On a metric on the set of uniformly discrete sets of $\mathbb{R}^n$ invariant by the group of rigid motions and densest packings of equal spheres

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Abstract

We give the construction of a metric, invariant by the group of rigid motions of $\mathbb{R}^n$, on the set of uniformly discrete point sets of $\mathbb{R}^n, n \geqslant 1$, of given constant, i.e. having the property that their minimal interpoint distance is greater than a given strictly positive real number. The corresponding metric space is complete and locally compact. As a consequence, we prove in a non-effective way the existence of at least one system of equal spheres of $\mathbb{R}^n$, called extreme, of density equal to the packing constant. By reduction to the space of lattices of $\mathbb{R}^n$, it implies the existence of extreme lattices, without invoking any theory of reduction for lattices.

1. Introduction

Sets of points sets, in particular sets of lattices, of $\mathbb{R}^n$ have been extensively studied for many problems of packing of spheres [Ca] [CS] [GL] [Ma] [Zo]. The recent extension by Muraz and Verger-Gaugry [MVG] of the classical selection theorem of Mahler [Mh] to the sets of uniformly discrete sets of given strictly positive constants of $\mathbb{R}^n$ offers new challenging questions, beyond the classical questions and conjectures concerning lattices. It is the purpose of this note, first to recall the construction of a metric on the space $\mathcal{UD}_r$ of uniformly discrete sets of $\mathbb{R}^n$ of minimum $r > 0$ (theorem 1.1) so that $\mathcal{UD}_r$ is compact, second to answer positively to the question whether there exists a metric on $\mathcal{UD}_r$ which is invariant by the group of rigid motions of $\mathbb{R}^n$ (theorem 1.2), then to give the important application of this result as for the existence of packings of equal spheres of maximal density in $\mathbb{R}^n$ (theorem 1.3).

Let $r > 0$. A uniformly discrete set Λ of $\mathbb{R}^n$ of constant r will be a discrete subset of $\mathbb{R}^n$ such that $\|x - y\| \geqslant r$ as soon as $x \neq y, x, y \in \Lambda$. Hence, Λ may be either the "empty set" element $\emptyset$ or a 1-point set $\{x\}$, with $x \in \mathbb{R}^n$ arbitrary, or any two distinct points of Λ satisfy the above inequality if Λ contains at least two points. We will denote by $\mathcal{UD}_r$ the

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set of uniformly discrete sets of constant $r > 0$ and by $\mathscr{UD}$ the union $\bigcup_{r>0} \mathscr{UD}_r$. An element of $\mathscr{UD}$ will be called a uniformly discrete set of $\mathbb{R}^n$. For all $\Lambda \in \mathscr{UD}$ having at least two distinct points, let us denote by $m(\Lambda) > 0$ its minimal interpoint distance which will be called *minimum* of Λ (so that $\Lambda \in \mathscr{UD}_{m(\Lambda)}$). Denote by $O(n,\mathbb{R})$ the n-dimensional orthogonal group of $n \times n$ matrices M such that $M^{-1} = {}^tM$. Let us define *a rigid motion* (or an *Euclidean displacement*) by an ordered pair (ρ, t) with $\rho \in O(n,\mathbb{R})$ and $t \in \mathbb{R}^n$ [Chp]. The composition of two rigid motions is given by $(\rho, t)(\rho', t') = (\rho\rho', \rho(t') + t)$ and the group of rigid motions is the split extension of $O(n,\mathbb{R})$ by $\mathbb{R}^n$. It is endowed with the usual topology. In section 2 we will give the proof of the following result, generalizing [MVG], which will open the way to the theorem 1.2, i.e. we will give the construction of the two metrics $d^{(r)}$ and $D^{(r)}$, noticing that, restricted to $\mathscr{UD}_r \setminus \{\emptyset\}$, the metric $D^{(r)}$ is equivalent to the Hausdorff metric.

THEOREM 1.1. — *Let $r > 0$. There exists a metric $d^{(r)}$ on $\mathscr{UD}_r$ such that: i) the space $(\mathscr{UD}_r, d^{(r)})$ is compact, ii) $d^{(r)}(\Lambda, \Lambda') = d^{(r)}(\rho(\Lambda), \rho(\Lambda'))$ for all $\rho \in O(n,\mathbb{R})$ and $\Lambda, \Lambda' \in \mathscr{UD}_r$.*

THEOREM 1.2. — *Let $r > 0$. There exists a metric $D^{(r)}$ on $\mathscr{UD}_r$ such that: i) $D^{(r)}(\Lambda_1, \Lambda_2) = D^{(r)}(\rho(\Lambda_1) + t, \rho(\Lambda_2) + t)$ for all $t \in \mathbb{R}^n, \rho \in O(n,\mathbb{R}^n)$ and all $\Lambda_1, \Lambda_2 \in \mathscr{UD}_r$, ii) the space $(\mathscr{UD}_r, D^{(r)})$ is complete and locally compact, iii) (pointwise pairing property) for all non-empty $\Lambda, \Lambda' \in \mathscr{UD}_r$ such that $D^{(r)}(\Lambda, \Lambda') < \epsilon$, each point $\lambda \in \Lambda$ is associated with a unique point $\lambda' \in \Lambda'$ such that $\|\lambda - \lambda'\| < \min\{r, 1\}\epsilon/2$, iv) the action of the group of rigid motions $O(n,\mathbb{R}) \ltimes \mathbb{R}^n$ on $(\mathscr{UD}_r, D^{(r)}) : ((\rho, t), \Lambda) \to (\rho, t) \cdot \Lambda = \rho(\Lambda) + t$ is such that its subgroup of translations $\mathbb{R}^n$ acts continuously on $\mathscr{UD}_r$.*

Let us turn to sphere packings [CS] [GL] [Oe] [Ro] [Zo] and let us denote by $B(c, r)$ the (generic) closed ball of $\mathbb{R}^n$ of centre c and of radius $r > 0$. For all $\Lambda \in \mathscr{UD}$ having at least two distinct points, let us denote by $\mathscr{B}(\Lambda)$ the system of balls $\Lambda + B(0, m(\Lambda)/2)$. This system of balls forms a packing whose density is defined by $\delta(\mathscr{B}(\Lambda)) = \limsup_{R\to+\infty} \mathrm{vol}(\mathscr{B}(\Lambda) \cap B(0,R))/\mathrm{vol}(B(0,R))$. The supremum $\delta = \sup_{\Lambda \in \mathscr{UD}} \delta(\mathscr{B}(\Lambda))$ will be called *the packing constant*. It depends only upon n. As a consequence of the theorems 1.1 and 1.2, we will prove the following result in section 3.

THEOREM 1.3. — *There exists an element $\Lambda \in \mathscr{UD}$ such that the following equality holds: $\delta(\mathscr{B}(\Lambda)) = \delta$.*

By restricting the density function $\Lambda \to \delta(\mathscr{B}(\Lambda))$ to the subspace of lattices of $\mathbb{R}^n$, we obtain the existence of extreme lattices, without invoking any theory of reduction for lattices [Ma] [Oe].

COROLLARY 1.4. — *Let $\mathscr{L}_n$ the locally compact topological space of lattices of $\mathbb{R}^n$. Let $\delta_L := \sup_{\Lambda \in \mathscr{L}_n} \delta(\mathscr{B}(\Lambda))$. Then, there exists $\Lambda \in \mathscr{L}_n$ such that the following equality holds: $\delta(\mathscr{B}(\Lambda)) = \delta_L$.*

2. Metrics on $\mathscr{UD}_r$, $r > 0$

2.1. Proof of the theorem 1.1

In [MVG], in the scope of generalizing the selection theorem of Mahler and its proof by Chabauty [Chy], the space $\mathscr{UD}_1$ was endowed with a metric $d^{(1)}$ so that it is compact. Now, for all $r \geqslant 1$, the subspace $\mathscr{UD}_r = \{\Lambda \in \mathscr{UD}_1 \mid m(\Lambda) \geqslant r\}$ of $\mathscr{UD}_1$ is obviously closed, hence compact. Hence, for all $r \geqslant 1$, it suffices to take for $d^{(r)}$ the restriction of $d^{(1)}$ to $\mathscr{UD}_r$ to obtain the metric we are looking for and the compactness of $(\mathscr{UD}_r, d^{(r)})$ as claimed. Therefore we will only consider that $0 < r < 1$ below and we will proceed by constructing exlicitely $d^{(r)}$ on $\mathscr{UD}_r$. We will construct $d^{(r)}$ as a kind of counting system normalized by a suitable distance function.

Since any $\Lambda \in \mathscr{UD}_r$ is countable, we denote by Λ_i its i-th element. Let $\mathscr{E} = \{(D,E) \mid D$ countable point set in $\mathbb{R}^n, E$ countable point set in $(0,1/2)\}$ and $f : \mathbb{R}^n \to [0,1]$ a continuous function with compact support in $B(0,1)$ which satisfies: i) $f(0) = 1$, ii) $f(\rho(t)) = f(t)$ for all $t \in \mathbb{R}^n$ and all $\rho \in O(n,\mathbb{R})$ and iii) $f(t) \leqslant \frac{1/2+\|\lambda - t/2\|}{1/2+\|\lambda\|}$ for all $t \in B(0,1)$ and $\lambda \in \mathbb{R}^n$ (for technical reasons [MVG]). Recall that a pseudo-metric δ on a space satisfies all the axioms of a distance except that $\delta(u,v) = 0$ does not necessarily imply $u = v$.

With each element $(D,E) \in \mathscr{E}$ and origin α of the affine euclidean space $\mathbb{R}^n$ we associate a real-valued function $d^{(r)}_{\alpha,(D,E)}$ on $\mathscr{UD}_r \times \mathscr{UD}_r$ in the following way (denoting by $\overset{o}{B}(c,v)$ the interior of the closed ball $B(c,v)$ of centre c and radius $v > 0$). Let $\mathscr{B}_{(D,E)} = \{\mathscr{B}^{(r)}_m\}$ denote the countable set of all possible finite collections $\mathscr{B}^{(r)}_m = \{\overset{o}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{o}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \ldots, \overset{o}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}$ (with i_m the number of elements $\#\mathscr{B}_m$ of $\mathscr{B}_m$) of open balls such that $c_q^{(m)} \in D$ and $\epsilon_q^{(m)} \in E$ for all $q \in \{1,2,\ldots,i_m\}$, and such that for all m and any two distinct balls in $\mathscr{B}^{(r)}_m$ of respective centers $c_q^{(m)}$ and $c_k^{(m)}$, we have $\|c_q^{(m)} - c_k^{(m)}\| \geqslant r$. Then we define the following function, with $\Lambda, \Lambda' \in \mathscr{UD}_r$,

$$d^{(r)}_{\alpha,(D,E)}(\Lambda,\Lambda') := \sup_{\mathscr{B}^{(r)}_m \in \mathscr{B}_{(D,E)}} \frac{\left|\phi^{(r)}_{\mathscr{B}^{(r)}_m}(\Lambda) - \phi^{(r)}_{\mathscr{B}^{(r)}_m}(\Lambda')\right|}{(r/2 + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{i_m}^{(m)}\|)} \quad (1)$$

where the real-valued function $\phi^{(r)}_{\mathscr{B}^{(r)}_m}$ is given by $\phi^{(r)}_{\mathscr{B}^{(r)}_m}(\Lambda) := \sum_{\overset{o}{B}(c,\epsilon)\in\mathscr{B}^{(r)}_m} \sum_i r\epsilon f\left(\frac{\Lambda_i - c}{r\epsilon}\right)$. By convention we put $\phi^{(r)}_{\mathscr{B}^{(r)}_m}(\emptyset) = 0$ for all $\mathscr{B}^{(r)}_m \in \mathscr{B}_{(D,E)}$ and all $(D,E) \in \mathscr{E}$. It is clear that, for all m and $\Lambda \in \mathscr{UD}_r$, inside each ball $\overset{o}{B}(c, r\epsilon)$, where $\overset{o}{B}(c,\epsilon) \in \mathscr{B}^{(r)}_m$, there is at most one point of Λ and therefore the summation $\sum_i r\epsilon f\left(\frac{\Lambda_i - c}{r\epsilon}\right)$ is reduced to at most one non-zero term. Therefore the sum $\phi^{(r)}_{\mathscr{B}^{(r)}_m}(\Lambda)$ is finite.

LEMMA 2.1. — *For all $(\alpha,(D,E))$ in $\mathbb{R}^n \times \mathscr{E}$, $d^{(r)}_{\alpha,(D,E)}$ is a pseudo-metric on $\mathscr{UD}_r$, valued in $[0,1]$.*

Proof. — Let $r > 0, \alpha \in \mathbb{R}^n$ and $(D,E) \in \mathscr{E}$. It is easy to check that $d^{(r)}_{\alpha,(D,E)}$ is a pseudo-metric on $\mathscr{UD}_r$. Let us show that it is valued in $[0,1]$. Let us consider $\mathscr{B}^{(r)}_m \in \mathscr{B}_{(D,E)}$ for which the centers of its constitutive balls are denoted by $c_1, c_2, \dots, c_{i_m}$. Then $\frac{i_m r}{2} \leqslant r/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\|$. Indeed, if there exists $j \in \{1, 2, \dots, i_m\}$ such that $\|c_j - \alpha\| \leqslant r/2$, then for all $k \neq j$, $\|c_k - \alpha\| \geqslant r/2$. Hence $r/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\| \geqslant r/2 + \|\alpha\| + \frac{(i_m - 1)r}{2} \geqslant \frac{i_m r}{2}$. If $\|c_k - \alpha\| \geqslant r/2$ for all $k \in \{1, 2, \dots, i_m\}$, then $r/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\| \geqslant r/2 + \|\alpha\| + \frac{i_m r}{2} \geqslant \frac{i_m r}{2}$. On the other hand, since the radii of the balls $\overset{o}{B}(c_j, \epsilon_j) \in \mathscr{B}_m^{(r)}$ are less than $1/2$ by construction, we have $0 \leqslant \phi^{(r)}_{\mathscr{B}_m^{(r)}}(\Lambda) \leqslant \frac{i_m r}{2}$ for all $\Lambda \in \mathscr{UD}_r$. Therefore $\left|\phi^{(r)}_{\mathscr{B}_m^{(r)}}(\Lambda) - \phi^{(r)}_{\mathscr{B}_m^{(r)}}(\Lambda')\right| \leqslant \frac{i_m r}{2} \leqslant r/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\|$, for all $\mathscr{B}_m^{(r)} \in \mathscr{B}_{(D,E)}$ and all $\Lambda, \Lambda' \in \mathscr{UD}_r$. We deduce the claim. □

LEMMA 2.2. — *The supremum* $d^{(r)} := \sup_{\substack{\alpha \in \mathbb{R}^n \\ (D,E) \in \mathscr{E}}} d^{(r)}_{\alpha,(D,E)}$ *is a metric on* $\mathscr{UD}_r$*, valued in* $[0,1]$.

Proof. — The supremum of the family of pseudo-metrics $d^{(r)}_{\alpha,(D,E)}$ is obviously a pseudo-metric which takes its values in $[0,1]$. We have only to show that $d^{(r)}$ is a metric. Let us assume that $\Lambda, \Lambda' \in \mathscr{UD}_r$ are are not empty, such that $d^{(r)}(\Lambda, \Lambda') = 0$, and let us show that $\Lambda = \Lambda'$. We will show that $\Lambda \not\subset \Lambda'$ and $\Lambda' \not\subset \Lambda$ are impossible. Assume that $\Lambda \neq \Lambda'$ and that $\Lambda \not\subset \Lambda'$. Then there exists $\lambda \in \Lambda$ such that $\lambda \notin \Lambda'$. Denote by $\epsilon := \frac{1}{2}\min\{\frac{1}{2}, \min\{\|\lambda - u\| \mid u \in \Lambda'\}\}$. We have $\epsilon > 0$. The ball $\overset{o}{B}(\lambda, \epsilon)$ contains no point of Λ' and only the point λ of Λ. Take $\alpha = \lambda$, $D = \{\lambda\}, E = \{\epsilon\}$. Then $d^{(r)}_{\lambda,(D,E)}(\Lambda, \Lambda') = \frac{r\epsilon}{r/2 + \|\lambda\|} > 0$. Hence $d^{(r)}(\Lambda, \Lambda')$ would be strictly positive. Contradiction. Therefore $\Lambda \subset \Lambda'$. Then, exchanging Λ and Λ', we have $\Lambda' \subset \Lambda$. We deduce the equality $\Lambda = \Lambda'$. If we assume that one of the sets Λ or Λ' is the empty set, we see that the above proof is still valid. □

LEMMA 2.3. — *For all* $0 < r \leqslant 1$*,* $(D,E) \in \mathscr{E}$*,* $\alpha \in \mathbb{R}^n$ *and* $\Lambda, \Lambda' \in \mathscr{UD}_r$*, the following equality holds:* $d^{(r)}_{\alpha,(D,E)}(\Lambda, \Lambda') = d^{(1)}_{\frac{\alpha}{r},(\frac{D}{r},E)}(\frac{\Lambda}{r}, \frac{\Lambda'}{r})$.

Proof. — Let $r \in (0,1]$. For all $(D,E) \in \mathscr{E}$ and $\mathscr{B}_m^{(r)} \in \mathscr{B}_{(D,E)}$ with $\mathscr{B}_m^{(r)} = \{\overset{o}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{o}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \dots, \overset{o}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}$, the following inequalities hold: $\|c_q^{(m)} - c_k^{(m)}\| \geqslant r$ for all $1 \leqslant q, k \leqslant i_m$ with $q \neq k$. The collection $\mathscr{B}_m^{(r)}$ is in one-to-one correspondance, by the dilation of fixed point the origin and scalar factor r, with the collection of open balls $\mathscr{B}_m^{(r),1} := \{\overset{o}{B}(c_1^{(m)}/r, \epsilon_1^{(m)}), \overset{o}{B}(c_2^{(m)}/r, \epsilon_2^{(m)}), \dots, \overset{o}{B}(c_{i_m}^{(m)}/r, \epsilon_{i_m}^{(m)})\} \in \mathscr{B}_{(D/r,E)}$, where now $\|c_q^{(m)}/r - c_k^{(m)}/r\| \geqslant 1$. Since, for a given $\alpha \in \mathbb{R}^n$,

$$\frac{\left|\phi^{(r)}_{\mathscr{B}_m^{(r)}}(\Lambda) - \phi^{(r)}_{\mathscr{B}_m^{(r)}}(\Lambda')\right|}{\frac{r}{2} + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \ldots + \|\alpha - c_{i_m}^{(m)}\|} = \frac{\left|\phi^{(1)}_{\mathscr{B}_m^{(r),1}}(\Lambda/r) - \phi^{(1)}_{\mathscr{B}_m^{(r),1}}(\Lambda'/r)\right|}{\frac{1}{2} + \|\frac{\alpha}{r}\| + \|\frac{\alpha}{r} - \frac{c_1^{(m)}}{r}\| + \ldots + \|\frac{\alpha}{r} - \frac{c_{i_m}^{(m)}}{r}\|}$$

we deduce, by taking the supremum over all the collections $\mathscr{B}_m^{(r)} \in \mathscr{B}_{(D,E)}$, the claimed equality. □

From the above lemma 2.3, by taking the supremum over all $\alpha \in \mathbb{R}^n$ and all $(D,E) \in \mathscr{E}$, we deduce the following fundamental identity:

$$d^{(r)}(\Lambda, \Lambda') = d^{(1)}(\Lambda/r, \Lambda'/r), \qquad \text{for all } 0 < r \leqslant 1 \text{ and } \Lambda, \Lambda' \in \mathscr{UD}_r.$$

This identity can be used as a definition of $d^{(r)}$ on $\mathscr{UD}_r$, when $r < 1$, since $d^{(1)}$ was already constructed in [MVG]. However, we have prefered to give a direct construction of $d^{(r)}$. The properties of $d^{(1)}$ on $\mathscr{UD}_1$ can now be invoked to deduce the completeness and the precompactness of the metric space $(\mathscr{UD}_r, d^{(r)})$ (see [MVG]), hence its compactness, as claimed.

Let us show that $d^{(r)}$ is invariant by the action of the orthogonal group $O(n,\mathbb{R})$.

LEMMA 2.4. — *For all* $r > 0, (D,E) \in \mathscr{E}, \alpha \in \mathbb{R}^n, \rho \in O(n,\mathbb{R})$, *and* $\Lambda, \Lambda' \in \mathscr{UD}_r$, *the following equality holds:* $d^{(r)}_{\alpha,(D,E)}(\Lambda,\Lambda') = d^{(r)}_{\rho(\alpha),(\rho(D),E)}(\rho(\Lambda),\rho(\Lambda'))$.

Proof. — Let $r > 0$. For all $(D,E) \in \mathscr{E}$ and $\mathscr{B}^{(r)}_m \in \mathscr{B}_{(D,E)}$ with $\mathscr{B}^{(r)}_m = \{\overset{o}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{o}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \ldots, \overset{o}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}$, the following inequalities hold: $\|c_q^{(m)} - c_k^{(m)}\| \geqslant r$ for all $1 \leqslant q,k \leqslant i_m$ with $q \neq k$. The collection $\mathscr{B}^{(r)}_m$ is in one-to-one correspondance, by a given isometry $\rho \in O(n,\mathbb{R})$, with the collection of open balls $\mathscr{B}^{(r),(\rho)}_m := \{\overset{o}{B}(\rho(c_1^{(m)}), \epsilon_1^{(m)}), \overset{o}{B}(\rho(c_2^{(m)}), \epsilon_2^{(m)}), \ldots, \overset{o}{B}(\rho(c_{i_m}^{(m)}), \epsilon_{i_m}^{(m)})\} \in \mathscr{B}_{(\rho(D),E)}$, where the following inequalities $\|\rho(c_q^{(m)}) - \rho(c_k^{(m)})\| \geqslant r$ are still true. Since the function f is invariant by construction by the action of ρ, the following equalities hold: $\phi^{(r)}_{\mathscr{B}^{(r)}_m}(\Lambda) = \phi^{(r)}_{\mathscr{B}^{(r),(\rho)}_m}(\rho(\Lambda))$. Hence, for a given $\alpha \in \mathbb{R}^n$, by taking the supremum over all the collections $\mathscr{B}^{(r)}_m \in \mathscr{B}_{(D,E)}$ of the following identity:

$$\frac{\left|\phi^{(r)}_{\mathscr{B}^{(r)}_m}(\Lambda) - \phi^{(r)}_{\mathscr{B}^{(r)}_m}(\Lambda')\right|}{\frac{r}{2} + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \ldots + \|\alpha - c_{i_m}^{(m)}\|} = \frac{\left|\phi^{(r)}_{\mathscr{B}^{(r),(\rho)}_m}(\rho(\Lambda)) - \phi^{(r)}_{\mathscr{B}^{(r),(\rho)}_m}(\rho(\Lambda')\right|}{\frac{1}{2} + \|\rho(\alpha)\| + \|\rho(\alpha) - \rho(c_1^{(m)})\| + \ldots + \|\rho(\alpha) - \rho(c_{i_m}^{(m)})\|}$$

we deduce the claimed equality. □

Taking now the supremum of $d^{(r)}_{\alpha,(D,E)}(\Lambda,\Lambda')$ over all $\alpha \in \mathbb{R}^n$ and $(D,E) \in \mathscr{E}$, we deduce from the above lemma 2.4 that $d^{(r)}(\Lambda,\Lambda') = d^{(r)}(\rho(\Lambda),\rho(\Lambda'))$ for all $\Lambda,\Lambda' \in \mathscr{UD}_r$ and $\rho \in O(n,\mathbb{R})$, as claimed.

2.2. Proof of the theorem 1.2

For all $r > 0$, the metric $d^{(r)}$ on $\mathscr{UD}_r$ has the advantage to make compact the metric space $(\mathscr{UD}_r, d^{(r)})$ but, by the way it is constructed, the disadvantage to specify a point (the origin) in the ambiant space $\mathbb{R}^n$. We will remove this disadvantage but the counterpart will be that we will loose the precompactness of the metric space $\mathscr{UD}_r$. In order to do this, let us first define, for all $x \in \mathbb{R}^n$, the new metric $d^{(r)}_x$ on $\mathscr{UD}_r$ by

$$d^{(r)}_x(\Lambda,\Lambda') = d^{(r)}(\Lambda - x, \Lambda' - x), \qquad \Lambda,\Lambda' \in \mathscr{UD}_r.$$

Let us remark that the metric spaces $(\mathscr{UD}_r, d^{(r)}_x), x \in \mathbb{R}^n$, are also compact (by the theorem 1.1).

DEFINITION 2.5. — *Let* $r > 0$. *Let* $D^{(r)}$ *the metric on* $\mathscr{UD}_r$, *taking its values in* $[0,1]$, *defined by* $D^{(r)}(\Lambda,\Lambda') := \sup_{x\in\mathbb{R}^n} d^{(r)}_x(\Lambda,\Lambda')$, *for all* $\Lambda,\Lambda' \in \mathscr{UD}_r$. *The metric* $D^{(r)}$ *will be called the metric of the proximity of points, or pp-metric.*

Let us show i), i.e. that $D^{(r)}$ is invariant by the group of rigid motions of $\mathbb{R}^n$. First, by construction, $D^{(r)}$ is invariant by the translations of $\mathbb{R}^n$. Therefore, we will just focus below on the proof of its invariance by the elements of the orthogonal group $O(n,\mathbb{R})$. For all $\Lambda,\Lambda' \in \mathcal{UD}_r$ and all $x \in \mathbb{R}^n, \rho \in O(n,\mathbb{R})$, since $d^{(r)}(\Lambda,\Lambda') = d^{(r)}(\rho(\Lambda),\rho(\Lambda'))$ by the lemma 2.4, we deduce that $d_x^{(r)}(\Lambda,\Lambda') = d^{(r)}(\Lambda - x, \Lambda' - x) = d^{(r)}(\rho(\Lambda)-\rho(x),\rho(\Lambda')-\rho(x)) = d_{\rho(x)}^{(r)}(\rho(\Lambda),\rho(\Lambda'))$. Hence, $\sup_{x\in\mathbb{R}^n} d_x^{(r)}(\Lambda,\Lambda') = \sup_{x\in\mathbb{R}^n} d_{\rho(x)}^{(r)}(\rho(\Lambda),\rho(\Lambda'))$, meaning exactly that $D^{(r)}(\Lambda,\Lambda') = D^{(r)}(\rho(\Lambda),\rho(\Lambda'))$ as claimed.

Let us show that $(\mathcal{UD}_r, D^{(r)})$ is complete in ii). It is obvious that any Cauchy sequence for the pp-metric $D^{(r)}$ will be a Cauchy sequence for all the metrics $d_x^{(r)}$ for all $x \in \mathbb{Q}^n$ in particular. But $\mathbb{Q}^n$ is countable. Therefore, we can extract from the initial Cauchy sequence, by a diagonalisation process over all $x \in \mathbb{Q}^n$, a subsequence which converges for all the metrics $d_x^{(r)}, x \in \mathbb{Q}^n$. Since $\mathbb{Q}^n$ is dense in the ambiant space $\mathbb{R}^n$, that $\sup_{x\in\mathbb{Q}^n} d_x^{(r)}(\Lambda,\Lambda') = \sup_{x\in\mathbb{R}^n} d_x^{(r)}(\Lambda,\Lambda')$ for all $\Lambda,\Lambda' \in \mathcal{UD}_r$ by continuity of the function f, this subsequence, extracted by diagonalization, also converges for the metric $D^{(r)}$. This prove the completeness of the metric space $(\mathcal{UD}_r, D^{(r)})$.

Let us prove the pointwise pairing property iii) (for $D^{(r)}$) for the uniformly discrete sets of $\mathcal{UD}_r$ (from the lemma 2.6 for $r \geqslant 1$ and from the lemma 2.7 for $0 < r < 1$).

Lemma 2.6. — *Let $r \geqslant 1$ and $x \in \mathbb{R}^n$. Let $\Lambda,\Lambda' \in \mathcal{UD}_r$ assumed non-empty and define $l_x := \inf_{\lambda\in\Lambda} \|\lambda - x\| < +\infty$. Let $\epsilon \in (0, \frac{1}{1+2l_x})$ and let us assume that $d_x^{(r)}(\Lambda,\Lambda') < \epsilon$. Then, for all $\lambda \in \Lambda$ such that $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$, (i) there exists a unique $\lambda' \in \Lambda'$ such that $\|\lambda' - \lambda\| < 1/2$, (ii) this pairing satisfies the inequality $\|\lambda' - \lambda\| \leqslant (1/2 + \|\lambda - x\|)\epsilon$.*

Proof. — This property comes from the pointwise pairing property (for the metric $d^{(r)}$) for the elements of $\mathcal{UD}_r$ [MVG]. (i) Let us assume that for all $\lambda' \in \Lambda'$ and all $\lambda \in \Lambda$ such that $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$ the inequality $\|\lambda' - \lambda\| \geqslant 1/2$ holds. This will lead to a contradiction. Assume the existence of an element $\lambda \in \Lambda$ such that $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$ and take $D = \{\lambda - x\}$ and let E be a countable dense subset in $(0,1/2)$. Each $\mathcal{B}_m$ in $\mathcal{B}_{(D,E)}$ is a set constituted by only one element: the ball (say) $\overset{o}{B}(\lambda - x, e_m)$ with $e_m \in E$. We deduce that $\phi_{\mathcal{B}_m}(\Lambda - x) = e_m$ and $\phi_{\mathcal{B}_m}(\Lambda') = 0$. Hence

$$d_{\lambda-x,(D,E)}(\Lambda - x, \Lambda' - x) = \sup_m \frac{e_m}{1/2 + \|\lambda - x\|} = \frac{1/2}{1/2 + \|\lambda - x\|} \leqslant d_x^{(r)}(\Lambda,\Lambda').$$

But $\epsilon < \frac{1}{1+2\|\lambda - x\|}$ is equivalent to $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$. Since we have assumed $d_x^{(r)}(\Lambda,\Lambda') < \epsilon$, we should obtain $\epsilon < d_{\lambda-x,(D,E)}(\Lambda - x, \Lambda' - x) \leqslant d_x^{(r)}(\Lambda,\Lambda') < \epsilon$. Contradiction. The uniqueness of λ' comes from the fact that Λ' is a uniformly discrete set of constant $r \geqslant 1$ allowing only one element λ' close to λ within distance $1/2$. (ii) Let us assume that $\lambda \neq \lambda'$ for all $\lambda \in \Lambda$ such that $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$, with $\lambda' \in \Lambda'$ that satisfies $\|\lambda' - \lambda\| < 1/2$ (if the equality $\lambda = \lambda'$ holds, there is nothing to prove). Then, for all $\lambda \in \Lambda$ such that $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$, let us take $\alpha = \lambda - x$ as base point, $D = \{\lambda - x\}$ and E a dense subset in $(0, \|\lambda - \lambda'\|] \subset (0, 1/2)$. Then $\phi_{\mathcal{B}_m}(\Lambda - x) - \phi_{\mathcal{B}_m}(\Lambda' - x) = e_m\left(1 - f\left(\frac{\lambda'-\lambda}{e_m}\right)\right)$. The restriction of the function $z \to z(1 - f(\frac{\lambda'-\lambda}{z}))$ to $(0, \|\lambda - \lambda'\|]$ is the identity function and is bounded above by $\|\lambda' - \lambda\|$, by continuity of the function f. Therefore, $d_{\lambda-x,(D,E)}(\Lambda - x, \Lambda' - x) = \sup_{\mathcal{B}_m} \frac{|\phi_{\mathcal{B}_m}(\Lambda - x) - \phi_{\mathcal{B}_m}(\Lambda' - x)|}{1/2 + \|\lambda - x\|} = \frac{\|\lambda' - \lambda\|}{1/2 + \|\lambda - x\|}$.

Since $d_{\lambda-x,(D,E)}(\Lambda - x, \Lambda' - x) \leqslant d_x^{(r)}(\Lambda, \Lambda') < \epsilon$, we obtain $\|\lambda' - \lambda\| \leqslant (1/2 + \|\lambda - x\|)\epsilon$ as claimed. □

Let $0 < \epsilon < 1$ and let us now suppose that $\Lambda, \Lambda' \in \mathscr{UD}_r$ are non-empty and satisfy $D^{(r)}(\Lambda, \Lambda') < \epsilon$. This means that for all $x \in \mathbb{R}^n$, in particular for all $\lambda \in \Lambda$, the inequality $d_x^{(r)}(\Lambda, \Lambda') < \epsilon$ holds. We deduce from the above lemma 2.6, restricting x to all the elements λ of Λ, that, for all $\lambda \in \Lambda$, there exists a unique $\lambda' \in \Lambda'$ such that $\|\lambda - \lambda'\| < \epsilon/2$ as claimed.

Let us now consider the case $0 < r < 1$. Since, for all $\Lambda, \Lambda' \in \mathscr{UD}_r$, we have the identity $d^{(r)}(\Lambda, \Lambda') = d^{(1)}(\frac{\Lambda}{r}, \frac{\Lambda'}{r})$, the following lemma is a reformulation of lemma 2.6.

Lemma 2.7. — *Let $0 < r < 1$ and $x \in \mathbb{R}^n$. Let $\Lambda, \Lambda' \in \mathscr{UD}_r$ assumed non-empty and define $l_x := \inf_{\lambda \in \Lambda} \|\frac{\lambda}{r} - x\| < +\infty$. Let $\epsilon \in (0, \frac{1}{1+2l_x})$ and let us assume that $d_x^{(1)}(\frac{\Lambda}{r}, \frac{\Lambda'}{r}) < \epsilon$. Then, for all $\lambda \in \Lambda$ such that $\|\frac{\lambda}{r} - x\| < \frac{1-\epsilon}{2\epsilon}$, (i) there exists a unique $\lambda' \in \Lambda'$ such that $\|\frac{\lambda'}{r} - \frac{\lambda}{r}\| < 1/2$, (ii) this pairing satisfies the inequality $\|\frac{\lambda'}{r} - \frac{\lambda}{r}\| \leqslant (1/2 + \|\frac{\lambda}{r} - x\|)\epsilon$.*

As a consequence, the assumption $D^{(r)}(\Lambda, \Lambda') < \epsilon$ for all $0 < \epsilon < 1$ and all non-empty $\Lambda, \Lambda' \in \mathscr{UD}_r$ ensures, as above with x restricted now to the point set Λ/r, the existence of the pointwise pairings of the points of Λ and Λ' with now $\|\lambda - \lambda'\| < \frac{r\epsilon}{2}$. We deduce the result (iii)).

Let us prove iv), i.e. that the subgroup of translations $\mathbb{R}^n$ acts continuously on $\mathscr{UD}_r$ for all $r > 0$. First, let us remark that, for all $\Lambda_0 \in \mathscr{UD}_r$, $\lim_{t \to 0} D^{(r)}(\Lambda_0 + t, \Lambda_0) = 0$ for the above point iii). Then, given an arbitrary $(t_0, \Lambda_0) \in \mathbb{R}^n \times \mathscr{UD}_r$, let us prove the continuity of this action at the element (t_0, Λ_0). Let us take $0 < \epsilon < 1$. Then, there exists $\eta > 0$ such that $|t - t_0| < \eta$ implies $D^{(r)}(\Lambda_0 + (t - t_0), \Lambda_0) < \epsilon/2$. Hence, for all $\Lambda \in \mathscr{UD}_r$ such that $D^{(r)}(\Lambda, \Lambda_0) < \epsilon/2$ and $t \in \mathbb{R}^n$ such that $|t - t_0| < \eta$, we obtain: $D^{(r)}(\Lambda + t, \Lambda_0 + t_0) = D^{(r)}(\Lambda + (t - t_0), \Lambda_0) \leqslant D^{(r)}(\Lambda + (t - t_0), \Lambda_0 + (t - t_0)) + D^{(r)}(\Lambda_0 + (t - t_0), \Lambda_0) = D^{(r)}(\Lambda, \Lambda_0) + D^{(r)}(\Lambda_0 + (t - t_0), \Lambda_0) \leqslant \epsilon/2 + \epsilon/2 = \epsilon$. We deduce the claim.

Let us prove that $\mathscr{UD}_r$ is locally compact (ii)). Let $r > 0$. Recall that the Hausdorff metric Δ is classically defined on the set $\mathscr{F}(\mathbb{R}^n)$ of the non-empty closed subsets of $\mathbb{R}^n$. If $\Lambda, \Lambda' \in \mathscr{UD}_r \setminus \{\emptyset\}$, then we define the metric h on $\mathscr{UD}_r \setminus \{\emptyset\}$ (it is in fact the restriction of the Hausdorff metric Δ to $\mathscr{UD}_r \setminus \{\emptyset\}$) by $h(\Lambda, \Lambda') := \max\{\inf\{\epsilon \mid \Lambda' \subset \Lambda + B(0,\epsilon)\}, \inf\{\epsilon \mid \Lambda \subset \Lambda' + B(0,\epsilon)\}\}$. Obviously, $\mathscr{UD}_r \setminus \{\emptyset\}$ is closed in the complete space $(\mathscr{F}(\mathbb{R}^n), \Delta)$. Then $\mathscr{UD}_r \setminus \{\emptyset\}$ is complete for h. On the space $\mathscr{UD}_r \setminus \{\emptyset\}$, the two metrics $D^{(r)}$ and h are obviously equivalent. The element $\emptyset$ is isolated in $\mathscr{UD}_r$ for $D^{(r)}$. Hence, it possesses a neighbourhood (reduced to itself) whose closure is compact. Now, if $\Lambda \in \mathscr{UD}_r \setminus \{\emptyset\}$ and $0 < \epsilon < 1$, the open neighbourhood $\{\Lambda' \in \mathscr{UD}_r \mid \Lambda' \subset \Lambda + \overset{o}{B}(0,\epsilon)\}$ of Λ admits $\{\Lambda' \in \mathscr{UD}_r \mid \Lambda' \subset \Lambda + B(0,\epsilon)\}$ as closure which is obviously precompact, hence compact, for $D^{(r)}$ or h. We deduce the claim.

3. Proof of the theorem 1.3 and of its corollary 1.4

We will say that $\Lambda \in \mathscr{UD}_r$ is *extreme* if the density $\delta(\mathscr{B}(\Lambda))$ of the system of balls $\Lambda + B(0, r/2)$ is equal to the packing constant δ.

In this section, we will show the existence of extreme packings of balls. Since the packing constant is defined by $\delta = \sup_{\Lambda \in \mathscr{UD}} \delta(\mathscr{B}(\Lambda)) > 0$ and that any non-singular affine transformation on the system of balls $\mathscr{B}(\Lambda)$ (theorem 1.7 in [Ro]) leaves it invariant, it suffices to restrict ourselves to systems of balls whose centres constitute a uniformly discrete set belonging to one of the spaces $\mathscr{UD}_r$, $r > 0$, to investigate the highest possible densities. Let us choose $\mathscr{UD}_1$. Then let us prove that there exists $\Lambda \in \mathscr{UD}_1$ such that the claim holds. This will prove theorem 1.3. Assume that it is not the case and this will lead to a contradiction. Then, by definition, there exists a sequence $(\Lambda_i)_{i \geqslant 1}$ such that $\Lambda_i \in \mathscr{UD}_1$, $m(\Lambda_i) = 1$ and $\lim_{i \to +\infty} \delta(\mathscr{B}(\Lambda_i)) = \delta$ (as a sequence of real numbers). We will prove that we can extract a subsequence from the sequence $(\Lambda_i)_{i \geqslant 1}$ which will converge for $D^{(1)}$.

Indeed, this sequence $(\Lambda_i)_{i \geqslant 1}$ may be viewed as a sequence in the compact space $(\mathscr{UD}_1, d_x^{(1)})$ for all $x \in \mathbb{Q}^n$. Therefore, for all $x \in \mathbb{Q}^n$, we can extract a subsequence from it which converges for the metric $d_x^{(1)}$. Iterating this extraction by a diagonalization process over all $x \in \mathbb{Q}^n$, since $\mathbb{Q}^n$ is countable, shows that we obtain a subsequence which converges for all the metrics $d_x^{(1)}$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we obtain a convergent sequence $(\Lambda_{i_j})_{j \geqslant 1}$ for $D^{(1)}$ since $\sup_{x \in \mathbb{R}^n} d_x^{(1)} = \sup_{x \in \mathbb{Q}^n} d_x^{(1)}$, by continuity of the function f.

LEMMA 3.1. — *Let $r > 0$. The density function $\Lambda \to \delta(\mathscr{B}(\Lambda))$ is continuous on $(\mathscr{UD}_r, D^{(r)})$ and locally constant.*

Proof. — Let $r > 0$, $\Lambda_0 \in \mathscr{UD}_r$, $T > 0$ large enough and $0 < \epsilon < 1$. By the lemmas 2.6, 2.7 and the pointwise pairing property iii) in the theorem 1.2, any $\Lambda \in \mathscr{UD}_r$ such that $D^{(r)}(\Lambda, \Lambda_0) < \epsilon$ is such that the number of elements $\#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\}$ of Λ within $B(0,T)$ satisfies the following inequalities: $\#\{\lambda \in \Lambda_0 \mid \lambda \in B(0, T - \min\{r,1\}\epsilon/2)\} \leqslant \#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\} \leqslant \#\{\lambda \in \Lambda_0 \mid \lambda \in B(0, T + \min\{r,1\}\epsilon/2)\}$. The density of the system of balls $\mathscr{B}(\Lambda)$ is equal to $\delta(\mathscr{B}(\Lambda)) = \limsup_{T \to +\infty} \#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\} \left(\frac{r}{2T}\right)^n$. Since the contribution - to the calculation of the density - of the points of Λ_0 which lie in the annulus $B(0, T + \min\{r,1\}\epsilon/2) \setminus B(0, T - \min\{r,1\}\epsilon/2)$ tends to zero when T tends to infinity by the theorem 1.8 in Rogers [Ro], we deduce that $\delta(\mathscr{B}(\Lambda)) = \delta(\mathscr{B}(\Lambda_0))$, hence the lemma. ☐

Let us finish the proof of theorem 1.3. Since the metric space $(\mathscr{UD}_1, D^{(1)})$ is complete, the subsequence $(\Lambda_{i_j})_{j \geqslant 1}$ is such that, by the lemma 3.1, there exists a limit point set $\Lambda = \lim_{j \to +\infty} \Lambda_{i_j} \in \mathscr{UD}_1$ that satisfies $\delta = \lim_{j \to +\infty} \delta(\mathscr{B}(\Lambda_{i_j})) = \delta(\Lambda)$. This gives the conclusion.

REMARK.– Let us make a comment about *saturation*, linked to the possible filling of holes of uniformly discrete sets of $\mathscr{UD}_r$ [MVG1]. We will say that $\Lambda \in \mathscr{UD}_r$ is *saturated*, or *maximal*, if it is impossible to add a sphere of radius $r/2$ to $\mathscr{B}(\Lambda)$ without destroying the fact that it is a packing of spheres, i.e. without creating an overlap of spheres. The set SS_r of systems of balls of radius $r/2$, is partially ordered by the relation $\prec$ defined by

$$\Lambda_1, \Lambda_2 \in \mathscr{UD}_r, \quad \mathscr{B}(\Lambda_1) \prec \mathscr{B}(\Lambda_2) \iff \Lambda_1 \subset \Lambda_2.$$

By Zorn's lemma, maximal sphere packings exist. The saturation operation of a sphere packing consists in adding spheres to obtain a maximal sphere packing. It is fairly arbitrary and may be finite or infinite. It is not because a sphere packing is maximal (saturated) that its density is equal to δ.

In the proof of the theorem 1.3, we did not need assume that the elements Λ_{i_j} are saturated. If the limit point set $\Lambda \in \mathscr{UD}_1$ is such that $\delta = \delta(\mathscr{B}(\Lambda))$, then any (partial or total) saturation process of Λ obtained by adding balls of radius $1/2$ to Λ leads to new systems of spheres of SS_1 which are of the same density δ, but whose centres constitute uniformly discrete sets which do not lie in a small neighbourhood of Λ in the metric space $(\mathscr{UD}_1, D^{(1)})$.

Let us prove the corollary 1.4. Since the subspace $\mathscr{L}_n \cap \mathscr{UD}_1$ is closed in the compact space $(\mathscr{UD}_1, d^{(1)})$ after [MVG], it is also closed in $(\mathscr{UD}_1, d_x^{(1)})$ for all $x \in \mathbb{R}^n$. Therefore, it is closed in $(\mathscr{UD}_1, D^{(1)})$. The proof of the corollary 1.4 comes now readily from the lemma 3.1 where the density function is restricted to the space of lattices which are uniform discrete sets of constant $r > 0$.

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