THE MULTIPLIERS OF $A_\omega^p(G)$

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Résumé

Soit G un groupe localement compact abélien, dx sa mesure de Haar et ω un poids de Beurling sur G. Ce travail est l'étude des espaces $A_\omega^p(G)$ des $A^p(G)$-modules, et leurs multiplicateurs.

Abstract

Let G be a locally compact abelian group with Haar measure dx and ω be a Beurling's weight function on G. This study is concerned with the space $A_\omega^p(G)$, $A^p(G)$-modules, and their multipliers.

I. Introduction and Preliminaries

Let G be a locally compact abelian group with Haar measure dx and the dual group $\widehat{G}$. A Banach space $(B, \|\cdot\|_B)$ is called a Banach G-module, if there exists a group representation L of G to the group of isometries of B with $L_e = I_B$ (identity) and $L_{x+y} = L_x \circ L_y$.

Let $(A, \|\cdot\|_A)$ be a Banach algebra, A Banach space $(B, \|\cdot\|_B)$ is called a Banach A module if there exists an algebra representation T (continuous) of A to $BL(B)$, the algebra of all continuous linear operators from B to B with

$$\|T_a\| \leqslant \|a\|_A$$

$$T a_1 \cdot a_2 = T a_1 \circ T a_2$$

where $\cdot$ is the multiplication on A. For $b \in B$, $T_a(b)$ is denoted by $a \cdot b$.

Such a module is order free if 0 is the only $b \in B$ for which $a \cdot b = 0$ for all $a \in A$

A Banach A-module B is called essential if the closed linear span of $A \cdot B$ coincides with B. If the Banach algebra $(A, \|\cdot\|_A)$ contains bounded approximate identity, *i.e.* a bounded net

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$(u_\alpha)_{\alpha\in I}$ such that $\lim_\alpha \|u_\alpha \cdot a - a\|_A = 0$ for all $a \in A$ then a Banach A-module B is an essential one, by Cohen's factorization theorem, if and only if $\lim_\alpha \|u_\alpha \cdot b - b\|_B = 0$ for all $b \in B$ (for more details see[4], [10]).

If B is Banach A-module

$$\mathrm{Hom}_A(B) = \{T \in BL(B) \mid \forall a \in A,\ \forall b \in B,\ T(a\cdot b) = a \cdot T(b)\}$$

is the space of all A-module homomorphisms. The elements of $\mathrm{Hom}_A(B)$ are traditionally called multipliers from B to B.

If B is a Banach G-module

$$\mathrm{Hom}_G(B) = \{T \in BL(B) \mid \forall x \in G,\ \forall b \in B,\ T(L_x b) = (L_x T)b\}$$

is the space of all G-module homomorphisms. The elements of $H_G(B)$ will be called centralizer. In the most cases, a multiplier is equal to a centralizer, however in the certain spaces, a centralizer need not be a multiplier [3, 16].

In this paper, we introduce the weighted Banach $A_p(G)$-modules and discuss some multiplier problems where $A_p(G)$ is the p-Fourier algebra. The space $A_p(G)$, $1 \leqslant p < \infty$, was introduced by Figa-Talamanca [7] and proved that the multiplier space of $L^p(G)$ (as a centralizer) is isometrically isomorphic to the dual space $A_p^*(G)$ of $A_p(G)$ provided G is abelian. For nonabelian locally compact group G, Eymard [6] studied the Fourier algebra $A_2(G) = A(G)$ and for general p, $1 \leqslant p < \infty$, Herz [9] proved that $A_p(G)$ is a Banach algebra under pointwise multiplication.

We are interested in the structure theory of the weighted $A_p(G)$, denoted by $A_\omega^p(G)$. Using by the technical developed by Spector [15] we show that $A_\omega^p(G)$ is a Banach $A_p(G)$-module under pointwise multiplication. Note that $A_p(G)$ is a Fourier algebra under pointwise multiplication but it is not an algebra under convolution. Moreover, it is obtained some necessary and sufficients conditions on operator to be an $A_\omega^p(G)$-multipliers.

II. The $A_\omega^p(G)$ spaces and their convolutors

Let G be a locally compact abelian group with Haar measure dx and ω be a non negative continuous function on G.

$L_\omega^p(G) = \{f \mid f\omega \in L^p(G)\}$ denote the Banach space under the natural norm $\|f\|_{p,\omega} = \|f\omega\|_p$, $1 \leqslant p \leqslant \infty$ and for $1 \leqslant p < \infty$ its dual space is $L_{\omega^{-1}}^{p'}(G)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

$C_{0,\omega}(G)$ denote the Banach subspace of $L_\omega^\infty(G)$ such that $f\omega \in C_0(G)$, the space of all continuous functions vanishing at infinity of G.

If ω is a weight function, *i.e.* continuous function satisfying $\omega(x) \geqslant 1$, $\omega(x+y) \leqslant \omega(x)\omega(y)$ then $L_\omega^1(G)$ is a Banach algebra under convolution called a Beurling algebra. It follows that $L_\omega^p(G)$ is an essential Banach $L_\omega^1(G)$-module with respect to convolution [2].

Throughout it will be assumed that ω is even function, *i.e.* satisfy $\omega(x) = \omega(-x)$ for all $x \in G$.

As the classical thecnic of harmonic analysis it follows that

PROPOSITION II.1. — *Let $f \in L^p_\omega(G)$ and $g \in L^{p'}_{\omega^{-1}}(G)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Then $f * g \in C_{0,\omega^{-1}}(G)$ with*

$$\|f * g\|_{\infty,\omega^{-1}} \leqslant \|f\|_{p,\omega}\, \|g\|_{p',\omega^{-1}}\,.$$

Thus by Proposition II.1 it can be defined a bilinear map from $L^p_\omega(G) \times L^{p'}_{\omega^{-1}}(G)$ into $C_{0,\omega^{-1}}(G)$ by

$$b(f,g) = \tilde{f} * g, \qquad \tilde{f}(x) = f(-x)$$

and $\|b\| \leqslant 1$. Then b lifts a linear map B from $L^p_\omega(G)$ and $L^{p'}_{\omega^{-1}}(G)$ as a Banach space, into $C_{0,\omega^{-1}}(G)$ and $\|B\| \leqslant 1$, [1, 5, 8].

DEFINITION II.2. — *It will be denoted by $A^p_\omega(G)$ the subspace of $C_{0,\omega^{-1}}(G)$ of all functions h satisfying $h = \sum_{i=1}^{\infty} \tilde{f}_i * g_i$, $f_i \in L^p_\omega(G)$, $g_i \in L^{p'}_{\omega^{-1}}(G)$ such that $\sum_{i=1}^{\infty} \|f_i\|_{p,\omega}\, \|g_i\|_{p',\omega^{-1}} < \infty$ equipped with the norm*

$$\|h\|_{A^p_\omega} = \inf\Big\{\sum_{i=1}^{\infty} \|f_i\|_{p,\omega}\, \|g_i\|_{p',\omega^{-1}},\ \ h = \sum_{i=1}^{\infty} \tilde{f}_i * g_i\Big\}.$$

$A^p_\omega(G)$ is a Banach space under this norm and identifies with the space of the quotient Banach space of $L^p_\omega(G) \otimes_\gamma L^{p'}_{\omega^{-1}}(G)$ with K, *i.e.*

$$A^p\omega(G) \cong L^p_\omega(G) \otimes_\gamma L^{p'}_{\omega^{-1}}(G)/K$$

where K is the kernel of the bilinear form B.

Since $L^p_\omega(G)$ is the reflexive Banach space, by the general operator theory [5], the algebra of the operators $BL(L^p_\omega(G), L^p_\omega(G))$ identifies with the dual space $(L^p_\omega(G) \otimes_\gamma L^{p'}_{\omega^{-1}}(G))^*$ and the subspace K is the annihilator of the translations it follows that

$$\mathrm{Hom}_G\left(L^p_\omega(G), L^p_{\omega^{-1}}(G)\right) \cong (L^p_\omega(G) \otimes_\gamma L^{p(}_{\omega^{-1}}(G)/K)^* \cong (A^p_\omega(G))^*\,.$$

Because $L^p_\omega(G)$ is an essential, order free $L^1_\omega(G)$-module then K is also the annihilator of the operators acted by $L^1_\omega(G)$, see for example [3, 10, 14], so

$$(A^p_\omega(G))^* \cong \mathrm{Hom}_G\left(L^p_\omega(G), L^p_\omega(G)\right) \cong \mathrm{Hom}_{L^1_\omega}\left(L^p_\omega(G), L^p_\omega(G)\right).$$

Since $\overline{C_c(G)} = L^p_\omega(G)$, $\overline{C_c(G)} = L^{p'}_{\omega^{-1}}(G)$ and the convolution of two support compact functions is support compact it follows that $C_c(G) \cap A^p_\omega(G)$ is dense in $A^p_\omega(G)$. Also $L^p_\omega(G)$ is the essential Banach $L^1_\omega(G)$-module under convolution, $A^p_\omega(G)$ is an essential Banach $L^1_\omega(G)$-module [13] . So, for every $u \in A^p_\omega(G)$ and $x \in G$, $x \to f_x$ is continuous function where $L_x f(y) = f_x(y) = f(y - x)$ for all $y \in G$.

THEOREM II.3. — *Let $T \in BL(A^p_\omega(G), A^p_\omega(G))$, $f \in L^p_\omega(G)$, $g \in L^{p'}_{\omega^{-1}}(G)$ and $h \in L^1_\omega(G)$. The following statements are equivalent* *(i)* $T(h * f * g) = h * T(f * g)$

(ii) $T((L_x f) * g) = L_x T \cdot (f * g)$

(iii) $\exists \widetilde{T} \in BL(L^p_\omega(G), L^p_\omega(G))$ *such that*

$$T(f * g) = (\widetilde{T} f) * g\,.$$

Proof. — $(i) \Leftrightarrow (ii)$: since $A^p_\omega(G)$ is the essential Banach $L^1_\omega(G)$-module, it is obtained the equivalence of (i) and (ii).

$(iii) \Leftrightarrow (i)$: it is obvious.

$(ii) \Leftrightarrow (iii)$: let T be in $BL(A^p_\omega(G), A^p_\omega(G))$, $f \in L^p_\omega(G)$ the map $g \in L^{p'}_{\omega^{-1}}(G) \to (T(f * g))(e)$ defines a linear, continuous functional. Because of the reflexivity of $L^p_\omega(G)$ there exists an element $\widetilde{T} f$ of $L^p_\omega(G)$ such that

$$\langle \widetilde{T} f, g\rangle = (T(f * g))(e)$$

and it is obtained the following equivality

$$(T(f * g))(x) = \langle \widetilde{T} f, L_x\rangle\,.$$

□

Corollary II.4. — $\mathrm{Hom}_{L^1_\omega}(A^p_\omega(G), A^p_\omega(G)) = \mathrm{Hom}_{L^1_\omega}(L^p_\omega(G), L^p_\omega(G))$.

Proof. — It is an interpretation of the equivalence of (ii) and (iii).

III. The multipliers of $A^p_\omega(G)$

If $\omega = 1$, Herz [9] proved that $A_p(G)$ is a Banach algebra under pointwise multiplication and $A_p(G)$ has a bounded approximate identity, *i.e.* there exists $(\alpha_i)_{i\in I} \subset A_p(G)$ such that $\lim_i \|u \cdot \alpha_i - u\|_{A_p} = 0$ for all $u \in A_p(G)$.

Using by the proof of Herz's theorem given by Spector [15] the next result follows. □

Theorem III.1. — $A^p_\omega(G)$ *is an essential, order free Banach* $A_p(G)$*-module under pointwise multiplication.*

Proof. — Let us consider two functions $u = \tilde{f} * g$ and $v = \tilde{h} * k$ where $f, g, j, k \in C_c(G)$. Then

$$\|u\|_{A^p_\omega} \leqslant \|f\|_{p,\omega}\|g\|_{p',\omega^{-1}}$$
$$\|v\|_{A^p_\omega} \leqslant \|h\|_{p,\omega}\|k\|_{p',\omega^{-1}}$$

$$u(x) = \int_G f(y^{-1}x^{-1})g(y^{-1})\,dy \quad\text{and}\quad v(x) = \int_G h(z^{-1}x^{-1})k(z^{-1})\,dz$$

we have

$$(uv)(x) = \int_G f(y^{-1}x^{-1})g(y^{-1})\,dy \int_G h(z^{-1}y^{-1}x^{-1})k(z^{-1}y^{-1})\,dz = \int_G A_z(x)\,dz$$

where $A_z = \tilde{a}_z * b_z$, $a_z, b_z \in C_c(G)$

$$a_z(x) = f(x)h(z^{-1}x)$$
$$b_z(x) = g(x)k(z^{-1}x)$$

since the mapping $z \to A_z$ is continuous as the Banach valued integral, it sufficies to show that

$$\int_G \|A_z\|_{A^p_\omega}\,dz \leqslant \|f\|_{p,\omega}\,\|g\|_{p',\omega^{-1}}\,\|h\|_p\,\|k\|_{p'}\ .$$

But

$$\begin{aligned}
\int_G \|A_z\|_{A^p_\omega}\,dz &\leqslant \int_G \|a_z\|_{p,\omega}\|b_z\|_{p',\omega^{-1}}\,dz \\
&\leqslant \Big(\int_G \|a_z\|^p_{p,\omega}\,dz\Big)^{1/p}\Big(\int_G \|b_z\|^{p'}_{p',\omega^{-1}}\,dz\Big)^{1/p'} \\
&= \Big(\int_G |f(x)\,\omega(x)|^p\,dx \int_G |h(z^{-1}x)|^p\,dz\Big)^{1/p} \\
&\qquad \cdot \Big(\int_G |g(x)\,\omega^{-1}(x)|^{p'}\,dx \int_G |k(t)|^{p'}\,dt\Big)^{1/p'} \\
&= \|f\|_{p,\omega}\,\|g\|_{p',\omega^{-1}}\,\|h\|_p\,\|k\|_{p'}\ .
\end{aligned}$$

So, it is obtained that for $u \in A_p(G)$ and $v \in A^p_\omega(G)$

$$\|uv\|_{A^p_\omega} \leqslant \|u\|_{A_p}\,\|v\|_{A^p_\omega}\ .$$

Since $C_c(G) \cap A^p_\omega(G)$ is dense in $A^p_\omega(G)$ and $A^p_\omega(G)$ is a Banach $A_p(G)$-module it is obtained that $A^p_\omega(G)$ is an essential Banach module and order free, *i.e.* $u \in A_p(G)$, $uv = 0$ for all $v \in A^p_\omega(G)$ then $v = 0$. □

Remark. — Since $A_2(G) \subset A_r(G) \subset A_p(G)$ for $1 < p < r < 2$ or $2 \leqslant r < p < \infty$, $A^p_\omega(G)$ is also $A_r(G)$-module for $1 < p < r \leqslant 2$ or $2 \leqslant r < p < \infty$, *i.e.*

$$A_r(G)\cdot A^p_\omega(G) \subset A^p_\omega(G)\ .$$

In particular, for $r = 2$, since $A_2(G)$ identifies with $\widehat{L^1(\widehat{G})} = F(L^1(\widehat{G}))$, $A^p_\omega(G)$ has a structure of $L^1(\widehat{G})$-module and the action φ is given by $\hat{\varphi}\cdot h \in A^p_\omega(G)$ for $\varphi \in L^1(\widehat{G})$, $h \in A^p_\omega(G)$.

Proposition III.2. — *$A^p_\omega(G)$ has a compatible structure $\widehat{G}$-module with the action of $A_p(G)$.*

Proof. — Since $A^p_\omega(G)$ is the essential Banach $L^1(\widehat{G})$-module $A^p_\omega(G)$ has a unique compatible structure $\widehat{G}$-module [11]. This action is given by $\gamma \in \widehat{G}$, $x \in G$, $u \in A^p_\omega(G)$,

$$(M_\gamma u)(x) = \overline{\gamma}(x)\ u(x)\ .$$

Indeed, $\forall \gamma_1, \gamma_2 \in \widehat{G}$

- $(M_{\gamma_1+\gamma_2}u)(x) = (\overline{\gamma_1+\gamma_2})(x)\ u(x) = \overline{\gamma}_1(x)\overline{\gamma}_2(x)\ u(x)$
$= (M_{\gamma_1}\cdot M_{\gamma_2}u)(x)$
- $M_e u = 1\cdot u = u$
- $\|M_\gamma u\| \leqslant \|u\| \leqslant \|M_{-\gamma}M_\gamma u\| \leqslant \|M_\gamma u\|$.

So M_γ is an isometry and also

$$\widehat{L_\gamma\varphi}\cdot u = M_\gamma\hat{\varphi}\cdot u = \hat{\varphi}(M_\gamma u) \ \text{for}\ \varphi \in L^1(\hat{G}) .$$

By the density of $A_2(G)$ in $A_p(G)$ it follows

$$(M_\gamma f)u = M_\gamma(fu) = f\cdot M_\gamma(u),\quad f \in L^1(\hat{G}) .$$

□

THEOREM III.3. — *Let G be a locally compact abelian group and $\hat{G}$ be a dual group,* $1 < p < \infty$ *for a continuous linear operator T the following statements are equivalent*

(i) For all $f \in L^1(\hat{G})$ and $u \in A^p_\omega(G)$ such that $fu = 0$ then $f\cdot T(u) = 0$.

(ii) T is $L^1(\hat{G})$-module homomorphism (or multiplier) i.e. $T \in \mathrm{Hom}_{L^1(\hat{G})}(A^p_\omega(G), A^p_\omega(G))$.

Proof. — Since $A^p_\omega(G)$ is an essential and order free Banach $L^1(\hat{G})$-module, it is obtained from [12]. □

PROPOSITION III.4. — *For* $1 < p \leqslant r \leqslant 2$ *or* $2 \leqslant r \leqslant p < \infty$,

$$\mathrm{Hom}_{L^1(\hat{G})}(A^p_\omega(G), A^p_\omega(G)) = \mathrm{Hom}_{\hat{G}}(A^p_\omega(G), A^p_\omega(G)) = \mathrm{Hom}_{A_r}(A^p_\omega(G), A^p_\omega(G)) .$$

So we consider that each multiplier on $A^p_\omega(G)$ as a centralizer.

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