DEFECTIVE THREEFOLDS

ANDREY INSHAKOV

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ABSTRACT. Threefolds for which the varieties of $(h+1)$-secant h-planes have the dimension less then expected are classified for $h > 1$. Geometry of such threefolds swept out by lines is described.

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1. Introduction

In this paper we work over complex numbers.

Let us start from the famous Waring problem for forms: for given m and n find a minimal number h such that a homogeneous form of degree m in $n+1$ variables could be represented as a sum of $h+1$ powers of linear forms. F. Palatini and A. Terracini at the beginning of XX century proposed a geometrical approach to the Waring problem for general forms. They introduced the notion of variety of $(h+1)$-secant h-planes. Namely, for a variety $X \subset \mathbb{P}^N$ of dimension n, variety of $(h+1)$-secants is

$$S^h(X) = \overline{\bigcup_{x_0,\dots,x_h \in X,\ \dim\langle x_0,\dots,x_h\rangle = h} \langle x_0,\dots,x_h\rangle}.$$

By the geometrical approach, the question "whether a general form of degree m in $n+1$ variables could be represented as a sum of $h+1$ powers of linear forms" is substituted by "does the variety $S^h(v_m(\mathbb{P}^n))$ coincides with the whole ambient space?". Here $v_m(\mathbb{P}^n)$ is the projectivization of the variety of all m-powers of linear forms on $\mathbb{C}^{n+1}$, and v_m is a Veronese embedding, i. e. the map corresponding to the full linear system of divisors of degree m. So, we came to calculating the dimensions of $S^h(v_m(\mathbb{P}^n))$.

One can see that if $X \subset \mathbb{P}^N$ is a non-degenerate irreducible variety of dimension n, then the expected dimension of $S^h(X)$ is $\min\{N, n(h+1)+h\}$. The number $\delta_h(X) = \min\{N, n(h+1)+h\} - \dim S^h(X)$ is called *h-defect of X*. So, in order to obtain the solution of the Waring problem for general forms it is enough to calculate all the numbers $\delta_h(v_m(\mathbb{P}^n))$, or, more generally, to find such m, n and h for which $\delta_h(v_m(\mathbb{P}^n)) > 0$. This problem was solved by J. Alexander and A. Hirschowitz [1], [2], [3], [4].

For "general" variety X and any h one has that $\delta_h(X) = 0$. If for some $h > 0$ one has $\delta_{h-1}(X) = 0$ and $\delta_h(X) > 0$, the variety X is called *h-defective.*

The main tool for studying defective varieties is the following lemma due to A. Terracini:

Lemma. *If $x_0,\dots,x_h \in X$ are general points and $z \in \langle x_0,\dots,x_h\rangle$ is a general point, then $T_zS^h(X) = \langle T_{x_0}X,\dots,T_{x_h}X\rangle$.*

It is more or less clear that there are no h-defective curves. Defective surfaces were classified by many authors. Classically such surfaces were considered by F. Palatini [12] and [13] whose classification theorem has a serious gap. Then A. Terracini [16] completed F. Palatini's classification. Also G. Scorza [15] and J. Bronowski [5] worked on this topic. Both F. Palatini's and A. Terracini's papers are obscure and difficult to read. L. Chiantini and C. Ciliberto [6] classified weakly defective surfaces, of which defective surfaces form a special case. Their approach is easier and faster then the previous ones. The result is as follows:

Theorem. *A surface X is h-defective iff X is one of the following*:

(1) *A non-degenerate surface $X \subset Cone_L C$, where C is a curve, L is a linear space of dimension $h-1$, $N \geq 3h+2$ such that for any linear subspace $l \subset L$ one has $\dim \pi_l(X) = 2$*;

(2) $X = v_2(Y) \subset \mathbb{P}^{3h+2}$, where $Y \subset \mathbb{P}^{h+1}$ is a non-degenerate surface of minimal degree.

1-defective threefolds were classified by G. Scorza [14].

Theorem. (*G. Scorza,* [14]) *An irreducible, non-degenerate, projective 3-fold $X \subset \mathbb{P}^N$ is 1-defective if and only if a pair of N and X is of one of the following*:
X is a cone;
$N = 7$ and X sits in a 4-dimensional cone over a curve;
$N = 7$ and X is contained in a 4-dimensional cone over the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$;
$N = 9$ and X is the Veronese variety $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$ or a projection of it in $\mathbf{P}^N$, $N = 7$ or 8;
$N = 7$ and X is a hyperplane section of the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$.

In more recent times F. Zak [17], T. Fujita and J. Roberts [10] and T. Fujita [9] considered smooth defective threefolds. L. Chiantini and C. Ciliberto [7] reworked the G. Scorza classification in an easy and fast way.

During the writing of the present paper the author received the paper of L. Chiantini and C. Ciliberto [8], where they proposed the classification of h-defective threefolds for all $h > 1$.

In the present paper we also build the classification of h-defective threefolds. Our way differs from the one used by C. Ciliberto and L. Chiantini and results where obtained independently. Big part of the present paper is devoted to studying threefolds that could be projected to a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. We prove that such threefolds often are covered by lines; we describe such families of lines as sub-surfaces of grassmannians; finally, we give the construction for obtaining such threefolds.

For higher dimensional defective varieties only general properties are known, see Zak [17].

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2. Notions and notation

2.1. h-secant varieties and their defects.

Definition. The variety $S^h(X) = \overline{\bigcup_{x_0,\dots,x_h \in X \;\; x_i \neq x_j \, for \;\; i \neq j} \langle x_0, \dots x_h \rangle}$ is called *h-secant variety of the variety X.*

By counting dimensions one can see that if N is big enough then the expected dimension of $S^h(X)$ is $\dim X \cdot (h+1) + h$. Hence for an arbitrary N the expected dimension of $S^h X$ is $\min\{N, \dim X \cdot (h+1) + h\}$.

Definition. The number $d_h(X) = \dim X(h+1) + h - \dim S^h(X)$ is called the *cumulative h-defect* of the variety $X \subset \mathbb{P}^N$. The number $\delta_h(X) = \min\{N, \dim X(h+1)+h\} - \dim S^h(X)$ is called the *h-defect* of the variety $X \subset \mathbb{P}^N$.

Definition. A variety $X \subset \mathbb{P}^N$ is called *h-defective*, if $\delta_h(X) > 0$ and $\delta_{h-1}(X) = 0$.

Remark 1. If X is h-defective then $d_h(X) > 0$ and $d_{h-1}(X) = 0$. If $d_h(X) > 0$ and $d_{h-1}(X) = 0$ then $S^h(X) = \mathbb{P}^N$ or X is h-defective.

One can see that $S^h(X) = S(S^k(X), S^{h-k-1}X)$, where $S(Y,Z) = \overline{\bigcup_{y \in Y, z \in Z, y \neq z} \langle y, z \rangle}$.

Definition. For a general point $z \in S^h X$ the variety

$$\Sigma_{X,h,z,k} = \overline{\{y | y \in S^k(X), \;\exists y' \in S^{h-k-1}X, \; y \neq y' : \; z \in \langle y, y' \rangle\}}$$

is called the *k-entry locus of z*. If $k = 0$, the k-entry locus of z is also called *entry locus of z*.

Further, if the value of h is fixed, we will denote $\Sigma_{X,h,z,0}$ by $\Sigma_z(X)$ or simply by Σ_z.

Remark 2. (1) $\dim \Sigma_{X,h,z,k} = \dim \Sigma_{X,h,z,h-k-1}$.

(2) If $S^h(X) \neq \mathbb{P}^N$, then $\dim \Sigma_{X,h,z,k} + d_k(X) + d_{h-k-1}(X) = d_h(X)$. In particular, $d_i(X) \geq d_j(X)$, if $i \geq j$.

(3) If X is h-defective, then $\forall k < h \;\; d_k(X) = 0$ and $\dim \Sigma_{X,h,z,k} = \dim \Sigma_{X,h,z,0} = d_h(X)$.

(4) Since $d_0(X) = 0$, one has $d_h(X) = \dim \Sigma_{X,h,z,0} + d_{h-1}(X) = \dim \Sigma_{X,h,z,0} + \ldots + \dim \Sigma_{X,1,z,0}$. This is the reason why $d_h(X)$ is a *cumulative* defect.

For a general point $z \in S^h(X)$ put

$$V_{X,h,z} = \overline{\{x | x \text{ is smooth}, T_x X \subset T_z S^h(X)\}}.$$

For the simplicity, we can write $V_z(X)$ or V_z instead of $V_{X,z,h}$, if the value of h is fixed. Denote the dimension of $V_{X,h,z}$ by $\nu_h(X)$.

Remark 3. If $z \in \langle x_0, \ldots, x_h \rangle$, where $x_0, \ldots, x_h \in X$ are general points, then, by Terracini lemma, $\langle T_{x_0}X, \ldots, T_{x_h}X \rangle = T_z S^h(X)$. Hence, $x_0, \ldots, x_h \in V_{X,h,z}$ and, thus, $\Sigma_z(X) \subset V_{X,h,z}$. Therefore, $\dim \Sigma_z(X) \leq \nu_h(X)$. If X is h-defective, one has $d_h(X) = \dim \Sigma_z(X) \leq \nu_h(X)$. More, $\mathbb{P}^N \neq T_z S^h(X)$ and, since X is non-degenerate, we obtain $X \neq V_{X,h,z}$ and $\nu_h(X) < \dim X$. Finally, $d_h(X) \leq \nu_h(X) < \dim X$.

Lemma 1. *Suppose that a variety $X \subset \mathbb{P}^N$ is non-degenerate, $q \in S^{m-1}(X)$ and $z \in S^{l+m}(X)$ $(l \geq 0)$ are general points, such that $q \in \Sigma_{X,l+m,z,m-1}$, and π is the projection with the center at $T_q S^{m-1}(X)$. Then*

(1) *$\pi(z)$ is a general point of $S^l(\pi(X))$ and $\pi^{-1}(T_{\pi(z)} S^l(\pi(X))) = T_z S^{l+m}(X)$;*
(2) $\dim \pi(X) = \dim X - d_m(X) + d_{m-1}(X)$;
(3) *for $k \geq 1$ one has $d_k(\pi(X)) = d_{k+m}(X) - d_m(X) - k(d_m(X) - d_{m-1}(X))$;*
(4) *for $k \geq 0$, $\pi(V_{X,k+m,z}) = V_{\pi(X),k,\pi(z)}$ and $\nu_k(\pi(X)) = \nu_{m+k}(X) - d_m(X) + d_{m-1}(X)$;*
(5) *if $d_m(X) = 0$ then $\dim \pi(X) = \dim X$ and for $k \geq 1$, $\delta_k(\pi(X)) = \delta_{k+m}(X)$.*

The proof of this Lemma easily follows from Terracini lemma; for the main part see [11, Proposition 1].

Corollary 1. *Suppose that a variety $X \subset \mathbb{P}^N$ is h-defective, $m \leq h-1$, $q \in S^{m-1}(X)$ is a general point and π is the projection with the center at $T_q S^{m-1}(X)$. Then $\pi(X)$ is $(h-m)$-defective, $\dim \pi(X) = \dim X$, for $k \geq 1$ one has $d_k(\pi(X)) = d_{k+m}(X)$, $\delta_k(\pi(X)) = \delta_{k+m}(X)$, $\nu_k(\pi(X)) = \nu_{m+k}(X)$.*

2.2. **Other notations.** We will denote by $G(k, N)$ the grassmannian of k-dimensional subspaces of $\mathbb{P}^N$. More, we will always assume that $G(k, N)$ is embedded in $\mathbb{P}^{\binom{N+1}{k+1}-1}$ by Plucker embedding. For a point $\alpha \in G(k, N)$, the corresponding subspace is denoted by $\mathbb{P}^k_\alpha$.

If $x \in X$ is a general point, the *osculating space of order k to X at the point x* is the linear span of all partial derivatives of order not more than k of some local parameterization of a neighbourhood of x. It is denoted by $T_x^k X$. For exact definitions and properties see, e. g. [11, Section 3].

If S is a set of linearly equivalent divisors on X, then the complete linear system of divisors, containing S, we will denote by $|S|$. If $\mathcal{L}$ is a linear system of divisors on X, $x_0, \ldots, x_k \in X$ are general points and $m_0, \ldots, m_k$ are positive integers, then $\mathcal{L}(-m_0 x_0 - \ldots - m_k x_k)$ is a subsystem of $\mathcal{L}$ that consists of divisors having their multiplicities at least m_i at x_i, $0 \leq i \leq k$.

Suppose that $\mathbb{P}^N = \langle L_1, \ldots, L_k \rangle$, where $L_1, \ldots, L_k \subset \mathbb{P}^N$ are linear subspaces of dimensions $a_1, \ldots, a_k$ respectively and $N = a_1 + \ldots + a_k + k - 1$, i. e. these subspaces are in general position. Let $\varphi_i : \mathbb{P}^1 \to L_i$ be a Veronese map of degree a_i, $1 \leq i \leq k$. Then the variety $\bigcup_{t \in \mathbb{P}^1} \langle \varphi_1(t), \ldots, \varphi_k(t) \rangle$ is called the *rational normal scroll of type* $(a_1, \ldots, a_k)$ or simply a *scroll of type* $(a_1, \ldots, a_k)$ and is denoted by $Scroll_{a_1, \ldots, a_k}$. We will call the curves $\varphi_i(\mathbb{P}^1) \subset L_i$, $1 \leq i \leq k$, *basic curves.*

Take non-negative integers a, b, c and consider two scrolls of types $(a, a+b)$ and $(a+c, a+b+c)$ such that their linear spans are skew. Take a map $\tau : Scroll_{a+c,a+b+c} \to Scroll_{a,a+b}$ which is the projection from c ruling lines. We will call the variety $M = \bigcup_{x \in Scroll_{a+c,a+b+c}} \langle x, \tau(x) \rangle$ *4-scroll of type* (a, b, c).

Remark 4.
(1) 4-scrolls of types (a, b, c) and (a, c, b) coincide.
(2) $\dim\langle M \rangle = \dim\langle Scroll_{a,a+b} \rangle + \dim\langle Scroll_{a+c,a+b+c} \rangle + 1 = 2a + b + 1 + 2a + b + 2c + 1 + 1 = 4a + 2b + 2c + 3$.

3. Main results

3.1. Main theorem.

Theorem 1. *Suppose that $X \subset \mathbb{P}^N$ is an irreducible, non-degenerate, projective threefold. Its defect $\delta_h(X)$ is not zero if and only if a triple of X, N and $d_h(X)$ is of one of the following:*

(1) *$X \subset Cone_L(Y)$, where L is a linear subspace, $\dim L \leq (h+1)(3 - \dim Y) - 2$, $1 \leq \dim Y \leq 2$ and $Y \subset \mathbb{P}^{N_Y}$ is non-h-defective, $N_Y \geq (h+1)(\dim Y + 1)$; $N = \dim L + N_Y + 1$, $d_h(X) \geq (h+1)(3 - \dim Y) - 1 - \dim L$, $\delta_h(X) \geq \min\{N_Y - (h+1)(\dim Y + 1) + 1, (h+1)(3 - \dim Y) - 1 - \dim L\}$;*
(2) *$X \subset Cone_L(v_2(Y))$, where $L \subset \mathbb{P}^N$ is a linear subspace having $\dim L \leq h$, $Y \subset \mathbb{P}^{h+1}$ is a non-degenerate surface of minimal degree, $N = 3h + 3 + \dim L$, $d_h(X) \geq h + 1 - \dim L$, $\delta_h(X) \geq 1$;*
(3) *$X = \pi_L(v_2(Y))$, where $Y \subset \mathbb{P}^{h+2}$ is a non-degenerate threefold of minimal degree, $L \subset \mathbb{P}^{4h+5}$ is a linear subspace of dimension*

not more than 1, $N = 4h+5-\dim L$, $d_h(X) = \delta_h(X) = 1$, $\delta_{h-1}(X) = 0$;

(4) $X \subset Cone_L(v_2(Y))$, *where* $L \subset \mathbb{P}^N$ *is a linear subspace*, $2 - d_h(X) \geq \dim L \geq \max\{-1, 2 - \frac{(h-1)(h-2)}{2} - d_h(X)\}$, $Y \subset \mathbb{P}^{h+1}$ *is a non-degenerate threefold such that* $\dim I_2(Y) = \binom{h}{2} - h - 1 + d_h(X) + \dim L$, $N = 4h+4-d_h(X)$, $d_h(X) = 1$ *or* 2, $\delta_h(X) = 1$, $\delta_{h-1}(X) = 0$;

(5) X *is covered by lines of form* $\langle \xi(x), \eta(x) \rangle$, $x \in Y$, *where* Y *is a rational normal scroll of type* (a,b), $a, b \geq 1$, $a+b = h$, $\xi : Y \to Scroll_{a+1,b+1}$ *is an isomorphism and* $\eta = \pi_{v_2(l)} \circ v_2$, *where* $l \subset Y$ *is a ruling line*, $N = 4h+3$, $d_h(X) = \delta_h(X) = 1$, $\delta_{h-1}(X) = 0$;

(6) X *is covered by lines of form* $\langle \xi(x), \eta(x) \rangle$, $x \in Y$, *where* $Y = Cone_p(C)$ *and* C *is a rational normal curve of degree* h, $\xi : Y \dashrightarrow Scroll_{1,h+1}$ *is a blowing-up of a line and* $\eta = \pi_{v_2(y)} \circ v_2$, *where* $y \in Y$ *is a smooth point, such that for a general point* $x \in \langle p, y \rangle$ *holds* $\xi(x) = \eta(x)$; $N = 4h+3$, $d_h(X) = \delta_h(X) = 1$, $\delta_{h-1}(X) = 0$;

(7) $X = v_3(\mathbb{P}^2) \times \mathbb{P}^1$, $N = 19$, $d_4(X) = \delta_4(X) = 1$, $\delta_3(X) = 0$;

(8) X *is a 4-scroll of type* (a,b,c), $2a+b+c = 2h$, $b, c \leq h$, b *and* c *are even iff* h *is*, $N = 4h+3$, $d_h(X) = \delta_h(X) = 1$, $\delta_{h-1}(X) = 0$.

3.2. **Other results.**

Proposition 1. *Suppose that* $X \subset \mathbb{P}^N$ *is an* h*-defective threefold*, $q \in S^{h-2}(X)$ *is a general point and* π *is the projection from* $T_q S^{h-2}(X)$.

If $\pi(X)$ *is a smooth hyperplane section of the Segre variety* $\mathbb{P}^2 \times \mathbb{P}^2$, *then*

(1) X *is covered by one irreducible family of lines* $U \subset G(1, N)$;

(2) $U = v_3(Y)$, *where* $Y \subset \mathbb{P}^{h+1}$ *is a surface of minimal degree*;

(3) *if* $Y \neq v_2(\mathbb{P}^2)$, *then for* $h+1$ *general points of* X *there exists unique rational normal scroll of type* $(h+1, 2h-1)$, *passing through them*;

(4) *if* $Y = v_2(\mathbb{P}^2)$, *then for* $h+1$ *general points of* X *there exists unique rational normal scroll of type* $(6,6)$, *passing through them*;

(5) *if* Y *is a rational normal scroll of type* (a,b), $a, b \geq 1$, $a+b = h$, *then* X *is covered by lines of form* $\langle \xi(x), \eta(x) \rangle$, $x \in Y$, *where* $\xi : Y \to Scroll_{a+1,b+1}$ *is an isomorphism and* $\eta = \pi_{v_2(l)} \circ v_2$, *where* $l \subset Y$ *is a ruling line*;

(6) *if* $Y = Cone_p(C)$ *and* C *is a rational normal curve of degree* h, *then a general line of* U *is of form* $\langle \xi(x), \eta(x) \rangle$, $x \in Y$, *where*

$\xi : Y \dashrightarrow Scroll_{1,h+1}$ *is a blowing-up of a line and* $\eta = \pi_{v_2(y)} \circ v_2$, *where* $y \in Y$ *is a smooth point, such that for a general point* $x \in \langle p, y\rangle$ *holds* $\xi(x) = \eta(x)$;

(7) *if* $Y = v_2(\mathbb{P}^2)$, *then* $X = v_3(\mathbb{P}^2) \times \mathbb{P}^1$;

Proposition 2. *Suppose that* $X \subset \mathbb{P}^N$ *is an* h*-defective threefold,* $q \in S^{h-2}(X)$ *is a general point and* π *is the projection from* $T_q S^{h-2}(X)$.

If $\pi(X)$ *is a singular irreducible hyperplane section of the Segre variety* $\mathbb{P}^2 \times \mathbb{P}^2$, *then*

(1) *If* X *is not covered by lines, then* X *is the image of the projection of* $v_2(Y)$ *from two different smooth points, where* $Y \subset \mathbb{P}^{h+2}$ *is a threefold of minimal degree*;

(2) *If* X *is covered by one irreducible family of lines* $U \subset G(1,N)$, *then*:

 (a) $U = \pi_{v_3(y)}(v_3(Y))$, *where* $Y \subset \mathbb{P}^{h+1}$ *is a surface of minimal degree and* $y \in Y$ *is a smooth point*;

 (b) *for* $h+1$ *general points of* X *there exists unique rational normal scroll of type* $(h, 2h)$, *passing through them*;

 (c) $X = \pi_{\langle v_2(p), v_2(y)\rangle}(v_2(Cone_p(Y)))$ *and* $Cone_p(Y) \subset \mathbb{P}^{h+2}$ *is a threefold of minimal degree*;

(3) *If* X *is covered by two irreducible family of lines and* $U \subset G(1,N)$ *is one of these families, then*:

 (a) *either* $U = \pi_{v_3(p)}(v_3(Y))$, *where* $Y = Cone_p(C)$ *is a rational normal scroll of type* (a,b), $a = 0$, $b = h$, *or* $U = \varphi(Y)$, *where* Y *is a rational normal scroll of type* (a,b), $a+b = h$, $1 \le a \le b$, *and* φ *is given by complete linear system* $|2C + (3a+b)L|$, *where* C *is the basic curve of degree* b *and* L *is a ruling line*;

 (b) X *is a 4-scroll of type* (a', b', c'), *where* $a' \ge a$, $b' = b - a$ *and* $2a' + b' + c' = 2h$;

 (c) *for* $h+1$ *general points of* X *there exists unique rational normal scroll of type* $(h+a'-a, 2h-a'+a)$ *passing through them and consisting of lines from* U.

The proofs of these Propositions are splitted into a lot of lemmas from Section 6.

3.3. **Proof of "if" part.**

Lemma 2. *Suppose that* $X \subset Cone_L(Y)$, *where* L *is a linear subspace,* $Y \subset \mathbb{P}^{N_Y}$ *is non-degenerate variety. Then* $d_h(X) \ge (h+1)(\dim X - \dim Y) - 1 - \dim L + d_h(Y)$ *and* $\delta_h(X) \ge \min\{N_Y - (h+1)(\dim Y + 1) + 1 + d_h(Y), (h+1)(\dim X - \dim Y) - 1 - \dim L + d_h(Y)\}$.

Proof. For a general point $z \in S^h(X)$ one has $\pi_L(T_zS^h(X)) = T_{\pi_L(z)}S^h(Y)$. We have $\dim T_{\pi_L(z)}S^h(Y) = \dim S^h(Y) = (h+1)(\dim Y + 1) - 1 - d_h(Y)$. Thus, $\dim T_zS^h(X) = \dim L \cap T_zS^h(X) + \dim \pi_L(T_zS^h(X)) + 1 = \dim L \cap T_zS^h(X) + (h+1)(\dim Y + 1) - d_h(Y) \leq \dim L + (h+1)(\dim Y+1) - d_h(Y)$. Hence, $d_h(X) = (h+1)\dim X + h - \dim S^h(X) \geq (h+1)\dim X + h - (h+1)(\dim Y + 1) - \dim L + d_h(Y) = (h+1)(\dim X - \dim Y) - 1 - \dim L + d_h(Y) > 0$. Since $N = N_Y + \dim L + 1$, we obtain $\delta_h(X) = \min\{N, (h+1)\dim X + h\} - \dim S^h(X) \geq \min\{N_Y + \dim L + 1 - (h+1)(\dim Y + 1) - \dim L + d_h(Y), (h+1)(\dim X - \dim Y) - 1 - \dim L + d_h(Y)\} = \min\{N_Y - (h+1)(\dim Y + 1) + 1 + d_h(Y), (h+1)(\dim X - \dim Y) - 1 - \dim L + d_h(Y)\}$. □

3.3.1. *Items 1 and 2 of Theorem 1.* Apply Lemma 2 to the item 1. Since Y is non-h-defective and $N_Y \geq (h+1)(\dim Y + 1)$, $d_h(Y) = 0$. Since $\dim L \leq (h+1)(\dim X - \dim Y) - 2$, $d_h(X) \geq (h+1)(\dim X - \dim Y) - 1 - \dim L > 0$ and $\delta_h(X) \geq \min\{N_Y - (h+1)(\dim Y + 1) + 1, (h+1)(\dim X - \dim Y) - 1 - \dim L\} \geq \min\{1, d_h(X)\} = 1$.

In the item 2, $X \subset Cone_L(v_2(Y))$, where Y is a surface of minimal degree in $\mathbb{P}^{h+1}$. By the classification of defective surfaces, $v_2(Y) \subset \mathbb{P}^{3h+2}$ is h-defective surface with $d_h(v_2(Y)) = 1$. Thus, by Lemma 2, one has $d_h(X) \geq (h+1)(3-2) - 1 - \dim L + 1 = h + 1 - \dim L$ and $\delta_h(X) \geq \min\{3h+2-(h+1)(2+1)+1+1, (h+1)(3-2)-1-\dim L+1\} = \min\{1, d_h(X)\}$. Since $\dim L \leq h$, we obtain $d_h(X) \geq 1$ and $\delta_h(X) \geq 1$.

3.3.2. *Item 3 of Theorem 1.* For the items 3 and 4, it is easier to use the following observation, which follows from Lemma 1. Suppose that $h \geq 2$. Take a general point $x \in X$ and consider the projection from T_xX. If $\pi_{T_xX}(X)$ is a threefold and $\pi_{T_xX}(X)$ satisfies the same conditions as X, but for $h-1$, then $d_h(X) = d_{h-1}(\pi_{T_xX}(X))$, $\delta_h(X) = \delta_{h-1}(\pi_{T_xX}(X))$, $\delta_{h-1}(X) = \delta_{h-2}(\pi_{T_xX})$. Hence, we can reduce the situation to $h = 1$. Also we will use the following fact. Let $Y \subset \mathbb{P}^{N_Y}$ be a non-degenerate variety, locally defined by quadratic equations, i. e. $v_2(Y)$ is a component of $\langle v_2(Y)\rangle \cap v_2(\mathbb{P}^{N_Y})$. If $y \in Y$ is a general point, then $\pi_{T_{v_2(y)}v_2(Y)} \circ v_2 = v_2 \circ \pi_y$. This fact is true, because both maps are given by the complete linear system of quadrics containing y with the multiplicity at least 2.

Suppose that $X = \pi_L(v_2(Y))$, where $Y \subset \mathbb{P}^{h+2}$ is a non-degenerate threefold of minimal degree. Note, that any variety of minimal degree is defined by quadratic equations. Consider a general point $x \in X$ and a point $y \in Y$ such that $x = \pi_L(v_2(y))$. Then $\pi_{T_xX}(X) = \pi_{L'}(\pi_{T_{v_2(y)}v_2(Y)}(v_2(Y)) = \pi_{L'}(v_2(\pi_y(Y)))$, where L' is an appropriate subspace, $\dim L' = \dim L$. Since $\pi_y(Y)$ is a threefold of minimal degree in $\mathbb{P}^{h+1}$, we are in the situation of $h-1$. Thus, it is suffisient to verify the case of $h = 1$, which is the classical one. First, let us check that for any line $L \subset \mathbb{P}^9$ and two general points $x, y \in v_2(\mathbb{P}^3)$ holds $L \cap \langle T_x v_2(\mathbb{P}^3), T_y v_2(\mathbb{P}^3)\rangle = \emptyset$. Assume the opposite and consider the projection from $T_x v_2(\mathbb{P}^3)$. Then we have that for a general point $y' \in \pi_{T_x v_2(\mathbb{P}^3)}$ holds $\pi_{T_x v_2(\mathbb{P}^3)}(L) \cap T_{y'}\pi_{T_x v_2(\mathbb{P}^3)}(v_2(\mathbb{P}^3)) \neq \emptyset$. Since $\pi_{T_x v_2(\mathbb{P}^3)}(v_2(\mathbb{P}^3)) = v_2(\pi_{v_2^{-1}(x)}(\mathbb{P}^3)) = v_2(\mathbb{P}^2)$, we obtain that for certain line $L' \subset \mathbb{P}^5$ and a general point $y' \in v_2(\mathbb{P}^2)$ holds $L' \cap T_{y'}v_2(\mathbb{P}^2) \neq \emptyset$. Taking the projection from $T_{y'}v_2(\mathbb{P}^2)$, one has that the image of L' is a point and through that point there pass all tangent lines to a conic $\pi_{T_{y'}v_2(\mathbb{P}^2)}(v_2(\mathbb{P}^2))$ on a plane, which is impossible. Thus, $L \cap \langle T_x v_2(\mathbb{P}^3), T_y v_2(\mathbb{P}^3)\rangle = \emptyset$. Hence, $d_1(\pi_L(v_2(\mathbb{P}^3))) = d_1(v_2(\mathbb{P}^3)) = 1$. Since $\dim\langle \pi_L(v_2(\mathbb{P}^3))\rangle = 9 - \dim L - 1 \geq 7$, one has $\delta_1(\pi_L(v_2(\mathbb{P}^3))) = 1$. Finally, we should notice that also $L \cap T_x v_2(\mathbb{P}^3) = \emptyset$. Thus, $T_x v_2(Y) \cap L = \emptyset$ and $\pi_L(v_2(Y))$ is a threefold. So, we were able to make the step from h to $h-1$.

3.3.3. *Item 4 of Theorem 1.* Suppose that $X \subset Cone_L(v_2(Y))$, where $L \subset \mathbb{P}^N$ is a linear subspace, $d = 1, 2$ is a parameter, $2 - d \geq \dim L \geq \max\{-1, 2 - \frac{(h-1)(h-2)}{2} - d\}$, $Y \subset \mathbb{P}^{h+1}$ is a non-degenerate threefold such that $\dim I_2(Y) = \binom{h}{2} - h - 1 + d + \dim L$. First, note that Y is locally defined by quadratic equations, if $\dim I_2(Y)$ is greater then the maximal possible for a variety of dimension 4 in $\mathbb{P}^{h+1}$. This maximal dimension is reached for four-folds of minimal degree, $h-2$. It is equal to $\binom{h-1}{2} - (h-2) = \binom{h}{2} - 2h + 3$. Thus, if $d_h(X) + \dim L + h > 4$ (in particular, if $h > 4$; if $h = 4$ and $\dim L + d > 0$; if $h = 3$ and $d + \dim L = 2$), Y is defined by quadratic equations. Take a general point $x \in X$ and the point $y = v_2^{-1}(\pi_L(x)) \in Y$. Then, $\pi_{T_x X}(X) \subset Cone_{L'}(\pi_{T_{v_2(y)}v_2(Y)}(v_2(Y)) = Cone_{L'}(v_2(\pi_y(Y)))$. If $h \geq 3$, the variety $\pi_{T_x X}(X)$ is three-dimensional, because $\pi_y(Y)$ is a threefold (the case $h = 2$ we will consider independently). Let us show that $\pi_{T_x X}(X)$ satisfies the conditions with the same value of the parameter d. Since $\dim L' = \dim L$, the inequality $2 - d \geq \dim L' \geq \max\{-1, 2 - \frac{(h-1)(h-2)}{2} - d\}$ holds unless $h = 4$, $d = 1$, $\dim L = -1$ and $h = 3$, $\dim L = 1$ (exactly the same exceptions as above!). In all other cases we also need to check $\dim I_2(\pi_y(Y))$. Since $\dim I_2(Y) + \dim\langle v_2(Y)\rangle = \dim H^0(\mathcal{O}(2), \mathbb{P}^{h+1})$, we have $\dim\langle v_2(Y)\rangle = \binom{h+3}{2} - 1 - \dim I_2(Y) = \binom{h+3}{2} - 1 - \binom{h}{2} + h + 1 - d - \dim L = 4h + 3 - d - \dim L$. Since $\dim\langle v_2(\pi_y(Y))\rangle = \dim\langle v_2(Y)\rangle - \dim T_{v_2(y)}v_2(Y) - 1 = \dim\langle v_2(Y)\rangle - 4$, one has $\dim I_2(\pi_y(Y)) = \dim H^0(\mathcal{O}(2), \mathbb{P}^h) - \dim\langle v_2(\pi_y(Y))\rangle = \binom{h+2}{2} - 1 - (4h+3-d-\dim L) + 4 = \binom{h-1}{2} - (h-1) - 1 + d + \dim L'$.

Let us consider special cases. If $h = 4$, $d = 1$, $\dim L = -1$, then $\dim I_2(Y) = 6-4-1+1-1 = 1$, i. e. Y is contained in unique quadric $Q \subset \mathbb{P}^5$ and $X = v_2(Y)$, $x = v_2(y)$. Since $T_xX \subset T_xv_2(Q)$, $\pi_{T_xX}(X)$ lays in the cone with the vertex $\pi_{T_xX}(T_xv_2(Q))$, which is a point, over $\pi_{T_xv_2(Q)}(X) \subset \pi_{T_xv_2(Q)}(v_2(Q)) = v_2(\pi_y(Q)) = v_2(\mathbb{P}^4)$. More, since Y is contained in unique quadric, $\dim I_2(\pi_y(Y)) = 0$. Since, L' is a point, the conditions are satisfied for $d = 1$.

The case $h = 3$, $\dim L + d = 1$. One has $\dim I_2(Y) = 3-3-1+1 = 0$. Thus, $\langle v_2(Y)\rangle = \langle v_2(\mathbb{P}^4)\rangle$. If $y' \in Y$ is a general point, one has that $\pi_{T_{v_2(y)}v_2(Y)}(v_2(Y))$ lays in a cone with the vertex at the point $\pi_{T_{v_2(y)}v_2(Y)}(T_{v_2(y)}v_2(\mathbb{P}^4))$ over $\pi_{T_{v_2(y)}v_2(\mathbb{P}^4)}(v_2(\mathbb{P}^4)) = v_2(\mathbb{P}^3)$. Hence, $\dim L' = \dim L + 1$, $\pi_y(Y) = \mathbb{P}^3$ and $\dim I_2(\pi_y(Y)) = 0$. All conditions are satisfied for the same value of d.

The case $h = 2$. In this case $X \subset Cone_L(v_2(\mathbb{P}^3))$ and $\dim L + d = 2$. Let us see that X is not 1-defective. Since $\dim\langle X\rangle = \dim\langle v_2(\mathbb{P}^3)\rangle + \dim L + 1 \geq 10$, X is not $v_2(\mathbb{P}^3)$ or its projections; X is not a section of $\mathbb{P}^2 \times \mathbb{P}^2$, X is not a subset of 4-dimensional cone over $v_2(\mathbb{P}^2)$. Thus, X could be either a cone, either a subset of a 4-dimensional cone over a curve. These two cases are impossible, because $\pi_L(X) = v_2(\mathbb{P}^3)$, which is not a cone and is not a subset of a 4-dimensional cone over a curve. Therefore, for a general point $x \in X$ the variety $\pi_{T_xX}(X)$ is a threefold. More, $\pi_{T_xX}(X) \subset Cone_{L'}(\pi_{T_{\pi_L(x)}v_2(\mathbb{P}^3)}(v_2(\mathbb{P}^3))) = Cone_{L'}(v_2(\mathbb{P}^2))$, where $L' = \pi_{T_xX}(L)$, $\dim L' = \dim L$. If $d = 2$, then L' is a point, $\pi_{T_xX}(X)$ is a cone over $v_2(\mathbb{P}^2)$. Thus, $d_2(X) = d_1(\pi_{T_xX}(X)) = 2 = d$ and $\delta_2(X) = \delta_1(\pi_{T_xX}(X)) = 1$; $\delta_1(X) = \delta_0(\pi_{T_xX}(X)) = 0$. If $d = 1$, then L' is a line. Thus, $d_1(\pi_{T_xX}(X)) \geq 1$. If $d_1(\pi_{T_xX}(X)) = 2$, then $\pi_{T_xX}(X)$ is a cone over 1-defective surface; $\pi_{T_xX}(X)$ is not a cone over $v_2(\mathbb{P}^2)$, because $\pi_{T_xX}(X)$ is non-degenerate; $\pi_{T_xX}(X)$ is not a cone over a curve because $\pi_{L'}(\pi_{T_xX}(X))$ is not. Hence, $d_2(X) = d_1(\pi_{T_xX}(X)) = 1 = d$ and $\delta_2(X) = \delta_1(\pi_{T_xX}(X)) = 1$; $\delta_1(X) = \delta_0(\pi_{T_xX}(X)) = 0$.

3.3.4. *Items 5–8 of Theorem 1.*

Lemma 3. *If X is one of the items 5–8 of Theorem 1, then for $h+1$ general points there exists a rational normal scroll of type (a,b), $a+b = 3h$, $a, b \geq h$, passing through them.*

Proof. Take $h+1$ general points and concider lines $l_0, \dots, l_h \subset X$, passing through them.

For the items 5 and 6 these lines give us certain points $y_0, \dots, y_h \in Y$. Since $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree, there exists a rational normal curve K of degree h passing through them. Then the surface S covered by lines $\langle\xi(x),\eta(x)\rangle$, $x\in K$, is a rational normal scroll. Really for the item 5, $\xi(K)$ is a rational normal curve of degree $h+1$, $\eta(K)$ is a projection of a rational normal curve of degree $2h$ from its point, which is a rational normal curve of degree $2h-1$; $\langle\xi(K)\rangle\cap\langle\eta(K)\rangle=\emptyset$. For the item 6, $\xi(K)$ is a rational normal curve of degree $h+1$ and $\eta(K)$ is a rational normal curve of degree $2h$; $\langle\xi(K)\rangle\cap\langle\eta(K)\rangle=\xi(K)\cap\eta(K)=\xi(K\cap\langle p,y\rangle)$. One can see that in this case S is a scroll of type $(h+1,2h-1)$.

For the item 7, lines $l_0,\dots,l_4$ give us 5 points on $v_3(\mathbb{P}^2)$, i. e. 5 points on the plane. If K is a conic, passing through them, then the surface $v_3(K)\times\mathbb{P}^1$ contains $l_0,\dots,l_4$. This surface is a scroll of type $(6,6)$.

For the item 8, lines $l_0,\dots,l_h$ give us $h+1$ points on $Scroll_{a+c,a+b+c}$. Through $h+1$ points on this scroll there passes a rational normal curve K of degree $a+c+\frac{b+h}{2}$. Under the projection ψ from c ruling lines this curve goes to a rational normal curve of degree $a+\frac{b+h}{2}$. By definition of 4-scroll, the surface S, covered by lines $\langle\psi(x),x\rangle$, $x\in K$, contains $l_0,\dots,l_h$. S is a scroll of type $(a+\frac{b+h}{2},a+c+\frac{b+h}{2})$; $a+\frac{b+h}{2}+a+c+\frac{b+h}{2}=2a+b+c+h=3h$; $a+\frac{b+h}{2}=\frac{2a+b+c+h-c}{2}=\frac{3h-c}{2}\geq\frac{3h-h}{2}=h$. □

By the classification of h-defective surfaces one can see that $Scroll_{a,b}$ is not h-defective, because $a,b\geq h$. Since $\dim\langle Scroll_{a,b}\rangle=a+b+1=3h+1<(h+1)\cdot\dim Scroll_{a,b}+h$, one has $S^h(Scroll_{a,b})=\langle Scroll_{a,b}\rangle$ and $d_h(Scroll_{a,b})=1$. Thus, if $z\in S^h(Scroll_{a,b})$, then $\Sigma_z(Scroll_{a,b})$ is a curve. More, $\Sigma_z(X)\supset\Sigma_z(Scroll_{a,b})$, and $d_h(X)\geq d_h(Scroll_{a,b})=1$.

Suppose that X is covered by an irreducible family of lines and $d_k(X)>0$. Take a general point $z'\in S^k(X)$ and consider $\Sigma_{z'}(X)$. By Terracini lemma, for a general point $x\in\Sigma_{z'}(X)$ one has $T_xX\subset T_{z'}S^k(X)$. So, if l is a line of the family passing through x, then $l\subset T_xX\subset T_{z'}S^k(X)$. Let $M_{z'}$ be (the closure of) a subvariety of X swept out by all such lines, passing through general points of $\Sigma_{z'}(X)$. Then $M_{z'}\subset T_{z'}S^k(X)\cap X$.

Take general points $x_0,\dots,x_h\in X$, $z\in\langle x_0,\dots,x_h\rangle$ and consider the corresponding surface S, that is a scroll of type (a,b), $a+b=3h$, $a,b\geq h$. Then $x_0,\dots,x_h\in\Sigma_z(Scroll_{a,b})$. If $d_{h-1}(X)>0$, then for a general point $z'\in S^{h-1}(X)$ one can define a surface $M_{z'}$ as above. Take a general point $z'\in\langle x_0,\dots,x_{h-1}\rangle$. Then by Terracini lemma, $T_{z'}S^{h-1}(X)\subset T_zS^h(X)$. Hence, $M_{z'}\subset T_zS^h(X)$. Since $\dim T_zS^h(X)=(h+1)\dim X+h-d_h(X)\leq 4h+2<N$, $X\not\subset T_zS^h(X)$. Thus, while we vary $x_0,\dots,x_{h-1}\in\Sigma_z(Scroll_{a,b})$ and z', the surface $M_{z'}$ cannot vary. Hence, $M_{z'}\supset\Sigma_z(Scroll_{a,b})$. Since by our constructions,

through a general point of X there passes only one line from the family, we obtain $M_{z'} \supset Scroll_{a,b}$ and the components of $\Sigma_{z'}(X)$ containing $x_0, \dots, x_{h-1}$, lay in $Scroll_{a,b}$. Since $Scroll_{a,b}$ is not $(h-1)$-defective and $\dim\langle Scroll_{a,b}\rangle = a+b+1 = 3h+1 > h \cdot \dim Scroll_{a,b} + (h-1)$, one has $d_{h-1}(Scroll_{a,b}) = 0$. Thus, $d_{h-1}(X) = 0$. Hence, X is h-defective.

If $d_h(X) = 2$, then, since $T_z S^h(X) \cap X \neq X$, $M_z = \Sigma_z(X)$. Thus, $Scroll_{a,b}$ is a component of $\Sigma_z(X)$, passing through $x_0, \dots, x_h$. But $d_h(Scroll_{a,b}) = 1$ and $\Sigma_z(Scroll_{a,b})$ is a curve. Thus, $d_h(X) = 1$ and $\delta_h(X) = 1$.

3.4. **Proof of "only if" part.** From now on we will suppose that X is an h-defective threefold, if the opposite is not written explicitly. Take a general point $q \in S^{h-2}(X)$. Denote the projection from $T_q S^h(X)$ by π. Note, that since $d_{h-2}(X) = 0$, one has $\dim T_q S^{h-2}(X) = \dim S^{h-2}(X) = 4h-5$. Since $d_{h-1}(X) = 0$, we also have $\dim \pi(X) = 3$. By Corollary 1, one has $\nu_1(\pi(X)) = \nu_h(X)$ and $d_1(\pi(X)) = d_h(X) > 0$. Denote by N' the dimension of the ambient space of $\pi(X)$, $N' = N - \dim T_q S^{h-2}(X) - 1 = N - 4h - 4$.

Since $\pi(X)$ is an 1-defective threefold, we can apply the theorem of Scorza to describe $\pi(X)$. Let us also calculate the numbers $d_h(X) = d_1(\pi(X))$, $\nu_h(X) = \nu_1(\pi(X))$ and N.

If $\pi(X)$ is a cone over a surface Y, we should distinguish the cases $d_1(Y) = \nu_h(Y) = 1$, $d_1(Y) = \nu_1(Y) = 0$ and $d_1(Y) = 0$, $\nu_1(Y) = 1$. In the first case $d_1(\pi(X)) = \nu_1(\pi(X)) = 2$. More, by the classification of 1-defective surfaces, $\dim\langle Y\rangle \geq 5$ and Y is either a cone over a curve, or $v_2(\mathbb{P}^2)$. Thus, $N' \geq 6$ and $\pi(X)$ is either a cone over a curve (see Section 4.2 and Corollary 2 for the classification of such X), or a cone over $v_2(\mathbb{P}^2)$ (see Lemma 4); $N \geq 4h+2$. In the second case $d_1(\pi(X)) = \nu_1(\pi(X)) = 1$, $N' \geq 7$ and $N \geq 4h+3$; see Proposition 4 for the classification.

In the third case, by [6, Theorem 1.3], non-1-defective surface Y with $\nu_1(Y) = 1$, which is called *1-weakly defective*, is either a subsurface of a cone with the vertex at a point over $v_2(\mathbb{P}^2)$, or a subsurface of a cone with the vertex at a line over certain curve. Hence, $\pi(X)$ contained either in a cone with the vertex at a line over $v_2(\mathbb{P}^2)$, or in a cone with the vertex at a plane over certain curve. Thus, the third case actually consists of two other cases from the list of the theorem of Scorza, and we will consider it as a part of further cases.

If $\pi(X)$ is a subvariety of a cone with the vertex at a plane over a curve, then $d_1(\pi(X)) = 1$, $\nu_1(X) = 2$, $N' \geq 7$, $N \geq 4h+3$. See Section 4.2 and Corollary 2 for the classification.

If $\pi(X)$ is a subvariety of a cone with the vertex at a line over $v_2(\mathbb{P}^2)$, then $d_1(\pi(X)) = 1$, $\nu_1(X) = 2$, $N' = 7$, $N = 4h + 3$. See Lemma 4 for the classification.

If $N' = 7$ and $\pi(X)$ is a hyperplane section of the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ then $d_1(\pi(X)) = \nu_1(\pi(X)) = 1$, variety $\Sigma_z(\pi(X)) = V_{\pi(X),1,z}$ is an irreducible conic, $N = 4h + 3$; see section 6 for classification.

If $N' = 7, 8, 9$ and $\pi(X)$ is the Veronese variety $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$ or a projection of it in $\mathbb{P}^{N'}$, then $d_1(\pi(X)) = \nu_1(\pi(X)) = 1$, the variety $\Sigma_z(\pi(X)) = V_{\pi(X),1,z}$ is an irreducible conic, $N = 4h+3, 4h+4, 4h+5$ respectively; see section 5.1 for classification.

4. $\nu_h(X) = 2$

4.1. $\pi(X) \subset Cone_L(v_2(\mathbb{P}^2))$, $\dim L = 2 - d_h(X)$.

Lemma 4. *If $\pi(X) \subset Cone_L(v_2(\mathbb{P}^2))$ and $\dim L = 2 - d_h(X)$, then X is one if the following threefolds:*

1. *$X \subset Cone_M(v_2(Y)) \subset \mathbb{P}^{4h+4-d_h(X)}$, where $M \subset \mathbb{P}^{4h+4-d_h(X)}$ is a linear subspace, $2 - d_h(X) \geq \dim M \geq \max\{-1, 2 - \frac{(h-1)(h-2)}{2} - d_h(X)\}$, $Y \subset \mathbb{P}^{h+1}$ is a non-degenerate variety, $\dim Y = 3$, $\dim I_2(Y) = \frac{(h+3)(h+2)}{2} - 4h - 2 - (2 - d_h(X) - \dim M)$.*
2. *$X \subset Cone_M(v_2(Y))$, where M is a linear subspace of dimension $h + 1 - d_h(X)$ and $Y \subset \mathbb{P}^{h+1}$ is a non-degenerate surface of minimal degree.*
3. *$X \subset Cone_M(Y)$, where M is a linear subspace of dimension $h - 1$ and $Y \subset \mathbb{P}^{3h+3}$ is a non-h-defective surface.*

Proof. First, note that in this case $N = 4h + 4 - d_h(X)$. Denote by $\mathcal{L}_h(X)$ a system of divisors on X of the form $V_z(X)$, $z \in S^h(X)$ is general. Through general points $x_0, \ldots, x_h \in X$ there passes exactly one such divisor, which corresponds to $z \in \langle x_0, \ldots, x_h \rangle$. Thus, $\dim \mathcal{L}_h = h + 1$.

Take general divisors $D_1, D_2 \in \mathcal{L}_h$, and general points $x_2, \ldots, x_h \in D_1 \cap D_2$. The divisors have the form $V_{z_1}(X)$ and $V_{z_2}(X)$ for certain general points $z_1, z_2 \in Cone_{\langle x_2,\ldots,x_h\rangle}(S(X)) \subset S^h(X)$. Then, by Lemma 1, $\pi(D_i) = V_{\pi(X),1,\pi(z_i)}$, $\pi(z_i) \in S(\pi(X))$, $i = 1, 2$, and thus $\pi(D_1), \pi(D_2) \in \mathcal{L}_1(\pi(X))$.

So, let us study the system $\mathcal{L}_1(\pi(X))$. Since $\pi(X) \subset Cone_L(v_2(\mathbb{P}^2))$, for a general point $w \in S(\pi(X))$ in order to obtain the divisor $V_{\pi(X),1,w}$, one has to consider a projection π_L from the vertex L of the cone ($\pi_L(\pi(X)) = v_2(\mathbb{P}^2)$). Then take the point $\pi_L(w) \in S(v_2(\mathbb{P}^2))$ and the corresponding divisor $V_{v_2(\mathbb{P}^2),1,\pi_L(w)} \subset v_2(\mathbb{P}^2)$, which is a conic, or, more concrete, $v_2(l)$, where $l \subset \mathbb{P}^2$ is a certain line. Further, $V_{\pi(X),1,w} =$

$\overline{\pi(X) \cap Cone_L(V_{v_2(\mathbb{P}^2),1,\pi_L(w)}) \setminus L}$. In other words, the projection π_L gives us one-to-one correspondence between the system $\mathcal{L}_1(\pi(X))$ and the system of conics on $v_2(\mathbb{P}^2)$, which is a system of lines in $\mathbb{P}^2$. So, the system $\mathcal{L}_1(\pi(X))$ is linear. Consider a map $\varphi : \pi(X) \dashrightarrow \mathbb{P}^2$, generated by this system. Then $v_2 \circ \varphi = \pi_L$. Hence, $\dim \varphi(\pi(X)) = 2$ and for general divisor $D \in \mathcal{L}_1(\pi(X))$ we have $2 \cdot D$ is a hyperplane section of $\pi(X)$, containing L.

So, the divisors $V_{\pi(X),1,\pi(z_1)}$ and $V_{\pi(X),1,\pi(z_2)}$ are linearly equivalent. So, divisors D_1 and D_2 are also linearly equivalent, and the system $\mathcal{L}_h(X)$ is linear. More, the divisor $2 \cdot D_1$ is a hyperplane section of X.

Consider a map $\varphi_h : X \dashrightarrow \mathbb{P}^{h+1}$, given by the linear system $\mathcal{L}_h(X)$. Since, as we saw, $\dim \varphi(\pi(X)) = 2$ for a map, constructed by a liner sub-system $\mathcal{L}_1(\pi(X)) \subset \mathcal{L}_h(X)$, we have that $\dim \varphi_h(X) \geq 2$. Consider a Veroneze embedding $v_2(\mathbb{P}^{h+1})$ and take its restriction $v_2 : \varphi_h(X) \to \mathbb{P}^{N'}$. Since the complete linear system of quadrics on $\mathbb{P}^{h+1}$ is generated by divisors of the form $2 \cdot H$, where H is a hyperplane in $\mathbb{P}^{h+1}$, and $2 \cdot D$ is a hyperplane section of X for a general $D \in \mathcal{L}_h(X)$, we have that any linear section of $v_2(\varphi_h(X))$ corresponds to a certain linear section of X, or, in other words, $v_2 \circ \varphi_h$ is a (rational) linear map on X and thus on $\mathbb{P}^N$.

Denote by M the center of projection $v_2 \circ \varphi_h$. So, $\pi_M = v_2 \circ \varphi_h$. Consider the cases of $\dim \pi_M(X)$ which coincides with $\dim \varphi_h(X)$, taking into account that $3 \geq \dim \varphi_h(X) \geq 2$.

The case $\dim \pi_M(X) = 3$. Since $N = 4h + 4 - d_h(X)$, one has that $\dim\langle \pi_M(X) \rangle = 4h + 4 - d_h(X) - \dim M - 1$. So, $\dim I_2(\varphi_h(X)) = \dim H^0(\mathcal{O}(2), \mathbb{P}^{h+1}) - \dim v_2(\varphi_h(X)) - 1 = \frac{(h+3)(h+2)}{2} - 4h - 4 + d_h(X) + \dim M$. Since $\dim I_2(\varphi_h(X)) \geq 0$, in particular, $\dim M \geq 4h + 4 - \frac{(h+3)(h+2)}{2} - d_h(X) = 2 - \frac{(h-1)(h-2)}{2} - d_h(X)$. Note, that for $h \geq 4$ the last unequality does not give us any restrictions on M; for $h = 2$ we have $\dim M \geq 2 - d_h(X)$; for $h = 3$ we have $\dim M \geq 1 - d_h(X)$. On the other hand, for a variety Y of dimension k in $\mathbb{P}^{h+1}$ the maximal possible value $\dim I_2(Y)$, which holds for varieties of minimal degree, is $\dim H^0(\mathcal{O}(2), \mathbb{P}^{h+1-k}) - (h + 1 - k + 1) = \frac{(h+1-k+2)(h+1-k+1)}{2} - h + k - 2 = \frac{(h+3)(h+2)}{2} - (k+1)h + \frac{(k-4)(k+1)}{2}$. For $k = 3$, we have that $\dim I_2(\varphi_h(X)) \leq \frac{(h+3)(h+2)}{2} - 4h - 2$. So, $\dim M \leq 2 - d_h(X)$. Finally, we have $X \subset Cone_M(v_2(Y)) \subset \mathbb{P}^{4h+4-d_h(X)}$, where $M \subset \mathbb{P}^{4h+4-d_h(X)}$ is a linear subspace, $2 - d_h(X) \geq \dim M \geq \max\{-1, 2 - \frac{(h-1)(h-2)}{2} - d_h(X)\}$, $Y \subset \mathbb{P}^{h+1}$ is a non-degenerate variety, $\dim Y = 3$, $\dim I_2(Y) = \frac{(h+3)(h+2)}{2} - 4h - 2 - (2 - d_h(X) - \dim M)$.

Remark 5. The condition $\dim I_2(Y) = \frac{(h+3)(h+2)}{2} - 4h - 2 - (2 - d_h(X) - \dim M)$ says that the number of linearly independent quadrics in $\mathbb{P}^{h+1}$ containing Y differs at most by 2 from the maximal possible for three-dimentional varieties. In particular, for $h > 3$, we have that $h - 1 \geq \deg Y \leq h + 1 - d_h(X) - \dim M \leq h + 1$.

Example 1. It is well-known, that the variety $X = v_4(\mathbb{P}^3) \subset \mathbb{P}^{34}$ is 8-defective with $d_8(X) = 2$. Since $X = v_2(Y)$, where $Y = v_2(\mathbb{P}^3) \subset \mathbb{P}^9$, X lays in a cone with the empty vertex over $v_2(Y)$. So, in our notation $\dim M = -1$. Let us calculate $\dim I_2(Y)$. If we interpret $\mathbb{P}^9$ as a projectivisation of symmetric 4×4 matrices and Y as a projectivisation of symmetric matrices of rank one, we can see, that $I_2(Y)$ is generated by 2×2 minors with exactly one linear relation among them. So, $\dim I_2(Y) = \frac{6 \cdot 6 - 6}{2} + 6 - 1 = 20$ and thus $\frac{(8+3)(8+2)}{2} - 4 \cdot 8 - 2 - (2 - d_8(X) - \dim M) = 55 - 32 - 2 - 1 = 20 = \dim I_2(Y)$.

The case $\dim \pi_M(X) = 2$. By the same reasons, $\dim I_2(\varphi_h(X)) = \frac{(h+3)(h+2)}{2} - 4h - 4 + d_h(X) + \dim M$. Since $\dim \varphi_h(X) = 2$, we have that $\dim I_2(\varphi_h(X)) \leq \frac{(h+3)(h+2)}{2} - 3h - 3$. So, $\dim M \leq h + 1 - d_h(X)$. On the other hand, since $\dim \pi_M(X) = 2$, for a general point $x \in X$ one has $\dim T_x X \cap M = 0$. Since the variety X is non-$(h-1)$-defective, we have that for general points $x_0, \dots, x_{h-1} \in X$ holds $\dim \langle T_{x_0} X \cap M, \dots, T_{x_{h-1}} X \cap M \rangle = h - 1$. Hence, $\dim M \geq h - 1$. If $d_h(X) = 2$, we obtain that $\dim M = h - 1 = h + 1 - d_h(X)$, and $\varphi(X)$ is a surface of minimal degree in $\mathbb{P}^{h+1}$. If $d_h(X) = 1$, either $\dim M = h = h + 1 - d_h(X)$ and $\varphi(X)$ is a surface of minimal degree in $\mathbb{P}^{h+1}$, or $\dim M = h - 1$ and $\dim I_2(\varphi_h(X)) = \frac{(h+3)(h+2)}{2} - 3h - 4$, i. e. is less than a maximal possible by 1. In the last case we can apply Proposition 3. To apply it, we need to check that for a general point $y \in v_2(\mathbb{P}^2)$ the variety $\pi_{T_y v_2(\mathbb{P}^2)}(v_2(\mathbb{P}^2))$ is not a cone, which is true, because $\pi_{T_y v_2(\mathbb{P}^2)}(v_2(\mathbb{P}^2)) = v_2(\mathbb{P}^1)$ is a conic. $\square$

4.2. $\pi(X) \subset Cone_{\mathbb{P}^{3-d_h(X)}}(C)$, where C is a curve, $N' \geq 8 - d_h(X)$, $N \geq 4h + 4 - d_h(X)$.

Proposition 3. *Suppose that X is non-degenerate irreducible h-defective variety, $h \geq 2$, $q \in S^{h-2}(X)$ is a general point and $\pi_{T_q S^{h-2}(X)}$ is the projection from $T_q S^{h-2}(X)$. If $\pi_{T_q S^{h-2}(X)}(X) \subset Cone_{L_q}(Y_q)$, where* $\dim L_q = 2(\dim X - \dim Y_q) - d_h(X) - 1$, *$Y_q$ is such that $d_1(Y_q) = 0$ and $\pi_{T_y Y_q}(Y_q)$ is not a cone for a general point $y \in Y_q$, then $X \subset Cone_L(Y)$, where* $\dim L = (h+1)(\dim X - \dim Y) - d_h(X) - 1$, $\dim Y = \dim Y_q$, *$d_h(Y) = 0$, $S^h Y$ is not a cone, $S^{h-1}(\pi_{T_y Y}(Y))$ is not a cone for a general point $y \in Y$, and $S^h(X) = Cone_L(S^h(Y))$.*

Proof. Let us use the induction on h.

Suppose that $h \geq 3$. Take a general point $x \in X$ and a general point $q \in Cone_x(S^{h-3}(X))$. By Corollary 1 the variety $\pi_{T_xX}(X)$ is $(h-1)$-defective and $d_{h-1}(\pi_{T_xX}(X)) = d_h(X)$. By Lemma 1, the point $q' = \pi_{T_xX}(q) \in S^{h-3}(\pi_{T_xX}(X))$ is general and $\pi^{-1}_{T_xX}(T_{q'}S^{h-3}(\pi_{T_xX}(X))) = T_qS^{h-2}(X)$. So, for the corresponding projections we have: $\pi_{T_qS^{h-2}(X)} = \pi_{T_{q'}S^{h-3}(\pi_{T_xX}(X))} \circ \pi_{T_xX}$. Hence, the variety $\pi_{T_xX}(X)$ satisfies the conditions of Proposition for $(h-1)$-defectivity, $L_{q'} = L_q$ and $Y_{q'} = Y_q$. So, by induction, we have that $\pi_{T_xX}(X) \subset Cone_{L_x}(Y_x)$, where $\dim Y_x = \dim Y_{q'} = \dim Y_q$, $d_{h-1}(Y_x) = 0$, $S^{h-1}(Y_x)$ is not a cone, $S^{h-2}(\pi_{T_yY_x}(Y_x))$ is not a cone for a general point $y \in Y_x$, and $S^{h-1}(\pi_{T_xX}(X))$ is a cone with the vertex L_x.

If $h = 2$ then $S^{h-2}(X) = X$ and for a general point $x \in X$ we, by our hypothesis, have that $\pi_{T_xX}(X) \subset Cone_{L_x}(Y_x)$, where $\dim L_x = 2(\dim X - \dim Y_q) - d_2(X) - 1$, Y_x is such that $d_1(Y_x) = 0$, $\pi_{T_yY_x}(Y_x)$ is not a cone for a general point $y \in Y_x$. So, in order to have the same basis for further constructions for all $h \geq 2$, we should show that $S(Y_x)$ is not a cone and that $S(\pi_{T_xX}(X)) = Cone_{L_x}(S(Y_x))$. Assume that $S(Y_x)$ is a cone with non-empty vertex R. Then by Terracini lemma, $\bigcup_{y,y_1\in Y,\ y,y_1 \text{ are general}} \langle T_yY, T_{y_1}Y\rangle = \bigcup_{q\in S(Y), q \text{ is general}} T_qS(Y) = R \neq \emptyset$. Hence, $R_y = \bigcup_{y_1\in Y,\ y_1 \text{ is general}} \langle T_yY, T_{y_1}Y\rangle \supset R$. Notice that $R_y \supset T_yY$. If $R_y \neq T_yY$, then for a general point $y_1 \in Y$ one has $\pi_{T_yY}(T_{y_1}Y) \supset \pi_{T_yY}(R_y) \neq \emptyset$. Since $\pi_{T_yY}(T_{y_1}Y) = T_{\pi_{T_yY}(y_1)}\pi_{T_yY}(Y)$, the variety $\pi_{T_yY}(Y)$ is a cone with the vertex $\pi_{T_yY}(R_y)$, which is excluded by the conditions. So, $R_y = T_yY$ and, thus, $R \subset T_yY$. Since the point $y \in Y$ is general, the variety Y is a cone with the vertex R, which is impossible because of $d_1(Y) = 0$. So, $S(Y_x)$ is not a cone. Let us show that $S(\pi_{T_xX}(X)) = Cone_{L_x}(S(Y_x))$. Since $d_1(Y_x) = 0$, we have $\dim S(Y_x) = 2\dim Y_x + 1$. Since $\dim X = \dim \pi_{T_xX}(X)$, we have $\dim S(Cone_{L_x}(Y_x)) = \dim Cone_{L_x}(S(Y_x)) = \dim L_x + \dim S(Y_x) + 1 = (2(\dim X - \dim Y_x) - d_h(X) - 1) + (2\dim Y_x + 1) + 1 = 2\dim \pi_{T_xX}(X) + 1 - d_1(\pi_{T_xX}(X)) = \dim S(\pi_{T_xX}(X))$. Since $S(\pi_{T_xX}(X)) \subset S(Cone_{L_x}(Y_x))$, we obtain $S(\pi_{T_xX}(X)) = S(Cone_{L_x}(Y_x)) = Cone_{L_x}(S(Y_x))$.

Now complete the proof for all $h \geq 2$. Put $M_x = \pi^{-1}_{T_xX}(L_x)$. Then, since $S^{h-1}(\pi_{T_xX}(X))$ is a cone with the vertex L_x, for a general point $q \in S^{h-1}(\pi_{T_xX}(X))$ one has $T_qS^{h-1}(\pi_{T_xX}(X)) \supset L_x$. Since $S^{h-1}(Y_x)$ is not a cone and $\pi_{L_x}(S^{h-1}(\pi_{T_xX}(X))) = S^{h-1}(Y_x)$, we obtain that

$$\bigcap_{q\in S^{h-1}(\pi_{T_xX}(X)),\ q \text{ is general}} T_qS^{h-1}(\pi_{T_xX}(X)) \subset L_x.$$

Thus,

$$\bigcap_{q\in S^{h-1}(\pi_{T_xX}(X)),\ q \text{ is general}} T_qS^{h-1}(\pi_{T_xX}(X)) = L_x,$$

or, by Lemma 1,

$$\bigcap_{q\in Cone_x(S^{h-1}(X)),\ q \text{ is general}} T_qS^h(X) = M_x.$$

Consider another general point $x' \in X$. Then both spaces M_x and $M_{x'}$ lay in $\bigcap_{q\in Cone_{\langle x,x'\rangle}(S^{h-2}(X)),\ q \text{ is general}} T_qS^h(X)$. Since the variety $S^{h-2}(\pi_{T_yY_x}(Y_x))$ is not a cone for a general point $y \in Y_x$, one has that

$$\bigcap_{q\in S^{h-2}(\pi_{T_yY_x}(Y_x)),\ q \text{ is general}} T_qS^{h-2}(\pi_{T_yY_x}(Y_x)) = \emptyset.$$

Taking into account that $\pi_{M_x} = \pi_{L_x} \circ \pi_{T_xX}$ and $\pi_{M_x}(X) = Y_x$, we put $y = \pi_{M_x}(x')$ and obtain that

$$\pi_{T_yY_x}(\pi_{M_x}(\bigcap_{q\in Cone_{\langle x,x'\rangle}(S^{h-2}(X)),\ q \text{ is general}} T_qS^h(X))) =$$

$$\pi_{T_yY_x}(\bigcap_{q\in Cone_y(S^{h-2}(Y_x)),\ q \text{ is general}} T_qS^h(Y_x)) =$$

$$\bigcap_{q\in S^{h-2}(\pi_{T_yY_x}(Y_x)),\ q \text{ is general}} T_qS^{h-2}(\pi_{T_yY_x}(Y_x)) = \emptyset.$$

Hence,

$$\bigcap_{q\in Cone_{\langle x,x'\rangle}(S^{h-2}(X)),\ q \text{ is general}} T_qS^h(X))) \subset \pi^{-1}_{M_x}(T_yY_x) = \langle M_x, T_{x'}X\rangle.$$

Since $M_{x'} \supset T_{x'}X$, we finally have $\langle M_x, M_{x'}\rangle = \langle M_x, T_{x'}X\rangle$. Using this fact, for a general point $x'' \in X$ and the point $y'' = \pi_{M_x}(x'') \in Y_x$, one can see that $\pi_{M_x}(M_{x'} \cap M_{x''}) \subset \pi_{M_x}(M_{x'}) \cap \pi_{M_x}(M_{x''}) = \pi_{M_x}(T_{x'}X) \cap \pi_{M_x}(T_{x''}X) = T_{y'}Y_x \cap T_{y''}Y_x = \emptyset$, because $d_{h-1}(Y_x) = 0$ and, thus, $d_1(Y_x) = 0$. So, $M_{x'} \cap M_{x''} \subset M_x$ and since all three points $x, x', x'' \in X$ are general, we have that there exists a subspace $L \subset \mathbb{P}^N$ such that for general points $x, x' \in X$ holds $M_x \cap M_{x'} = L$.

Put $Y = \pi_L(X)$. Then $X \subset Cone_L(Y)$. Since

$$M_x = \bigcap_{q\in Cone_x(S^{h-1}(X)),\ q \text{ is general}} T_qS^h(X),$$

we have that

$$L = \bigcap_{q\in S^h(X),\ q \text{ is general}} T_qS^h(X).$$

Thus, $S^h(X)$ is a cone with the vertex L. Since $\pi_L(S^h(X)) = S^h(Y)$, we obtain that $S^h(X) = Cone_L(S^h(Y))$. Further,

$$\bigcap_{q\in S^h(Y),\ q \text{ is general}} T_qS^h(Y) =$$

$$\bigcap_{q\in S^h(X),\ q \text{ is general}} T_{\pi_L(q)}S^h(\pi_L(X)) = \bigcap_{q\in S^h(X),\ q \text{ is general}} \pi_L(T_qS^h(X)).$$

Since $L \subset T_qS^h(X)$ for any point $q \in S^h(X)$, we have

$$\bigcap_{q\in S^h(X),\ q \text{ is general}} \pi_L(T_qS^h(X)) =$$

$$\pi_L(\bigcap_{q\in S^h(X),\ q \text{ is general}} T_qS^h(X)) = \pi_L(L) = \emptyset.$$

Hence,

$$\bigcap_{q\in S^h(Y),\ q \text{ is general}} T_qS^h(Y) = \emptyset$$

and, thus, $S^h(Y)$ is not a cone. More, since $L \subset T_qS^h(X)$ for a general point $q \in S^h(X)$ and $\pi_L(T_qS^h(X)) = T_{\pi_L(q)}S^h(Y)$, one has $\dim T_qS^h(X) = \dim T_{\pi_L(q)}S^h(Y) + \dim L + 1$. So, $\dim L = \dim S^h(X) - \dim S^h(Y) - 1 = ((h+1)\cdot \dim X + h - d_h(X)) - ((h+1)\cdot \dim Y + h - d_h(Y)) - 1 = (h+1)(\dim X - \dim Y) + d_h(Y) - d_h(X) - 1$.

Since $\langle M_x, M_{x'}\rangle = \langle M_x, T_{x'}X\rangle$ and $T_{x'}X \subset M_{x'}$, one has that $M_{x'} = \langle T_{x'}X, M_x \cap M_{x'}\rangle = \langle T_{x'}X, L\rangle$. So, if $y = \pi_L(x)$ and $y' = \pi_L(x')$, then $T_yY \cap T_{y'}Y = \pi_L(T_xX) \cap \pi_L(T_{x'}X) = \pi_L(\langle L, T_xX\rangle \cap \langle L, T_{x'}X\rangle) = \pi_L(M_x \cap M_{x'}) = \pi_L(L) = \emptyset$. Since the points $x, x' \in X$ are general, we have $d_1(Y) = 0$. On the other hand, $\pi_{L_x} \circ \pi_{T_xX} = \pi_{M_x} = \pi_{\langle T_xX, L\rangle} = \pi_{\pi_L(T_xX)} \circ \pi_L = \pi_{T_yY} \circ \pi_L$ and thus $Y_x = \pi_{L_x}(\pi_{T_xX}(X)) = \pi_{T_yY}(\pi_L(X)) = \pi_{T_yY}(Y)$. Since $d_1(Y) = 0$, by Lemma 1, we have that $\dim Y = \dim Y_x = \dim Y'$ and $d_h(Y) = d_{h-1}(Y_x) = 0$. Hence, $\dim L = (h+1)(\dim X - \dim Y) + d_h(Y) - d_h(X) - 1 = (h+1)(\dim X - \dim Y) - d_h(X) - 1$. Also since $S^{h-1}(Y_x)$ is not a cone, we obtain that $S^{h-1}(\pi_{T_yY}(Y))$ is not a cone. □

Proposition 4. *Suppose that X is non-degenerate irreducible h-defective variety, $h \geq 2$, $q \in S^{h-2}(X)$ is a general point and $\pi_{T_qS^{h-2}(X)}$ is the projection from $T_qS^{h-2}(X)$. If $\pi_{T_qS^{h-2}(X)}(X) \subset Cone_{L_q}(Y_q)$, where* $\dim L_q = 2(\dim X - \dim Y_q) - d_h(X) - 1$ *and Y_q is such that $\nu_1(Y_q) = 0$, then $X \subset Cone_L(Y)$, where* $\dim L = (h+1)(\dim X - \dim Y) - d_h(X) - 1$, $\dim Y = \dim Y_q$ *and $\nu_h(Y) = 0$, in particular, $d_h(Y) = 0$.*

Proof. Let us use Proposition 3. To apply it we need to show that $d_1(Y_q) = 0$ and $\pi_{T_yY_q}(Y_q)$ is not a cone for a general point $y \in Y_q$. Really, $0 = \nu_1(Y_q) \geq d_1(Y_q) \geq 0$ and, thus, $d_1(Y_q) = 0$. If $\pi_{T_yY_q}(Y_q)$ is a cone with the vertex R_y for a general point $y \in Y_q$, then for a general point $y' \in Y_q$ and a general point y'' of the fiber $F_{y,y'} = \overline{(\pi^{-1}_{T_yY_q}(\langle R_y, \pi_{T_yY_q}(y')\rangle \setminus R_y) \cap Y_q) \setminus T_yY_q}$ one has $T_{\pi_{T_yY_q}(y'')}\pi_{T_yY_q}(Y_q) = T_{\pi_{T_yY_q}(y')}\pi_{T_yY_q}(Y_q)$ and, thus, $\pi_{T_yY_q}(T_{y'}Y_q) = \pi_{T_yY_q}(T_{y''}Y_q)$. Hence, $T_{y''}Y_q \subset \langle T_yY_q, T_{y'}Y_q\rangle$ and, thus, for a general point $q' \in \langle y, y'\rangle$ one has $V_{Y_q,1,q'} \supset F_{y,y'}$ and $\nu_1(Y_q) = \dim V_{Y_q,1,q'} \geq \dim F_{y,y'} > 0$, which is not the case.

To complete the proof we should slightly modify the proof of Proposition 3. During that proof we had that $\pi_{T_yY}(Y) = Y_x$ in the corresponding notation ($x \in X$ is a general point, $y = \pi_L(x) \in Y$, Y_x is such that $\pi_{T_xX}(X) \subset Cone_{L_x}(Y_x)$). So, by induction on h we have that $\nu_{h-1}(Y_x) = 0$. Since $d_1(Y) = 0$, by Lemma 1, $\nu_h(Y) = \nu_{h-1}(Y_x) = 0$. □

Corollary 2. *Suppose that $h \geq 2$ and $\pi(X) \subset Cone_{L'}(Y')$, where Y' is a curve and $\dim L' = 2\dim X - d_h(X) - 3$. Then $X \subset Cone_L(Y)$, where Y is a curve and $\dim L = (h+1)\dim X - d_h(X) - h - 2$.*

Proof. Since X is h-defective variety, by Corollary 1, the variety $\pi(X)$ is 1-defective, $\dim \pi(X) = \dim X$ and $d_1(\pi(X)) = d_h(X)$. So, $\dim\langle\pi(X)\rangle > 2\dim \pi(X) + 1 - d_1(\pi(X)) = 2\dim X + 1 - d_h(X)$. Hence, $\dim\langle Y'\rangle = \dim\langle\pi(X)\rangle - \dim L' - 1 > (2\dim X + 1 - d_h(X)) - (2\dim X - d_h(X) - 3) - 1 = 3$. Since for a general point $q' \in S(Y')$ one has $\dim T_{q'}S(Y') \leq 3$, we have that $T_{q'}S(Y') \neq \langle Y'\rangle$ and, thus, $V_{Y',1,q'}$ is just a finite number of points. Hence, $\nu_1(Y') = 0$. So, we can apply Proposition 4, which finishes the proof. □

5. $\nu_h(X) = 1$ and $\pi(X)$ is not a cone

Further we will consider the cases when $\pi(X)$ is $v_2(\mathbb{P}^3)$ or its projection or a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. For both these cases for a general point $z' \in S(\pi(X))$ the variety $V_{\pi(X),1,z'} = \Sigma_{z'}(\pi(X))$ is an irreducible conic.

Lemma 5. *Suppose that X is an h-defective threefold and for a general point $z' \in S(\pi(X))$ the variety $V_{\pi(X),1,z'} = \Sigma_{z'}(\pi(X))$ is an irreducible conic. Then*

(1) *for a general point $z \in S^h(X)$ the curve $\Sigma_z(X)$ is a rational normal curve of degree $2h$;*
(2) *If $w \in S^k(X)$, $k \leq h-2$, then the projection from $T_wS^k(X)$ restricted to X is a birational isomorphism.*

Proof. By Lemma 1 and Corollary 1, we have that $\nu_h(X) = \nu_1(\pi(X)) = d_h(X) = d_1(\pi(X)) = 1$ and $\pi(V_{X,h,z}) = V_{\pi(X),1,\pi(z)}$ for a general point $z \in S^h(X)$ such that $q \in \Sigma_{X,h,z,h-2}$. Since $d_h(X) = \nu_h(X) = 1$, we obtain that $\Sigma_z(X) = V_{X,h,z}$ is a curve and $\pi(\Sigma_z(X)) = \Sigma_{\pi(z)}(\pi(X))$ is an irreducible conic. Since $q \in \Sigma_{X,h,z,h-2}$, $q \in \langle x_0, \dots, x_{h-2}\rangle$ for some (general) points of $\Sigma_z(X)$. More, by Terracini lemma, π is the projection from $\langle T_{x_0}X, \dots, T_{x_{h-2}}X\rangle$. Thus, it is sufficient to show that if $h \geq 2$, for a general point $x \in X$ and a general point $z \in S^h(X)$ such that $x \in \Sigma_z(X)$ the curve $\pi_{T_xX}(\Sigma_z(X)) = \Sigma_{\pi_{T_xX}(z)}(\pi_{T_xX}(X))$ is a rational normal curve of degree $2h-2$, then $\Sigma_z(X)$ is a rational normal curve of degree $2h$.

Since $\Sigma_{\pi(z)}(\pi(X))$ is an irreducible conic, for a general point $x' \in \Sigma_{\pi(z)}(\pi(X))$ holds $T_{x'}\pi(X) \cap \langle\Sigma_{\pi(z)}(\pi(X))\rangle = T_{x'}\Sigma_{\pi(z)}(\pi(X))$. Thus, for a general point $x \in \Sigma_z(X)$ holds $T_xX \cap \langle\Sigma_z(X)\rangle = T_x\Sigma_z(X)$. Therefore, the projection π_{T_xX} restricted to $\langle\Sigma_z(X)\rangle$, is the projection from $T_x\Sigma_z(X)$.

Let us show that the projection from T_xC for a general point $x \in C$ is a birational isomorphism. Assume the opposite. Then for a general point $y \in C$ the plane $\langle T_xC, y\rangle$ contains other point $y' \in C$. If we take the projection from T_yC, then $\pi_{T_yC}(y') \in T_{\pi_{T_yC}(x)}\pi_{T_yC}(C)$. Since $\pi_{T_yC}(C)$ is a rational normal curve of degree $2h-2$, one has $\pi_{T_yC}(y') = \pi_{T_yC}(x)$. Thus, $y' \in \langle T_yC, x\rangle$. Since $T_yC \cap T_xC = \emptyset$, $y' \in \langle y, x\rangle$, i. e. general secant line of C is a three-secant. Hence, the same is true for $\pi_{T_xX}(C)$, which is a rational normal curve of degree $2h-2$. Thus, it is impossible, and the plane $\langle T_xC, y\rangle$ meets C only by x and y. Therefore, the projection from T_xC for a general point $x \in C$ is a birational isomorphism. If H' is a general hyperplane containing T_xC, then $H' \cap C$ consists of $2h-2$ points, which are the preimages of the points $\pi_{T_xC}(H') \cap \pi_{T_xC}(C)$, and the point x with the multiplicity 2. Thus, $\deg C = \#(H' \cap C) = 2h - 2 + 2 = 2h$.

Further, if $y \in X$ is a general point and $y' \in \langle T_xX, y\rangle$ is another point of $X \setminus T_xX$, then $T_{y'}X \subset \langle T_xX, T_yX\rangle$. Hence, if $z \in S^h(X)$ is a general point such that $x, y \in \Sigma_z(X)$, then $x, y, y' \in V_{X,h,z}$. Since π_{T_xX} is a birational isomorphism restricted to $V_{X,h,z}$, one has $y' = y$, and π_{T_xX} is a birational isomorphism on X, if $h \geq 2$. If $w \in S^k(X)$, $k \leq h-2$, then the projection from $T_wS^k(X)$, by Terracini lemma, is a composition of $k+1$ projections of type π_{T_xX}. Thus, the projection from $T_wS^k(X)$ is a birational isomorphism on X. □

5.1. $\nu_h(X) = 1$, **$\pi(X)$ is the Veronese variety $v_2(\mathbb{P}^3)$ or one of its projections.** The possible cases in this situation are the following:

(1) $\pi(X) = v_2(\mathbb{P}^3)$;

(2) $\pi(X) = \pi_x(v_2(\mathbb{P}^3))$, where x is a point;
(3) $\pi(X) = \pi_L(v_2(\mathbb{P}^3))$, where a line L intersects $v_2(\mathbb{P}^3)$ with multiplicity at most 1;
(4) $\pi(X) = \pi_L(v_2(\mathbb{P}^3))$, where a line L intersects $v_2(\mathbb{P}^3)$ in two different points;
(5) $\pi(X) = \pi_L(v_2(\mathbb{P}^3))$, where a line L intersects $v_2(\mathbb{P}^3)$ with multiplicity 2 at one point.

The cases 1-3 are covered by Lemma 7. For the case 4 see Lemma 8 and further the section for $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$. In the case 5 one has $\pi_L(v_2(\mathbb{P}^3)) \subset Cone_l(v_2(\mathbb{P}^2))$, where $l = \pi_L(T_x v_2(\mathbb{P}^3))$ is a line and $v_2(\mathbb{P}^2) = \pi_{T_x v_2(\mathbb{P}^3)} v_2(\mathbb{P}^3)$. Hence, in this case $\nu_1(X) \neq 1$ and we do not consider it here.

Lemma 6. *Suppose that* $\dim X \geq 2$*, for a general point* $x \in X$ *and the projection* π' *from* $T_x^k X$*, the normalization of* $\pi'(X)$ *is a birational projection of a linearly normal variety* Y_x *from its point* y_x*. Let* D_x *be the image on* $\pi'(X)$ *of the exceptional divisor of the projection* $\pi_{y_x} : Y_x \dashrightarrow \pi'(X)$*. If* $\dim T_x^k X$ *is the maximal possible and the preimage* $D = \pi'^{-1}(D_x)$ *does not depend on* x*, then there exist a linearly normal variety* X' *and its point* x' *such that* $\pi_{x'}(X')$ *is a normalization of* X *and for a general point* $y \in X'$ *the variety* $\pi_{T_y^k X'}(X')$ *is linearly isomorphic to* $Y_{\pi_{x'}(y)}$.

Proof. We will denote the linear system of hyperplane sections of a variety Z by $\mathcal{H}(Z)$.

Since D_x is the exceptional divisor of the projection π_{y_x}, the pullback of $\mathcal{H}(Y_x)$ to $\pi'(X)$ is a sub-system of the system $|\mathcal{H}(\pi'(X)) + mD_x|$, where m is an appropriate multiplicity. Since Y_x is linearly normal, these systems are equal. By definition of an osculating space of order k, the pullback of the system $\mathcal{H}(\pi'(X))$ to X is $\mathcal{H}(X)(-(k+1)x)$. Since D does not depend on x and, thus, does not contain x, the pullback of the system $\mathcal{H}(Y_x)$ to X is equal to $|\mathcal{H}(X) + mD|(-(k+1)x)$.

Consider the system $|\mathcal{H}(X) + mD|$ and the map $\varphi : X \dashrightarrow X'$, given by this system. Consider the point $y = \varphi(x)$, which is general for X'. Then the projection $\pi_{T_y^k X'}$ is given by the linear system $\mathcal{H}(X')(-(k+1)y)$ and, thus, the composition $\pi_{T_y^k X'} \circ \varphi$ is given by $|\mathcal{H}(X) + mD|(-(k+1)x)$. Hence, $\pi_{T_y^k X'}(X')$ is linearly isomorphic to Y_x and $\dim\langle X' \rangle = \dim\langle Y_x \rangle + \dim T_y^k X' + 1$. Further, since the linear system $|\mathcal{H}(X)|$ is complete, the system $|\mathcal{H}(X)|(-(k+1)x)$ is also complete and, thus, if X'' is a normalization of X and $x'' \in X''$ is a point, corresponding to x, then $\pi_{T_{x''}^k X''}(X'')$ is a normalization of $\pi'(X)$. Therefore, $\dim\langle Y_x \rangle = \dim\langle \pi_{T_{x''}^k X''}(X'') \rangle + 1 = \dim\langle X'' \rangle - \dim T_{x''}^k X''$, we obtain $\dim\langle X'\rangle = \dim\langle X''\rangle + (\dim T^k_y X' - \dim T^k_{x''} X'') + 1$. Since $|\mathcal{H}(X) + mD| = ||\mathcal{H}(X)| + mD|$, one can construct a linear projection $\pi'' : X' \dashrightarrow X''$. Thus, $\dim T^k_y X' \geq \dim T^k_{x''} X'' \geq \dim T^k_x X$. If the dimension of $T^k_x X$ is the maximal possible for an osculating space of order k to a variety of dimension $\dim X$, then $\dim T^k_y X' = \dim T^k_{x''} X''$ and $\dim\langle X'\rangle = \dim\langle X''\rangle + 1$, i. e. X'' is a projection of X' from its point. □

Proposition 5. *Suppose that* $\dim X = n \geq 2$*, for a general point* $x \in X$ *the dimension of* $T^k_x X$ *is maximal possible, and for the projection* π' *from* $T^k_x X$ *the variety* $\pi'(X)$ *is a birational projection of* $v_{k+1}(Y')$ *from* L'*, where* $Y' \subset \mathbb{P}^m$ *is a variety of minimal degree,* $m \geq n$*,* $L' \cap v_k(Y') = \{y_1, \ldots, y_l\}$ *is a finite number of points and* $\forall\ 1 \leq i \leq l$ *holds* $T_{y_i} v_{k+1}(Y') \cap L' = \{y_i\}$*. If the set of divisors* $\{D_1, \ldots, D_l\}$ *on* X*, which are the preimages under* π' *of the exceptional divisors corresponding to points* $y_0, \ldots, y_l$*, does not depend on* x*, then* X *is a birational projection of* $v_{k+1}(Y)$ *from* L*, where* $Y \subset \mathbb{P}^{m+1}$ *is a variety of minimal degree,* $L \cap v_{k+1}(Y)$ *is a finite number of points and* $\forall\ y \in L \cap v_{k+1}(Y)$ *holds* $T_y v_{k+1}(Y) \cap L = \{y\}$*.*

Proof. Let use induction over l.

The base, $l = 0$. Thus, we need to show that for a general point $x \in X$ and the projection π' from $T^k_x X$ the normalization of $\pi'(X)$ is $v_{k+1}(Y')$, where $Y' \subset \mathbb{P}^m$ is a variety of minimal degree, then the normalization of X is $v_{k+1}(Y)$, where $Y \subset \mathbb{P}^{m+1}$ is a variety of minimal degree. By the hypothesis and the definition of an osculating space of order k, the pullback of $\mathcal{H}(\pi'(X))$ to X equals to $\mathcal{H}(X)(-(k+1)x)$. Hence, the pullback of $|\mathcal{H}(\pi'(X))|$ equals to $|\mathcal{H}(X)|(-(k+1)x)$. Thus, we can suppose that X and $\pi'(X)$ are linearly normal, i. e. $\pi'(X) = v_{k+1}(Y')$. Denote by $\mathcal{L}_x$ the linear system corresponding to $v^{-1}_{k+1} \circ \pi'$. One has $\mathcal{H}(X)(-(k+1)x) \subset |(k+1)\mathcal{L}_x|$.

If $y' \in Y'$ is a general point, then $\pi_{T^k_{v_{k+1}(y')} v_{k+1}(Y')} \circ v_{k+1} = v_{k+1} \circ \pi_{y'}$, because both maps age given by the system $H^0(\mathcal{O}(k+1), Y')(-(k+1)y')$. Therefore, if we take general points $x_1, \ldots, x_{m-n} \in X$, then the projection π'' from $\langle T^k_x X, \ldots, T^k_{x_{m-n+1}} X\rangle$ is a composition of the projections from $T^k_x X$ and $\langle T^k_{\pi'(x_1)} \pi'(X), \ldots, T^k_{\pi'(x_{m-n+1})} \pi'(X)\rangle$. By our hypothesis, $\pi'(X) = v_{k+1}(Y')$. Thus, $\pi''(X) = v_{k+1}(\pi_{\langle v^{-1}_{k+1}(\pi'(x_1)), \ldots, v^{-1}_{k+1}(\pi'(x_{m-n+1}))\rangle}(Y'))$. Since $Y' \subset \mathbb{P}^m$ is a variety of minimal degree, $\pi''(X) = v_{k+1}(\mathbb{P}^{n-1})$ and $\mathcal{L}_x(-x_1 - \ldots - x_{m-n+1})$ is the pullback of the system of $v_{k+1}(\mathbb{P}^{n-2})$, $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$. If we exchange the roles of x and x_1, the projection π'' will not change. Thus, $\mathcal{L}_{x_1}(-x - x_2 - \ldots - x_{m-n+1}) = \mathcal{L}_x(-x_1 - \ldots - x_{m-n+1})$. Hence,

$\mathcal{L}_{x_1}(-x) = \mathcal{L}_x(-x_1)$. Therefore, there exists a linear system $\mathcal{L}$ on X such that for a general point $x \in X$ holds $\mathcal{L}_x = \mathcal{L}(-x)$, $\dim \mathcal{L} = m+1$. Consider the map $\varphi : X \dashrightarrow \mathbb{P}^{m+1}$ given by $\mathcal{L}$. Put $Y = \varphi(X)$. By definition, $\pi_{\varphi(x)} \circ \varphi = v_{k+1}^{-1} \circ \pi'$. Therefore, for a general point $y \in Y$ the projection of Y from y is a variety of minimal degree, and, thus, Y is a variety of minimal degree and $\mathcal{L}$ is complete. Since $\mathcal{H}(X)(-(k+1)x) \subset |(k+1)\mathcal{L}_x|$ and the system $\mathcal{H}(X)$ is complete, we have $\mathcal{H}(X) = |(k+1)\mathcal{L}|$ and $X = v_{k+1}(Y)$.

The step is given by Lemma 6. □

Lemma 7. *Suppose that $\pi(X) = \pi_{L'} v_2(\mathbb{P}^3)$, where L' is a linear subspace intersecting $v_2(\mathbb{P}^3)$ with multiplicity at most 1, $\dim L' \leq 1$. Then $X = \pi_L(v_2(Y))$, where $Y \subset \mathbb{P}^{h+2}$ is a threefold of minimal degree, $\dim L = \dim L'$ and the multiplicity of intersection L with $v_2(Y)$ equals to the one of L' and $v_2(\mathbb{P}^3)$.*

Proof. Let us use induction over h. For the base, $h = 1$, there is nothing to prove. The step is given by Proposition 5. To apply it, we need to show that if for a general point $x \in X$ holds $\pi_{T_xX}(X) = \pi_{L_x}(v_2(Y_x))$, and L_x intersects $v_2(Y_x)$ with multiplicity 1, then the preimage D of the corresponding exceptional divisor D_x under $\pi_{T_xX}^{-1}$ does not depend on x. Consider general points $x_0, ldots, x_{h-2} \in X$. Then the projection π' from $\langle T_xX, T_{x_0}X, \dots, T_{x_{h-2}}X\rangle$ is a composition of the projections from T_xX and from $\langle T_{\pi_{T_xX}(x_0)}\pi_{T_xX}(X), \dots, T_{\pi_{T_xX}(x_{h-2})}\pi_{T_xX}(X)\rangle$. Since $\pi_{T_xX}(X) = \pi_{L_x}(v_2(Y_x))$, the last projection restricted on $\pi_{T_xX}(X)$ takes it to $\pi_{L'_x}(Y'_x)$, where $Y'_x = \pi_{\langle y_0, \dots, y_{h-2}\rangle}(Y_x)$ and $y_i = v_2^{-1}(\pi_{T_xX}(x_i))$. Since the points $y_0, \dots, y_{h-2} \in Y_x$ are general and $Y \subset \mathbb{P}^{h+1}$ is a threefold of minimal degree, $Y'_x = \mathbb{P}^2$. Since L_x intersects $v_2(Y_x)$ with multiplicity 1, L'_x intersects $v_2(Y'_x)$ with miltiplicity 1. Thus, the exceptional divisor of the projection from L'_x could be uniquely determined as a minimal line on a rational normal scroll of type $(1,2)$ or on its projection to $\mathbb{P}^3$. Thus, it preimage D will not change if we will exchange the roles of the points x and x_0. Hence, D does not depend on x. □

Lemma 8. *If H is tangent to $\mathbb{P}^2 \times \mathbb{P}^2$ at the point x_H, then $\mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is linearly isomorphic to $\pi_l(v_2(\mathbb{P}^3))$, where l is a line meeting $v_2(\mathbb{P}^3)$ at two different points.*

Proof. Let l_1 and l_2 be two lines in the multiplied planes of $\mathbb{P}^2 \times \mathbb{P}^2$. Take the subvariety $l_1 \times l_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ and consider the projection $\varphi : \mathbb{P}^8 \dashrightarrow \mathbb{P}^4$ from $\langle l_1 \times l_2\rangle$. Then the restriction of φ on $\mathbb{P}^2 \times \mathbb{P}^2$ is a birational isomorphism. The images $\varphi(l_1 \times \mathbb{P}^2)$ and $\varphi(\mathbb{P}^2 \times l_2)$ are lines and the map $\varphi^{-1} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ is generated by the full linear system of quadrics containing these lines.

So, for any hyperplane $H \subset \mathbb{P}^8$ containing $l_1 \times l_2$ one has that $H \cap \mathbb{P}^2 \times \mathbb{P}^2$ is am image of the map φ^{-1} restricted on $\varphi(H)$, i. e. the map generated by the full linear system of quadrics on $\mathbb{P}^3 = \varphi(H)$ containing $\varphi(H) \cap \varphi(l_1 \times \mathbb{P}^2)$ and $\varphi(H) \cap \varphi(\mathbb{P}^2 \times l_2)$. Since the hyperplane H does not contain any of subvarieties $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^2$, varieties $\varphi(H) \cap \varphi(l_1 \times \mathbb{P}^2)$ and $\varphi(H) \cap \varphi(\mathbb{P}^2 \times l_2)$ are points. Hence, $H \cap \mathbb{P}^2 \times \mathbb{P}^2$ is a projection of $v_2(\mathbb{P}^3)$ from two its different points.

On the other hand, if H is tangent to $\mathbb{P}^2 \times \mathbb{P}^2$, then H contains a plane $\mathbb{P}^2 \times \{x\}$. Thus, for any line $l_2 \in \mathbb{P}^2$, containing x, the intersection $H \cap \mathbb{P}^2 \times l_2$ is reducible and consists of the components $\mathbb{P}^2 \times \{x\}$ and $l_1 \times l_2$ for certain line l_1. Applying the construction described above we obtain that $H \cap \mathbb{P}^2 \times \mathbb{P}^2$ is a projection of $v_2(\mathbb{P}^3)$ from two its different points. □

6. $\nu_h(X) = 1$, $\pi(X)$ is a hyperplane section of the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$.

6.1. **General properties.** Let $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$. First, note that there are three different types of disposition of H and $\mathbb{P}^2 \times \mathbb{P}^2$:

(1) H does not contain any plane from the families $\{*\} \times \mathbb{P}^2$ and $\mathbb{P}^2 \times \{*\}$ or, in other words, H is not tangent to $\mathbb{P}^2 \times \mathbb{P}^2$;
(2) H contains a unique plane from one of the families $\{*\} \times \mathbb{P}^2$ or $\mathbb{P}^2 \times \{*\}$. Then H contains a unique plane from another family also and thus H is tangent to $\mathbb{P}^2 \times \mathbb{P}^2$ at unique point x_H;
(3) H contains one-dimensional sub-family of planes from one of the families $\{*\} \times \mathbb{P}^2$ or $\mathbb{P}^2 \times \{*\}$. Then this sub-family is a family of planes of type $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}^2 \times \mathbb{P}^1$. More, in this case H contains also one-dimensional sub-family of planes from another family, and, finally, $H = \langle l_1 \times \mathbb{P}^2, \mathbb{P}^2 \times l_2 \rangle$, where l_1 and l_2 are certain lines.

In the third case the section $H \cap \mathbb{P}^2 \times \mathbb{P}^2 = l_1 \times \mathbb{P}^2 \cup \mathbb{P}^2 \times l_2$ is reducible and both irreducible components are degenerate (actually, the dimension of the linear span of each equals to 5). Thus we do not consider this case.

Remark 6. Actually, all threefolds from the first item are linearly isomorphic to each other. The same is true for the second item also.

The following lemma will be used often in this section as motivation for induction over the order of defectivity.

Lemma 9. *If $h \geq 2$ and $x \in X$ is a general point, then for a general point $q'' \in S^{h-3}(X)$ the threefolds $\pi_{T_{q''}S^{h-3}(\pi_{T_xX}(X))}\pi_{T_xX}(X)$ and $\pi(X)$ are linearly isomorphic.*

Proof. By Lemma 1, the threefold $\pi_{T_xX}(X)$ is $(h-1)$-defective and for a general point $q' \in S^{h-2}(X)$ such that $x \in \Sigma_{q'}$, one has that $q'' = \pi_{T_xX}(q') \in S^{h-3}(\pi_{T_xX}(X))$ is a general point, $T_{q''}S^{h-3}(\pi_{T_xX}(X)) = \pi_{T_xX}(T_{q'}S^{h-2}(X))$. Hence, the projection from $T_{q'}S^{h-2}(X)$ is a composition of the projections from T_xX and from $T_{q''}S^{h-3}(\pi_{T_xX}(X))$. Thus, $\pi_{T_{q''}S^{h-3}(\pi_{T_xX}(X))}(\pi_{T_xX}(X)) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H'$. □

Our target is to find family of lines covering X. Suppose that X is covered by an irreducible family of lines and $d_h(X) > 0$. Take a general point $z \in S^h(X)$ and consider $\Sigma_z(X)$. By Terracini lemma, for a general point $x \in \Sigma_z(X)$ one has $T_xX \subset T_zS^h(X)$. So, if l is a line of the family passing through x, then $l \subset T_xX \subset T_zS^h(X)$. Let M_z be (the closure of) a subvariety of X swept out by all such lines, passing through general points of $\Sigma_z(X)$. Then $M_z \subset T_zS^h(X) \cap X$.

So, we will study surfaces lying in $T_zS^h(X) \cap X$ for a general point $z \in S^h(X)$.

Lemma 10. *If $z \in S(\pi(X))$ is a general point, then $T_zS(\pi(X)) \cap \pi(X)$ is reducible and consists of two copies of $Scroll_{1,2}$, which intersect each other by the conic $\Sigma_z(\pi(X))$.*

The family of such scrolls consists of two linear systems of dimension 2 each. Scrolls from different linear systems are not linearly equavialent.

Proof. First, notice that $z \in S(\mathbb{P}^2 \times \mathbb{P}^2)$ is a general point. Hence, its entry locus $\Sigma_z(\mathbb{P}^2 \times \mathbb{P}^2)$ is $l_1 \times l_2$ for certain lines l_1 and l_2. Since for a general point $(x_1, x_2) \in l_1 \times l_2$ one has $T_{(x_1,x_2)}\mathbb{P}^2 \times \mathbb{P}^2 = \{x_1\} \times \mathbb{P}^2 \cup \mathbb{P}^2 \times \{x_2\}$ and by Terracini lemma $T_{(x_1,x_2)}\mathbb{P}^2 \times \mathbb{P}^2 \subset T_zS(\mathbb{P}^2 \times \mathbb{P}^2)$, we obtain that $\{x_1\} \times \mathbb{P}^2 \cup \mathbb{P}^2 \times \{x_2\} \subset T_zS(\mathbb{P}^2 \times \mathbb{P}^2)$. Varying $x_1 \in l_1$ and $x_2 \in l_2$ we get that $l_1 \times \mathbb{P}^2 \cup \mathbb{P}^2 \times l_2 \subset T_zS(\mathbb{P}^2 \times \mathbb{P}^2)$. Finally, since $\deg \mathbb{P}^2 \times \mathbb{P}^2 = 6$ and $\deg l_1 \times \mathbb{P}^2 = \deg \mathbb{P}^2 \times l_2 = 3$, one has $l_1 \times \mathbb{P}^2 \cup \mathbb{P}^2 \times l_2 = T_zS(\mathbb{P}^2 \times \mathbb{P}^2)$. More, $l_1 \times \mathbb{P}^2 \cap \mathbb{P}^2 \times l_2 = l_1 \times l_2 = \Sigma_z(\mathbb{P}^2 \times \mathbb{P}^2)$.

Since $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$, one has that $\Sigma_z(\pi(X)) = \Sigma_z(\mathbb{P}^2 \times \mathbb{P}^2) \cap H$ is a conic and $S(\pi(X)) = S(\mathbb{P}^2 \times \mathbb{P}^2) \cap H$. So, $T_zS(\pi(X)) = T_zS(\mathbb{P}^2 \times \mathbb{P}^2) \cap H$ and $\pi(X) \cap T_zS(\pi(X)) = X \cap T_zS(\mathbb{P}^2 \times \mathbb{P}^2) \cap H = (l_1 \times \mathbb{P}^2 \cup \mathbb{P}^2 \times l_2) \cap H = (l_1 \times \mathbb{P}^2 \cap H) \cup (\mathbb{P}^2 \times l_2 \cap H)$. But the only hyperplane sections of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$ are $Scroll_{1,2}$ and $\mathbb{P}^1 \times l \cup \{x\} \times \mathbb{P}^2$ for certain line l and point x. Since the point z is general and H does not contain one-dimensional family of planes from $\mathbb{P}^2 \times \mathbb{P}^2$, the only possibility for the intersection is $Scroll_{1,2}$. More, the intersection of two obtained scrolls is $(l_1 \times \mathbb{P}^2 \cap H) \cap (\mathbb{P}^2 \times l_2 \cap H) = (l_1 \times \mathbb{P}^2 \cap \mathbb{P}^2 \times l_2) \cap H = l_1 \times l_2 \cap H = \Sigma_z(\mathbb{P}^2 \times \mathbb{P}^2) \cap H = \Sigma_z(\pi(X))$.

There are two natural families of described scrolls: the scrolls of type $l_1 \times \mathbb{P}^2 \cap H$ and the scrolls of type $\mathbb{P}^2 \times l_2 \cap H$. Since the system of divisors of type $l_1 \times \mathbb{P}^2$ on $\mathbb{P}^2 \times \mathbb{P}^2$ gives the natural projection $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$, it is a linear system. Thus, the system of scrolls of type $l_1 \times \mathbb{P}^2 \cap H$ is also linear. Two scrolls from different systems are not linearly equivalent, because for a general line of type $\{*\} \times \mathbb{P}^2 \cap H$ the intersection with a general divisor of the first family is empty, but the intersection with a general divisor of the second family is a point. □

Lemma 11. *For a general point $z \in S^h(X)$ and a general point $q \in \Sigma_{X,h,z,h-2}$ the intersection $T_zS^h(X) \cap X$ consists of two surfaces S_1 and S_2, which are mapped to $Scroll_{1,2}$ under the projection from $T_qS^{h-2}(X)$; $S_1 \cap S_2 = \Sigma_z(X) \cup C_q$, where $C_q \subset T_qS^{h-2}(X)$.*

Proof. Take a general point $z \in S^h(X)$ and consider the intersection $T_zS^h(X) \cap X$. Since X is a non-degenerate threefold in $\mathbb{P}^{4h+3}$ and $\dim T_zS^h(X) = \dim S^h(X) = \dim X \cdot (h+1) + h - d_h(X) = 4h+2$, the variety $T_zS^h(X) \cap X$ is a (reducible) surface. Suppose also that $q \in \Sigma_{X,h,z,h-2}$. Then by Terracini lemma $T_qS^{h-2}(X) \subset T_zS^h(X)$ and, thus, $\pi(X \cap T_zS^h(X)) = \pi(X) \cap \pi(T_zS^h(X))$. By Lemma 1, $\pi(T_zS^h(X)) = T_{\pi(z)}S(\pi(X))$. So, by Lemma 10, we have that $\pi(X \cap T_zS^h(X))$ consists of two copies of $Scroll_{1,2}$, which intersect each other by the conic $\Sigma_{\pi(z)}(\pi(X))$. Since the projection π is a birational isomorphism and the point $\pi(z)$ is general in $S(\pi(X))$, $X \cap T_zS^h(X)$ consists of irreducible surfaces $S_{q,1}$ and $S_{q,2}$, which are mapped (birationally) to the scrolls by π, and a (possibly reducible) surface $S_{q,3}$, which loses the dimension under π. Note, that the surface $S_{q,3}$ depends on q and does not depend on z such that $q \in \Sigma_{X,h,z,h-2}$. Hence, $S_{q,3} \subset T_zS^h(X)$ for such general point z. Thus, $\pi(S_{q3}) \subset \bigcap_{z' \in S(\pi(X)),\ z' \text{ is general}} T_{z'}S(\pi(X))$, which is empty. So, $S_{q,3} \subset T_qS^{h-2}(X)$.

By Remark 2, $\dim \Sigma_{X,h,z,h-2} = d_h(X) = 1$, thus we may vary $q \in \Sigma_{X,h,z,h-2}$ continously. At the same time the intersection $T_zS^h(X) \cap X$ contains only finite number of irreducible components. Hence, $S = S_{q,3}$ does not depend on $q \in \Sigma_{X,h,z,h-2}$. Consider the points $x_0, \dots, x_{h-2} \in X$ such that $q \in \langle x_0, \dots, x_{h-2} \rangle$ and general $x_{h-1}, x_h \in X$ such that $z \in \langle q, x_{h-1}, x_h \rangle$. Since q and z are general, $x_0, \dots, x_h$ are also general in X. By Terracini lemma, $S \subset \langle T_{x_0}X, \dots, T_{x_{h-2}}X \rangle$. Changing q we obtain that $S \subset \langle T_{x_1}X, \dots, T_{x_{h-1}}X \rangle$. Since X is non-$(h-1)$-defective, one has $S \subset \langle T_{x_1}X, \dots, T_{x_{h-2}}X \rangle$. Following this procedure, we finally obtain that $S \subset T_{x_i}X$, $i = 0, \dots, h$. So, $S = \emptyset$.

So, there exist exactly two irreducible surfaces $S_1, S_2 \subset T_zS^h(X) \cap X$, which are mapped birationally to the scrolls of type $(1,2)$ by $\pi_{T_qS^{h-2}(X)}$ (or, in our notation, π for q fixed). More, $S_1 \cap S_2$ maps to $\pi(S_1) \cap \pi(S_2)$, which is $\Sigma_{\pi(z)}(\pi(X))$. On the other hand, since in our case under consideration $d_h(X) = \nu_h(X) = d_1(\pi(X)) = \nu_1(\pi(X)) = 1$, we have that $V_{X,h,z} = \Sigma_z(X)$ and $V_{\pi(X),1,\pi(z)} = \Sigma_{\pi(z)}(\pi(X))$. By Lemma 1, one has $\pi(V_{X,h,z}) = V_{\pi(X),1,\pi(z)}$. So, $\pi(\Sigma_z(X)) = \Sigma_{\pi(z)}(\pi(X)) = \pi(S_1 \cap S_2)$. Since z is general for q, all the points of $\Sigma_{\pi(z)}(\pi(X))$ are general for $\pi(X)$. Taking into account that π is a birational isomorphism and $\Sigma_z(X)$ is irreducible, we obtain that $S_1 \cap S_2 = \Sigma_z(X) \cup C_q$, where $C_q \subset T_q S^{h-2}(X)$. $\square$

Lemma 12. (1) *For a general point $z' \in S^h(S_i)$ and a general point $q' \in S^{h-2}(S_i)$ one has $T_{z'}S^h(X) \supset S_i$ and the variety $\pi_{T_q S^{h-2}(S_i)}(S_i)$ is linearly isomorphic to $Scroll_{1,2}$.*

(2) *The family of components of $T_z S^h(X) \cap X$, $z \in S^h(X)$ is general, has the dimension $h+1$ and consists of two linear systems $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$. Through $h+1$ general points of X there pass exactly one surface of every of these systems.*

(3) *The dimension of the linear span of such general surface is $3h+1$.*

(4) *If $S_1 \in \mathcal{L}_1(X)$ and $S_2 \in \mathcal{L}_2(X)$ are general, then $S_1 \cap S_2 = \Sigma_z(X)$ for certain (general) point $z \in S^h(X)$.*

(5) *For a general point $x \in X$ and a general surface $S \in \mathcal{L}_i(X)(-x)$ one has $\pi_{T_x X}(S) \in \mathcal{L}_i(\pi_{T_x X}(X))$, $i = 1$ or 2.*

Proof. First, let us take the point $z \in S^h(X)$ for which the surface S_i was constructed. As we saw, $\Sigma_z \subset S_i$. So, we can take points $x_0, \ldots, x_h \in S_i$ such that $z \in \langle x_0, \ldots, x_h \rangle$. Consider the point $q \in \langle x_0, \ldots, x_{h-2} \rangle$, $q \in \Sigma_{X,h,z,h-2}$ and the projection π from $T_q S^{h-2}(X)$. Then, as we saw, $\pi(S_i)$ is a $Scroll_{1,2}$ and $\pi(z) \in S(\pi(S_i))$. Further, if we take two general points $x'_{h-1}, x'_h \in S_i$ and a general point $z' \in \langle x_0, \ldots, x_{h-2}, x'_{h-1}, x'_h \rangle$, we have, by Lemma 1, that $\pi(T_{z'}S^h(X)) = T_{\pi(z')}S(\pi(X))$ and $\pi(z') \in S(\pi(S_i))$. Since $S(\pi(S_i))$ coincides with $\langle \pi(S_i) \rangle$, one has $T_{\pi(z')}S(\pi(X)) \supset T_{\pi(z')}S(\pi(S_i)) = S(\pi(S_i)) \supset \pi(S_i)$. Since $T_{z'}S^h(X) = \pi^{-1}(T_{\pi(z')}S(\pi(X)))$, we obtain that $S_i \subset T_{z'}S^h(X)$.

So, starting from the points $x_0, \ldots, x_h$, which depend on z and the construction of S_i, after some iterations of substituting some points by general points as above, we obtain that for general points $x_0, \ldots, x_h \in S_i$, $z' \in \langle x_0, \ldots, x_h \rangle$ holds $T_{z'}S^h(X) \supset S_i$. More, for the point $q' \in \langle x_0, \ldots, x_{h-2} \rangle$, $q' \in \Sigma_{X,h,z',h-2}$ the projection $\pi_{T_{q'}S^{h-2}(X)}$ takes S_i to $Scroll_{1,2}$. Since z' is general, q' is also general in $\langle x_0, \ldots, x_{h-2} \rangle$.

Let $\mathcal{F}$ be the family of components of $T_{z'}S^h(X) \cap X$, $z' \in S^h(X)$. For $h+1$ general points $x_0, \ldots, x_h \in X$ and a general point $z \in \langle x_0, \ldots, x_h \rangle$ there exist exactly two components $S_1, S_2 \in \mathcal{F}$ passing through $x_0, \ldots, x_h$, which are the components of $T_z S^h(X) \cap X$. On the other hand, if some general surface S from $\mathcal{F}$ contains the points $x_0, \dots, x_h$, these points are general for S. Thus, S is one of two components of $T_zS^h(X) \cap X$. Hence, $\dim \mathcal{F} = h+1$ and through $h+1$ general points of X there pass exactly two surfaces of this family. More, by Lemma 11, $S_1 \cap S_2 = \Sigma_z(X) \cup C_q$, where $C_q \subset T_qS^{h-2}(X)$, $C_q \not\subset \Sigma_z(X)$ and $q \in \langle x_0, \dots, x_{h-2}\rangle \cap \Sigma_{X,h,z,h-2}$. Hence, doing in the same way as in the proof of Lemma 11, we may obtain that $C_q \subset T_{x_i}X$, $0 \le i \le h$, and, thus, is empty.

Further, take a general point $z \in S^h(X)$ and two surfaces $S_1, S_2 \subset T_zS^h(X) \cap X$. Let $\mathcal{L}_2(X)$ be the system of components $T_{z'}S^h(X) \cap X$ other than S_1, where $z' \in S^h(S_1)$ is a general point. Since $T_{z'}S^h(X)$ is a hyperplane, $\mathcal{L}_2(X)$ is a sub-system of a linear system. On the other hand, if $z', z'' \in S^h(S_1)$ are two general points such that $T_{z'}S^h(X) = T_{z''}S^h(X)$, then $\Sigma_{z'}(X) = V_{X,h,z'} = V_{X,h,z''} = \Sigma_{z''}(X)$. At the same time, for general points $x_0, \dots, x_h \in S_1$ and a general point $z''' \in \langle x_0, \dots, x_h\rangle \subset S^h(S_1)$ one has $\Sigma_{z'''}(X) \ni x_0, \dots, x_h$. So, the dimension of the family of entry loci in S_1 is equal to $h+1$, and, thus, the dimension of different spaces of type $T_{z'}S^h(X)$, $z' \in S^h(X)$ is equal to $h+1$. Hence, $\mathcal{L}_2(X)$ is a linear system. More, if $z', z'' \in S^h(S_1)$ are two general points such that $T_{z'}S^h(X) \cap X = T_{z''}S^h(X) \cap X = S_2'$, then $\Sigma_{z'}(X) = S_1 \cap S_2' = \Sigma_{z''}(X)$. So, the dimension of $\mathcal{L}_2(X)$ is equal to the dimension of the family of entry loci on S_1, which is $h+1$.

The linear system $\mathcal{L}_1(X)$ is defined in the same way based on S_2. One has $S_1 \in \mathcal{L}_1(X)$, $S_2 \in \mathcal{L}_2(X)$. By Lemma 10, S_1 and S_2 are not linearly equivalent. So, $\mathcal{L}_1(X) \cap \mathcal{L}_2(X) = \emptyset$. Since $\dim \mathcal{L}_1(X) = \dim \mathcal{L}_2(X) = h+1$, through $h+1$ general points of X there passes at least one surface from $\mathcal{L}_1(X)$ and one surface from $\mathcal{L}_2(X)$. As we saw, through $h+1$ general points of X there pass exactly two surfaces from $\mathcal{F}$. So, through these points there passes exactly one surface of every of $\mathcal{L}_i$, $i = 1, 2$.

Consider general points $x_0, \dots, x_h \in X$ and $z \in \langle x_0, \dots, x_h\rangle$ and a component S of the intersection $T_zS^h(X) \cap X$. Then the points $x_0, \dots, x_h$ are general for S. In particular, the spaces $T_{x_0}S, \dots, T_{x_{h-1}}S$ are in a general position. Thus, for a general point $q' \in \langle x_0, \dots, x_{h-2}\rangle$ one has $\dim T_{q'}S^{h-2}(S) = \dim\langle T_{x_0}S, \dots, T_{x_{h-2}}S\rangle = (h-1)\cdot \dim S + h - 2 = 3h - 4$. Since $\dim\langle \pi_{T_{q'}S^{h-2}(S)}(S)\rangle = 4$, one has $\dim\langle S\rangle = 4 + (3h-4) + 1 = 3h+1$.

Consider a general point $x \in X$. The family of components of $T_{z'}S^{h-1}(\pi_{T_xX}(X)) \cap \pi_{T_xX}(X)$, $z' \in S^{h-1}(\pi_{T_xX}(X))$, consists of two linear systems $\mathcal{L}_1(\pi_{T_xX}(X))$ and $\mathcal{L}_2(\pi_{T_xX}(X))$. Take a general surface $S \in \mathcal{L}_1(X)$ such that $x \in S$. Then for a general point $z \in S^h(S)$ such that $x \in \Sigma_z(X)$, one has $S \subset T_z S^h(X) \cap X$. Since $\pi_{T_x X}(z)$ is a general point of $S^{h-1}(\pi_{T_x X}(X))$ and $T_{\pi_{T_x X}(z)} S^{h-1}(\pi_{T_x X}(X)) = \pi_{T_x X}(T_z S^h(X))$, we have that $\pi_{T_x X}(S) \subset T_{\pi_{T_x X}(z)} S^{h-1}(\pi_{T_x X}(X)) \cap \pi_{T_x X}(X)$ is one of the components. Hence, we can choose the numbers of the linear systems in order to have $S \in \mathcal{L}_1(\pi_{T_x X}(X))$. □

6.2. **Lines on X.**

6.2.1. *General facts.*

Lemma 13. *Let D be an irreducible surface such that for a general point $x \in D$ the projection $\pi_{T_x D}$ from $T_x D$ of D is a birational isomorphism to its image and the projected surface $\pi_{T_x D}(D)$ satisfies the following condition: for a general point $y \in \pi_{T_x D}(D)$ the tangent space $T_y \pi_{T_x D}(D)$ meets $\pi_{T_x D}(D)$ by a line. Then one of the following conditions holds:*

1. *D contains one-dimensional family of lines and for a general point $x \in D$ one has that $T_x D \cap D$ is a line;*
2. *for two general points $x_0, x_1 \in D$ there exists a twisted cubic in D containing them.*

Suppose in addition that for two general points $x_0, x_1 \in D$ the projection π' from $\langle T_{x_0} D, T_{x_1} D \rangle$ is a birational isomorphism to its image and the projected surface $\pi'(D)$ satisfies the following condition: for a general point $y \in \pi'(D)$ the tangent space $T_y \pi'(D)$ meets $\pi'(D)$ by a line. Then D contains one-dimensional family of lines and for a general point $x \in D$ one has that $T_x D \cap D$ is a line.

Proof. Take two general points $x_0, x_1 \in D$. Let i, j be indeces such that $\{i, j\} = \{0, 1\}$. If $T_{x_i} D \cap D$ is a curve, then this curve is mapped to a line $T_{\pi_{T_{x_j} D}(x_i)} \pi_{T_{x_j} D}(D) \cap \pi_{T_{x_j} D}(D)$ under the projection $\pi_{T_{x_j} D}$. Since $\pi_{T_{x_j} D}$ is a birational isomorphism on D, one has $T_{x_i} D \cap T_{x_j} D = \emptyset$. So, the restriction of $\pi_{T_{x_j} D}$ to $T_{x_i} D$ is isomorphism, and $T_{x_i} D \cap D$ is a line.

Let us suppose now that $T_{x_i} D \cap D$ is a finite set of points, $i = 0, 1$. Since $T_{\pi_{T_{x_i} D}(x_j)} \pi_{T_{x_i} D}(D) \cap \pi_{T_{x_i} D}(D)$ is a line, the variety $D \cap \langle T_{x_0} D, T_{x_1} D \rangle$ is a curve. If this curve is singular at x_i, its tangent space at x_i coincides with $T_{x_i} D$. Since the curve itself is projected to a line under $\pi_{T_{x_j} D}$, its tangent space, which is $T_{x_i} D$, is also projected to a line. The last is not the case because, as we saw, $T_{x_i} D \cap T_{x_j} D = \emptyset$. So, the curve $D \cap \langle T_{x_0} D, T_{x_1} D \rangle$ is smooth at the points x_0, x_1. Hence, it has unique irreducible one-dimensional component C_i, passing through x_i $(i = 0, 1)$. Note, that C_i is smooth at x_i and $\pi_{T_{x_j} D}(C_i)$ is a line, passing through $\pi_{T_{x_j} D}(x_i)$. Now let us consider two cases.

$C_0 \neq C_1$. Consider the variety $\pi_{T_{x_i}D}(C_i)$, which is a subvariety of the line $\pi_{T_{x_i}D}(D \cap \langle T_{x_0}D, T_{x_1}D\rangle)$. Also since $T_{x_i}D \cap D$ is a finite number of points, one has $C_i \not\subset T_{x_i}D$ and, thus, $\pi_{T_{x_i}D}(C_i) \neq \emptyset$. If $\pi_{T_{x_i}D}(C_i)$ is a line, then $\pi_{T_{x_i}D}(x_j) \in \pi_{T_{x_i}D}(C_i)$. Since $\pi_{T_{x_i}D}$ is a birational isomorphism, one has $x_j \in C_i$, which is impossible. If $\pi_{T_{x_i}D}(C_i)$ is a point, then $\dim\langle T_{x_i}D, C_i\rangle = 3$. Since the codimension of $\pi_{T_{x_j}D}(\langle C_i\rangle)$ in $\pi_{T_{x_j}D}(T_{x_i}D)$ equals to 1, we obtain that $\langle C_i\rangle$ has the codimension 1 in $\langle T_{x_i}D, C_i\rangle$, and, thus, C_i is a plane curve. So, for a general point $y \in C_i$ one has $T_yC_i \cap T_{x_i}D \neq \emptyset$. But the subvariety $\{y \in D | T_yD \cap T_{x_i}D \neq \emptyset\}$ is one-dimensional at most, and the curve C_i is a component of it. Hence, the curve C_i does not depend on the choise of x_j, $C_i = C(x_i)$. Since the projection $\pi_{T_{x_j}D}$ restricted to $C(x_i)$ is a birational isomorphism to a line, one has $T_{x_j}D \cap C(x_i) \neq \emptyset$. Since $C(x_i) \subset D$ the point $T_{x_j}D \cap C(x_i) \in T_{x_j}D \cap D$, which is a finite set of points. So, the point $T_{x_j}D \cap C(x_i)$ does not depend on the choise of x_i. Hence, $\bigcap_{x\in D, x \text{ is general}} C(x) \neq \emptyset$. But for a general point $x \in D$ the curve $C(x)$ is irreducible by definition. So, the set $\bigcap_{x\in D, x \text{ is general}} C(x)$ is a finite number of points. Therefore, finally, the point $T_{x_j}D \cap C(x_i)$ does not depend on the choise of x_i and x_j. Hence, in particular, this point is contained in $T_{x_j}D \cap T_{x_i}D$, which is impossible, because $T_{x_j}D \cap T_{x_i}D = \emptyset$.

$C_0 = C_1$. Since $\pi_{T_{x_i}D}(C_0)$ is a line passing through $\pi_{T_{x_i}D}(x_j)$, it lays in the 2-dimensional space $T_{\pi_{T_{x_i}D}(x_j)}\pi_{T_{x_i}D}(D)$. Thus, $\dim\langle\pi_{T_{x_i}D}(C_0)\rangle \cap T_{\pi_{T_{x_i}D}(x_j)}\pi_{T_{x_i}D}(D) = \dim \pi_{T_{x_i}D}(\langle C_0\rangle \cap \pi_{T_{x_i}D}(T_{x_j}D) = 1$. Since $T_{x_i}D \cap T_{x_j}D = \emptyset$, one has that $\dim\langle C_0\rangle \cap T_{x_j}D \leq 1$. Since $\langle C_0\rangle \cap T_{x_j}D \supset T_{x_j}C_0$, we obtain that $\langle C_0\rangle \cap T_{x_j}D = T_{x_j}C_0$. Hence, the projection from $T_{x_i}D$ restricted to $\langle C_0\rangle$ is the projection from $T_{x_i}C_0$, i. e. a birational isomorphism to a line, and $\dim\langle C_0\rangle = 3$. Further, since $\pi_{T_{x_i}D}(C_0)$ is a line, for a general point $y \in C_0$ one has $T_{\pi_{T_{x_i}D}(y)}\pi_{T_{x_i}D}(D) \cap \pi_{T_{x_i}D}(D) = \pi_{T_{x_i}D}(C_0)$ and the irreducible component of the curve $\langle T_{x_i}D, T_yD\rangle \cap D$, passing through y, is again C_0. So, the curve C_0 does not depend on the choise of the point $x_j \in C_0$. Therefore, by the symmetry, it does not depend on the choise of $x_i \in C_0$ and, thus, for a general point $y \in C_0$ the projection from T_yC_0 of C_0 is a birational isomorphism to a line. Finally, since a general point of C_0 could not be a flex point, we have that C_0 is a twisted cubic in certain $\mathbb{P}^3$.

Consider a projection $\pi' = \pi_{\langle T_{x_0}D, T_{x_1}D\rangle}$ and suppose that for a general point $y' \in \pi'(D)$ one has $T_{y'}\pi'(D) \cap \pi'(D)$ is a line. Let us show that it is impossible for D to satisfy the condition that trough two general points there passes a cubic. Assume the converse. Take a general point $y \in D$. Then for the pairs (y, x_0) and (y, x_1) there exist two cubics, passing through the points of the pair. Both these cubics are mapped to lines, passing through $\pi'(y)$ under the projection π'. Since π' is a birational isomorphism, we obtain that $T_{\pi'(y)}\pi'(X) \cap \pi'(X)$ contains at least two different lines, which contradicts the hypothesis. $\square$

Lemma 14. *If $S \subset \mathbb{P}^k$ is irreducible nondegenerate surface, which contains one-dimensional family of lines with a rational base and l-dimensional family ($l \geq 1$) of curves of degree $m > 1$, $\frac{k-1}{2} \leq m < k-1$ and $k = 2m - l + 2$, then S is a rational normal scroll of type (a,b), $a + b = k - 1$, $a, b \leq m$.*

Proof. By Grothendieck theorem, there exists a normal scroll of type (a,b), $a \leq b$, and a linear subspace $L \subset \langle Scroll_{a,b}\rangle$, which does not intersect this scroll, such that $S = \pi_L(Scroll_{a,b})$, where π_L is the projection from L, and π_L is a birational isomorphism on $Scroll_{a,b}$. So, $k = \dim\langle S\rangle = \dim\langle Scroll_{a,b}\rangle - \dim L - 1 = (a + b + 1) - \dim L - 1 \leq (a+b+1) - (-1) - 1 = a+b+1$. More, $Scroll_{a,b}$ has to contain an l-dimensional family of curves of degree m. Since $m < k-1 \leq a+b \leq 2b$, any such curve intersects a general ruling line of our scroll by one point. Therefore, the dimension of full linear system of curves of degree m on a normal scroll of type (a,b) is $2m-(a+b)+1$, if $m \geq b$; otherwise it is 0 or -1. Hence, $l \leq 2m-(a+b)+1$ or $a+b \leq 2m-l+1$. On the other hand, $k = \dim\langle S\rangle = \dim\langle Scroll_{a,b}\rangle - \dim L - 1 = (a+b+1) - \dim L - 1 \leq (a + b + 1) - (-1) - 1 = a + b + 1$. Hence, $k \leq 2m - l + 2$. Since by our hypothesis $k = 2m - l + 2$, one has that $\dim L = -1$, and S is a normal scroll of type (a,b), $a + b = k - 1$ and $a, b \leq m$. $\square$

Corollary 3.
1. *If a general surface of the linear system $\mathcal{L}_i(X)$ (here i is equal to 1 or 2) contains one-dimensional family of lines, then this surface is a rational normal scroll of type (a,b), $a + b = 3h$, $h \leq a, b \leq 2h$.*
2. *Suppose that $h \geq 3$. Then the following conditions are equivalent:*
   - (a) *for a general point x a general surface $S' \in \mathcal{L}_i(\pi_{T_xX}(X))$ is a rational normal scroll of type (a',b'), $a' \leq b'$;*
   - (b) *a general surface from $\mathcal{L}_i(X)$ is a rational normal scroll of type $(a'+1, b'+2)$ or $(a'+2, b'+1)$, if $a' = b' - 1$.*

Proof. Denote that surface by S. Then, by Lemma 12, one has that $\dim\langle S\rangle = 3h + 1$. For a general point $z \in S^h(S)$, by Lemma 11, holds $\Sigma_z(X) \subset S$ and, by Lemma 5, $\Sigma_z(X)$ is a rational normal curve of degree $2h$. Thus, through $h + 1$ general points of S there passes one rational normal curve of degree $h + 1$, which lays in S. So, S contains $(h+1)$-dimensional family of rational normal curves of degree

$2h$. Hence, the base of the family of lines has to be rational. More, the surface S satisfies the conditions of Lemma 14 for $k = 3h+1$, $l = h+1$ and $m = 2h$. Therefore, S is a normal scroll of type (a, b), $a + b = 3h$, $a, b \leq 2m$ (and, thus, $a, b \geq h$).

By Lemma 12, surfaces of the system $\mathcal{L}_i(\pi_{T_xX}(X))$ are the images under π_{T_xX} of surfaces of $\mathcal{L}_i(X)$ passing through x. Thus, if a general surface $S \in \mathcal{L}_i(X)(-x)$ is covered by lines, then $\pi_{T_xX}(S) \in \mathcal{L}_i(\pi_{T_xX}(X))$, which is a general surface of this system, is covered by lines too. More, S is a rational normal scroll of type (a, b), $a + b = 3h$, $a \leq b$. For a general point $x \in S$ one has that $\pi_{T_xX}(S)$ is a scroll of type $(a-1, b-2)$, which equals to (a', b'), where $a' \leq b'$. Thus, if $a < b$, then the only possibility is $a' = a - 1$, $b' = b - 2$ $(a = a' + 1,\ b = b' + 2)$. If $a = b$, then it is possible also that $a' = b - 2$, $b' = a - 1$, i. e. $a' = b' - 1$, $a = a' + 2$, $b = b' + 1$.

Suppose now that a general surface of $\mathcal{L}_i(\pi_{T_xX}(X))$ is covered by lines. Moving x in X, we obtain (without loss of generality) that a general surface of the family $\mathcal{L}_i(X)$, passing through x is projected by π_{T_xX} to $Scroll_{a',3h-3-a'}$. So, for a general surface S of the system $\mathcal{L}_i(X)$ and a general point $x \in S$ one has that $\pi_{T_xX}(S)$ is a scroll of type (a', b') $(h - 1 \leq a', b' \leq 2h - 2,\ a' + b' = 3h - 3,\ a' \leq b')$. By Lemma 12, $\dim\langle S\rangle = 3h + 1$, $\dim\langle \pi_{T_xX}(S)\rangle = 3h - 2$. Hence, $\dim T_xX \cap \langle S\rangle = 2$ and, thus, $T_xX \cap \langle S\rangle = T_xS$ and $\pi_{T_xS}(S)$ is a scroll of type (a', b'). So, S satisfies the conditions of the first part of Lemma 13. More, for a general point $x' \in \pi_{T_xS}(S) = Scroll_{a',b'}$ the image of $\pi_{T_xS}(S)$ after the projection from $T_{x'}\pi_{T_xS}(S)$ is a rational normal scroll of type $(a' - 1, b' - 2)$. Since $a' \geq h - 1 \geq 2$ and $b' \geq \frac{3}{2}(h - 1) \geq 3$, one has $a' - 1 \geq 1$, $b' - 2 \geq 1$ and $(a' - 1) + (b' - 2) = a' + b' - 3 = 3h - 6 \geq 3$. Thus, for the scroll of type $(a' - 1, b' - 2)$ the intersection of a general tangent plane with the scroll itself is exactly one line. Therefore, the surface S satisfies the conditions of the second part of Lemma 13 and, thus, S contains one-dimensional family of lines. □

6.2.2. *Families of lines on X as surfaces in $G(1, N)$.*

Lemma 15. *Suppose that a general surface $S \in \mathcal{L}_1(X)$ contains one-dimensional family of lines $C(S) \subset G(1, N)$, and $U \subset G(1, N)$ is a corresponding two-dimensional family of lines that sweeps out X, $C(S) \subset U$. Then:*

(1) *Through a general point $x \in X$ there passes exactly one line α_x from U, which is contained in all $S \in \mathcal{L}_1(X)(-x)$.*
(2) *For a general surface $S \in \mathcal{L}_1(X)$ the curve $C(S)$ is a rational normal curve of degree $3h$.*

(3) *The system* $\mathcal{L}_1(U)$ *of curves* $C(S)$, $S \in \mathcal{L}_1(X)$, *is a linear system of dimension* $h+1$, *and, thus, gives us the map* $\widetilde{\varphi} : U \dashrightarrow \mathbb{P}^{h+1}$.
(4) *The surface* $Y = \widetilde{\varphi}(U)$ *is a surface of minimal degree.*
(5) *If* $x \in X$ *is a general point and* $L_x = \langle \{\alpha | \alpha \in G(1,N),\ \mathbb{P}^1_\alpha \cap T_xX \neq \emptyset\}\rangle$, *then* $\pi_{L_x}(U) \subset G(1, N-4)$ *is the family of lines that sweeps out* $\pi_{T_xX}(X)$ *and all ruling lines of a general surface from* $\mathcal{L}_1(\pi_{T_xX}(X))$ *belong to this family.*
(6) *If* $\widetilde{\varphi}_x : \pi_{L_x}(U) \dashrightarrow Y_x$ *is a map given by* $\mathcal{L}_1(\pi_{L_x}(U))$, *then* $\pi_{\widetilde{\varphi}(\alpha_x)} \circ \widetilde{\varphi} = \widetilde{\varphi}_x \circ \pi_{L_x}$ *(restricted to* U*).*
(7) L_x *contains the osculating space* $T^2_{\alpha_x}U$ *of order* 2 *to* U *at the point* α_x.

Proof. For a general point $x \in X$ and a genral surface $S \in \mathcal{L}_1(X)$ passing through x, S contains also certain line, passing through x. Since the number of lines passing through x is finite (because the same is true for $\pi(X)$), we have that there exists a line $\alpha \in U$, passing throgh x and containing in all $S \in \mathcal{L}_1(X)(-x)$. Thus, for general $\alpha \in U$ and a general point $x \in \mathbb{P}^1_\alpha$ holds $\mathbb{P}^1_\alpha \subset S$ for all $S \in \mathcal{L}_1(X)(-x)$. If through a general point $x \in X$ there pass two lines from U, then they are both general in U and, thus, they are cointained in all $S \in \mathcal{L}_1(X)(-x)$. By Lemma 3, such general S is a rational normal scroll of type (a,b), $h \le a, b \le 2h$ and $a+b = 3h$. Thus, if $a \le b$, then $a \ge h \ge 1$ and $b \ge \frac{3}{2}h$, which is equivalent to $b \ge 2$. Hence, S does not contain two different lines passing through its general point x. Therefore through a general point $x \in X$ there passes exactly one line from U.

By Lemma 3, a general surface $S \in \mathcal{L}_1(X)$ is a rational normal scroll of type $(a, 3h-a)$, where $h \le a \le 2h$, and, thus, $C(S)$ is a rational normal curve of degree $3h$.

Consider general points $u_0, \ldots, u_h \in U$ and $x_0, \ldots, x_h \in X$ such that $x_i \in \mathbb{P}^1_{u_i}$, $0 \le i \le h$. Then by Lemma 12, there exists unique surface $S \in \mathcal{L}_1(X)$ passing through $x_0, \ldots, x_h$. Thus, this surface, which is a scroll, contains the lines $\mathbb{P}^1_{u_0}, \ldots, \mathbb{P}^1_{u_h}$ or $C(S) \ni u_0, \ldots, u_h$. On the other hand, if for some $S' \in \mathcal{L}_1(X)$ holds $u_0, \ldots, u_h \in C(S')$, then $S' \ni x_0, \ldots, x_h$, and, by Lemma 12, $S' = S$. Denote by $\mathcal{L}(U)$ the family of curves on U of type $C(S)$, $S \in \mathcal{L}_1(X)$. Then we have that through $h+1$ general points of U there passes exactly one curve from $\mathcal{L}(U)$. By [6, Theorem 5.10], either $\mathcal{L}(U)$ is a linear system, or it is composed with a pensil. The latter is not the case, because a general curve of type $C(S)$ is irreducible. So, $\mathcal{L}(U)$ gives us a map $\widetilde{\varphi} : U \dashrightarrow \mathbb{P}^{h+1}$, because $\dim \mathcal{L}_1(X) = h+1$. Put $Y = \widetilde{\varphi}(U)$.

Let us show that $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree. Really, if $\deg Y \geq h+1$, then for two general surfaces $S_1, S_2 \in \mathcal{L}_1(X)$ the curves $C(S_1)$ and $C(S_2)$ intersect each other by at least $h+1$ points, which are general for U, i. e. $S_1 \cap S_2$ contains at least $h+1$ ruling lines. Since S_1 and S_2 are rational normal scrolls of type (a,b), $h \leq a, b \leq 2h$, any $h+1$ ruling lines are in general position and $\dim\langle S_1 \cap S_2\rangle \geq 2h+1$. Thus, $\dim\langle S_1, S_2\rangle = \dim\langle S_1\rangle + \dim\langle S_2\rangle - \dim\langle S_1\rangle \cap \langle S_2\rangle \leq (3h+1)+(3h+1)-2h+1 = 4h+1$ and $\langle S_1, S_2\rangle$ does not coincide with the ambient space. On the other hand, for a general surface $S_3 \in \mathcal{L}_1(X)$ the intersection $S_3 \cap (S_1 \cup S_2)$ contains at least $2(h+1)$ ruling lines. Since S_3 is a rational normal scroll of type (a,b) for $a, b \leq 2h$, the linear span of these lines contains S_3. Thus, $S_3 \subset \langle S_1, S_2\rangle$ and, therefore, $X \subset \langle S_1, S_2\rangle$, which is not true because X is nondegenerate.

If $L_x = \langle\{\alpha | \alpha \in G(1,N),\ \mathbb{P}^1_\alpha \cap T_xX \neq \emptyset\}\rangle$, then for the projection $\pi_{L_x} : G(1,N) \dashrightarrow G(1,N-4)$ we have that for any $\alpha \in G(1,N) \setminus L_x$ holds $\pi_{T_xX}(\mathbb{P}^1_\alpha) = \mathbb{P}^1_{\pi_{L_x}(\alpha)}$, i. e. the projection π_{L_x} is induced by π_{T_xX}. Therefore, $\pi_{L_x}(U)$ is a family of lines for $\pi_{T_xX}(X)$. The linear system of curves $\mathcal{L}_1(\pi_{L_x}(U))$ contains curves of type $C(S')$, $S' \in \mathcal{L}_1(\pi_{T_xX}(X))$. By Lemma 12, $S' = \pi_{T_xX}(S)$ for a certain $S \in \mathcal{L}_1(X)$, $S \ni x$. Since $C(\pi_{T_xX}(S)) = \pi_{L_x}(C(S))$, the map given by $\mathcal{L}_1(U)(-\alpha_x)$ (i. e. all divisors from $\mathcal{L}_1(U)$ that contain the point α_x), which is $\pi_{\widetilde{\varphi}(\alpha_x)} \circ \widetilde{\varphi}$, coincides with the composition $\widetilde{\varphi}_x \circ \pi_{L_x}$.

To show that $L_x \supset T^2_\alpha U$ it is sufficient to check that $T^2_\alpha C(S) \subset L_x$ for a general $S \in \mathcal{L}_1(X)$, $S \ni x$. The latter fact could be shown by local computations. □

Lemma 16. (1) *If $\mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is not singular, then both families of lines on this threefold in $G(1,7)$ are linearly isomorphic to $v_3(\mathbb{P}^2)$.*

(2) *If $\mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is singular, then both families of lines on this threefold in $G(1,7)$ are linearly isomorphic to $\pi_\alpha(v_3(\mathbb{P}^2))$, where $\alpha \in v_3(\mathbb{P}^2)$; the exeptional divisor of the projection from α is a line corresponding to all lines laying in $\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ and passing trough the point of tangency of H to $\mathbb{P}^2 \times \mathbb{P}^2$.*

Proof. Consider the family of planes $U' \subset G(2,8)$, which are of type $\mathbb{P}^2 \times \{*\}$. It is not hard to see that $U' = v_3(\mathbb{P}^2)$. Put $L = \langle\{\alpha | \alpha \in G(2,8),\ \mathbb{P}^2_\alpha \subset H\}\rangle$. Then the projection $\pi_L : G(2,8) \dashrightarrow G(1,7)$ satisfies the following condition: for any $\alpha \in G(2,8)$ such that $\alpha \notin L$ holds $\mathbb{P}^2_\alpha \cap H = \mathbb{P}^1_{\pi_L(\alpha)}$. Thus, $U = \pi_L(U')$.

If $\mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is not singular, then one can show that $L \cap \langle U'\rangle = \emptyset$. Hence, $U = v_3(\mathbb{P}^2)$ is a linearly normal surface.

If $\mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is singular, then $L \cap \langle U' \rangle = \{\alpha\}$, where $\mathbb{P}^2_\alpha$ is a plane laying in H. Hence, $U = \pi_\alpha(v_3(\mathbb{P}^2))$ is a linearly normal surface. The exeptional divisor then corresponds to $\mathbb{P}^2_\alpha$ and it describes all lines in $\mathbb{P}^2_\alpha$, passing through the point of tangency of H to $\mathbb{P}^2 \times \mathbb{P}^2$. □

Lemma 17. *Suppose that a general surface $S \in \mathcal{L}_1(X)$ contains one-dimensional family of lines, $U \subset G(1, N)$ is a corresponding two-dimensional family that sweeps out X and $U = \varphi(Y)$, where $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree and $\varphi = \widetilde{\varphi}^{-1}$. Then for a general point $\beta \in U$ one has $\dim T^2_\beta U = 5$.*

(1) *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is not singular, then $\dim\langle U \rangle = 6h + 3$ and $\varphi = v_3$. U is linearly normal.*
(2) *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is singular, then $\dim\langle U \rangle \geq 6h + 2$. If $\dim\langle U \rangle = 6h + 3$, then $\varphi = v_3$ and U is linearly normal. If $\dim\langle U \rangle = 6h + 2$, then either $\varphi = \pi_x \circ v_3$ and $x \in \langle v_3(Y) \rangle$, or $Y = Scroll_{a,b}$, where $1 \leq a \leq b \leq h - 1$, $a + b = h$ and φ is given by the complete system $|2C + (3a+b)L|$. U is not linearly normal iff $x \notin v_3(Y)$.*

Proof. Use an induction by h. The base, $h = 1$. By Lemma 16, U is linearly isomorphic to $v_3(\mathbb{P}^2)$ or $\pi_\alpha(v_3(\mathbb{P}^2))$. In both these cases for general $\beta \in U$ holds $\dim T^2_\beta U = 5$. Also $\dim\langle v_3(\mathbb{P}^2) \rangle = 9 = 6 \cdot 1 + 3$, $\dim\langle \pi_\alpha(v_3(\mathbb{P}^2)) \rangle = 8 = 6 \cdot 1 + 2$.

The step. For a general point $x \in X$ we construct $L_x = \langle \{\alpha | \alpha \in G(1, N),\ \mathbb{P}^1_\alpha \cap T_x X \neq \emptyset\} \rangle$. By Lemma 15, $\pi_{L_x}(U)$ is a family of lines for $\pi_{T_x X}(X)$, for which by induction we have that for general point $\beta \in \pi_{L_x}(U)$ holds $\dim T^2_\beta \pi_{L_x}(U) = 5$. Thus, for general point $\gamma \in U$ one has $\dim T^2_\gamma U \geq 5$. Since the maximal possible dimension for $T^2_\gamma U$ is 5, we obtain $\dim T^2_\gamma U = 5$. More, if $\alpha_x \in U$ corresponds to a line passing through x, then, by Lemma 15, $L_x \supset T^2_{\alpha_x} U$. Hence, $\dim L_x \cap \langle U \rangle \geq \dim T^2_{\alpha_x} U = 5$, and $\dim\langle U \rangle = \dim\langle \pi_{L_x}(U) \rangle + \dim L_x \cap \langle U \rangle + 1 \geq \dim\langle \pi_{L_x}(U) \rangle + 6$. By induction, $\dim\langle \pi_{L_x}(U) \rangle \geq 6(h-1) + 3$ (or $6(h-1) + 2$). Thus, $\dim\langle U \rangle \geq 6h + 3$ (or $6h + 2$).

By Lemma 15, $U = \varphi(Y)$, where $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree. Also, a general hyperplane section of Y is taken by $\varphi = \widetilde{\varphi}^{-1}$ to a rational normal curve of degree $3h$. By the classification of surfaces of minimal degree, Y is either $Scroll_{a,b}$, where $a + b = h$, or $v_2(\mathbb{P}^2)$. If $Y = Scroll_{a,b}$, $1 \leq a \leq b \leq h-1$, $a+b = h$, then φ is given by subsystem of $|kC + lL|$, where $C \subset Scroll_{a,b}$ is a rational normal curve of degree b. If H' is a hyperplane section, then $H' \cdot (kC + lL) = kb + l = 3h$. By Lemma 18, $\dim |kC + lL| = \frac{k(k+1)}{2}(b - a) + (k + 1)l + k$. More, $6h + 2 \leq \dim\langle U \rangle \leq \dim |kC + lL|$. From these inequalities follows

$k \in [2, 3 + \frac{2}{h}]$. Thus, k could be equal to 2 or 3 or, if $h = 2$, to 4. In the last case we have $a = b = 1$ and $l = 2$, and by changing the ruling family on quadric, we will have $k = 2$. If $k = 2$, $\dim |2C + (3a+b)L| = \frac{2\cdot 3}{2}(b-a) + (2+1)(3a+b) + 2 = 6a + 6b + 2 = 6h + 2$. If $k = 3$, $\dim |3C + 3aL| = \frac{3\cdot 4}{2}(b-a) + (3+1)3a + 3 = 6a + 6b + 3 = 6h + 3$, and $|3C + 3aL|$ coincides with the complete system of cubic. Thus, $U = v_3(Scroll_{a,b})$ or $U = \pi_x(v_3(Scroll_{a,b}))$, where $x \in \langle v_3(Scroll_{a,b})\rangle$ is a point. In the first case U is linearly normal, in the second one U is linearly normal iff $x \in v_3(Scroll_{a,b})$.

If $Y = Cone_p(C)$, $\deg C = h$, then φ is given by a subsystem of $|lL|$. If H' is a hyperplane section, then $H' \cdot (lL) = l = 3h$. Thus, φ is given by a subsystem of the complete linear system of cubics. By Lemma 18, $\dim |3hL| = \frac{3\cdot 4}{2}h + 3 = 6h + 3$. Thus, $U = v_3(Cone_p(C))$ or $U = \pi_x(v_3(Cone_p(C)))$, where $x \in \langle v_3(Cone_p(C))\rangle$ is a point. In the first case U is linearly normal, in the second one U is linearly normal iff $x \in v_3(Cone_p(C))$.

By the similar reasons, if $Y = v_2(\mathbb{P}^2)$, then for a general conic $Q \subset \mathbb{P}^2$, holds $\varphi(v_2(Q))$ is a rational normal curve of degree $3h = 12$. Thus, the map $\varphi \circ v_2$ is given by a susbsystem of complete linear system of sixtics on $\mathbb{P}^2$, the dimension of which is equal to $\binom{6+2}{2} - 1 = 27 = 6 \cdot 4 + 3 = 6h + 3$. Thus, $U = v_3(v_2(\mathbb{P}^2))$ or $U = \pi_x(v_3(v_2(\mathbb{P}^2)))$, where $x \in \langle v_3(v_2(\mathbb{P}^2))\rangle$ is a point. In the first case U is linearly normal, in the second one U is linearly normal iff $x \in v_3(v_2(\mathbb{P}^2))$. □

Lemma 18. (1) *If $a, b \geq 1$, then $Pic(Scroll_{a,b}| = \mathbb{Z}^2$. Suppose that $a \leq b$. Let C be a rational normal curve of degree b, which is one of the generators of the scroll, and L is a ruling line. Then for $l \geq -1$ and $k \geq 0$ one has $\dim |kC + lL| = \frac{k(k+1)}{2}(b-a) + (k+1)l + k$. If $-(m-1)(b-a) - 2 \geq l \geq -m(b-a) - 1$ and $1 \leq m \leq k$, then $\dim |kC + lL| = \frac{(k-m)(k-m+1)}{2}(b-a) + (l + m(b-a) + 1)(k-m+1) - 1$.*

(2) *For the scroll of type $(0, b)$ and $l \geq 0$ one has $\dim |lL| = \frac{k(k+1)}{2}b + (k+1)l' + k$, where k and l' are integers such that $l = kb + l'$ and $0 \leq l' < b$.*

Proof. Since $L \cdot L = 0$, $L \cdot C = 1$, $C \cdot C = b - a$, one has $L \cdot (kC + lL) = k(L \cdot C) + l(L \cdot L) = k \cdot 1 + l \cdot 0 = k$ and $C \cdot (kC + lL) = k(C \cdot C) + l(C \cdot L) = k \cdot (b-a) + l \cdot 1 = k(b-a) + l$. Thus, if $|kC + lL| \neq \emptyset$, then $k \geq 0$ and $k(b-a) + l \geq 0$.

More, if for $D \in |kC + lL|$ holds $D \cdot L \geq k + 1$, then $D = L + D'$. If $l \geq 0$, then for k general points of L it is easy to find such reducible divisor, which consists of l ruling lines and certain k rational normal curves of degree b each. Hence the map from $|kC + lL|$ to the linear system of k points on L is an epimorphism, and, thus, $\dim|kC+lL| = \dim|kC + (l-1)L| + k + 1$. Similarly, if for $D \in |kC + lL|$ holds $D \cdot C \geq k(b-a) + l + 1$, then $D = C + D''$. For the same reason, if $l \geq 0$, one has $\dim|kC + lL| = \dim|(k-1)C + lL| + k(b-a) + l + 1$.

Therefore, if $l \geq 0$, one has $\dim|kC + lL| = \dim|kC| + (k+1)l = (k + (k-1) + \ldots + 1)(b-a) + k + (k+1)l = \frac{k(k+1)}{2}(b-a) + (k+1)l + k$.

If $l = -1$, then $\dim|kC+lL| = \dim|kC| - (k+1) = \frac{k(k+1)}{2}(b-a) - 1 = \frac{k(k+1)}{2}(b-a) + (k+1)l + k$. In order to decrease l below -1, we can use the same observations. For example, if $-1 \geq l \geq b-a$, then for $k-1$ general points of L it is easy to find such reducible divisor, which consists of $b - a + l$ ruling lines, one minimal curve and certain $k-1$ rational normal curves of degree b each. Hence, the map from $|kC + lL|$ to the linear system of k points on L contains in its image the sub-system, divisors of which contains the point of intersection of L and the minimal curve. Thus, $\dim|kC + lL| \geq \dim|kC + (l-1)L| + k$. More generally, if $-(m-1)(b-a) - 1 \geq l \geq -m(b-a)$, $(1 \leq m \leq k)$, then for $k-m$ general points of L it is easy to find such reducible divisor, which consists of $m(b-a)+l$ ruling lines, m copies of minimal curve and certain $k-m$ rational normal curves of degree b each. Hence, the map from $|kC + lL|$ to the linear system of k points on L contains in its image the sub-system, divisors of which contains the point of intersection of L and the minimal curve with multiplicity m. Thus, $\dim|kC+lL| \geq \dim|kC+(l-1)L| + (k-m) + 1$. Finally, $\dim|kC-1L| \geq \dim|kC - (b-a+1)L| + k(b-a) \geq \ldots \geq \dim|kC - ((k-1)(b-a) + 1)L| + (b-a)(k + \ldots + 2) \geq \dim|kC - k(b-a)L| + (b-a)(k + \ldots + 2) + (b-a-1) \cdot 1 = \dim|kC - k(b-a)L| + (b-a)\frac{k(k+1)}{2} - 1$. Hence, $\dim|kC - k(b-a)L| \leq \dim|kC - 1L| - (b-a)\frac{k(k+1)}{2} + 1 = \frac{k(k+1)}{2}(b-a) + (k+1)(-1) + k - (b-a)\frac{k(k+1)}{2} + 1 = 0$. Since the system $|kC - k(b-a)L|$ contains at least k times minimal curve, its dimension is not less than 0. Therefore all inequalities here are equalities.

If $a = 0$, then $bL = C$, $L \cdot C = 1$, $C \cdot C = b$. Since $Scroll_{0,b}$ is a cone over the curve C, it has a singularity at the point p of its vertex. Therefore we will calculate all the indices of intersection outside p. Thus, for divisors C_1, C_2, D_1, D_2 such that C_i is equivalent to D_i, $i = 1, 2$, $p \notin C_1$, $p \in C_2$, $p \notin D_1$, $p \in D_2$, one has $C_1 \cdot D_1 = C_1 \cdot D_2 = C_2 \cdot D_1 \geq C_2 \cdot D_2$ and $L \cdot L = 0$. Hence, for a general divisor D of the system $|kC + lL|$, where $0 \leq l < b$, holds $D \cdot C = kb + l$, $D \cdot L = k$. So, all the equalities written for the case $a \geq 1$ hold in the case $a = 0$. In particular, if $0 \leq l < b$, $\dim|kC + lL| = \frac{k(k+1)}{2}b + (k+1)l + k$. Thus, for $l \geq 0$ one has $\dim |lL| = \frac{k(k+1)}{2}b + (k+1)l' + k$, where k and l' are such integers, that $l = kb + l'$ and $0 \leq l' < b$. □

6.2.3. *$\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is smooth.*

Lemma 19. *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$, where H is not tangent to $\mathbb{P}^2 \times \mathbb{P}^2$, then for one of the systems $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$ a general surface of that system is a scroll of type $(a, 3h-a)$, where $h+1 \leq a \leq 2h-1$; a general surface of another system contains only finite number of lines.*

Proof. Take a general surface $S_i \in \mathcal{L}_i(X)$, $i = 1, 2$, and a general point $q' \in S^{h-2}(S_i)$. The projection from $T_{q'}S^{h-2}(X)$ is a birational isomorphism by Lemma 5. By Lemma 12, it maps S_i to $Scroll_{1,2}$.

Let us use an induction over h.

The base, $h = 2$. Since for a general point y in $Scroll_{1,2}$ the tangent space at y meets the scroll by a single line, we can apply Lemma 13 to the surface S_i. So, we have that either S_i contains one-dimensional family of lines or it contains two-dimensional family of cubics. In the first case by Corrolary 3, S_i is either $Scroll_{2,4}$ or $Scroll_{3,3}$. Also we should note that the preimage of the line, which is a minimal section in $Scroll_{1,2}$, is either a minimal conic or a certain cubic respectively.

If S_i contains two-dimensional family of cubics, take a general point $x \in S_i$ and consider the family of all entry loci (for $S^2(S_i)$), which pass through x. By Lemma 5, these curves are in general rational normal curves of degree 4, which under the projection π' from T_xS_i are mapped to conics on $Scroll_{1,2}$. Since the family of conics on $Scroll_{1,2}$ contains in its closure reducible conics, the family of curves on S_i also contains reducible curves in its closure. More, a reducible conic on $Scroll_{1,2}$ consists of a minimal curve (a line) and one of ruling lines. Hence its reduced preimage consists of the preimage of a minimal curve and a cubic. Since this reduced curve lays in the closure of the family of curves of degree 4, the preimage of a minimal curve on $Scroll_{1,2}$ under π' is a line.

Since $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is a non-singular section, the only lines it contains are lines of types $\{*\} \times \mathbb{P}^2 \cap H$ or $\mathbb{P}^2 \times \{*\} \cap H$. Any of these lines is a ruling line for one-dimensional family of surfaces linearly isomorphic to $Scroll_{1,2}$. Hence the preimage under the projection π of such general line is a line or a cubic. If the preimage of a general line of the first family is a line, take a general scroll of type $(1, 2)$, which is ruled by lines of first family. The minimal curve, which is a line, for this scroll belongs to the second family of lines on $\pi(X)$. Hence, its preimage under π, as we saw, is either a conic or a cubic. Since it cannot be a conic, it is a cubic and, thus, $T_zX \cap X$ contains $Scroll_{3,3}$.

More, the preimage of a general line from the second family is a cubic. If the preimage of a general line from the first family is a cubic, by the same arguments we can obtain that the preimage of a general line from the second family is a line and, thus, $T_zX \cap X$ again contains $Scroll_{3,3}$.

The step is given by Lemma 3. □

Lemma 20. *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$, where H is not tangent to $\mathbb{P}^2 \times \mathbb{P}^2$, then X is swept out by a family of lines $U \subset G(1, N)$ such that $U = v_3(Y)$, where $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree, and U is linearly normal.*

Proof. By Lemma 19, a general surface of the system $\mathcal{L}_1(X)$ is a rational normal scroll of type $(a, 3h - a)$, where $h + 1 \le a \le 2h - 1$. Hence, we can apply Lemma 15. Thus, $U = \varphi(Y)$, where Y is a surface of minimal degree in $\mathbb{P}^{h+1}$, $\varphi = \widetilde{\varphi}^{-1}$. There exist two ways to see that ψ is given by the complete linear system of cubics. The first is to apply Lemma 17. The second is by induction over h.

The base, $h = 1$, holds by Lemma 16. The step. In the notations from Lemma 15, $\widetilde{\varphi}(U) = Y$ and $\pi_{\widetilde{\varphi}(\alpha_x)} \circ \widetilde{\varphi} = \widetilde{\varphi}_x \circ \pi_{L_x}$. Since all these maps are birational isomorphisms and, by induction, $\varphi_x = \widetilde{\varphi}_x^{-1} = v_3$, we have $\pi_{L_x} \circ \varphi = v_3 \circ \pi_{\widetilde{\varphi}(\alpha)}$. Therefore, the map φ is given by linear system of cubics on Y. For calculating the dimensions of complete linear systems of cubics on surfaces of minimal degree in $\mathbb{P}^{h+1}$ and the possible dimensions of $\langle U \rangle$ see Lemma 17. □

6.2.4. $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ *is singular.*

Lemma 21. *If a general surface of the system $\mathcal{L}_2(X)$ contains only finite number of lines, then there exists a plane $P_2 \subset X$ such that*

(1) *for general points $x_0, \dots, x_{h-2} \in X$ and the projection π' from $\langle T_{x_0}X, \dots, T_{x_{h-2}}X \rangle$ hold*
   (a) *$\pi'(P_2) = \mathbb{P}^2 \times \{x_{H'}\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \cap H' = \pi'(X)$, where $x_{H'}$ is the component of the point of tangency of H' to $\mathbb{P}^2 \times \mathbb{P}^2$,*
   (b) *$\pi'(S_2) = l_1 \times \mathbb{P}^2 \cap H'$, $\pi'(S_1) = \mathbb{P}^2 \times l_2 \cap H'$ for general $S_i \in \mathcal{L}_i(X)(-x_0 - \dots - x_{h-2})$, $i = 1, 2$ (l_1, l_2 are lines).*

(2) *a general surface of the system $\mathcal{L}_1(X)$ does not intersect P_2*
(3) *a general surface of the system $\mathcal{L}_2(X)$ intersects P_2 by a line.*

Proof. This proof is similar to one of Lemma 19. By Lemma 12, for a general surface $S_i \in \mathcal{L}_i(X)$, $i = 1, 2$, and general points $x_0, \dots, x_{h-2} \in S_i$ the projection from $\langle T_{x_0}S_i, \dots, T_{x_{h-2}}S_i \rangle$, which is a birational isomorphism by Lemma 5, maps S_i to $Scroll_{1,2}$.

Let us use an induction over h.

The base, $h = 2$. Since for a general point y in $Scroll_{1,2}$ the tangent space at y meets the scroll by a single line, we can apply Lemma 13 to the surface S_2. So, we have that either S_2 contains one-dimensional family of lines or it contains two-dimensional family of cubics. The first is not the case by our hypothesis. Thus, S_2 contains two-dimensional family of cubics. Take a general point $x \in S_2$ and consider the family of all entry loci (for $S^2(S_2)$), which pass through x. By Lemma 5, these curves are in general rational normal curves of degree 4, which under the projection $\pi_{T_xS_2}$ are mapped to conics on $Scroll_{1,2}$. Since the family of conics on $Scroll_{1,2}$ contains in its closure reducible conics, the family of curves on S_i also contains reducible curves in its closure. More, a reducible conic on $Scroll_{1,2}$ consists of a minimal curve (a line) and one of ruling lines. Hence its reduced preimage consists of the preimage of a minimal curve and a cubic. Since this reduced curve lays in the closure of the family of curves of degree 4, the preimage of a minimal curve on $Scroll_{1,2}$ under $\pi_{T_xS_2}$ (and, thus, under π_{T_xX}) is a line.

Let $x \in X$ be a general point and π' be the projection from T_xX. Then $\pi'(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H'$, where H' is a hyperplane tangent to $\mathbb{P}^2 \times \mathbb{P}^2$ at the point $(y_{H'}, x_{H'})$. By Lemma 12, we can choose the order of P^2's such that $\pi'(S_2) = l_1 \times \mathbb{P}^2 \cap H'$, $\pi'(S_1) = \mathbb{P}^2 \times l_2 \cap H'$ for general $S_i \in \mathcal{L}_i(X)(-x)$, $i = 1, 2$ (l_1, l_2 are lines). For the singular threefold $\mathbb{P}^2 \times \mathbb{P}^2 \cap H'$ surfaces of type $l_1 \times \mathbb{P}^2 \cap H'$, where $l_1 \subset \mathbb{P}^2$ is a line, are scrolls of type $(1, 2)$ and their minimal curves are lines $l_1 \times \{x_{H'}\}$. Thus, a general surface $S_2 \in \mathcal{L}_2(X)(-x)$ contains a line, which is projected by π' to a line on the plane $\mathbb{P}^2 \times \{x_{H'}\}$. Hence, this family of lines on X is two-dimensional and these lines sweep out a surface, which is projected by π' to $\mathbb{P}^2 \times \{x_{H'}\}$. So, this surface is a plane $P_2 \subset X$ and $T_xX \cap P_2 = \emptyset$. By construction, $S_2 \cap P_2$ is exactly a line. More, for a general surface $S_1 \in \mathcal{L}_1(X)(-x)$ the intersection $S_1 \cap P_2$ is projected by π' to $(\mathbb{P}^2 \times l_2) \cap (\mathbb{P}^2 \times \{x_{H'}\}) \cap H'$, which is empty. Thus, $S_1 \cap P_2 = \emptyset$.

The step. Take a general point $x \in X$. Then a general surface of the system $\mathcal{L}_2(\pi_{T_xX}(X))$ contains only finite number of lines, and, thus, there exists a plane $P_2' \subset \pi_{T_xX}(X)$. Put $P_2 = \pi_{T_xX}^{-1}(P_2')$. For general points $x_0, \ldots, x_{h-3} \in X$ and corresponding points $x_0', \ldots, x_{h-3}' \in \pi_{T_xX}(X)$, the projection π' from $\langle T_xX, T_{x_0}X, \ldots, T_{x_{h-3}}X\rangle$ is a composition of the projections from T_xX and π'' from $\langle T_{x_0'}\pi_{T_xX}(X), \ldots, T_{x_{h-3}'}\pi_{T_xX}(X)\rangle$. More, $\pi'(P_2) = \pi''(P_2') = \mathbb{P}^2 \times \{x_{H''}\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \cap H'' = \pi''(X)$, where $x_{H''}$ is the component of the point of tangency of H'' to $\mathbb{P}^2 \times \mathbb{P}^2$. Therefore, if we exchange the roles of x and x_0, the image $\pi'(P_2)$ will not change, and P_2 does not depend on x. Further, since π'' takes the plane P_2' to a plane, the center of projection does not intersect

P_2'. Hence, $T_{x_0'}\pi_{T_xX}(X)\cap P_2' = \emptyset$ and, thus, $T_{x_0}X\cap\langle P_2\rangle = \emptyset$. Therefore, $T_xX\cap\langle P_2\rangle = \emptyset$ and P_2 is a plane. All other reqired observations easily follow from the same for $\pi_{T_xX}(X)$. □

Lemma 22. *If general surfaces from $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$ contain only finite number of lines, then $X = \pi_{\langle y_1,y_2\rangle}v_2(Y)$, where $Y\subset\mathbb{P}^{h+2}$ is a threefold of minimal degree and $y_1, y_2\in v_2(Y)$ are different smooth points.*

Proof. Let us use the induction over h. The base, $h = 1$, (actually, this is not the base, because there are a lot of lines for $h = 1$) is given by Lemma 8. If $X = \mathbb{P}^2\times\mathbb{P}^2\cap H$, then the exceptional divisors are two planes of form $\mathbb{P}^2\times\{y_H\}$ and $\{x_H\}\times\mathbb{P}^2$, where (x_H, y_H) is the point of tangency of H' to $\mathbb{P}^2\times\mathbb{P}^2$.

The step. By Lemma 21, there exist two planes $P_1, P_2\subset X$ such that for general points $x_0,\dots,x_{h-2}\in X$ and the projection π' from $\langle T_{x_0}X,\dots,T_{x_{h-2}}X\rangle$ hold $\pi'(X) = \mathbb{P}^2\times\mathbb{P}^2\cap H'$, $\pi'(P_2) = \mathbb{P}^2\times\{y_{H'}\}$ and $\pi'(P_1) = \{x_{H'}\}\times\mathbb{P}^2$, where $(x_{H'}, y_{H'})$ is the point of tangency of H' to $\mathbb{P}^2\times\mathbb{P}^2$. Thus, the planes P_1, P_2 after the projection from $T_{x_{h-2}}X$ will satisfy the similar condition for $h-2$ general points and, by induction, are corresponding exceptional divisors. Thus, we can apply Proposition 5.

Since the exceptional divisors are planes, the points of $v_2(Y)$ laying in the center of projection are smooth. □

Remark 7. A point $v_2(y)\in v_2(Y)$ is smooth iff $y\in Y$ is smooth. If Y is a variety of minimal degree and $y\in Y$ is a singular point, then Y is a cone with the vertex at y. Hence, Y is covered by lines passing through y. Under the map $\pi_{v_2(y)}\circ v_2$ every such line goes to a line. Thus, $\pi_{v_2(y)}(v_2(Y))$ is covered by lines.

So, if $X = \pi_{\langle y_1,y_2\rangle}v_2(Y)$, where y_1 is a smooth and y_2 is a singular points, then X is covered by one irreducible family of lines. If both y_1 and y_2 are singular, then X is covered by two different irreducible families of lines.

Lemma 23.
1. *Suppose that a general surface from $\mathcal{L}_1(X)$ contains one-dimensional family of lines and a general surface from $\mathcal{L}_2(X)$ contains only finite number of lines. If $U\subset G(1,N)$ is a two-dimensional corresponding family of lines on X, then $U = \pi_x(v_3(Y))$, where $Y\subset\mathbb{P}^{h+1}$ is a surface of minimal degree and $x\in v_3(Y)$ is a smooth point.*
2. *If for both linear systems $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$ a general surface contains one-dimensional family of lines and $U\subset G(1,N)$ is a family of lines, corresponding to $\mathcal{L}_1(X)$, then U is either*

$\pi_{v_3(p)} v_3(Cone_p(C))$, where C is a rational normal curve of degree h, either $\varphi(Scroll_{a,b})$, where $1 \le a \le b \le h-1$ and φ is given by the system $|(3a+b)L+2C|$. In both cases U is linearly normal.

Proof. Concider threefold $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$, where H is tangent to $\mathbb{P}^2 \times \mathbb{P}^2$ at unique point (x_H, y_H). Take a general line $l_1 \ni x_H$. Then $l_1 \times \mathbb{P}^2 \cap H$ contains $\{x_H\} \times \mathbb{P}^2$. Thus, there exists a line $l_2 \ni y_H$ such that $l_1 \times \mathbb{P}^2 \cap H = l_1 \times l_2 \cup \{x_H\} \times \mathbb{P}^2$. The quadric $l_1 \times l_2$ is covered by lines from both families $\mathcal{L}_1(\pi(X))$ and $\mathcal{L}_2(\pi(X))$, and $l_1 \times \{y_H\} \in \mathcal{L}_1(\pi(X))$, $\{x_H\} \times l_2 \in \mathcal{L}_2(\pi(X))$. Hence, $\mathcal{L}_1(\pi(X))$ contains all lines in $\mathbb{P}^2 \times \{y_H\}$ passing through (x_H, y_H). Also we have an isomorphism $\psi : x_H^* \to y_H^*$ such that $\forall\ \alpha \in x_H^*$ holds $l_\alpha \times l_{\psi(\alpha)} \subset H$. Thus, there exists one-dimensional family of surfaces covered by lines from both families.

If a general surface from $\mathcal{L}_2(X)$ contains only finite number of lines, by Lemma 21, there exists a plane $P_2 \subset X$ such that a general surface of the system $\mathcal{L}_1(X)$ does not intersect P_2. Since $P_2 \subset X$, it should be covered by lines from U. Since such general line does not intersect P_2, there is one-dimensional subfamily $C \subset U$ of lines laying in P_2. Since for general points $x_0, \dots, x_{h-2} \in X$ and the projection π' from $\langle T_{x_0}X, \dots, T_{x_{h-2}}X\rangle$ hold

(1) $\pi'(P_2) = \mathbb{P}^2 \times \{y_{H'}\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \cap H' = \pi'(X)$, where $(x_{H'}, y_{H'})$ is the point of tangency of H' to $\mathbb{P}^2 \times \mathbb{P}^2$,
(2) $\pi'(S_1) = \mathbb{P}^2 \times l_2 \cap H'$ for general $S_1 \in \mathcal{L}_1(X)(-x_0 - \dots - x_{h-2})$, l_2 is a line.

So, the family C goes under π' to the family of lines in $\mathbb{P}^2 \times \{y_{H'}\}$ passing through $(x_{H'}, y_{H'})$. Since the projection π' restricted to P_2 is an isomorphism, C is a family of lines on P_2 passing through ${\pi'}^{-1}(y_{H'})$, i. e. a line in $G(1, N)$.

By Lemma 17, U is either $v_3(Y)$, either $\pi_x(v_3(Y))$, where $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree and $x \in \langle v_3(Y)\rangle$, or $U = \varphi(Scroll_{a,b})$, where $1 \le a \le b \le h-1$, $a+b = h$ and φ is given by the complete system $|2C + (3a+b)L|$. Since U contains a line, the only possibility is $U = \pi_x v_3(Y)$, where $x \in v_3(Y)$ and C is the exceptional divisor of the projection from x. Since C is a line, x is a smooth point of $v_3(Y)$. More, we obtain that U is linearly normal.

Consider the case when for both linear systems $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$ a general surface contains one-dimensional family of lines. On $\pi(X)$ there exists one-dimensional family of surfaces covered by lines from both families. The preimages of these lines under π are again lines. Hence, X contains one-dimensional family of surfaces, which contain two one-dimensional families of lines from both $\mathcal{L}_1(X)$ and $\mathcal{L}_2(X)$. So, those surfaces are nonsingular quadrics in $\mathbb{P}^3$. The family of lines laying on such quadric is a conic in $G(1,N)$. Thus, U contains one-dimensional family of conics. By Lemma 17, either $U = v_3(Y)$, either $U = \pi_x v_3(Y)$, where $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree and $x \in \langle v_3(Y)\rangle$, or $U = \varphi(Scroll_{a,b})$, where $1 \leq a \leq b \leq h-1$, $a+b=h$ and φ is given by the complete system $|2C+(3a+b)L|$. If $U = v_3(Y)$ or $U = \pi_x v_3(Y)$, where $x \notin v_3(Y)$, U does not contain conics. If $U = \pi_x v_3(Y)$, where $x \in v_3(Y)$, then the only conics it contains are the images of lines, passing through $v_3^{-1}(x)$. Thus, $v_3^{-1}(x)$ should be the vertex of a cone and $Y = Cone_p(C)$. If $U = \varphi(Scroll_{a,b})$, where $1 \leq a \leq b \leq h-1$, $a+b=h$ and φ is given by the complete system $|2C+(3a+b)L|$, then U contains one-dimensional family of conics, which are the images of ruling lines of $Scroll_{a,b}$. In all these cases U is linearly normal. □

6.3. **Geometry of X in the case, when X is swept out by a family of lines.** Our further considerations will be based on the following fact:

Lemma 24. [[8], Proposition 6.9] *Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate, projective, regular threefold. Let $k \geq 2$ and assume that a general tangential projection X_1 of X is linearly normal and contained in the Segre embedding of $\mathbb{P}^k \times \mathbb{P}^k$. Suppose that each of the two projections of X_1 to $\mathbb{P}^k$ spans $\mathbb{P}^k$. Then X is linearly normal and contained in the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$. Moreover each of the two projections of X spans $\mathbb{P}^{k+1}$.*

Denote by p_1 and p_2 the projections of $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$ to the first and the second factors. If we know, that X is covered by lines and $X \subset \mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$, then one (or both) of the projections p_1, p_2 takes X to a surface. The other will take X to a threefold covered by lines. Further we will study all possible cases.

Lemma 25. [[8], Corollary 6.17] *Let X be an irreducible, non-degenerate threefold in $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$, $k \geq 2$, which does not lie in the 2-uple embedding of $\mathbb{P}^{k+1}$. Assume that each of the two projections of X to $\mathbb{P}^{k+1}$ spans $\mathbb{P}^{k+1}$. Then X spans a space of dimension at least $4k+3$.*

Furthermore, if X spans a $\mathbb{P}^{4k+3}$, then given $k+1$ general points of X, there is a rational normal curve C of degree $2k$ on X containing the given points, and X is k-defective.

Lemma 26. *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ and X is swept out by a family of lines, then*

(1) *$X \subset \mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$;*

(2) *for a general point* $x \in X$ *one has* $T_xX = T_x\mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap \langle X \rangle$, *i. e.* X *is a component of* $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap L$, *where* $\dim L = 4h+3$;
(3) $p_i(X) \subset \mathbb{P}^{h+1}$ *are varieties of minimal degree;*
(4) *the projections* p_1 *and* p_2 *are given by linear systems* $\mathcal{L}_1(X)$ *and* $\mathcal{L}_2(X)$ *respectively;*
(5) *Suppose also that a general surface of* $\mathcal{L}_1(X)$ *is covered by lines,* $U \subset G(1,N)$ *is the corresponding family of lines covering* X*, and* $U = \varphi(Y)$*, where* $Y \subset \mathbb{P}^{h+1}$ *is a surface if minimal degree and* φ *is as in Lemma 17. Then* Y *is linearly isomorphic to* $p_1(X)$.

Proof. The first statement immediately follows from Lemma 24.

Take a general point $x = (x_1, x_2) \in \mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$. For the projection from $\{x_1\} \times \mathbb{P}^{h+1}$ restricted to $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$ holds $p_1 \circ \pi_{\{x_1\}\times\mathbb{P}^{h+1}} = \pi_{x_1} \circ p_1$. Thus, for the projection from $T_x\mathbb{P}^{h+1} \times \mathbb{P}^{h+1} = \langle \{x_1\} \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times \{x_2\} \rangle$ we have $p_i \circ \pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}} = \pi_{x_i} \circ p_i$, $i = 1, 2$. So, for the variety $\pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}}(X) \subset \mathbb{P}^h \times \mathbb{P}^h$ hold $p_i(\pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}}(X)) = \pi_{x_i} \circ p_i(X)$. Hence, $\langle p_i(\pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}}(X)) \rangle = \mathbb{P}^h$. Since X, and, thus, $\pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}}(X)$ are covered by lines, $\pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}}(X)$ is not a subset of the diagonal, and we can apply Lemma 25. So, $\dim\langle \pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}}(X) \rangle \geq 4(h-1)+3 = 4h-1$. Thus, $\dim T_x\mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap \langle X \rangle = \dim\langle X \rangle - \dim\langle \pi_{T_x\mathbb{P}^{h+1}\times\mathbb{P}^{h+1}}(X) \rangle - 1 \leq 4h+3-(4h-1)-1 = 3$. If we take $x \in X$ general point, then $T_xX \subset T_x\mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$. Since $\dim T_xX = 3$, we obtain $T_xX = T_x\mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap L$, where $L = \langle X \rangle$. Thus, X is a component of $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap L$.

To show that $p_i(X)$ is a variety of minimal degree, it is sufficient to prove that for general points $x_0, \dots, x_m \in X$, $m = h + 1 - \dim p_i(X)$, the projection π' of $p_i(X)$ from $\langle p_i(x_0), \dots, p_i(x_m) \rangle$ is a birational isomorphism. Since X is a component of the linear section, for a general point $x \in p_i(X)$ the fiber $p_i^{-1}(x)$ is a linear subspace. The same is true for the projected threefold $\pi_{\langle T_{x_0}X, \dots, T_{x_m}X \rangle}(X)$. Thus, if π' is not a birational isomorphism and for two general points $y_0, y_1 \in p_i(X)$ holds $\pi'(y_0) = \pi'(y_1)$, the projection $\pi_{\langle T_{x_0}X, \dots, T_{x_m}X \rangle}$ should take the points $p_i^{-1}(y_0)$ and $p_i^{-1}(y_1)$ to one point. Since $m \leq h-1$, by Lemma 5, the last projection is a birational isomorphism, and, therefore, pi' is a biratinal isomorphism.

Take a general point $z \in S^h(X)$ and general points $x_0, \dots, x_h \subset \Sigma_z(X)$ such that $z \in \langle x_0, \dots, x_h \rangle$. Then

$$T_z S^h(X) = \langle T_{x_0} X, \dots, T_{x_h} X \rangle = \langle T_{x_0} \mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap L, \dots, T_{x_h} \mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap L \rangle \subset \langle T_{x_0} \mathbb{P}^{h+1} \times \mathbb{P}^{h+1}, \dots, T_{x_h} \mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \rangle \cap L =$$
$$\langle \langle p_1(x_0) \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times p_2(x_0) \rangle, \dots, \langle p_1(x_h) \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times p_2(x_h) \rangle \rangle \cap L =$$
$$\langle \langle p_1(x_0) \times \mathbb{P}^{h+1}, \dots, p_1(x_h) \times \mathbb{P}^{h+1} \rangle, \langle \mathbb{P}^{h+1} \times p_2(x_0), \dots, \mathbb{P}^{h+1} \times p_2(x_h) \rangle \rangle \cap L =$$
$$\langle \langle p_1(x_0), \dots, p_1(x_h) \rangle \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times \langle p_2(x_0), \dots, p_2(x_h) \rangle \rangle \cap L = \langle L_1 \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times L_2 \rangle \cap L,$$

where L_1, L_2 are hyperplanes. Since $\langle L_1 \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times L_2 \rangle \cap \mathbb{P}^{h+1} \times \mathbb{P}^{h+1} = L_1 \times \mathbb{P}^{h+1} \cup \mathbb{P}^{h+1} \times L_2$, $X \not\subset \langle L_1 \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times L_2 \rangle$. Since $T_z S^h(X)$ is a hyperplane in L, we obtain $T_z S^h(X) = \langle L_1 \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times L_2 \rangle \cap L$. Thus, $T_z S^h(X) \cap X = \langle L_1 \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times L_2 \rangle \cap X = \langle L_1 \times \mathbb{P}^{h+1}, \mathbb{P}^{h+1} \times L_2 \rangle \cap \mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap X = (L_1 \times \mathbb{P}^{h+1} \cup \mathbb{P}^{h+1} \times L_2) \cap X = (L_1 \times \mathbb{P}^{h+1} \cap X) \cup (\mathbb{P}^{h+1} \times L_2 \cap X)$. Therefore, $L_1 \times \mathbb{P}^{h+1} \cap X \in \mathcal{L}_1(X)$, $\mathbb{P}^{h+1} \times L_2 \cap X \in \mathcal{L}_2(X)$. Varying the point z, we obtain that $\mathcal{L}_1(X)$ coincides with $L_1 \times \mathbb{P}^{h+1} \cap X$, $L_1 \subset \mathbb{P}^{h+1}$ is a hyperplane. Thus, $\mathcal{L}_1(X)$ is the system of preimages of hyperplane sections of p_1. Also, $\mathcal{L}_2(X)$ gives us p_2.

By Lemma 15, the map from U to Y is given by the linear system $C(S)$, $S \in \mathcal{L}_1(X)$, where $C(S)$ is a family of ruling lines on S. Since a p_1 contracts lines from U, one can construct a map $U \dashrightarrow \pi_1(X)$. Since the preimages of hyperplane sections of $p_1(X)$ are given again by $C(S)$, $S \in \mathcal{L}_1(X)$, we have $Y = p_1(X)$. □

6.3.1. *X is covered by only one irreducible family of lines, which are ruling lines of surfaces from the system $\mathcal{L}_1(X)$.*

Lemma 27. (1) *There exists a map $\psi : p_1(X) \dashrightarrow G(1, h+1)$ such that X is covered by lines of type $\{p_1(x)\} \times \mathbb{P}^1_{\psi(p_1(x))}$, $x \in X$ is a general point.*
(2) *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is smooth, then ψ is a linear isomorphism.*
(3) *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is singular, then ψ is the projection from a smooth point $y \in p_1(X) = Y$ such that $U = \pi_{v_3(y)} v_3(Y)$, where $U \subset G(1, N)$ is a family of lines covered X.*

Proof. Since X is a component of the linear section $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1} \cap L$, for a general point $x \in X$ one has $p_1^{-1}(p_1(x))$ is a linear subspace. Since p_1

contracts lines from U, $p_1^{-1}(p_1(x))$ is a line of U. Thus, $p_2(p_1^{-1}(p_1(x)))$ is a line on $p_2(X)$ and $p_1^{-1}(p_1(x)) = \{p_1(x)\} \times p_2(p_1^{-1}(p_1(x)))$. Denote by $W \subset G(1, h+1)$ the corresponding family of lines covering $p_2(X)$. Then we have a required map $\psi : p_1(X) \dashrightarrow W$.

Let us study ψ. Take a general surface $S \in \mathcal{L}_1(X)$. By Lemma 26, $p_1(S)$ is a hyperplane section of $p_1(X)$, which is a rational normal curve of degree h. Since p_2 is the restriction of $\pi_{\langle L_1 \times \mathbb{P}^{h+1}\rangle}$, where $L_1 \subset \mathbb{P}^{h+1}$ is a hyperplane, one has $p_2(S) = \pi_{\langle L_1 \times \mathbb{P}^{h+1}\rangle}(S)$. More, $L_1 \times \mathbb{P}^{h+1} \cap S \subset (L_1 \cap p_1(S)) \times \mathbb{P}^{h+1} \cap L$. Since $L_1 \cap p_1(S))$ is h points, $(L_1 \cap p_1(S)) \times \mathbb{P}^{h+1} \cap L$ is exactly h lines laying in general position. By Lemma 3, S is a rational normal scroll of type (a, b), $h \le a \le b \le 2h$, $a+b = 3h$. Hence, the projection of $Scroll_{a,b}$ from h its ruling lines gives us $Scroll_{a-h,b-h}$. Thus, $p_2(S)$ is a rational normal scroll of type $(a-h, b-h)$. The family of ruling lines of this scroll is a rational normal curve of degree h in $G(1, h+1)$. Hence, ψ takes general hyperplane section of $p_2(X)$, which is a rational normal curve of degree h, to a rational normal curve of degree h on W.

Concider a hyperplane section $K \subset W$. Then $K \cap \psi(p_1(S))$ is a set of h points as a hyperplane section of $\psi(p_1(S))$. Thus, $\psi^{-1}(K) \cap p_1(S)$ is a set of h points. Since $p_1(S)$ is a hyperplane section of $p_1(X)$, one has $\deg \psi^{-1}(K) = h$. But the only curves of degree h on a surface $p_1(X) \subset \mathbb{P}^{h+1}$ of minimal degree h are its hyperplane sections. Thus, the preimages under ψ of hyperplane sections are hyperplane sections, and ψ is a projection from certain linear subspace $M \subset \mathbb{P}^{h+1}$. On the other hand, since $\psi(p_1(S))$ is a rational normal curve of degree h, $\dim\langle W\rangle \ge h$. Thus, $\dim M \le 0$. If M is a point and $M \notin p_1(X)$, then for h general points of X there exists $S \in \mathcal{L}_1(X)$ such that S is not linearly normal and its normalization is a rational normal scroll of type (a, b), $a + b = 3h$. Take the projection from the linear span of tangent spaces of $h-1$ general points of X, which maps X to $\mathbb{P}^2 \times \mathbb{P}^2 \cap H$. We obtain, that there exists one-dimensional family of not linearly normal surfaces in $\mathbb{P}^2 \times \mathbb{P}^2 \cap H$, whose normalizations are rational normal scrolls of type $(1, 2)$, which is not true. If $M \in p_1(X)$ is a singular point, then $\pi_M(p_1(X))$ is a curve, which is not the case. If $M \in p_1(X)$ is a smooth point, then X contains lines $\{M\} \times \mathbb{P}^1_\alpha$, where α is a point of the exceptional divisor, which is a line. Thus, these lines form a line in $U \subset G(1, N)$. By Lemma 23, $U = \pi_{v_3(y)}(v_3(Y))$ and the only line on it is the exceptional divisor of the projection from $v_3(y)$. Therefore, using the isomorphism of Y and $p_1(X)$, we obtain that M corresponds to y. □

Lemma 28. *Suppose that X is covered by one irreducible family of lines $U \subset G(1, N)$.*

(1) *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is smooth, then*
   (a) *if $U = v_3(Y)$, where Y is a rational normal scroll of type (a, b), $a, b \geq 1$, $a + b = h$, then X is covered by lines of form $\langle \xi(x), \eta(x)\rangle$, $x \in Y$, where $\xi : Y \to Scroll_{a+1,b+1}$ is an isomorphism and $\eta = \pi_{v_2(l)} \circ v_2$, where $l \subset Y$ is a ruling line;*
   (b) *if $U = v_3(Y)$, where $Y = Cone_p(C)$ and C is a rational normal curve of degree h, then a general line of U is of form $\langle \xi(x), \eta(x)\rangle$, $x \in Y$, where $\xi : Y \dashrightarrow Scroll_{1,h+1}$ is a blowing-up of a line and $\eta = \pi_{v_2(y)} \circ v_2$, where $y \in Y$ is a smooth point, such that for a general point $x \in \langle p, y\rangle$ $\xi(x) = \eta(x)$;*
   (c) *if $U = v_3(v_2(\mathbb{P}^2))$, then $X = v_3(\mathbb{P}^2) \times \mathbb{P}^1$.*

(2) *If $\pi(X) = \mathbb{P}^2 \times \mathbb{P}^2 \cap H$ is singular and $U = \pi_{v_3(y)}(v_3(Y))$, where $Y \subset \mathbb{P}^{h+1}$ is a surface of minimal degree and $y \in Y$ is a smooth point, then $X = \pi_{\langle v_2(p), v_2(y)\rangle}(v_2(Cone_p(Y)))$.*

In all these cases the embedding $X \to \mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$ is unique upto an automorphism.

Proof. Let us describe families of lines covering $p_2(S)$ in $G(1, h + 1)$. By Lemma 26, $p_2(X) \subset \mathbb{P}^{h+1}$ is a threefold of minimal degree. By the classification, $p_2(X)$ is either a rational normal scroll of type (c, d, e), $c \leq d \leq e$, $c+d+e = h-1$, or a cone $Cone_p(v_2(\mathbb{P}^2))$, $h = 5$. The family of lines containing in planes of $Scroll_{c,d,e}$ is a rational normal scroll of type $(c+d, c+e, d+e)$ in $G(1, h+1)$ $(c+d \leq c+e \leq d+e)$. If $Scroll_{c,d,e}$ contains a family of lines which do not belong to its planes, then the only non-trivial case is $c = d = e$, $h = 4$. In this case the family of lines is the family of fibers in $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The corresponding surface in $G(1, h+1)$ is $v_2(\mathbb{P}^2)$. The family of lines on $Cone_p(v_2(\mathbb{P}^2))$ also form $v_2(\mathbb{P}^2) \subset G(1, h + 1)$.

Now we will study the ways how $p_1(X)$ (or its projection) could be included in these families. By Lemma 26, $p_1(X) \subset \mathbb{P}^{h+1}$ is a surface of minimal degree. By the classification, $p_1(X)$ is either a rational normal scroll of type (a, b), $a \leq b$, $a + b = h$, or $v_2(\mathbb{P}^2)$, $h = 4$. First, consider the case when ψ is an isomorphism. Thus, $p_1(X)$ is a subvariety of the family of lines on $p_2(X)$. If $Scroll_{a,b} \subset Scroll_{c+d,c+e,d+e}$, then $c+d \leq a$, $c + e \leq b$. Thus, $c + (c + d + e) \leq a + b$. Since $c + d + e = h - 1$ and $a + b = h$, we obtain $c \leq 1$. If $c = 1$, then $a = d + 1$, $b = e + 1$. Hence, $p_2(X) = Scroll_{1,a-1,b-1}$ and for points of the basic curve of

$Scroll_{a,b}$ of degree a there correspond ruling lines of a natural sub-scroll type $(1, a-1)$; for points of the basic curve of $Scroll_{a,b}$ of degree b there correspond ruling lines of a natural sub-scroll type $(1, b-1)$; for points of a ruling line there correspond lines in a plane of $p_2(X)$ passing through the point of this plane laying on the minimal line. If $c = 0$, then either $a > d$ and $b = e$, or $a = d$ and $b > e$. In both cases $Scroll_{a,b}$ is not contained in $Scroll_{d,e} \subset Scroll_{d,e,d+e}$. Thus, $b \geq d+e$. Since $d+e = h-1$ and $a+b = h$, one has either $b = h-1$, $a = 1$, $e = h-2$, $d = 1$, either $b = h-1$, $a = 1$, $e = h-1$, $d = 0$, or $b = h$, $a = 0$, $e = h-1$, $d = 0$. In the first case to points of basic curves of $Scroll_{a,b}$ there correspond ruling lines of natural sub-scrolls of $Scroll_{c,d,e}$: $Scroll_{0,1}$ and $Scroll_{1,h-1}$ $(c = 0, d = 1, e = h-1)$. In the second case the scroll of type $(c+d, c+e, d+e) = (0, h-1, h-1)$ should contain a line, which does not belong to its planes. Thus, $h-1 = 1$, and situation is similar to the previous case. Let us describe X. Concider the maps ψ_a, ψ_b, ψ_c from $\mathbb{P}^1$ to corresponding basic curves of $Scroll_{a,b}$ and $Scroll_{c,d,e}$ and the map $\psi_{de} : Scroll_{a,b} \to Scroll_{d,e}$, which are coherent with ψ: if $t \in \mathbb{P}^1$ and $x \in \langle \psi_a(t), \psi_b(t) \rangle \subset Scroll_{a,b}$, then $\mathbb{P}^1_{\psi(x)} = \langle \psi_c(t), \psi_{de}(x) \rangle$. So, X is covered by lines of form $\langle (x, \psi_c(t)), (x, \psi_{de}(x)) \rangle$. The map $x \mapsto (x, \psi_c(t))$ takes $p_1(X) = Scroll_{a,b}$ to a scroll of type $(a+1, b+1)$ and coincides with blowing-up of one ruling line. Since ψ_{de} is equivalent to a projection from a ruling line, the map $x \mapsto (x, \psi_{de}(x))$ is equivalent to composition of v_2 and a projection from a conic, which is the image of a ruling line.

If $b = h$, $a = 0$, $e = h-1$, $d = 0$, the situation is more complicated. First, to points of a ruling line of $p_1(X) = Scroll_{0,h} = Cone_p(C)$ there correspond lines in a plane of $p_2(X) = Scroll_{0,0,h-1} = Cone_l(C')$ passing trough one point on l. Denote by $\psi_0 : p_1(X) \dashrightarrow l$ the corresponding map. One has, that for the vertex p of $p_1(X)$ there corresponds the vertex l of $p_2(X)$, i. e. ψ_0 is a blowing-up of p. For points of the basic curve of degree h there correspond lines from $Scroll_{1,h-1}$, whose minimal curve is l and whose basic curve coincides with one of $Scroll_{0,0,h-1}$. It is hard to give nice geometrical description of X. But one should note, that X contains one special line $\{p\} \times l$, which corresponds in $U = v_3(Cone_p(C)) \subset G(1, N)$ to a singular point. Also X is covered by one-dimensional family of rational normal scrolls of type $(1, 2)$, containing the special line as one of ruling lines. Minimal curves of these scrolls are ruling lines of $Scroll_{1,h+1}$, whose minimal curve is again the special line. More precisely, we can decribe X as follows. Take a sub-scroll $M = Scroll_{0,h-1} \subset Scroll_{0,0,h-1}$ with the vertex $r \in l$. Then one has a map $\psi_1 : p_1(X) \dashrightarrow M$, $\psi_1(x) = \mathbb{P}^1_{\psi(x)} \cap M$. This map is well-defined for all points except $y \in p_1(X)$ such that $\mathbb{P}^1_{\psi(y)} \subset M$. Thus, geometrically, ψ_1 is the projection from y. More, $\psi_0(x) = \psi_1(x) = r$, if $x \in \langle p, y\rangle$ is a general point. So, X is swept out by lines of form $\langle (x, \psi_0(x)), (x, \psi_1(x))\rangle$. The map $x \mapsto (x, \psi_0(x))$ gives us a blowing-up of a line to $Scroll_{1,h+1}$. The map $x \mapsto (x, \psi_1(x))$ is geometrically $\pi_{v_2(y)} \circ v_2$. $Scroll_{1,h+1}$ intersects $\pi_{v_2(y)}(v_2(p_1(X)))$ by the image of $\langle p, y\rangle$.

If $p_1(X) = v_2(\mathbb{P}^2)$ and $h = 4$, then to points of $v_2(\mathbb{P}^2)$ there correspond fibers of $\mathbb{P}^1 \times \mathbb{P}^2 = p_2(X)$. To describe X, we should take two different planes $P_1, P_2 \subset \mathbb{P}^1 \times \mathbb{P}^2$, consider two maps $\psi_1 : p_1(X) \to P_1$, $\psi_2 : p_1(X) \to P_2$. Then X is covered by lines of form $\langle (x, \psi_1(x)), (x, \psi_2(x))\rangle$, $x \in p_1(X) = v_2(\mathbb{P}^2)$. Since the map $x \mapsto (x, \psi_i(x))$ has $v_3(\mathbb{P}^2)$ its image, we have that $X = v_3(\mathbb{P}^2) \times \mathbb{P}^1$.

Consider the case, when ψ is the projection from the smooth point $y \in p_1(X)$. If $p_1(X)$ is a rational normal scroll of type (a, b), then $\pi_y(p_1(X))$ is a rational normal scroll of type (a', b'), where (a', b') equals to $(a-1, b)$ or $(a, b-1)$. If $p_1(X) = v_2(\mathbb{P}^2)$, then $\pi_y(p_1(X))$ is $Scroll_{1,2}$. In any case, $a' + b' = h - 1$. Using the same analysis as above, we obtain that $a' \geq c + d$, $b' \geq c + e$. Since $a' + b' = h - 1 = c + d + e$, the only possibility is $c = 0$, $d = a'$, $e = b'$. Thus, $p_2(X) = Scroll_{0,a',b'}$ and for points of the basic curve of $Scroll_{a,b}$ of degree a there correspond ruling lines of a natural sub-scroll of type $(0, a')$; for points of the basic curve of $Scroll_{a,b}$ of degree b there correspond ruling lines of a natural sub-scroll of type $(0, b')$. Denote the vertex of $p_2(X)$ by p. Let $\psi_1 : p_1(X) \dashrightarrow Scroll_{0,a',b'}$ be a map, that puts a point x on the line $\psi(x)$, and $\psi_1(p_1(X)) = Scroll_{a',b'}$. Note, that ψ_1 geometrically is the projection from y. Then X is covered by lines of form $\langle (x, p), (x, \psi_1(x))\rangle$. The map $x \mapsto (x, p)$ is just a linear embedding. The map $x \mapsto (x, \psi_1(x))$ is equivalent to $\pi_{v_2(y)} \circ v_2$. Consider a cone $M = Cone_r(p_1(X)) \subset \mathbb{P}^{h+2}$ and an embedding $i : p_1(X) \to M$. Since $p_1(X)$ is a surface of minimal degree, M is a threefold of minimal degree. Take a threefold $X' = \pi_{\langle v_2(r), v_2(i(y))\rangle}(v_2(M))$. Then the exceptional divisor, corresponding to r, is linearly isomorphic to $p_1(X)$. More, $\pi_{\langle v_2(r), v_2(i(y))\rangle} \circ v_2 \circ i$ restricted to $p_1(X)$ is equivalent to $\pi_{v_2(y)} \circ v_2$. Any line on M, passing through the vertex r, goes to a line under $\pi_{\langle v_2(r), v_2(i(y))\rangle} \circ v_2$. Thus, X' is covered by the same family of lines as X. Therefore, $X = X'$. So, $X = \pi_{\langle v_2(r), v_2(y')\rangle}(v_2(M))$, where $M \subset \mathbb{P}^{h+2}$ is a singular threefold of minimal degree, r is its vertex and y' is a smooth point.

Since all the data, i. e. $p_1(X)$, $p_2(X)$, ψ, could be uniquely upto an automorphism of X could be determined, X has unique embedding into $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$ upto an automorphism. □

Lemma 29. *If $\pi(X)$ is smooth and $X \neq v_3(\mathbb{P}^2) \times \mathbb{P}^1$, then a general surface of $\mathcal{L}_1(X)$ is a rational normal scroll of type $(h+1, 2h-1)$. If $X = v_3(\mathbb{P}^2) \times \mathbb{P}^1$, then a general surface of $\mathcal{L}_1(X)$ is a rational normal scroll of type $(6,6)$.*

If $\pi(X)$ is singular, then a general surface of $\mathcal{L}_1(X)$ is a rational normal scroll of type $(h, 2h)$.

Proof. If $\pi(X)$ is smooth and $\psi : p_1(X) \to G(1, h+1)$ is the corresponding map, then a general surface $S \in \mathcal{L}_1(X)$ is swept out by lines $\{x\} \times \mathbb{P}^1_{\psi(x)}$, $x \in p_1(X) \cap H'$, where H' is a hyperplane, and, thus, $p_1(X) \cap H'$ is a rational normal curve of degree h. If $p_1(X)$ is a scroll, then the surface in $p_2(X)$ covered by lines $\psi(x)$, $x \in p_1(X) \cap H'$, is always a scroll of type $(1, h-1)$. Thus, S is a scroll of type $(h+1, 2h-1)$. If $p_1(X) = v_2(\mathbb{P}^2)$, then the corresponding sub-scroll in $p_2(X) = \mathbb{P}^2 \times \mathbb{P}^1$ is of type $(2,2)$. Thus, S is a scroll of type $(6,6)$. Of $\pi(X)$ is singular, then the sub-scroll in $p_2(X) = Scroll_{0,a',b'}$ corresponding to a hyperplane section of $p_1(X)$ is of type $(0,h)$. Thus, S is a scroll of type $(h, 2h)$. □

6.3.2. *X is covered by two families of lines, which are ruling lines of surfaces from $\mathcal{L}_1(X)$, $\mathcal{L}_2(X)$.*

Lemma 30. *Suppose that X is covered by two irreducible families of lines, $U \subset G(1,N)$ is one of these families and $U = \varphi(Y)$, where Y is a rational normal scroll of type (a', b'), $a' \leq b'$, $a' + b' = h$, and φ is given by the complete linear system $|2C + (3a' + b')L|$ for $a' > 0$ and $\varphi = \pi_{v_3(p)} \circ v_3$, for $a' = 0$ and p being the vertex of Y. Then X is a 4-scroll of type (a,b,c), $b = b' - a'$, $2a + b + c = 2h$, $a \geq a'$. The number of non-isomorphic embedding U into $G(1,N)$ equals to $[\frac{h}{2}]$. For such fixed embedding the corresponding embedding of X into $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$ is unique upto an automorphism.*

Proof. Since X is covered by two families of lines, $p_1(X)$ and $p_2(X)$ are surfaces, covered by lines. By Lemma 26, $p_1(X) = Y$ and $p_2(X)$ is a surfaces of minimal degree, i. e. a rational normal scroll of type (a'', b''), $a'' \leq b''$, $a'' + b'' = h$.

As in the case of one family of lines one can construct two maps $\psi_1 : p_1(X) \dashrightarrow G(1, h+1)$ and $\psi_2 : p_2(X) \dashrightarrow G(1, h+1)$ such that X is covered by lines of form $\{x\} \times \mathbb{P}^1_{\psi_1(x)}$, $x \in p_1(X)$, and by lines of form $\mathbb{P}^1_{\psi_2(y)} \times \{y\}$, $y \in p_2(X)$. More, if $(x,y) \in X$ is a general point, then $y \in \mathbb{P}^1_{\psi_1(x)}$ and $x \in \mathbb{P}^1_{\psi_2(y)}$. Hence, for any point $y' \in \mathbb{P}^1_{\psi_1(x)}$ holds $(x, y') \in X$, and, thus, $x \in \mathbb{P}^1_{\psi_2(y')}$. Since there is only one ruling line of $p_1(X)$ passing through x, we obtain that $\psi_2(y) = \psi_2(y')$, i. e. ψ_2 is a constant along a ruling line. The same is true for ψ_1. Therefore, there exist two maps $\psi_1', \psi_2' : \mathbb{P}^1 \to G(1, h+1)$ such that X is covered by quadrics of form $\mathbb{P}^1_{\psi_2'(t)} \times \mathbb{P}^1_{\psi_1'(t)}$, $t \in \mathbb{P}^1$.

Further, one can construct four maps $\xi_1, \xi_2, \eta_1, \eta_2 : \mathbb{P}^1 \to \mathbb{P}^{h+1}$, which give us the basic curves of the scrolls $p_1(X)$, $p_2(X)$ and $\mathbb{P}^1_{\psi_2'(t)} = \langle \xi_1(t), \xi_2(t) \rangle$, $\mathbb{P}^1_{\psi_1'(t)} = \langle \eta_1(t), \eta_2(t) \rangle$. The degree of these curves are a', b', a'', b'' respectively. Consider $Scroll_{a'+a'',a'+b''}$ given by its basic curves $(\xi_1(t), \eta_1(t))$, $(\xi_1(t), \eta_2(t))$, $t \in \mathbb{P}^1$, and $Scroll_{b'+a'',b'+b''}$ given by its basic curves $(\xi_2(t), \eta_1(t))$, $(\xi_2(t), \eta_2(t))$, $t \in \mathbb{P}^1$. By the projection p_2 these scrolls are naturally isomorphic to $p_2(X) = Scroll_{a'',b''}$. Thus, there exists a natural isomorphism between the scrolls τ such that $p_2 = p_2 \circ \tau$. Thus, for a general point $x \in Scroll_{a'+a'',a'+b''}$ the line $\langle x, \tau(x) \rangle$ belongs to X. Hence, X is 4-scroll of type $(a' + a'', b' - a', b'' - a'')$, $2(a' + a'') + (b' - a') + (b'' - a'') = a' + b' + a'' + b'' = 2h$. So, we can determine b, but cannot determine a and c. One has: $a' \leq a \frac{2h-b-c}{2} \leq \frac{2h-b}{2} = \frac{2h-(b'-a')}{2} = \frac{h}{2} + a'$. Thus, the number of different solutions for a and c equals to $[\frac{h}{2}]$. If we know, that X is a 4-scroll of type (a, b, c), then the system $a' + a'' = a$, $b' - a' = b$, $b'' - a'' = c$, $a' + b' = h$, $a'' + b'' = h$ has unique solution. Thus, the embedding of X into $\mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$ is unique. □

Remark 8. If $X = \pi_{\langle v_2(x), v_2(y) \rangle}(v_2(Y))$, where $Y \subset \mathbb{P}^{h+2}$ is a threefold of minimal degree and $x, y \in Y$ are singular points, then Y is a rational normal scroll of type $(0, 0, h)$. X is covered by two families of lines, which are the images of lines on Y passing through x or y. Thus, $X \subset \mathbb{P}^{h+1} \times \mathbb{P}^{h+1}$, $p_1(X) = p_2(X) = Scroll_{0,h}$, and X is a 4-scroll of type $(0, h, h)$.

Lemma 31. *A general surface of* $\mathcal{L}_1(X)$ *is a rational normal scroll of type* $(h + a - a', 2h - a + a')$.

Proof. A general surface of $S' \in \mathcal{L}_1(X)$ is swept out by lines $\{x\} \times \mathbb{P}^1_{\psi_1(x)}$, $x \in p_1(X) \cap H'$, where H' is a hyperplane. Thus, $p_1(X) \cap H'$ is a rational normal curve of degree h. Let $\mu : \mathbb{P}^1 \to p_1(X) \cap H'$ be a map such that $\mu(t) \in \mathbb{P}^1_{\psi_2(t)}$. Then the curves $(\mu, \eta_1(t))$ and $(\mu, \eta_2(t))$ are basic curves for S'. These curves have the degree $h + a''$ and $h + b''$ respectively. Since $a'' = a - a'$ and $b'' = h - a''$, one has that S' is a rational normal scroll of type $(h + a - a', 2h - a + a')$. □

References

[1] J. Alexander, Singularités imposables en position générale à une hypersurface projective, Compositio Math., 68 (1998), 305–354.

[2] J. Alexander and A. Hirschowitz, La méthode d'Horace éclatée: application à l'interpolation en degré quatre, Inventiones Math., 107 (1992), 585–602.

[3] J. Alexander and A. Hirschowitz, Un lemme d'Horace différentiel: application aux singularités hyperquartiques de $\mathbb{P}^6$, J. Algebraic Geom., 1 (1992), 411–426.

[4] J. Alexander and A. Hirschowitz, Polynomial interpolation in several variables, J. Algebraic Geom., 4 (1995), 201–222.

[5] J. Bronowski, Surfaces whose prime sections are hyperelliptic, J. London Math. Soc., 8 (1933), 308–312.

[6] L. Chiantini and C. Ciliberto, Weakly defective varieties, Trans. Amer. Math. Soc. 354 (2002), no. 1, 151–178.

[7] L. Chiantini and C. Ciliberto, Threefolds with degenerate secant variety: on a theorem of G. Scorza, Lecture Notes in Pure and Appl. Math., 217, Dekker, New York, 2001.

[8] L. Chiantini and C. Ciliberto, On the classification of defective threefolds, preprint, arXiv/math.AG/0312518.

[9] T. Fujita, Projective threefolds with small secant varieties, Sci. Papers College Gen. Ed. Univ. Tokyo, 32 (1982), 33–46.

[10] T. Fujita and J. Roberts, Varieties with small secant varieties: the extremal case, Amer. J. of Math., 103 (1981), 1, 953–976.

[11] A. Inshakov, Surfaces with defective tangential varieties, Le Matematiche, vol. LVI (2001), fasc. II, 203–231.

[12] F. Palatini, Sulle superficie algebriche i cui S_h $(h+1)$-seganti non riempionto lo spazio ambiente, Atti. Accad. Torino, (1906), 634–640.

[13] F. Palatini, Sulle varietà algebriche per le quali sono di dimesione minore dell'ordinario, senza riempire lo spazio ambiente, una o alcune delle varietà formate da spazi seganti, Atti. Accad. Torino, 44 (1909), 362–374.

[14] G. Scorza, Determinazione delle varietà a tre dimensioni di S_r, $(r \geq 7)$, i cui S_3 tangenti si tagliano a due a due, Rend. Circ. Mat. Palermo, 25 (1907), 193–204.

[15] G. Scorza, Un problema sui sistemi lineari di curve appartenenti a una superficie algebrica, Rend. R. Ist. Lombardo, (2) 41 (1908), 913–920.

[16] A. Terracini, Su due problemi, consernenti la determinazione di alcune classi di superficie, considerati da G. Scorza e F. Palatini, Atti. Soc. Natur. e Matem. Modena, V, 6 (1921–22), 3–16.

[17] F. Zak, Tangents and secants of algebraic varieties, Transl. Math. Monogr., 127 (1993).

INSTITUT FOURIER
Laboratoire de Mathématiques
UMR 5582 (UJF–CNRS)
BP 74
38402 St-Martin d'Hères Cedex
France

E-mail address: inshakov@mozart.ujf-grenoble.fr

Independent University of Moscow
B. Vlas'evskiy, 11
119002, Moscow
Russia
E-mail address: inshakov@mccme.ru