslant \frac{\lfloor \beta \rfloor}{1 - \beta^{-1}} \beta^{-s_n} \qquad n = 1, 2, \ldots. \tag{8}$$

Putting together (7) and (8), we deduce that

$$\frac{\beta^{s_n}}{\left(1 + \sum_{j=0}^{n-1} Q_j(1)\right)^{d-1} \left(s(P_\beta)\right)^{m_n}} \leqslant \frac{\lfloor \beta \rfloor}{1 - \beta^{-1}} \tag{9}$$

should be satisfied for $n = 1, 2, 3, \ldots$. Denote $u_n := \frac{\beta^{s_n}}{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1}\left(s(P_\beta)\right)^{m_n}}$, for all $n \geqslant 1$.

Proof of assertion (i): From Inequality (3) assumed to be true there exists a sequence of integers (n_i) which tends to infinity such that $\lim_{i\to+\infty} \frac{s_{n_i}}{m_{n_i}} > \frac{\log(s(P_\beta))}{\log(\beta)}$. Hence, there exists i_0 such that $\frac{s_{n_i}}{m_{n_i}} > \frac{\log(s(P_\beta))}{\log(\beta)}$ for all $i \geqslant i_0$. Now,

$$\left(\frac{1}{1+\lfloor\beta\rfloor m_{n_i}}\right)^{d-1}\left(\frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{s(P_\beta)}\right)^{m_{n_i}} \leqslant \frac{1}{\left(1+\sum_{j=0}^{n_i-1} Q_j(1)\right)^{d-1}}\left(\frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{s(P_\beta)}\right)^{m_{n_i}} \leqslant u_{n_i}. \tag{10}$$

For $i \geqslant i_0$ the inequality

$$1 = \frac{\beta^{\frac{\log(s(P_\beta))}{\log(\beta)}}}{s(P_\beta)} < \frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{s(P_\beta)} \tag{11}$$

holds. This implies that the left-hand member of (10), hence u_{n_i} also, tends exponentially to infinity when i tends to infinity. The contradiction now comes from (9) since the sequence (u_n) should be uniformly bounded.

Proof of assertion (ii): For $n = 1, 2, \ldots$, let us rewrite the n-th quotient

$$\frac{u_{n+1}}{u_n} = \frac{\beta^{s_{n+1}-s_n}}{s(P_\beta)^{m_{n+1}-m_n}}\frac{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1}}{\left(1+\sum_{j=0}^{n} Q_j(1)\right)^{d-1}} \tag{12}$$

as

$$\frac{u_{n+1}}{u_n} = \frac{\left(\frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{s(P_\beta)}\right)^{m_{n+1}-m_n}}{(m_{n+1}-m_n+1)^{(d-1)}}\left[(m_{n+1}-m_n+1)^{(d-1)}\frac{\left(1+\sum_{j=0}^{n-1} Q_j(1)\right)^{d-1}}{\left(1+\sum_{j=0}^{n} Q_j(1)\right)^{d-1}}\right] \tag{13}$$

and denote

$$U_n := \frac{1}{(m_{n+1}-m_n+1)^{(d-1)}}\left(\frac{\beta^{\frac{s_{n+1}-s_n}{m_{n+1}-m_n}}}{s(P_\beta)}\right)^{m_{n+1}-m_n} \tag{14}$$

and

$$W_n := (m_{n+1}-m_n+1)^{(d-1)}\left(\frac{1+\sum_{j=0}^{n-1} Q_j(1)}{1+\sum_{j=0}^{n} Q_j(1)}\right)^{d-1} \tag{15}$$

so that $u_{n+1}/u_n = U_n W_n$.

Let us prove that $\limsup_{n\to+\infty} U_n = +\infty$. If we assume that $\liminf_{n\to+\infty}(m_{n+1}-m_n) = +\infty$ and that $\limsup_{n\to+\infty}(s_{n+1}-s_n)/(m_{n+1}-m_n) > \log(s(P_\beta))/\log(\beta)$ then

$$1 = \frac{\beta^{\frac{\log(s(P_\beta))}{\log(\beta)}}}{s(P_\beta)} < \frac{\beta^{\frac{s_{n_i+1}-s_{n_i}}{m_{n_i+1}-m_{n_i}}}}{s(P_\beta)} \tag{16}$$

for some sequence of integers (n_i) which tends to infinity so that $\lim_{i\to+\infty} U_{n_i} = +\infty$ exponentially, by (14).

LEMMA 2.2. —

$$0 < \liminf_{n\to+\infty} W_n \tag{17}$$

Proof. — Assume the contrary. Then there exists a subsequence (n_i) of integers which tends to infinity such that $\lim_{i\to+\infty} W_{n_i} = 0$. In other terms, for all $\epsilon > 0$, there exists i_1 such that $i \geqslant i_1$ implies $W_{n_i} \leqslant \epsilon$, equivalently

$$(m_{n_i+1} - m_{n_i} + 1)\left(1 + \sum_{j=0}^{n_i-1} Q_j(1)\right) \leqslant \epsilon^{\frac{1}{d-1}} \times \left(1 + \sum_{j=0}^{n_i} Q_j(1)\right). \tag{18}$$

Since, by hypothesis, $t_{s_n} \geqslant 1$ and $t_{m_n+1} \geqslant 1$ for all $n \geqslant 1$, we have: $n_i \leqslant 1 + \sum_{j=0}^{n_i-1} Q_j(1)$. On the other hand, $Q_{n_i}(1) \leqslant \lfloor\beta\rfloor(m_{n_i+1} - m_{n_i} + 1)$. Then, from (18) with ϵ taken equal to 1, we would have

$$n_i \leqslant 1 + \sum_{j=0}^{n_i-1} Q_j(1) \leqslant \frac{Q_{n_i}(1)}{(m_{n_i+1} - m_{n_i} + 1) - 1} \leqslant \lfloor\beta\rfloor \times \frac{m_{n_i+1} - m_{n_i} + 1}{m_{n_i+1} - m_{n_i}} \leqslant \frac{3}{2}\lfloor\beta\rfloor. \tag{19}$$

But the left-hand member of (19) tends to infinity which is impossible. Contradiction. □

From Lemma 2.2 there exists $r > 0$ such that $W_n \geqslant r$ for all n large enough. Therefore, $u_{n+1}/u_n = U_n W_n \geqslant rU_n$ for all n large enough. Since $\limsup_{n\to+\infty} U_n = +\infty$ we conclude that $\limsup u_{n+1}/u_n = +\infty$, hence that $\limsup u_n = +\infty$. This contradicts (9) and proves the assertion (ii) of Theorem 1.1.

3. Comments and examples

1.— The class C_3 in the classification of Blanchard of real numbers > 1 is badly known [Bl] [Sc] [CK]. Conjecture 1 asserts that the union of the classes C_1, C_2 and C_3 contains all Perron numbers $\beta > 1$. Let us give, after Solomyak ([So], p 483), the example of a Perron number which is not a beta-number therefore which is not in the class C_2, without knowing whether it is in the class Q_0, Q_1 or Q_2. This will allow to estimate the sharpness of the upper bound $\log(s(P_\beta))/\log(\beta)$ of the "quotient of the gap" value in Theorem 1.1. Recall that a real number $\beta > 1$ is a beta-number if the orbit of $x = 1$ under the transformation $T_\beta : x \to \beta x \pmod 1$ is finite [Lo] [PF]. A beta-number β is such that $d_\beta(1)$ is finite (class C_1) or eventually periodic (class C_2). The set of all conjugates of all beta-numbers is the union of the closed unit disc in the complex plane and the set of reciprocals of zeros of the function class $\{f(z) = 1 + \sum a_j z^j \mid 0 \leqslant a_j \leqslant 1\}$. This domain, say Φ, was studied by Flatto, Lagarias and Poonen [FLP] and Solomyak [So]. After [So], the Perron number $\beta = \frac{1}{2}(1+\sqrt{13})$ is not a beta-number, though its only conjugate $\beta' = \frac{1}{2}(1 - \sqrt{13})$ lies in the interior $\mathrm{int}(\Phi)$. The real algebraic integer β is the dominant root of the irreducible polynomial, of degree 2, $P_\beta(X) = X^2 - X - 3$. With the notation $d_\beta(1) = \sum_{j=1}^{+\infty} t_j \beta^{-j}$, we have the following factorization of the corresponding analytic functions, for $|z| > 1$, where $T_\beta = T_\beta^1$ and $T_\beta^{j+1} = T_\beta(T_\beta^j)$ for $j \geqslant 1$:

$$1 - \sum_{j=1}^{+\infty} t_j z^{-j} = (1 - \beta/z)\left(1 + \sum_{j=1}^{+\infty} T_\beta^j(1) z^{-j}\right). \tag{1}$$

It is easy to check that a long string of zeros given by $t_j = 0$ for $m_n < j < s_n$ corresponds to a value $T_\beta^{m_n}(1)$ very close to zero, followed by a limited "geometric progression"

$T_\beta^j(1) = \beta^{j-m_n} T_\beta^{m_n}(1)$ for $m_n < j < s_n$. Since $T_\beta^j(1) = \alpha_j \beta + \gamma_j \beta'$, for $\alpha_j, \gamma_j \in \mathbb{Z}$, where $|\gamma_{j+1}| > |\gamma_j|$ with $|\beta\beta'| = 3$ an integer, we conclude, by easy diophantine approximation arguments, that the occurrence of arbitrary long strings of zeros in the nonperiodic Rényi expansion $d_\beta(1)$ is not unavoidable a priori .... As for the "quotient of the gap" value, since $s(P_\beta) = 5$, it is bounded above by $\log(5)/\log(\beta)$ which is roughly equal to $1.929515...$. This upper bound is slightly smaller than the degree $d = 2$ of β, but $\neq 1$ (As shown below, a natural upper bound is $d \log(1+\beta)/\log(\beta)$). This does not suffice to deduce that $\beta = \frac{1}{2}(1+\sqrt{13})$ belongs to Q_0.

Let us investigate whether the conjecture could be true for Perron numbers β large enough, that is which lie in a neighbourhood of $+\infty$. Let $\beta > 1$ be a Perron number of degree $d \geqslant 2$ and denote by $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d-1)}$ the conjugates of $\beta = \beta^{(0)}$. Let $P_\beta(X)$ be the minimal polynomial of β. Let $K_\beta := \max\{\beta^{-1}|\beta^{(i)}| \mid i = 1, 2, \ldots, d-1\}$. After Lemma 2 in [Li2] the following inequality holds:

$$K_\beta < K_\beta^{\max} := 1 - \frac{1}{(d\beta)^{6d^3}}. \tag{2}$$

A simple relation between K_β, the degree d and $s(P_\beta)$ is given by the following

LEMMA 3.1. —

$$s(P_\beta) \leqslant (\beta+1)(K_\beta \beta + 1)^{d-1}. \tag{3}$$

Proof. — This comes from the relations between the coefficients α_i and the roots of $P_\beta(X) = \sum_{i=0}^{d} \alpha_i X^i = \prod_{i=0}^{d-1}(X - \beta^{(i)})$. We have: $\alpha_d = 1$, $|\alpha_{d-1}| = |\sum_{i=0}^{d-1} \beta^{(i)}| \leqslant \beta + (d-1)K_\beta \beta$ and $|\alpha_0| \leqslant \beta (K_\beta \beta)^{d-1}$. For $j = 1, 2, \ldots, d-1$, $\alpha_{d-j} = (-1)^j \sum_{0 \leqslant i_1 < i_2 < \ldots < i_j \leqslant d-1} \beta^{(i_1)} \beta^{(i_2)} \ldots \beta^{(i_j)}$.

Hence, for $j = 1, 2, \ldots, d-1$,

$$|\alpha_{d-j}| \leqslant \beta \left| \sum_{1 \leqslant i_2 < i_3 < \ldots < i_j \leqslant d-1} \beta^{(i_2)} \beta^{(i_3)} \ldots \beta^{(i_j)} \right| + \left| \sum_{1 \leqslant i_1 < i_2 < \ldots < i_j \leqslant d-1} \beta^{(i_1)} \beta^{(i_2)} \ldots \beta^{(i_j)} \right|$$

$$\leqslant \beta \binom{d-1}{j-1} (K_\beta \beta)^{j-1} + \binom{d-1}{j} (K_\beta \beta)^j.$$

We deduce:

$$\sum_{j=0}^{d} |\alpha_{d-j}| \leqslant \sum_{j=1}^{d} \beta \binom{d-1}{j-1} (K_\beta \beta)^{j-1} + \sum_{j=0}^{d-1} \binom{d-1}{j} (K_\beta \beta)^j$$

$$\leqslant (\beta+1) \sum_{j=0}^{d-1} \binom{d-1}{j} (K_\beta \beta)^j = (\beta+1)(K_\beta \beta + 1)^{d-1}.$$

□

COROLLARY 3.2. —

$$\frac{\log(s(P_\beta))}{\log(\beta)} \leqslant \frac{(d-1)\log(1+K_\beta \beta) + \log(1+\beta)}{\log(\beta)}. \tag{4}$$

If β is a Pisot number, then $K_\beta \beta < 1$ and we know from a Theorem of Parry [Par] that Pisot numbers are beta-numbers (see also [Be1] and [Sch]). The "quotient of the gap" value in $d_\beta(1)$ is of course bounded since $d_\beta(1)$ is then eventually periodic (see also [Bo1]). On the other hand, we observe that the above upper bound of (4), in the case where β is assumed large enough, is bounded by $1 + o(1)$. This is coherent with the conjecture. This reasoning can be extrapolated to beta-numbers for which the conjugates are known to be bounded in modulus by $(1+\sqrt{5})/2$ [So] [FLP] or to Perron numbers for which all the conjugates lie within a closed disk $D(0, M)$ centered at the origin of fixed radius $M \geqslant 1$ in the complex plane (see [Bo2] for Salem numbers), so that: $|K_\beta \beta| \leqslant M$. In these three cases,

$$\frac{(d-1)\log(1+K_\beta \beta)}{\log(\beta)} = o(1), \text{ when } \beta \text{ tends to } +\infty.$$

In this numerical context, when β belongs to a neighborhood of $+\infty$, the proximity to 1 of the upper bound of (4) is used as a test. By extrapolation we obtain the following assertion: for all Perron numbers β such that the following assumption

$$(H): \qquad \frac{(d-1)\log(1+K_\beta \beta)}{\log(\beta)} = o(1), \text{ when } \beta \text{ tends to } +\infty,$$

holds, then β would belong to $\mathbb{Q}_0$ and Conjecture 1 has chances to be true, without being able to discriminate the case "$\beta \in \mathbb{Q}_0^{(3)}$" supposed never to occur. Indeed, from Corollary 3.2, then we have:

$$\frac{\log(s(P_\beta))}{\log(\beta)} \leqslant \frac{(d-1)\log(1+K_\beta \beta) + \log(1+\beta)}{\log(\beta)} = o(1) + \frac{\log(1+\beta)}{\log(\beta)} = 1 + o(1).$$

If (H) is not satisfied, the upper bound of the "quotient of the gap" value in Theorem 1.1 is not sharp enough or, if not, the conjecture is not true. Else, this could reveal two different behaviours of Perron numbers: the first one being characterized by conjugates $\beta^{(i)}$ which do not go too quickly to infinity in modulus (as stated by (H)), when β tends to infinity, and the second one, where one conjugate (at least) has a modulus ($\simeq K_\beta \beta$) which becomes prominent in the upper bound of (4), for which the conjecture would perhaps be false. In this last case, when $|K_\beta \beta| \simeq \beta$, the upper bound $\frac{(d-1)\log(1+K_\beta \beta)+\log(1+\beta)}{\log(\beta)}$ in (4) equals $d\frac{\log(1+\beta)}{\log(\beta)} = d + o(1) > 1 + o(1)$, when β tends to infinity. In any case, the following inequality holds:

$$\frac{\log(s(P_\beta))}{\log(\beta)} \leqslant d\frac{\log(1+\beta)}{\log(\beta)}. \tag{5}$$

2.— Rational numbers $p/q > 1$: assume that p and q are two coprime integers > 1 such that $p/q > 1$. Then $d_{p/q}(1)$ is obviously infinite. It is also non-periodic. Recall the proof of the non-periodicity [Fr2]: if we assume that it is eventually periodic, then the $(\frac{p}{q})$-shift would be sofic and p/q would be a Perron number by a Theorem of Lind [Li1] and Denker, Grillenberger and Sigmund [DGS]. But a Perron number is an algebraic integer and we have assumed that $q \neq 1$ preventing p/q from being a rational integer. Contradiction.

Since $P_{p/q}(X) = qX - p$, we have:

$$\frac{\log(s(P_{p/q}))}{\log(p/q)} = \frac{\log(p+q)}{\log(p/q)}. \tag{6}$$

When q is fixed and p is large enough, this bound is $1+o(1)$, but it may take large values for large q's. What are the rational numbers $p/q > 1$ which belong to $Q_0^{(1)}$, $Q_0^{(2)}$, $Q_0^{(3)}$, Q_1 or Q_2 ?

3.— The Komornik-Loreti constant: it is defined as follows [KL] [AC].

THEOREM 3.3. — *There exists a smallest $q \in (1,2)$ for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$, with $\delta_n \in \{0,1\}$. Furthermore, for this smallest q, the coefficient δ_n is equal to 0 (respectively, 1) if the sum of the binary digits of n is even (respectively, odd). This number q can then be obtained as the unique positive solution of $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$. It is equal to* $1.787231650\ldots$.

This constant q is named Komornik-Loreti constant. Allouche and Cosnard [AC] have shown more.

THEOREM 3.4. — *The constant q is a transcendental number, where the sequence of coefficients $(\delta_n)_{n\geqslant 1}$ is the Prouhet-Thue-Morse sequence on the alphabet $\{0,1\}$.*

The uniqueness of the development of 1 in base q given by Theorem 3.3 allows to write

$$d_q(1) = 0.\delta_1\delta_2\delta_3\ldots,$$

the coefficients δ_n being the digits of the Rényi q-expansion of 1. Since the strings of zeros and 1's in the Prouhet-Thue-Morse sequence are known (Thue, 1906/1912; [AS]) and uniformly bounded, the constant q belongs to the class $Q_0^{(1)}$.

4.— Sturmian numbers in $(1,2)$ (in the sense of [CK]): The real number $\beta > 1$ is called a Sturmian number if $d_\beta(1)$ is a Sturmian word over a binary alphabet $\{a,b\}$, with $0 \leqslant a < b = \lfloor\beta\rfloor$. Chi and Kwon [CK] have shown the following theorem.

THEOREM 3.5. — *Every Sturmian number is transcendental.*

Let us consider all the Sturmian numbers $\beta \in (1,2)$ for which the two-letter alphabet is $\{0,1\}$. For such numbers lacunarity appears in $d_\beta(1)$ (in the sense of Theorem 1.1). By Theorem 3.3 in [CK] strings of zeros, resp. of 1's, cannot be arbitrarily long. Hence, all Sturmian numbers in $(1,2)$ are in the class $Q_0^{(1)}$.

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