

INTEGRABILITY OF INDUCTION COCYCLES FOR KAC-MOODY GROUPS

BERTRAND RÉMY

Prépublication de l'Institut Fourier n^o **637** (2004)
<http://www-fourier.ujf-grenoble.fr/prepublications.html>

Abstract: We prove that whenever a Kac-Moody group over a finite field is a lattice of its buildings, it has a fundamental domain with respect to which the induction cocycle is L^p for any $p \in [1; +\infty)$. The proof uses elementary counting arguments for root group actions on buildings. The applications are the possibility to apply some lattice superrigidity, and the normal subgroup property for Kac-Moody lattices.

Keywords: Kac-Moody lattice, fundamental domain, induction cocycle, superrigidity, normal subgroup property.

Mathematics Subject Classification (2000): 22F50, 22E20, 51E24, 53C24, 22E40, 17B67.

INTRODUCTION

Let Λ be an infinite (possibly twisted) Kac-Moody group over a finite field. It acts diagonally on the product $X_- \times X_+$ of its twinned buildings. We may, and shall, assume that the action is faithful (the kernel of the action lies in the finite center of Λ). The Λ -action on a simple factor is not discrete, and we call *geometric completion* of positive (resp. negative) sign the closure $\overline{\Lambda}_+$ (resp. $\overline{\Lambda}_-$) of the image of Λ in the action on the positive (resp. negative) building [RR03, 1.B].

If we set $G := \overline{\Lambda}_- \times \overline{\Lambda}_+$, then Λ can be seen as a discrete subgroup of G via the diagonal embedding. If we denote by $W(t)$ the growth series of the common Weyl group W of X_- and X_+ , then the finiteness of $W(\frac{1}{q})$ implies that Λ is a lattice of G [CG99], [Rem99]. By construction the lattice Λ is *irreducible*, i.e. its projections on the factors $\overline{\Lambda}_\pm$ are dense. Moreover the group Λ is generated by finitely many finite subgroups, which provides a length function ℓ_Λ . To any fundamental domain X for G/Λ is attached a cocycle $\alpha_X : G \times X \rightarrow \Lambda$ by: $\alpha_X(g, x) = \lambda \Leftrightarrow gx\lambda \in X$. This cocycle is useful to induce representations of lattices in Lie groups, and Y. Shalom's work shows that it is a powerful tool to prove deep rigidity results (where the ambient topological groups needn't be Lie groups) [Sha00a], [Sha00b]. Our main purpose is to prove the following.

Theorem. *Let Λ , G and W be as above. Then, there is a fundamental domain D for G/Λ , which is a countable union of compact open subsets $\{D_w\}_{w \in W}$ and such that for any $p \in [1; +\infty)$ and any $g \in G$, we have:*

$$\int_D \ell_\Lambda(\alpha_D(g, d))^p d\mu(d) < +\infty,$$

whenever the minimal order q of the root groups satisfies $W(\frac{1}{q}) < +\infty$.

In other words, for the cocycle α_D to be L^p , there is no further assumption on a Kac-Moody group Λ over a finite field than being a lattice of its buildings. Note that for $p = 2$, the above integrability is nothing else than condition (1.5) of [Sha00a, 1.II p.14]. This enables us to deduce:

Corollary. *All the results valid under the hypothesis (0.1) in the above cited paper by Y. Shalom, are still valid when the uniform lattice is replaced by a Kac-Moody group over a finite field, provided the latter group is a lattice of its twinned buildings, e.g. when $W(\frac{1}{q}) < +\infty$.*

We note that the idea to replace the cocompactness of a closed subgroup by representation-theoretic conditions (and in particular by integrability conditions) appears in [Mar91, III.1]. The results alluded to in the Corollary contain a superrigidity theorem for irreducible lattices, an arithmeticity theorem, a superrigidity theorem for actions on trees... The square integrability is also one ingredient needed in a recent paper by N. Monod, providing a very general superrigidity theorem for actions of irreducible lattices on arbitrary complete CAT(0)-spaces [Mon04], see Subsect. 3.1 for further details. We note that the only so far available superrigidity theorem for Kac-Moody lattices was a commensurator superrigidity [Bon03], which is easier to obtain than a lattice superrigidity. Conversely, we can see Kac-Moody lattices as a substantial enrichment of the list of examples for which one really needs the new rigidity results (with respect to those previously proved in [Mar91]).

Still, the main application of the square-integrability we have in mind is the normal subgroup property for Kac-Moody lattices, a well-known property for irreducible lattices of higher-rank Lie groups over local fields [Mar91, VIII.2]. In our case, this is a joint work with U. Bader and Y. Shalom which uses a general result due to them about amenability of factor groups of irreducible lattices [BS03], and a result due to Y. Shalom about property (T) for the same quotients [Sha00a]. This provides the following (see [BS03, Theorem 1.5] and Theorem 21 of the present paper):

Theorem (with U. Bader and Y. Shalom). *Let Λ be a Kac-Moody group over a finite field, with irreducible Weyl group. Assume it is a lattice of the product of its twinned buildings. Then any normal subgroup of Λ either has finite index or lies in the finite center $Z(\Lambda)$.*

Using a result on just infinite groups due to J.S. Wilson, we can then prove (Corollary 22):

Corollary. *If Λ is center-free, e.g. because it is adjoint, and if it is not residually finite, then Λ is virtually simple.*

Note that many Kac-Moody groups over finite fields are not linear over any field [Rem03b], therefore are potentially non-residually finite, but no example of a non-residually finite Kac-Moody group has been given yet.

This paper is organized as follows. In the first section, we recall the basic combinatorial notions for groups with twin root data and their geometric completions, and we describe the fundamental domain D in these terms. In the second section, we prove the above L^p -integrability. The proof uses a quantitative version of the fact that a fundamental domain for the diagonal action of a group with twin root datum on the product of its twinned buildings, is the product of a negative chamber by a suitable positive apartment. In the third section, we provide applications of the integrability. We first show that this enables to apply superrigidity results proved by Y. Shalom [Sha00a], and leads to ask whether N. Monod's more recent work [Mon04] can be applied. The second application is group-theoretic, it proves that Kac-Moody lattices enjoy the normal subgroup property.

It is a great pleasure to thank Uri Bader, Nicolas Monod and Yehuda Shalom. Their comments and encouragements were extremely useful to write down this paper.

1. TWIN BUILDING, AUTOMORPHISM GROUPS, FUNDAMENTAL DOMAIN AND COCYCLE

We introduce some automorphism groups of twin buildings, and we show that their combinatorial properties enable to construct nice fundamental domains.

1.1. Twin building and automorphism groups. Let (W, S) be a Coxeter system [Bou81, IV.1]. Our starting point is a group Λ admitting a twin root datum $(\Lambda, \{U_a\}_{a \in \Phi}, H)$ whose root groups U_a are indexed by the root system Φ of (W, S) [Tit87], [Rem02, 5.4.1]. We assume that the root groups U_a , as well as the subgroup H , are all finite and that the Weyl group W is infinite. This is the context chosen in [RR03, Sect. 1] to define topological groups generalizing semisimple groups over local fields of positive characteristic. We will recall briefly the main notions, properties and references; further details and motivations are given in [Rem03a, §3].

The notion of a twin root datum was given in a paper by J. Tits [Tit92]. In the axioms, the Weyl group W appears as the image of a quotient map $\nu : N \rightarrow W$ with kernel H , where N is a subgroup of Λ . The group H normalizes each root group U_a and we have: $nU_a n^{-1} = U_{\nu(n).a}$ for any root $a \in \Phi$ and any $n \in N$. It is also required that any element in N lifting a reflection s in the canonical generating set S of W , lies in the finite subgroup $\langle U_{a_s}, U_{-a_s}, H \rangle$, where a_s is the simple root attached to s . Each group $M_s := \langle U_{a_s}, U_{-a_s}, H \rangle$ may be seen as a generalized rank one finite group of Lie type (it is, strictly speaking, a finite group of Lie type when Λ is a Kac-Moody group over a finite field). The root groups $\{U_a\}_{a \in \Phi}$ and H are required to satisfy other additional properties for which we refer to [Rem02, 1.6]. At last, Λ is generated by H and the U_a 's, so in view of the previous remarks, the group Λ is finitely generated.

Definition 1. (i) We denote by ℓ_W the length function on W associated to the canonical generating set S .
 (ii) We denote by ℓ_Λ the length function on Λ associated to the finite set of generators given by the union of the groups M_s , when s ranges over S .

In what follows, we are mainly interested in the geometric counterpart to this, which involves the structure of a *building* for which we adopt the viewpoint of chamber systems [Ron89, §2]. To Λ is associated a twin building (X_+, X_-, w^*) together with a twin apartment of reference $\Sigma = \Sigma_- \sqcup \Sigma_+$ and a pair of opposite chambers $\{c_-; c_+\}$ in Σ [Tit92], [Abr97], [Rem02, §2]. By non-triviality and finiteness of the root groups, the buildings X_\pm are thick and locally finite. «Thickness» means that for any *panel* (i.e. any codimension one simplex) Π , the set of chambers whose closure contains Π has at least three elements. We denote by B_+ (resp. B_-) the fixator of the positive (resp. negative) chamber c_+ (resp. c_-) in Λ ; it contains as a finite index subgroup the group U_+ (resp. U_-) generated by the root groups indexed by the positive (resp. negative) roots.

The codistance w^* is a map $(X_- \times X_+) \sqcup (X_+ \times X_-) \rightarrow W$ defined thanks to the Birkhoff decomposition of Λ [Abr97, §2]. The group of twin building automorphisms of (X_+, X_-, w^*) is the subgroup A of couples $(g_-, g_+) \in \text{Aut}(X_-) \times \text{Aut}(X_+)$ which satisfy $w^*(g_- . c_-, g_+ . c_+) = w^*(c_-, c_+)$ for any couple of chambers $(c_-, c_+) \in X_- \times X_+$. We have: $\Lambda < A$.

The twin building (X_+, X_-, w^*) has the *Moufang property* [Ron89, §6]: if we identify Φ with the set of twin roots in Σ , this roughly means that for any twin root $a \subset \Sigma$ and any chamber c having a panel Π in the wall ∂a , the root group U_a fixes the half twin apartment $a \subset \Sigma$ and acts simply transitively on the chambers different from c and containing Π . This explains why the local finiteness of the buildings X_\pm amounts to the finiteness of the root groups.

We can now turn to topology. We denote by $\text{Aut}(X_\pm)$ the group of all type-preserving building automorphisms of X_\pm . For the compact open topology, in which a fundamental system of neighborhoods of the identity is given by fixators of finite subsets of chambers, the group $\text{Aut}(X_\pm)$ is locally compact.

Definition 2. We denote by $\overline{\Lambda}_\pm$ the closure in $\text{Aut}(X_\pm)$ of the image of the Λ -action on X_\pm and we call it the geometric completion of Λ of sign \pm . The fixator of the chamber c_\pm in $\overline{\Lambda}_\pm$ is called the standard Iwahori subgroup of $\overline{\Lambda}_\pm$ and is denoted by \mathcal{B}_\pm . We denote by \widehat{U}_\pm the closure of U_\pm in $\overline{\Lambda}_\pm$, and for any $w \in W$ we introduce the group $\widehat{U}_-^w := \widehat{U}_- \cap w^{-1} \widehat{U}_- w$.

These groups were introduced in [RR03, 1.B], where it was checked that $(\overline{\Lambda}_+, N, \widehat{U}_+, U_-, H, S)$ and $(\overline{\Lambda}_-, N, \widehat{U}_-, U_+, H, S)$ both satisfy the axioms of a *refined Tits system* whenever Λ is an infinite Kac-Moody group over a finite field. This notion is due to V. Kac and D. Peterson [KP85] and its basic properties will be used in the sequel with appropriate references.

Assumption 3. *Until the end of the paper, the group Λ is assumed to be an infinite Kac-Moody group over a finite field.*

All the results of the paper remain valid for groups with twin root data with infinite Weyl groups and finite root groups and whose geometric completions are refined Tits systems. This is the case when $H = \{1\}$, which is possible for some exotic Moufang twin buildings [AR03, Example 69].

1.2. Fundamental domain and cocycle. We keep the twin apartment of reference $\Sigma = \Sigma_- \sqcup \Sigma_+$, the standard pair of opposite chambers $\{c_-; c_+\}$, and we now introduce a remarkable subset of G .

Definition 4. *For each $w \in W$, we denote by D_w the subset $\widehat{U}_-^w \times \mathcal{B}_+ w$ of G . We denote by D the disjoint union $\bigsqcup_{w \in W} D_w$.*

Here is the main property of D .

Proposition 5. *For any $g = (g_-, g_+) \in G$, there is a unique $\lambda \in \Lambda$ and a unique $w \in W$ such that $g_- \lambda \in \widehat{U}_-^w$ and $g_+ \lambda \in \mathcal{B}_+ w$, i.e. D is a fundamental domain for G/Λ .*

Proof. We use the right action on $X_- \times X_+$ defined by: $(d_-, d_+).(h_-, h_+) = (h_-^{-1}.d_-, h_+^{-1}.d_+)$ for any $(h_-, h_+) \in \text{Aut}(X_-) \times \text{Aut}(X_+)$ and any pair of chambers of opposite signs $\{d_-; d_+\}$. We argue on the G -transforms of the standard couple of chambers (c_-, c_+) . Since any pair of chambers is contained in a twin apartment [Abr97, Lemma 2 p.24] and since the diagonal Λ -action is transitive on the set of twin apartments [Abr97, Lemma 4 p.29], there exist $\delta \in \Lambda$ and $w \in W$ such that $(c_-, c_+).(g_-, g_+).\delta = (c_-, w^{-1}.c_+)$, so we have $g_- \delta \in \mathcal{B}_-$ and $g_+ \delta \in \mathcal{B}_+ w$.

By standard properties of refined Tits systems [Rem02, 1.2.3], we have $\mathcal{B}_- = \widehat{U}_- \rtimes T = \widehat{U}_-^w.U_{-w}.T$, with $U_{-w} = \widehat{U}_- \cap w^{-1}U_+w = U_- \cap w^{-1}U_+w$ and $\widehat{U}_-^w = \widehat{U}_- \cap w^{-1}\widehat{U}_-w$. We can thus write: $g_- \delta = \widehat{u}_-^w u_{-w} t$, with $\widehat{u}_-^w \in \widehat{U}_-^w$, $u_{-w} \in U_{-w}$ and $t \in T$. If we set $\lambda := \delta t^{-1}(u_{-w})^{-1} \in \Lambda$, we have that $g_- \lambda = g_- \delta t^{-1}(u_{-w})^{-1}$ lies in \widehat{U}_-^w . Moreover writing $g_+ \delta = b_+ w$ with $b_+ \in \mathcal{B}_+$, we get: $g_+ \lambda = b_+(w t^{-1} w^{-1})(w u_{-w} w^{-1})w$. Since T is normalized by W , we finally obtain: $g_+ \lambda \in \mathcal{B}_+ w$ by definition of U_{-w} .

This proves that D contains a representative for each class of G/Λ . It remains to check that $\lambda \in \Lambda$ and $w \in W$ are uniquely determined by (g_-, g_+) and the conditions $g_- \lambda \in \widehat{U}_-^w$ and $g_+ \lambda \in \mathcal{B}_+ w$. Assume there exist $\delta \in \Lambda$ and $z \in W$ such that $g_- \delta \in \widehat{U}_-^z$ and $g_+ \delta \in \mathcal{B}_+ z$. We have $\lambda^{-1} g_-^{-1} c_- = c_-$ and $\lambda^{-1} g_+^{-1} c_+ = w^{-1} c_+$; and since the diagonal Λ -action preserves the codistance w^* between chambers of opposite signs, this gives $w^*(g_-^{-1}.c_-, g_+^{-1}.c_+) = w^*(\lambda^{-1} g_-^{-1}.c_-, \lambda^{-1} g_+^{-1}.c_+) = w^*(c_-, w^{-1}.c_+) = w^{-1}$. We can do the same computation with λ replaced by δ and w replaced by z , to get $w = z$.

It remains to compute: $g_- \delta = (g_- \lambda)(\lambda^{-1} \delta)$, which implies $\lambda^{-1} \delta \in \widehat{U}_-^w$. Moreover we have: $\lambda^{-1} g_+^{-1}.c_+ = w^{-1}.c_+$, but also: $\lambda^{-1} g_+^{-1}.c_+ = (\lambda^{-1} \delta).(\delta^{-1} g_+^{-1}.c_+) = (\lambda^{-1} \delta).(w^{-1}.c_+)$. This shows that $\lambda^{-1} \delta$ fixes the chamber $w^{-1}.c_+$, hence belongs to $w^{-1}\mathcal{B}_+ w$. We have: $\lambda^{-1} \delta \in \widehat{U}_-^w \cap w^{-1}\mathcal{B}_+ w$, which provides: $w(\lambda^{-1} \delta)w^{-1} \in \mathcal{B}_+ \cap w\widehat{U}_-^w w^{-1} \cap \Lambda < \mathcal{B}_+ \cap \widehat{U}_- \cap \Lambda = \mathcal{B}_+ \cap U_- = \{1\}$ [KP85, Axiom (RT3)]. We finally obtain: $\lambda = \delta$. \square

This enables to recover a basic result on the existence of lattices for Kac-Moody buildings [Rem99]. We normalize the right Haar measure μ_{\pm} on $\text{Aut}(X_{\pm})$ so that $\mu_{\pm}(\widehat{U}_{\pm}) = 1$, and set $\mu := \mu_- \otimes \mu_+$.

Corollary 6. *The μ -volume of D_w is $\frac{|T|}{|U_{-w}|}$; the group Λ is a lattice of G whenever $\sum_{w \in W} \frac{1}{|U_{-w}|}$ converges. The latter condition is fulfilled when the ground field \mathbf{F}_q satisfies $W(\frac{1}{q}) < +\infty$.*

Proof. We have: $\text{Vol}(D_w, \mu) = \mu_-(\widehat{U}_-^w) \cdot \mu_+(\mathcal{B}_+ w) = \frac{1}{|U_{-w}|} \cdot \mu_-(\widehat{U}_-) \cdot \mu_+(\mathcal{B}_+)$. The second equality follows from $\widehat{U}_- = \widehat{U}_-^w \cdot U_{-w}$ [Rem02, 1.2.3], and the first assertion follows from $\mathcal{B}_+ = T \times \widehat{U}_+$ [loc. cit.]. The second assertion is then obvious, and the third one follows from the existence of a bijection between U_{-w} and the product of $\ell_W(w)$ suitable root groups, all having at least q elements (see also the proof of Lemma 16 for details). \square

We can now introduce the cocycle we are interested in.

Definition 7. *We define $\alpha = \alpha_D : G \times D \rightarrow \Lambda$ by setting $\alpha((g_-, g_+), (u_-^w, b_+ w)) = \lambda$ if and only if $g_- u_-^w \lambda \in \widehat{U}_-^z$ and $g_+ b_+ w \lambda \in \mathcal{B}_+ z$ for some $z \in W$. In other words, denoting by d the element $(u_-^w, b_+ w)$ of D_w , we set $\alpha(g, d) = \gamma$ if and only if $gd\gamma \in D$.*

Remark 8. In terms of this cocycle, the first two paragraphs of the proof of Proposition 5 show that, given $g = (g_-, g_+) \in G$, an element $\delta \in \Lambda$ such that $\delta^{-1} g_-^{-1} \cdot c_- = c_-$ and $\delta^{-1} g_+^{-1} \cdot c_+ = w^{-1} \cdot c_+$ for some $w \in W$, is an approximation modulo TU_{-w} of $\alpha(g, 1_G)$. This is used to prove Lemma 16.

Remark 9. What we call «cocycle» leads to the relation: $\alpha(gh, x) = \alpha(h, x)\alpha(g, h.x)$; in order to have a true cocycle, i.e. $\alpha(gh, x) = \alpha(g, h.x)\alpha(h, x)$, we should set $\alpha(g, d) = \gamma^{-1}$ if and only if $gd\gamma \in D$. Since we want to apply Y. Shalom's results, we adopted his «anticocycle» convention.

2. INTEGRABILITY

We can now proceed to the proof of the integrability of the length of the cocycle. We first prove some geometric inequalities, basically following from counting root group actions. Then we use these inequalities to show that the integrability of the cocycle only depends on the convergence of a power series which is very close to the one computing the covolume in Corollary 6.

We keep the infinite Kac-Moody group Λ over \mathbf{F}_q with twin root datum $(\Lambda, \{U_a\}_{a \in \Phi}, H)$. We will still use (X_+, X_-, w^*) its thick, locally finite, Moufang twin building, as well as the twin apartment of reference $\Sigma = \Sigma_- \sqcup \Sigma_+$ and the standard pair of opposite chambers $\{c_-; c_+\}$ in Σ .

2.1. Geometric inequalities. The following result is a quantitative version of the fact that the diagonal Λ -action on pairs of chambers of opposite signs admits $\{c_-\} \sqcup \Sigma_+$ as fundamental domain.

Proposition 10. *Let $\{d_-, d_+\}$ be a pair of chambers of opposite signs. We introduce the combinatorial distances $L_- := \text{dist}(c_-, d_-)$ and $L_+ := \text{dist}(c_+, d_+)$, and their sum $L := L_- + L_+$. Then, there exist $\lambda \in \Lambda$ and $w \in W$ with: $\lambda^{-1} \cdot d_- = c_-$ and $\lambda^{-1} \cdot d_+ = w^{-1} \cdot c_+$, whose lengths satisfy: $\ell_\Lambda(\lambda) \leq 2L^2 + 3L$ and $\ell_W(w) \leq L$.*

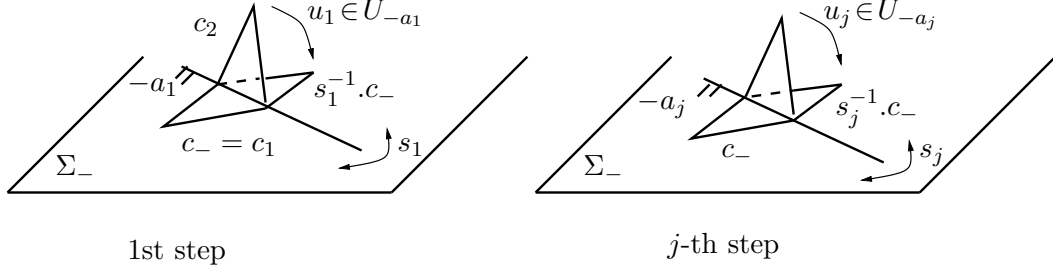
Proof. The proof is divided into three steps.

Step 1: negative chambers (see Picture 11). Let $c_- = c_1, c_2, \dots, c_i, c_{i+1}, \dots, c_n = d_-$ be a minimal gallery from c_- to d_- , such that $\{c_i; c_{i+1}\}$ is a pair of s_i -adjacent chambers for some $s_i \in S$. We have $n = L_- + 1$. For each i , we denote by a_i the simple root attached to the reflection s_i . By minimality we have $c_1 \neq c_2$, so the Moufang property [Ron89, 6.4] implies that there is a unique $u_1 \in U_{-a_1}$ such that $s_1 u_1 \cdot c_2 = c_1$. Then $s_1 u_1 \cdot c_2, \dots, s_1 u_1 \cdot c_i, s_1 u_1 \cdot c_{i+1}, \dots, s_1 u_1 \cdot c_n$ is a minimal gallery from c_- . Therefore, there is a unique $u_2 \in U_{-a_2}$ such that $s_2 u_2 s_1 u_1 \cdot c_3 = c_-$. We iterate the procedure. After the $(j-1)$ -th step, we have: $s_{j-1} u_{j-1} \dots s_2 u_2 s_1 u_1 \cdot c_j = c_-$, and we are interested in the chamber $s_{j-1} u_{j-1} \dots s_2 u_2 s_1 u_1 \cdot c_{j+1}$. It is a chamber s_j -adjacent to c_- and different from c_- in X_- , so by the Moufang property there is a unique $u_j \in U_{-a_j}$ such that $u_j s_{j-1} u_{j-1} \dots s_2 u_2 s_1 u_1 \cdot c_{j+1}$ is the unique

chamber $s_j^{-1}.c_-$ to be s_j -adjacent to c_- and different from c_- in Σ_- . We eventually obtain an element $\delta = (s_{n-1}u_{n-1})\dots(s_2u_2)(s_1u_1)$ such that

$$\delta.d_- = c_- \quad \text{and} \quad \ell_\Lambda(\delta) \leq 2L_- \leq 2L.$$

Picture 11.



Step 2: signs of roots. We set $z := s_{n-1}\dots s_2s_1$.

Claim 12. *We have: $\delta \in U_+.z$.*

Proof of the claim. Let us write:

$$\delta = (s_{n-1}u_{n-1}s_{n-1}^{-1}) \cdot (s_{n-1}s_{n-2}u_{n-2}s_{n-2}^{-1}s_{n-1}^{-1}) \dots (s_{n-1}\dots s_j u_j s_j^{-1} \dots s_{n-1}^{-1}) \dots (s_{n-1}\dots s_1 u_1 s_1^{-1} \dots s_{n-1}^{-1})z.$$

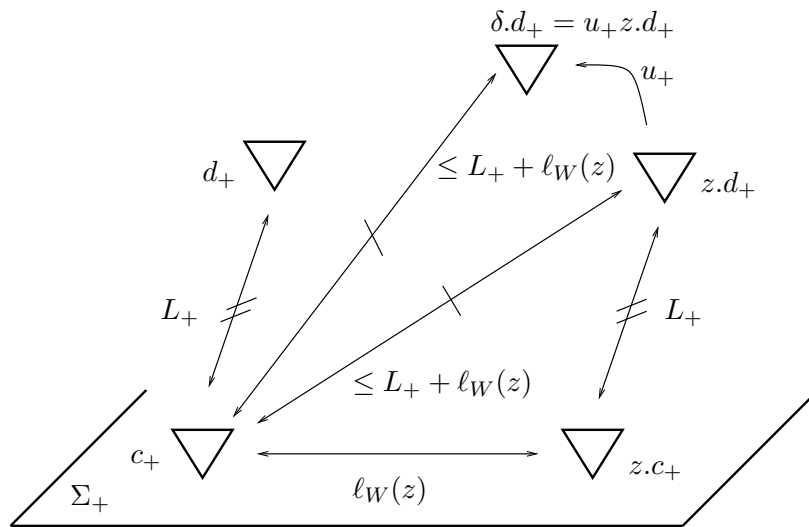
For each j , we set $\delta_j = s_{n-1}\dots s_j u_j s_j^{-1} \dots s_{n-1}^{-1} \in U_{s_{n-1}\dots s_{j+1}.a_j}$. It is enough to show that each factor δ_j lies in a positive root group. Recall the combinatorial definition of the root system Φ [Tit87, 5.1]: the simple root attached to the canonical reflection $s_j \in S$ is $a_j = \{w \in W \mid \ell_W(s_j w) > \ell_W(w)\}$, and its opposite is $-a_j = \{w \in W \mid \ell_W(s_j w) < \ell_W(w)\}$; an arbitrary root $w.a_s$ is defined by translation of a simple root by a suitable element of the Weyl group W . In this description, a root is positive if and only if it contains 1_W . Now the fact that $s_{n-1}\dots s_2s_1$ is a reduced word implies that $s_{n-1}\dots s_{j+1}.a_j$ is a positive root for any j . \square

We can therefore write $\delta = u_+.z$ for some $u_+ \in U_+$, and because u_+ fixes c_+ we have:

$$\text{dist}(c_+, \delta.d_+) = \text{dist}(c_+, z.d_+) \leq \text{dist}(c_+, z.c_+) + \text{dist}(z.c_+, z.d_+) = \ell_W(z) + L_+ = L,$$

see also Picture 13.

Picture 13.

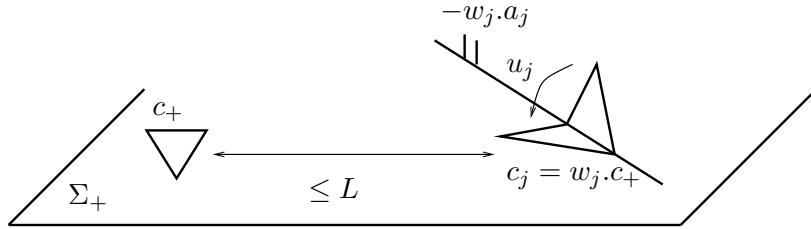


Step 3: negative root groups acting on the positive side.

Claim 14. *There is an element $u_- \in U_-$ such that $u_-u_+z.d_+ \in \Sigma_+$ and $\ell_\Lambda(u_-) \leq L(2L+1)$.*

Proof of the claim (see Picture 15). First, if $u_+z.d_+ \in \Sigma_+$, there is nothing to do because $u_+z.d_+$ is a chamber at combinatorial distance $\leq L$ from c_+ , and as such can be written $w^{-1}.c_+$ for $w \in W$ with $\ell_W(w) \leq L$. If the combinatorial distance $\text{dist}(\Sigma_+, u_+z.d_+)$ from Σ_+ to $u_+z.d_+$ is positive, then we choose a gallery $c_1 = c_+, c_2, \dots, c_n = u_+z.d_+$ of length $\leq L$, and we denote by j the smallest index such that for any $i > j$ we have $c_i \notin \Sigma_+$. The chambers c_j and c_{j+1} are s_j -adjacent, and there is a unique $w_j \in W$ such that $c_j = w_j.c_+$ and $\ell_W(w_j) = \text{dist}(c_+, c_j) \leq L$. By the Moufang property, there is $u_j \in w_j U_{-a_j} w_j^{-1}$ such that $u_j.c_{j+1} = c_j$. Note that $\ell_\Lambda(u_j) \leq \ell_W(w_j) + 1 + \ell_W(w_j) \leq 2L + 1$. We obtain a new gallery $c_1 = c_+, c_2, \dots, c_j, u_j.c_{j+2}, \dots, u_j u_+z.d_+$, with $\text{dist}(\Sigma_+, u_j u_+z.d_+) \leq \text{dist}(\Sigma_+, u_+z.d_+) - 1$. Iterating the use of a suitable element u_i of length $\leq 2L + 1$ in a negative root group $w_i U_{-a_i} w_i^{-1}$ ($i > j$), we obtain a sequence of at most $L - j$ elements of U_- of length $\leq 2L + 1$ whose product, say u_- , sends $u_+z.d_+$ to a chamber in Σ_+ , at distance $\leq L$ from c_+ . \square

Picture 15.



Proof of Proposition, conclusion. The chamber $u_-u_+z.d_+$ is at combinatorial distance $\leq L$ from c_+ , so it can be written $w^{-1}.c_+$ with $\ell_W(w) \leq L$. Therefore, setting $\lambda^{-1} := u_-u_+z$, we have:

$$\ell_\Lambda(\lambda) \leq 2L + L(2L + 1) = 2L^2 + 3L \quad \text{and} \quad \ell_W(w) \leq L,$$

as well as:

$$\lambda^{-1}.d_- = (u_-u_+z).d_- = u_-.c_- = c_- \quad \text{and} \quad \lambda^{-1}.d_+ = w^{-1}.c_+.$$

\square

2.2. Computation. We can now turn to the proof of the main result, i.e. the finiteness of the integral:

$$\int_D \ell_\Lambda(\alpha(g, d))^p d\mu(d),$$

for any $p \in [1; +\infty)$ and any $g \in G$.

First, since 1_G belongs to the fundamental domain D and since $\lambda = \alpha(g, d) \Leftrightarrow gd\lambda \in D$, we have: $\alpha(g, d) = \alpha(gd, 1_G)$ for any $g \in G$ and any $d \in D$. Therefore it is enough to show:

$$\int_D \ell_\Lambda(\alpha(gd, 1_G))^p d\mu(d) < +\infty,$$

for any $p \in [1; +\infty)$ and any $g \in G$. We start with two lemmas.

Lemma 16. *Let $h = (h_-, h_+) \in G$. Let us set: $L_+(h) = \text{dist}(c_+, h_+^{-1}.c_+)$, $L_-(h) = \text{dist}(c_-, h_-^{-1}.c_-)$ and $L(h) = L_+(h) + L_-(h)$. Then we have: $\ell_\Lambda(\alpha(h, 1_G)) \leq P(L(h))$, with $P(X) = 3X^2 + 3X + 1$.*

Proof. We take $\lambda \in \Lambda$ and $w \in W$ given by Proposition 10 and the choices $d_- = h_-^{-1}.c_-$ and $d_+ = h_+^{-1}.c_+$. We have $\alpha(h, 1_G)\lambda^{-1} \in TU_{-w}$ by Remark 8. From standard facts on twin root data

[Rem02, Lemma 1.5.2], the group U_{-w} is in bijection with a product $\prod_{\beta} U_{\beta}$, where β runs in a suitable order over the $\ell_W(w)$ negative roots such that $w.\beta > 0$. If $w = s_n \dots s_2 s_1$ is a reduced word, i.e. if $n = \ell_W(w)$, the roots β under consideration are $-a_1, -s_1.a_2, \dots, -s_1 s_2 \dots s_{n-1}.a_n$, where a_i is the simple root attached to the canonical reflection s_i . Therefore we have:

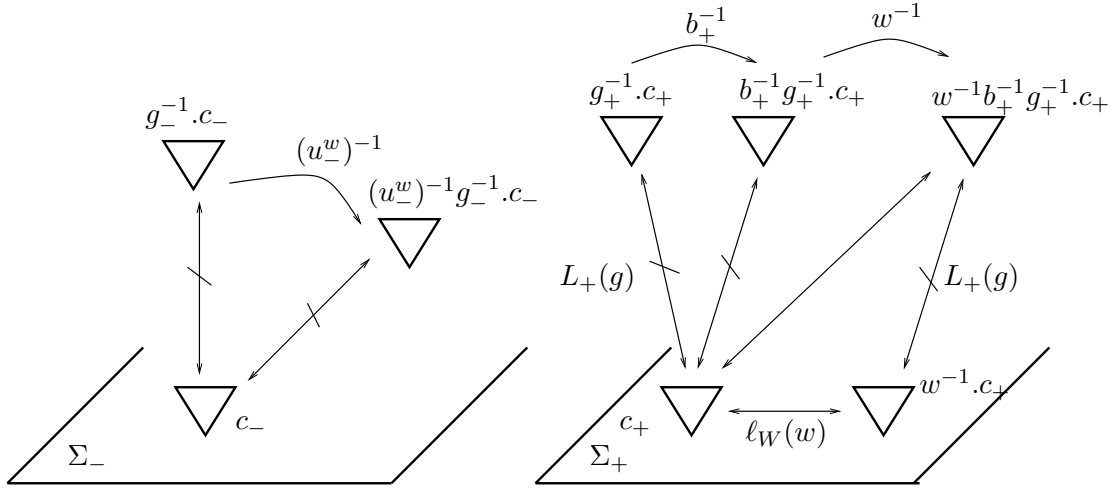
$$\ell_{\Lambda}(\alpha(h, 1_G)\lambda^{-1}) \leq 1 + \sum_{j=0}^{\ell_W(w)-1} (2j+1) = 1 + \ell_W(w)^2 \leq 1 + L^2$$

(the first term comes from the factor in T), and we finally obtain the required inequality. \square

Lemma 17. *Let $g = (g_-, g_+) \in G$. Let $d = (u_-^w, b_+ w)$ be in the subset D_w of D . Then we have: $\ell_{\Lambda}(\alpha(gd, 1_G)) \leq Q(\ell_W(w))$, with $Q(X) = 3X^2 + (6L(g) + 3)X + (3L(g)^2 + 3L(g) + 1)$.*

Proof (see Picture 18). Since u_-^w belongs to \widehat{U}_-^w , hence to \mathcal{B}_- , it fixes c_- so that we have the equality: $L_-(gd) = \text{dist}(c_-, (u_-^w)^{-1}g_-^{-1}.c_-) = \text{dist}(u_-^w.c_-, g_-^{-1}.c_-) = L_-(g)$. We have $b_+ \in \mathcal{B}_+$, so $L_+(gd) = \text{dist}(c_+, w^{-1}b_+^{-1}g_+^{-1}.c_+) \leq L_+(g) + \ell_W(w)$. This finally implies $L(gd) \leq L(g) + \ell_W(w)$, and it remains to apply the previous lemma. \square

Picture 18.



At last, we can go back to the integral.

Proof of the main theorem. We can now compute:

$$\begin{aligned} \int_D \ell_{\Lambda}(\alpha(gd, 1_G))^p d\mu(d) &= \sum_{w \in W} \left(\int_{D_w} \ell_{\Lambda}(\alpha(gd, 1_G))^p d\mu(d) \right) \\ &\leq \sum_{w \in W} \left(\int_{D_w} Q(\ell_W(w))^p d\mu(d) \right) \\ &= \sum_{w \in W} \left(Q(\ell_W(w))^p \cdot \text{Vol}(D_w, \mu) \right) \\ &\leq |T| \cdot \sum_{w \in W} \left(\frac{Q(\ell_W(w))^p}{q^{\ell_W(w)}} \right) \\ &= |T| \cdot \sum_{n \in \mathbf{N}} c_n \frac{Q(n)^p}{q^n}. \end{aligned}$$

The first equality follows from $D = \bigsqcup_{w \in W} D_w$. The first inequality follows from Lemma 17. The second inequality follows from the equality $\mu(D_w) = \frac{|T|}{|U_{-w}|}$ (Corollary 6) and the existence of a bijection between U_{-w} and the product of $\ell_W(w)$ root groups, all of order at least q . The last equality follows from rearranging the elements of the Weyl group with respect their length, which makes appear the growth series $W(t) = \sum_{n \in \mathbf{N}} c_n t^n$. \square

3. APPLICATIONS AND QUESTIONS

We mention the two main applications of the square-integrability of the cocycle with respect to the fundamental domain provided by our main result. The point is that this property enables to use induction for 1-cohomology with unitary coefficients (in the spirit of [Sha00a]) and more generally in the context of actions on CAT(0)-spaces (as introduced in [Mon04]). We first deal with applications to rigidity theory, and then state a normal subgroup theorem for Kac-Moody lattices.

3.1. Lattice superrigidity. The square-integrability condition:

$$\int_D \ell_\Lambda(\alpha_D(g, d))^2 d\mu(d) < +\infty$$

is a sufficient condition to induce the first (reduced) cohomology of the unitary representations of the irreducible lattice Λ to the first (reduced) continuous cohomology of the ambient topological group $G = \overline{\Lambda}_- \times \overline{\Lambda}_+$ [Sha00a, Proposition 1.11]. Induction in cohomology is the starting point of Y. Shalom's proof of the series of results alluded to in the first corollary of the introduction, i.e. all the results in [loc. cit.] for which the hypotheses are **(0.1)**. The datum of a uniform irreducible lattice Γ in a compactly generated product of topological groups $G = G_1 \times \dots \times G_n$ has to be replaced by the datum of a Kac-Moody lattice Λ in the product of its two geometric completions $\overline{\Lambda}_- \times \overline{\Lambda}_+$. The fact that the groups $\overline{\Lambda}_\pm$ are compactly generated follows easily from a suitable Bruhat decomposition [Rem03b, Corollary 1.B.1]. The so-obtained results are (see [Sha00a, Introduction]): property (T) for proper quotients (Theorem 0.1), superrigidity of homomorphisms to groups containing some rank one lattices (Theorem 0.3), arithmeticity of some images (Theorem 0.5), some strong rigidity (Theorem 0.6), superrigidity of actions on trees (Theorem 0.7) and of characters (Theorem 0.8).

The idea to replace a cocompactness by an integrability assumption appears in [Mar91, III.1] and in [Sha00a]. The proof of the square-integrability of induction cocycles for non-uniform irreducible lattices in products of algebraic groups over local fields [Sha00a, §2] uses reduction theory for S -arithmetic groups (as summed up in [Mar91, VIII.1]), and the relation between word and Riemannian metrics for lattices in higher-rank semisimple Lie groups [LMR01]. Thus, from this point of view, Sect. 2 is a light substitute for the latter two deep results, in the Kac-Moody case. (The intersection between Kac-Moody groups over finite fields and S -arithmetic groups is non-empty; it contains the affine Kac-Moody type, corresponding to the points over $\mathbf{F}_q[t, t^{-1}]$ of simply connected Chevalley groups, i.e. $\{0; \infty\}$ -arithmetic groups.)

We conclude this subsection by a potential generalization of the previously quoted results to superrigidity results for actions on CAT(0)-spaces. In [Mon04], such actions of uniform irreducible lattices are induced to actions of the ambient product of topological groups on much bigger (non-proper) CAT(0)-spaces. This, together with geometric splitting theorems in order to suitably generalize the Zariski density assumption, leads to the rigidity theorems.

Question 19. *To what extent can irreducible uniform lattices be replaced by Kac-Moody non-uniform lattices in N. Monod's work?*

Under the square-integrability condition, there is no further obstruction to induce actions in the context of metric spaces. Still, uniformness of the lattice is used elsewhere in [Mon04], namely

to control the behaviour under induction of *evanescence*, a notion introduced to generalize the existence of a fixed point in $\partial_\infty X$ for actions on non proper CAT(0)-spaces. The results of [loc. cit.] are valid when the irreducible uniform lattices are replaced by irreducible square-integrable and weakly cocompact lattices. Recall that a lattice Γ in G is called *weakly cocompact* if the orthogonal complement $L_0^2(G/\Gamma)$ of the constant functions in $L^2(G/\Gamma)$ doesn't weakly contain the trivial one-dimensional representation [Mar91, p.111]. Weak cocompactness is fulfilled when the lattice has property (T), and the latter property is shown to often hold for Kac-Moody lattices [DJ02]. For instance, applying [Mon04, Theorem 6] gives:

Corollary 20. *Let $\Lambda < \overline{\Lambda}_- \times \overline{\Lambda}_+$ be a Kac-Moody lattice, which is assumed to be Kazhdan. Let $H < \text{Isom}(X)$ be a closed subgroup, where X is any complete CAT(0)-space. Let $\tau : \Lambda \rightarrow H$ be a homomorphism with reduced unbounded image. Then τ extends to a continuous homomorphism $\tilde{\tau} : \overline{\Lambda}_- \times \overline{\Lambda}_+ \rightarrow H$.*

We refer to [loc. cit., Appendix B] for a detailed proof of the fact that the main results of this article can be applied to Kac-Moody lattices with property (T).

3.2. Normal subgroup theorem. The main application of our main theorem is group-theoretic; it is the following statement, proved with U. Bader and Y. Shalom.

Theorem 21. *Let Λ be a Kac-Moody group over a finite field, with irreducible Weyl group. Assume it is a lattice of the product of its twinned buildings. Then any normal subgroup of Λ either has finite index or lies in the finite center $Z(\Lambda)$.*

Reference. The details of the proof are provided in [BS03, §4 p.27]. The strategy at large scale is the same as Margulis': in order to prove that a factor group is finite, one proves that it is both amenable and Kazhdan. The amenability half follows from [loc. cit., Theorem 1.3] and it doesn't use any uniformness or square-integrability assumption. The property (T) half is [Sha00a, Theorem 0.1], and it does need the square-integrability property, referred to as **S-I** in [BS03]. \square

From our point view, the normal subgroup property has the following interesting consequence, which reduces (virtual) simplicity to non-residual finiteness for finitely generated Kac-Moody groups with irreducible Weyl groups.

Corollary 22. *If Λ is center-free, e.g. because it is adjoint, and if it is not residually finite, then Λ is virtually simple.*

Proof. The group Λ is just infinite, i.e. all its proper quotients are finite. Set $N := \bigcap_{[\Lambda:\Delta] < \infty} \Delta$. Since Λ is not residually finite we have $N \neq \{1\}$, and since Λ is just infinite we have $[\Lambda : N] < \infty$. By [Wil71, Proposition 1], the group N is the direct product of finitely many pairwise isomorphic simple groups. Therefore it is enough to show that there is only one factor in this product. Assume there are two, say H and G . Topological simplicity of $\overline{\Lambda}$ [Rem03b, 2.A.1] first implies $\overline{N} = \overline{\Lambda}$ because $N \triangleleft \Lambda$, and then $\overline{G} = \overline{H} = \overline{\Lambda}$. Let us pick two non-commuting elements g and h in $\overline{\Lambda}$ and write them $g = \lim_{n \rightarrow \infty} g_n$ and $h = \lim_{n \rightarrow \infty} h_n$, with $g_n \in G$ and $h_n \in H$ for each $n \geq 1$. We get a contradiction when writing: $[g, h] = \lim_{n \rightarrow \infty} [g_n, h_n] = 1$. \square

It is not known whether non-residually finite finitely generated Kac-Moody groups exist. On the other hand, partial non-linearity results are available [Rem03b] and can probably be extended to wider classes of Kac-Moody groups.

REFERENCES

- [Abr97] Peter Abramenko, *Twin buildings and applications to S-arithmetic groups*, Lecture Notes in Mathematics, vol. 1641, Springer Verlag, 1997.

- [AR03] Peter Abramenko and Bertrand Remy, *Commensurators of some nonuniform tree lattices and Moufang twin trees*, Institut Fourier preprint **627**, 2003.
- [Bon03] Patrick Bonvin, *Strong boundaries and commensurator super-rigidity*, appendix to [Rem03b], 2003.
- [Bou81] Nicolas Bourbaki, *Groupes et algèbres de Lie IV-VI*, Éléments de Mathématique, Masson, 1981.
- [BS03] Uri Bader and Yehuda Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, preprint, 2003.
- [CG99] Lisa Carbone and Howard Garland, *Lattices in Kac-Moody groups*, Math. Res. Letters **6** (1999), 439–448.
- [DJ02] Jan Dymara and Tadeusz Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Inventiones Mathematicæ **150** (2002), 579–627.
- [KP85] Victor Kac and Dale Peterson, *Defining relations for certain infinite-dimensional groups*, Élie Cartan et les mathématiques d’aujourd’hui. The mathematical heritage of Élie Cartan. Lyon, 25-29 juin 1984 (Société Mathématique de France, ed.), Astérisque Hors-Série, 1985, pp. 165–208.
- [LMR01] Alex Lubotzky, Shahar Mozes, and M.S. Raghunathan, *The word and Riemannian metrics on lattices of semisimple groups*, Publications Mathématiques de l’IHÉS **91** (2001), 5–53.
- [Mar91] Gregory A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 17, Springer, 1991.
- [Mon04] Nicolas Monod, *Superrigidity for irreducible lattices and geometric splitting*, preprint, 2004.
- [Rem99] Bertrand Remy, *Construction de réseaux en théorie de Kac-Moody*, Comptes-Rendus de l’Académie des Sciences de Paris **329** (1999), 475–478.
- [Rem02] Bertrand Remy, *Groupes de Kac-Moody déployés et presque déployés*, Astérisque, vol. 277, Société Mathématique de France, 2002.
- [Rem03a] Bertrand Remy, *Sur les propriétés algébriques et géométriques des groupes de Kac-Moody*, habilitation à diriger les recherches, Institut Fourier, Grenoble 1, 2003.
- [Rem03b] Bertrand Remy, *Topological simplicity, commensurator super-rigidity and non-linearities of Kac-Moody groups*, Institut Fourier preprint **590**, 2003.
- [Ron89] Mark R. Ronan, *Lectures on Buildings*, Perspectives in Mathematics, vol. 7, Academic Press, 1989.
- [RR03] Bertrand Remy and M. Ronan, *Topological groups of Kac-Moody type, Fuchsian twinings and their lattices*, Institut Fourier preprint **563**, shortened version, 2003.
- [Sha00a] Yehuda Shalom, *Rigidity of commensurators and irreducible lattices*, Inventiones Mathematicæ **141** (2000), 1–54.
- [Sha00b] Yehuda Shalom, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Annals of Mathematics **152** (2000), 113–182.
- [Tit87] Jacques Tits, *Uniqueness and presentation of Kac-Moody groups over fields*, Journal of Algebra **105** (1987), 542–573.
- [Tit92] Jacques Tits, *Twin buildings and groups of Kac-Moody type*, Groups, Combinatorics & Geometry (LMS Symposium on Groups and Combinatorics, Durham, July 1990) (M. Liebeck and J. Saxl, eds.), London Mathematical Society Lecture Notes Series, vol. 165, Cambridge University Press, 1992, pp. 249–286.
- [Wil71] John S. Wilson, *Groups with every proper quotient finite*, Proceedings of the Cambridge Philosophical Society **69** (1971), 373–391.

Institut Fourier – UMR 5582 du CNRS
 Université de Grenoble 1 – Joseph Fourier
 100, rue des maths – BP 74
 38402 St Martin d’Hères Cedex – France
 bremy@ujf-grenoble.fr