WEIGHTED ALMOST PERIODICITY

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Prépublication de l'Institut Fourier n° 628 (2003)

http://www-fourier.ujf-grenoble.fr/prepublications.html

Résumé

Soit G un groupe abélien localement compact et ω un poids défini sur G ; le but de ce travail est l'étude des éléments presque périodiques à poids dans les G-modules et $L^1_\omega(G)$-modules.

Abstract

Let G be a locally compact abelian group and ω be a weight function on G; this study is concerned with the weighted almost periodic elements of G-modules and $L^1_\omega(G)$-modules and the relationship between the various definitions.

1. INTRODUCTION

In the study of almost periodic functions the translation is generally an isometry ([1], [2], [5], [9]). But in the weighted normed space it is impossible to obtain an isometry by the translation operators.

Let G be a locally compact abelian group, a Banach space E is said to be a G-module if G acts on E *i.e.* there exists a map L_x from G into the invertible bounded operators of E such that L_0 = Identity and $L_{x+y} = L_x \circ L_y$. Whenever $\|L_x\| = 1$ for every $x \in G$, in the study of the almost periodicity, $L^1(G)$-modules are the natural spaces ([4], [9], [10]).

In the present work, L_x is not necessarily to be an isometry and the correspondent spaces are the $L^1_\omega(G)$-modules where $L^1_\omega(G)$ is a Banach algebra under the convolution and ω is a weight function, for example, $\omega(x) = \max\{1, \|L_x\|\}$, ([11], [13]). First the fundamental definitions and the properties of the G-modules and the $L^1_\omega(G)$-modules are presented, then the definition of weighted almost periodicity is given. The relationships between various definitions in G-modules of $L^1_\omega(G)$-modules are studied.

As an application, it is shown that if G is not compact and $\overline{\lim}_{x\to\infty}\omega(x) = +\infty$ there is no non-null weighted almost periodic function in $L^p_\omega(G)$, $1 \leqslant p \leqslant \infty$.

2000 Mathematics Subject Classification: 43A20, 43A22, 43A60.
Keywords: Weight function, Banach module, almost periodicity, multiplier.

2. NOTATIONS AND GENERAL FRAMEWORK

2.1. G-module

A Banach space E is called a G-module if there exists a group representation L of G into the group of the invertible operators of E with $L_0 = I$ (identity) and $L_{x+y} = L_x \circ L_y$.

Given such a G-module E, denote by

$$E_b = \{e \in E \mid \sup\{\|L_x e\|\ , x \in G\} < \infty\}$$
$$E_c = \{e \in E \mid x \to L_x e \text{ is continuous}\}$$
$$E_{uc} = \{e \in E \mid x \to L_x e \text{ is uniformly continuous}\}$$

$$E_{cb} = E_c \cap E_b\ ,\ E_{ucb} = E_{uc} \cap E_b\ .$$

All these spaces are Banach sub-G-modules of E.

Let ω be a (continuous, measurable) nonnegative function defined on G; by Proposition 1 of Spector ([13]) ω can be continuous without loss of generality. Denote by $L^p_\omega(G) = \{f \mid f\omega \in L^p(G)\}\ ,\ 1 \leqslant p \leqslant \infty$, with the natural norm $\|f\|_{p,\omega} = \|f\omega\|_p$, $L^p_\omega(G)$ is a Banach space. Its dual space is $L^{p'}_{\omega^{-1}}(G)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 \leqslant p < \infty$.

Let ω be a weight (continuous, measurable) function on G, *i.e.*

$$\omega(x) \geqslant 1\ ,\ \forall x \in G$$

$$\omega(x+y) \leqslant \omega(x)\omega(y)\ ,\ \forall x,y \in G\ .$$

In this case $L^p_\omega(G) \subset L^p(G)$ and $L^p_\omega(G)$ is a G-module, $1 \leqslant p < \infty$, for $(L_x f)(y) = f(y-x)$ with $\|L_x\| \leqslant \omega(x)$. For $p = 1$, $L^1_\omega(G)$ is a Banach algebra under the convolution.

Throughout this paper, ω always denotes a weight function.

2.2. $L^1_\omega(G)$-module

A Banach space E is called a $L^1_\omega(G)$-module if there exists an algebra representation T (continuous) of $L^1_\omega(G)$ into $L(E,E)$ the algebra of bounded operators of E into itself with

$$\|Tf\| \leqslant \|f\|_{1,\omega}$$

$T_{f*g} = T_f T_g$ where $*$ is the convolution product in $L^1_\omega(G)$. For e in E, $(T_f)(e)$ is denoted $f*e$.

A $L^1_\omega(G)$-module E is order free if for each $e \in E$, $e \neq 0$ there exists $f \in L^1_\omega(G)$ such that $f * e \neq 0$. The subspace $E_{\text{deg}} = \{e \in E,\ f * e = 0, \forall f \in L^1_\omega(G)\}$ is called the degenerate part of E.

For an $L^1_\omega(G)$-module E its essential part is defined by

$$\begin{aligned} E_{\text{ess}} &= \left\{f * e\ ,\ f \in L^1_\omega(G)\ , e \in E\right\} \\ &= \left\{e \in E\ ,\ \lim_\alpha \mu_\alpha * e = e\right\}\ , \end{aligned}$$

where $\{\mu_\alpha\}_\alpha$ is a bounded approximate identity (b.a.i.) of $L^1_\omega(G)$. For the existence of a b.a.i. see Gaudry ([8, Lemma 3]).

E_{ess} is a Banach subspace of E and the precedent equivalence is obtained by Cohen's factorization theorem. A $L^1_\omega(G)$-module E is called "essential" if $E_{\text{ess}} = E$. The subspace E_{ess} is an order free $L^1_\omega(G)$-module.

2.3. Compatible translation

A $L^1_\omega(G)$-module E has a G-action if E is also a G-module with $L_x T_f = T_f L_x = T(f_x)$ where $f_x(y) = f(y-x) = L_x f$, $\forall x, y \in G$, $\forall f \in L^1_\omega(G)$.

Let us remark that if E is a G-module then

$$\omega(x) = \max\{1, \|L_x\|\}$$

is a weight (measurable) function, E_c is a $L^1_\omega(G)$-module with G-action, the $L^1_\omega(G)$-action is given by

$$f \in L^1_\omega(G) \to f * e = \int_G L_x e(f(x)\, dx$$

with

$$\begin{aligned} \|f * e\|_E &\leqslant \int_G \|L_x e\| |f(x)|\, dx \\ &\leqslant \|e\| \int_G |f(x)|\, \|L_x\|\, dx \\ &\leqslant \|e\| \int_G |f(x)|\omega(x)\, dx = \|f\|_{1,\omega}\, \|e\|\,. \end{aligned}$$

And also if E is a G-module and ω is a weight function such that $\|L_x\| \leqslant \omega(x)$ then E_c is a $L^1_\omega(G)$-module with G-action.

In the same way an essential $L^1_\omega(G)$-module has a G-action defined by if $e = f_1 * e_1$, $f_1 \in L^1_\omega(G)$, $e_1 \in E$ then:

$$L_x e = (L_x f_1) * e_1 = \lim_\alpha (L_x \mu_\alpha) * e$$

where $(\mu_\alpha)_\alpha$ is a b.a.i. of $L^1_\omega(G)$.

In this case it follows $\|L_x\| \leqslant \omega(x)$.

Comparison between E_c and E_{ess}:

LEMMA 2.3.1. — *Let E be an order free $L^1_\omega(G)$-module with a G-action then:*

i) If e_1 is a cluster point of $\{\mu_\alpha * e\}_\alpha$ where $\{\mu_\alpha\}_\alpha$ is a b.a.i. then $e_1 = e$ and $e \in E_{\text{ess}}$.

ii) If e_1 is a cluster point of $\{L_x e_1\}$ when $x \to 0$ then $e_1 = e$ and $e \in E_c$.

Proof. — *(i)* Let $f \in L^1_\omega(G)$. There exists a subnet μ_β such that $\lim_\beta f * \mu_\beta * e = f * e_1 = \lim_\beta(\mu_\beta * f) * e = f * e$. It follows that $f * (e - e_1) = 0$ *i.e.* $e - e_1 = E_{\text{deg}} = \{0\}$.

(ii) can be deduced in the same way. □

DEFINITION 2.3.2. — *A G-action and a $L^1_\omega(G)$-action are said to be compatible if there exist $a,b \in \mathbb{R}^{+*}$ such that*

i) $L_x(f * e) = f_x * e = f * L_x e,\ x \in G,\ e \in E,\ f \in L^1_\omega(G)$.
ii) $a\|L_x\| \leqslant \omega(x) \leqslant b\|L_x\|,\ x \in G$.

THEOREM 2.3.3. — *Assume that $x \to \|L_x\|$ is locally bounded on G, then E_c is an essential $L^1_\omega(G)$-module with a compatible G-action for the weight $\omega(x) = \max\{1,\{L_x\|\}$.*

Proof. — It is clear that E_c is a $L^1_\omega(G)$-module with a compatible G-action. Let us show that E_c is essential. It suffices, by Cohen's factorization theorem, to show that for a b.a.i. $\{\mu_\alpha\}_\alpha$ of $L^1_\omega(G)$, every $e \in E_c$ satisfies

$$\lim_\alpha \|\mu_\alpha - e - e\| = 0\,.$$

Let V be a compact (symmetric) neighborhood of zero in G. There exists a non-negative (continuous) function f_V with compact support V such that $f_V(0) = 1$ and $\|f_V\|_1 = 1$; the family $\{\mu_V = f_V,\ V \subset K\}$ where K is a fixed compact neighborhood is a b.a.i. with $\|\mu_V\|_{1,\omega} \leqslant \sup\{\omega(x), x \in K\}$. It follows that:

$$\begin{aligned}\|\mu_V * e - e\| = \|\int_G L_x e\, d\mu_V - e\| &= \|\int_G (L_x e - e)\, d\mu_V\| \\ &\leqslant \|\mu_V\|_{1,\omega} \sup\{\|L_x e - e\|, x \in V\} \\ &\leqslant \sup\{\omega(x), x \in V\} \sup\{\|L_x e - e\|, x \in V\}.\end{aligned}$$

Since $x \to \|L_x\|$ is locally bounded and $e \in E_c$,

$$\lim_V \|\mu_\alpha * e - e\| = 0.$$

□

COROLLARY 2.3.4. — *Let ω be a measurable, locally bounded weight function and E be a $L^1_\omega(G)$-module with a compatible G-action. Then the equality $E_{\text{ess}} = E_c$ holds.*

Proof. — Since ω is a locally bounded weight function then $x \in G \to L_x f \in L^1_\omega(G)$ is a continuous map and then $E_{\text{ess}} \subset E_c$.

Conversely, since the G-action is compatible then the function $x \to \|L_x\|$ is locally bounded, so that $E_c \subset E_{\text{ess}}$. □

2.4. Dual module

The dual E' of a $L^1_\omega(G)$-module E is also on $L^1_\omega(G)$-module for the action defined by:

$$\langle f * e', e\rangle = \langle e', f * e\rangle,\ f \in L^1_\omega(G), e' \in E', e \in E.$$

It is the same for a G-module.

The degenerate part $(E')_{\text{deg}}$ in E' is the orthogonal of E_{ess} and E' is order free if and only if E is essential.

2.5. Examples

- $L^p_\omega(G) = \{f \in L^1_{\text{loc}},\ f\omega \in L^p_\omega(G),\ \|f\|_{p,\omega} = \|f\omega\|_p\}$, $1 \leqslant p < \infty$ is an essential-module and the translation is a compatible G-action, here ω is a locally bounded measurable weight function.
- $L^r_\omega(G) = \{f \in L^1_{\text{loc}},\ \frac{f}{\omega} \in L^r(G),\ \|f\|_{r,\omega^{-1}} = \|\frac{f}{\omega}\|_r\}$, $1 \leqslant r \leqslant \infty$ is an order free $L^1_\omega(G)$-module and the translation is a compatible G-action.
- For $1 \leqslant r < \infty$, $L^r_{\omega^{-1}}$ is an essential $L^1_\omega(G)$-module.
- If $\omega(x)$ is continuous then $\{f$ bounded and uniformly continuous $\} \subset (L^\infty_{\omega^{-1}})_{\text{ess}}$.
- If $\omega(x)$ is continuous and $\lim_{x\to 0}\omega(x) = \infty$ then $\{f$ bounded and continuous $\} \subset (L^\infty_{\omega^{-1}})_{\text{ess}}$.
- Let E be a Banach space and T is an invertible, bounded operator of E, then E is an essential $\ell^1_\omega(\mathbb{Z})$-module with compatible G-action where $\omega(x) = \max\{\|T^n\|,\|T^{-n}\|\}$ and the $\ell^1_\omega(G)$-action is given by $a = a_n \in \ell^1_\omega(\mathbb{Z})$, $a * e = \sum_{n\in\mathbb{Z}} a_n T^n e$, the G-action is given by $L_n a = T^n e$.

3. HOMOMORPHISMS

3.1. General case

Because of the two structures on the essential part of a Banach $L^1_\omega(G)$-module there exist also two homomorphism notions for an operator: Let E be a $L^1_\omega(G)$-module; a bounded linear operator T from E into E is:

- invariant if T commutes with the "translation operators", $TL_x = L_xT$, $\forall x \in G$, that is $T \in \text{Hom}_G(E,E)$;
- multiplier if T commutes with the "convolution operators" $T * f = f * T \quad (TT_f = T_f T)$, $\forall f \in L^1_\omega(G)$, that is $T \in \text{Hom}_{L^1\omega}(E,E)$.

The equivalence of this two notions in the general case is a classical problem of harmonic analysis.

Theorem 3.1.1. — *Let E and F be two $L^1_\omega(G)$-modules with G-compatible translation then*

$$\text{Hom}_{L^1_\omega}(E,F) \subset \text{Hom}_G(E,F).$$

If F is an order free $L^1_\omega(G)$-module then

$$\text{Hom}_{L^1_\omega}(E,F) = \text{Hom}_G(E,F).$$

Proof. — The proof may be found, for example in [4]. □

3.2. Particular case

The following statement is important for the definition of the weighted almost periodic function.

Let B be an order free $L^1_\omega(G)$-module with compatible G-action. Throughout this part the space

$$\mathrm{Hom}_G(L^1_\omega(G),E) = \mathrm{Hom}_{L^1\omega}(L^1_\omega(G),E)$$

is denoted by $(L^1_\omega(G),E)$. It is obvious that $(L^1_\omega(G),E)$ is a $L^1_\omega(G)$-module with a compatible G-action and since $L^1_\omega(G) * L^1_\omega(G) = L^1_\omega(G)$, $(L^1_\omega(G),E)$ is order free.

Since E is an order free $L^1_\omega(G)$-module, the natural continuous homomorphism $j : E \to (L^1_\omega(G),E)$, with $j(e)(f) = f * e$ is injective.

In the general case $j(E)$ is different from $(L^1_\omega(G), E)$ (for example $(L^1_\omega(G), L^1_\omega(G)) \cong M_\omega(G)$, where $M_\omega(G)$ is the space of measures μ such that $\mu\omega$ is bounded), ([8]).

Lemma 3.2.2. — *$j(E)$ is dense in $(L^1_\omega(G),E)$ for the strong operator topology with*

$$(L^1_\omega(G),E) \subset (L^1_\omega(G),E_{\mathrm{ess}}).$$

Proof. — Let T be in $(L^1_\omega(G),E)$, $(\mu_\alpha)_\alpha$ be a b.a.i. of $L^1_\omega(G)$ then

$$(Tf) = \lim_\alpha T(\mu_\alpha * f) = \lim_\alpha f * (T\mu_\alpha),$$

since $(T\mu_\alpha) \in E$ and $(T\mu_\alpha) * f \in E_{\mathrm{ess}}$ the result follows. ☐

Theorem 3.2.3. — *Let E be a $L^1_\omega(G)$-module, the $L^1_\omega(G)$-module isomorphisms follow:*

i) $(L^1_\omega(G),E) \cong (L^1_\omega(G),E_{\mathrm{ess}})$
ii) $(L^1_\omega(G),E)_{\mathrm{ess}} \cong E_{\mathrm{ess}}$
iii) $(L^1_\omega(G),(L^1_\omega(G),E)) \cong (L^1_\omega(G),E)$
iv) $(L^1_\omega(G),E') \cong (E_{\mathrm{ess}})'$.

Proof. — By the Lemma 2.3.1, *(i)* and *(ii)* are obvious. From *(i)* and *(ii)* it follows

$$(L^1_\omega(G),(L^1_\omega(G),E)) \cong (L^1_\omega(G),(L^1_\omega(G),E)_{\mathrm{ess}}) \cong (L^1_\omega(G),E_{\mathrm{ess}}) \cong (L^1_\omega(G),E).$$

The classical result $H_A(X,Y') \cong (X \otimes_A Y)'$, (see, for example [11]) gives here:

$$\mathrm{Hom}_{L^1_\omega}(L^1_\omega(G),E') \cong (L^1_\omega(G) \otimes_{L^1_\omega} E)' \cong (L^1_\omega * E)' = (E_{\mathrm{ess}})'.$$

☐

Corollary 3.2.4. — *If E is an essential $L^1_\omega(G)$-module, then $(L^1_\omega(G),E') \cong E'$.*

If E is an order free and reflexive $L^1_\omega(G)$-module then E is essential with $(L^1_\omega(G),E) \cong E$.

Proof.

- $(L^1_\omega(G),E') \cong (E_{\mathrm{ess}})' \cong E'$
- $(L^1_\omega(G),E) \cong (L^1_\omega,(E')') = ((E')_{\mathrm{ess}})' = (E') = E$.

☐

3.3. Examples

(i) $(L^1_\omega(G), L^p_\omega(G)) \cong L^p_\omega(G)$, $1 \leqslant p < \infty$.
(ii) $(L^1_\omega(G), L^\infty_{\omega^{-1}}(G)) \cong (L^1_\omega(G))' = L^\infty_{\omega^{-1}}(G)$.

4. WEIGHTED ALMOST PERIODICITY

The classical concept of the Bochner almost periodicity (normal function, [1]) for $e \in E^\omega_c$ is defined for a G-module as e is almost periodic if $\{L_x e, x \in G\}$ is relatively compact. But when the G-action L_x is not an isometry, it will be defined as the following:

Definition 4.1. — *Let E be a G-module and $e \in E_c$, e is called weighted G-almost periodic element if $\{\frac{L_x e}{\|L_x\|}, x \in G\}$ is relatively compact in E.*

Denote by E^ω_{ap} is the set of all the weighted G-almost periodic elements. It is clear that E^ω_{ap} is a G-submodule of E. The definition of E^ω_{ap} can be defined in the frame of the $L^1_\omega(G)$ with a compatible G-action by the following equivalent conditions

(i) $\left\{\frac{L_x e}{\|L_x\|}, x \in G\right\}$ is relatively compact;
(ii) $\left\{\frac{L_x e}{\omega(x)}, x \in G\right\}$ is relatively compact.

Indeed, *(i)* and *(ii)* are equivalent to (respectively)

$$\left\{\lambda \frac{L_x e}{\|L_x\|}, x \in G, \lambda \in I \text{ compact interval of } \mathbb{R}\right\},$$

$$\left\{\lambda \frac{L_x e}{\omega(x)}, x \in G, \lambda \in I \text{ compact interval of } \mathbb{R}\right\}$$

are relatively compact. Since the G-action is compatible it follows

$$\left\{\lambda \frac{L_x e}{\omega(x)}, x \in G, \lambda \in [0,a]\right\} \subset \left\{\frac{L_x e}{\|L_x\|}, x \in G\right\} \subset \left\{\lambda \frac{L_x e}{\omega(x)}, x \in G, \lambda \in [0,b]\right\}$$

by the $a\|L_x\| \leqslant \omega(x) \leqslant b\|L_x\|$.

Remark 4.2. — Note that if $\{L_x e, x \in G\}$ is relatively compact than $\left\{\frac{L_x e}{\omega(x)}, x \in G\right\}$ is also relatively compact.

As the case $\omega = 1$, the following classical result is obtained ([5], [9]):

Theorem 4.3. — *Let E be an order free $L^1_\omega(G)$-module with a compatible G-action where ω is a continuous weight function, for $e \in E$ the following statements are equivalent:*

(i) $e \in E$ satisfies $\left\{\frac{L_x e}{\omega(x)}, x \in G\right\}$ is relatively compact;
(ii) $j(e) : f \in L^1_\omega(G) \to f * e \in E$ is a compact multiplier.

Remark 4.4. — Note that the condition *(i)* implies that e is in E_c by Lemma 2.3.1 and *(ii)* that e is in E_{ess}, by the Theorem 2.3.3 and Corollary 2.3.4, the existence of a compatible G-action in the hypothesis is redundant.

Clearly, E_{ap} is an essential $L^1_\omega(G)$-module with a compactible G-action contained in E_{cb}.

COROLLARY 4.5. — *Let E be an $L^1_\omega(G)$-module with the compatible G-action and T be a compact multiplier from $L^1_\omega(G)$ into E, then there exists $e \in E_{ap}$ such that $T = f(e)$.*

Proof of the corollary. — Let $\{\mu_\alpha\}_\alpha$ be a b.a.i. of $L^1_\omega(G)$, then for every α, $T(\mu_\alpha) = e_\alpha$ is in E_{ap} and

$$T(\mu_\alpha * f) = f * e_\alpha = j(e_\alpha)(f).$$

Since $\{\mu_\alpha\}_\alpha$ is bounded then $\{T(\mu_\alpha)\}_\alpha$ is contained in a compact of E and let e be an adherent element of $\{e_\alpha\}$, so

$$\lim_\alpha f * e_\alpha = \lim_\alpha T(\mu_\alpha * f) = Tf = f * e.$$

Since E_{ap} is essential, it is order free and e is unique.

□

Proof of the theorem. — The condition *(i)* is equivalent to the convex hull $\left\{\sum\limits_i a_i \frac{L_{x_i}e}{\omega(x_i)},\right.$ $\left.\sum |a_i| \leqslant 1\right\}$ is relatively compact or $\left\{\sum\limits_i a_i L_{x_i} e,\ \sum |a_i| \omega(x_i) \leqslant 1\right\}$ is relatively compact.

To show the theorem it is sufficient to prove that $A = \left\{\sum\limits_i a_i L_{x_i} e, \sum |a_i| \omega(x_i) \leqslant 1\right\}$ and $B = \left\{f * e,\ f \in L^1_\omega(G), \|f\|_{1,\omega} \leqslant 1\right\}$ have the same closure in E. This statement is classic for the case $\omega(x) = 1$ (see [5], [9], [12]).

(i) ⇒ (ii): Let f be in $L^1_\omega(G)$ with $\|f\|_{1,\omega} \leqslant 1$, without lost of generality, f can be chosen with support compact K.

Note that by the remark, e is in $E_{\text{ess}} = E_c$ and on the compact K, the maps $x \to L_x e$ and $x \to \omega(x)$ are uniformly continuous then for all $\varepsilon > 0$ there exists a finite family of $x_i \in K$, $i = 1,2,\dots,n$, K_i Borel sets, $i = 1,\dots,n$ such that:

- $x_i + K_i \cap x_j + K_j = \emptyset,\ i \neq j$
- $\bigcup\limits_{i=1}^n x_i + K_i = K$
- $|\omega(x_i + h) - \omega(x_i)| \leqslant \varepsilon,\quad h \in K_i$
- $\|L_{x_i+h} e - L_{x_i} e\| \leqslant \varepsilon,\quad h \in K_i.$

Let us start to show

$$\left\|f * e - \sum_{i=1}^n \int_{x_i+K_i} L_{x_i} e f(x)\, dx\right\| = \left\|\sum_{i=1}^n \int_{x_i+K_i} (L_x e - L_{x_i} e) f(x)\, dx\right\|$$
$$\leqslant \varepsilon \cdot \sum_{i=1}^n \int_{x_i+K_i} |f(x)|\, dx \leqslant \varepsilon \|f\|_1 \leqslant \varepsilon \cdot \|f\|_{1,\omega} \leqslant \varepsilon.$$

Denote $\alpha_i = \int_{x_i+K_i} f(x)\ dx$, it follows:

$$\begin{aligned}\sum_{i=1}^{n} |\alpha_i|\omega(x_i) &\leqslant \sum_i \int_{x_i+K_i} |f(x)|\omega(x_i)\ dx \\ &\leqslant \sum_i \int_{x_i+K_i} |f(x)|\,|\omega(x_i)-\omega(x)|\ dx + \int_{x+K} |f(x)|\omega(x)\ dx \\ &\leqslant \|f\|_{1,\omega} + \varepsilon\|f\|_1 \leqslant \|f\|_{1,\omega}(1+\varepsilon).\end{aligned}$$

Denote $a_i = \frac{\alpha_i}{1+\varepsilon}$, it follows:

$$\sum |a_i|\omega(x_i) \leqslant \sum \frac{|\alpha_i|\omega(x_i)}{1+\varepsilon} \leqslant \|f\|_{1,\omega} \leqslant 1.$$

$$\begin{aligned}\|f * e - \sum a_i L_{x_i} e\| &\leqslant \|f * e - \sum \alpha_i L_{x_i} e\| + \|\sum \alpha_i L_{x_i} e - \frac{\alpha_i}{1+\varepsilon} L_{x_i} e\| \\ &\leqslant \varepsilon + \|\sum \alpha_i L_{x_i} e\| \left(1 - \frac{1}{1+\varepsilon}\right) \\ &\leqslant \varepsilon + \sum_i |\alpha_i|\omega(x_i) \frac{\|L_{x_i} e\|}{\omega(x_i)} \left(\frac{\varepsilon}{1+\varepsilon}\right) \\ &\leqslant \varepsilon + \frac{1+\varepsilon}{1+\varepsilon}\varepsilon\|e\| \sup_x \frac{\|L_x\|}{\omega(x)}.\end{aligned}$$

Since the action is compatible there exist a and b such that $0 < a \leqslant \frac{\|L_x\|}{\omega(x)} \leqslant b$ and so $\|f * e - \sum a_i L_{x_i} e\| \leqslant \varepsilon(1 + b\|e\|)$ and then the closure of A contains B.

As the same way, mutadis mutandis

(ii) ⇒ *(i)*: Let $\sum a_i L_{x_i} e$ be an element of A with $\sum |a_i|\omega(x_i) \leqslant 1$, without loss generality, it is accepted that the sum is finite. As the precedent there exists a finite family x_i, $1 = 1,2,\ldots,n$ and K_i open, relatively compact set $i = 1,2,\ldots,n$ such that:

- $x_i + K_i \cap x_j + K_j = \varnothing$
- $|\omega(x_i + h) - \omega(x_i)| \leqslant \varepsilon, \quad h \in K_i$
- $\|L_{x_i+h} e - L_{x_i}\| \leqslant \varepsilon, \quad h \in K_i.$

Hence

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i L_{x_i} e - \sum_{i=1}^{n} a_i \int_{x_i+K_i} L_x e \frac{\chi_{x_i+K_i}(x)}{|K_i|}[dx\right\| &\leqslant \sum_{i=1}^{n} |a_i| \int_{x_i+K_i} \|L_{x_i} e - L_x e\| \frac{\chi_{x_i+K_i}(x)}{|K_i|}\ dx \\ &\leqslant \varepsilon \sum_{i=1}^{n} |a_i| \leqslant \varepsilon \sum_{i=1}^{n} |a_i|\omega(x_i) \leqslant \varepsilon.\end{aligned}$$

Denote $g(x) = \sum_{i=1}^{n} a_i \frac{\chi_{x_i+K_i}(x)}{|K_i|}$, it follows:

$$\begin{aligned}\int |g(x)|\omega(x)\ dx &= \sum_{i=1}^{n} |a_i| \int_{x_i+K_i} \frac{\chi_{x_i+K_i}(x)}{|K_i|}(\omega(x) - \omega(x_i))\ dx + \sum_{i=1}^{n} |a_i|\omega(x_i) \\ &\leqslant \varepsilon \sum_{i=1}^{n} |a_i| + 1 \leqslant \sum_{i=1}^{n} |a_i|\omega(x_i)(\varepsilon + 1) \leqslant 1 + \varepsilon.\end{aligned}$$

Denote $f(x) = \frac{g(x)}{1+\varepsilon}$, it follows with $\|f\|_{1,\omega} \leqslant 1$:

$$\left\|\sum a_i L_{x_i} e - f * e\right\| \leqslant \left\|\sum_{i=1}^{n} a_i L_{x_i} e - g * e\right\| + \|g * e - f * e\|$$
$$\leqslant \varepsilon + \|g\|_{1,\omega}(1 - \frac{1}{1+\varepsilon}) \cdot \|e\|$$
$$\leqslant \varepsilon(1+\varepsilon) - \frac{\varepsilon}{1+\varepsilon} = \varepsilon(1 + \|e\|),$$

so the closure of B contains A. □

5. BOHR WEIGHTED ALMOST PERIODIC ELEMENT

The classical definition of Bohr almost periodic function is based on the notion of "almost period" ([1], [2]).

It is known that Bohr and Bocher continuous almost periodic functions are the same ([5], [6]). The notation of Furshtenberg ([6]) for the uniformly recurrent function motivates the following definition :

DEFINITION 5.1. — *An element e of a G-module is said "weighted uniformly recurrent" if for every $\varepsilon > 0$ there exists a compact set K of G such that for every $x \in G$ there exists $k \in K$ such that*

$$\left\|\frac{L_x e}{\|L_x\|} - \frac{L_k e}{\|L_k\|}\right\| < \varepsilon.$$

Denote the set of these elements by E^{ω}_{ur}.

THEOREM 5.2. — *Let E be a G-module. Then*

$$E^{\omega}_{ap} = E^{\omega}_{ur} \cap E_c\,.$$

Proof. — Let us show the inclusion $E^{\omega}_{ap} \subset E_{ur} \cap E_c$. First, by Remark 4.4 E^{ω}_{ap} is contained in E_c. Now, let $e \in E^{\omega}_{ap}$. Since $\left\{\frac{L_x e}{\|L_x\|}, x \in G\right\}$ is relatively compact, then, for every ε there exists a finite family $\{x_i \in E, i = 1, \ldots, n\}$ such that for every $x \in G$ there exists i with

$$\left\|\frac{L_x e}{\|L_x\|} - \frac{L_{x_i} e}{\|L_{x_i}\|}\right\| < \varepsilon$$

and take $K = \{x_i, i = 1, \ldots, n\}$.

To show that $E^{\omega}_{ur} \cap E_c \subset E^{\omega}_{ap}$, let $e \in E^{\omega}_{ur} \cap E_c$, and $\varepsilon > 0$, there exists a compact set K such that for every $x \in G$ there exists $k \in K$ with

$$\left\|\frac{L_x e}{\|L_x\|} - \frac{L_k e}{\|L_k\|}\right\| < \varepsilon;$$

since $e \in E_c$ the map $x \to L_x e$ is continuous then $A = \{\lambda L_x e, x \in K, \lambda \in [0,1]\}$ is a compact set. So for every $\varepsilon > 0$ there exists a compact set A such that

$$\left\{\frac{L_x e}{\|L_x\|}, x \in G\right\} \subset \left\{\frac{L_x e}{\|L_x\|}, x \in K\right\} + B(0,\varepsilon) \subset A + \varepsilon$$

where $B(0,\varepsilon) = \{e_1 \in E, \|e_1\| \leqslant \varepsilon\}$ it follows that $\left\{\frac{L_x e}{\|L_x\|}, x \in G\right\}$ is paracompact, *i.e.* it is relatively compact. $\square$

Remark 5.3. — *Note that if* $\left\{\frac{L_x e_1}{\|L_x\|}, x \in G\right\}$ *and* $\left\{\frac{L_x e_2}{\|L_x\|}, x \in G\right\}$ *are relatively compact then* $\{\frac{L_x(e_1+e_2)}{\|L_x\|}, x \in G\}$ *is also relatively compact but if* $e_1, e_2 \in E_{ur}$ *perhaps* $e_1 + e_2$ *may not be in* E_{ur}.

Remark 5.4. — *Let* E *be a* $L^1_\omega(G)$*-module with compatible* G*-action, if* $e \in E_{ur}$ *and* $f \in L^1_\omega(G)$ *then* $f * e$ *is in* $E_{ur} \cap E_c = E^\omega_{ap}$. *It follows* $E^\omega_{ap} \subsetneq E^\omega_{ur} \subsetneq (L^1_\omega, E^\omega_{ap})$.

6. APPLICATION

Let $E = L^p_\omega$, $1 \leqslant p \leqslant +\infty$, it is known that $L^p_\omega(G)$ is a $L^1_\omega(G)$-module with compatible G-action. If $\omega = 1$, it is shown that $L^p_\omega(G)$, $(1 \leqslant p < \infty)$ have no non null almost periodic element if and only if G is compact ([3], [5], [14]). Now it will be denoted that:

Theorem 6.1. — *Let* G *be locally compact, non-compact, abelian group and* ω *be a (continuous) weight function with*

$$\overline{\lim}_{x\to\infty}\omega(x) = +\infty$$

then $(L^p_\omega)^\omega_{ap} = \{0\}, 1 \leqslant p \leqslant \infty$.

Remark 6.2. — *Note that in the case* $\omega = 1$, $(L^\infty(G))_{ap} \neq \{0\}$. *The condition* $\overline{\lim}_{x\to\infty}\omega(x) = \infty$ *is essential.*

Proof. — Let f be in $(L^p_\omega)^\omega_{ap}$, $1 \leqslant p < \infty$, since $\left\{\frac{L_x f}{\omega(x)}, x \in G\right\}$ is relatively compact in $L^p_\omega(G)$ then the set

$$\left\{h(x,y) = \left|\frac{f(x-y)}{\omega(x)}\right|^p, x \in G\right\}$$

is relatively compact in $L^1(G)$ and also by the Fourier transform the set $\{\hat{h}(x,y), x \in G\}$ is also relatively compact in $C_0(\widehat{G})$. It follows

$$\begin{aligned}\hat{h}(x,\gamma) &= \int_G \left(\frac{|f(y-x)|}{\omega(x)}\right)^p \overline{(\gamma,y))}\, dy \\ &= \int_G \frac{|f(u)|}{\omega^p(x)}\langle\overline{\gamma,u}\rangle\langle\overline{\gamma,x}\rangle\, du.\end{aligned}$$

$$\hat{h}(x,\gamma) = \frac{\langle\overline{\gamma,x}\rangle}{\omega^p(x)}(\widehat{|f|^p})(\gamma).$$

If $g = \lim_{x_n\to\infty}\frac{L_{x_n}f}{\omega(x_n)}$ with $\lim_{x_n\to\infty}\omega(x_n) = +\infty$ then $|g|^p$ is in $L^1(G)$ and

$$\left|\widehat{|g|^p(\gamma)}\right| = \lim_{x_n\to\infty}|\hat{h}(x_n,\gamma)| \leqslant \||f|^p\|_1 \lim_{x_n\to\infty}\frac{1}{\omega^p(x_n)} = 0.$$

i.e. $g \equiv 0$.

On the other hand it will be shown that $\underline{\lim}_{x_n\to\infty}\left\|\frac{L_{x_n}f}{\omega(x_n)}\right\|_{p,\omega} \neq 0$. It follows:

$$\int\left(\frac{|L_x f|}{\omega(x)}\right)^p\omega^p(y)\,dy = \left(\int\frac{f(u)\omega(u+x)}{\omega(x)}|^p du\right)^{1/p} \geqslant \left(\int\left(\frac{1}{\omega(-u)}\right)^p|f(u)|^p\,du\right)^{1/p}$$

and for every compact K of G:

$$\left\|\frac{L_x f}{\omega(x)}\right\|_{p,\omega} \geqslant \frac{1}{\sup\{\omega(-k),\, k\in K\}}\left(\int_K |f(u)|^p\,du\right)^{1/p}.$$

It is easy to choose K such that the second side is not zero. By the hypothesis $\overline{\lim}_{x\in G}\omega(x) = +\infty$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_{x_n\to\infty}\omega(x_n) = +\infty$ and the sequence $\left\{\frac{L_{x_n} f}{\omega(x_n)},\right.$ $n\in\mathbb{N}\}$ have not limit point.

So the only weight almost periodic function in $L^p_\omega(G)$ is the null function.

Case $p = +\infty$, let f be in $(L^\infty_\omega)^\omega_{ap}$ and $\varphi \in L^1_\omega \subset L^1$ the Fourier transform of $h(x,y) = \varphi(y)\omega(y)\frac{f(y-x)}{\omega(x)}$ satisfies

$$\hat{h}(x,y) = \int \frac{\varphi(y)\, f(y-x)\langle\overline{y,y}\rangle\omega(y)}{\omega(x)}\,dy = \langle\overline{y,x}\rangle\int \varphi(u+x)\, f(u)\frac{\omega(u+x)}{\omega(x)}\,\overline{\langle y,u\rangle}\,du.$$

$$|\hat{h}(x,y)| \leqslant \frac{1}{\omega(x)}\int |\varphi(u+x)||f(u)|\omega(u+x)\,du$$

$$\|\hat{h}(x,y)\|_\infty \leqslant \frac{1}{\omega(x)}\|f\|_\infty\,\|\varphi\|_{1,\omega}.$$

Let x_n a sequence such that $\lim_{x_n\to\infty}\omega(x_n) = +\infty$ then the only possible limit of $h(x_n,y)$ is zero.

If $g = \lim_{x_n\to\infty}\frac{L_{x_n} f}{\omega(x_n)}$ it follows for every $\varphi \in L^1_\omega(G)$ $0 = \lim_{x_n\to\infty}\varphi\omega\frac{L_{x_n} f}{\omega(x_n)} = \varphi\omega g$ *i.e.* $g = 0$. On the other hand

$$\frac{|f(u)|}{\omega(u)} \leqslant \left|\frac{f(y-x)}{\omega(x)}\omega(y)\right| = \left|f(u)\frac{\omega(u+x)}{\omega(x)}\right| \leqslant \left\|\frac{L_x f}{\omega(x)}\right\|_{\infty,\omega}$$

and if $f \neq 0$ then it is not possible that

$$\lim_{x_n\to\infty}\frac{\|L_{x_n} f\|}{\omega(x_n)} = 0.$$

So the only weight almost periodic function in $L^\infty_\omega(G)$ is the null function. □

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