# Vanishing Twist near Focus-Focus Points

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## Abstract

We show that near a focus-focus point in a Liouville integrable Hamiltonian system with two degrees of freedom lines of locally constant rotation number in the image of the energy-momentum map are spirals determined by the eigenvalue of the equilibrium. From this representation of the rotation number we derive that the twist condition for the isoenergetic KAM condition vanishes on a curve in the image of the energy-momentum map that is transversal to the line of constant energy. In contrast to this we also show that the frequency map is non-degenerate for *every* point in a neighborhood of a focus-focus point.

Math. Class.: 37J35, 37J15, 37J40, 70H06, 70H08, 37G20
PACS: 02.30.Ik, 45.20.Jj, 05.45.-a
Keywords: Completely Integrable Systems; Focus-Focus Point; KAM; isoenergetic non-degeneracy; Vanishing Twist;

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# 1 Introduction

Consider a neighborhood of an equilibrium point of a Hamiltonian system with two degrees of freedom. In local symplectic coordinates $(x, y, p_x, p_y) \in \mathbb{R}^4$ with standard symplectic structure $\Omega = \mathrm{d}x \wedge \mathrm{d}p_x + \mathrm{d}y \wedge \mathrm{d}p_y$ the Hamiltonian is $H(x, y, p_x, p_y)$, where the equilibrium is taken to be the origin. The Hamiltonian vector field in these coordinates is

$$\mathscr{X}_H = J \mathrm{d}H, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1}$$

or in components

$$\dot{x} = \frac{\partial H}{\partial p_x}, \ \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{p}_x = -\frac{\partial H}{\partial x}, \ \dot{p}_y = -\frac{\partial H}{\partial y}. \tag{2}$$

Since an equilibrium of a dynamical system is a zero of the vector field one has $\mathrm{d}H(0) = 0$. The critical value $H(0)$ of the critical point 0 is assumed to be 0. The equilibrium is characterized by the eigenvalues $\lambda$ of its linearization $J\mathrm{d}^2 H(0)$. If $\lambda = \alpha + \mathrm{i}\omega$ is an eigenvalue, so is $\bar{\lambda}$, $-\lambda$, and $-\bar{\lambda}$. Real pairs of non-zero eigenvalues $\pm\alpha$ are called hyperbolic, non-zero pure imaginary pairs $\pm\mathrm{i}\omega$ are called elliptic, and quadruples of eigenvalues with non-zero real and non-zero imaginary part $\pm\alpha \pm \mathrm{i}\omega$ are called loxodromic. Near a loxodromic equilibrium there exist symplectic coordinates $(\tilde{x}, \tilde{y}, \tilde{p}_x, \tilde{p}_y)$ such that the quadratic part $H_2$ of $H$ is [23, 1]

$$H_2 = \alpha J_1 + \omega J_2, \qquad J_1 = \tilde{x}\tilde{p}_x + \tilde{y}\tilde{p}_y, \quad J_2 = \tilde{x}\tilde{p}_y - \tilde{y}\tilde{p}_x. \tag{3}$$

Introducing $z = \tilde{x} + \mathrm{i}\tilde{y}$ and $p_z = \tilde{p}_x - \mathrm{i}\tilde{p}_y$ gives $\Omega = \Re(\mathrm{d}z \wedge \mathrm{d}p_z)$, and $J_1 - \mathrm{i}J_2 = zp_z$. Outside $z = 0$ this extends to a multivalued symplectic coordinate system $(\ln z, zp_z)$, and the flow of $H_2$ is easily found to be $(z, p_z) = (\exp(\lambda t)z_0, \exp(-\lambda t)p_{z0})$. The quadratic Hamiltonian has a two-dimensional stable eigenspace spanned by the eigenvectors of eigenvalues with negative real part, and a two-dimensional unstable eigenspace spanned by the eigenvectors of eigenvalues with positive real part. In each eigenspace the dynamics is that of a focus point in the plane. The invariant manifolds of the equilibrium point of the full Hamiltonian are tangent to these eigenspaces at the equilibrium. In general the stable and unstable invariant manifolds will have transverse intersections, and the system is non-integrable.

Here we are concerned with the dynamics near an equilibrium with loxodromic eigenvalues of a Liouville integrable Hamiltonian system with two degrees of freedom. Under some additional hypothesis (see below) we call

such an equilibrium focus-focus point. Let $L$ be an independent second integral that commutes with $H$, i.e. the Poisson bracket $\{H, L\} = \Omega(JdH, JdL)$ vanishes. The system is called Liouville integrable if the energy-momentum map $F = (H, L)$ is regular almost everywhere, and $\{H, L\} = 0$. The values in the image of $F$ are denoted by $c = (h, l)$. The Liouville-Arnold theorem states that near any compact connected component of regular points $\mathbb{T}_c$ of $F^{-1}(c)$ there are so called action-angle coordinates $(\theta_1, \theta_2, I_1, I_2)$ with $\Omega = d\theta_1 \wedge dI_1 + d\theta_2 \wedge dI_2$ and $I_1$ and $I_2$ are commuting constants of motion with periodic flows on (and near) the two-dimensional torus $\mathbb{T}_c \subseteq F^{-1}(c)$. In these coordinates the Hamiltonian is a function of $I_1$ and $I_2$ alone, and the quasiperiodic flow of $H$ is the solution of $\dot{\theta}_i = \partial H / \partial I_i = \omega_i(I)$, $i = 1, 2$. In other words, the Hamiltonian vectorfield $\mathscr{X}_H$ is a linear combination of the periodic flows of the actions

$$\mathscr{X}_H = \omega_1 \mathscr{X}_{I_1} + \omega_2 \mathscr{X}_{I_2}\,. \tag{4}$$

The coefficients of the linear combination are the frequencies $\omega_i(I)$ of the angles $\theta_i$, $i = 1, 2$. The frequencies $\omega = (\omega_1, \omega_2)$ in general depend on the actions $I = (I_1, I_2)$. The map from actions $I$ to frequencies $\omega(I)$ is called the frequency map.

The KAM theorem (see e.g. [1, 18]) asserts that under small perturbations the invariant torus $\mathbb{T}_c$ with frequencies $\omega(I)$ persists if 1) the frequency map is non-degenerate and 2) the frequencies $\omega(I)$ are Diophantine. In the isoenergetic KAM theorem persistence of invariant tori is considered for the same energy $h$. Then a torus is characterized by its rotation number $W(I)$ which is the frequency ratio $[\omega_1 : \omega_2] \in \mathbb{R}P^1$. The iso-energetic non-degeneracy condition $\partial W / \partial l \neq 0$, where the derivative is taken at constant energy, replaces the non-degeneracy condition on the frequency map, see e.g. [1]. In a semi-global Poincaré section transversal to $\mathscr{X}_H$ on $\mathbb{T}_c$ the iso-energetic non-degeneracy condition ensures that the Poincaré map is a twist map near the invariant curve corresponding to $\mathbb{T}_c$. The twist condition $\partial W / \partial l \neq 0$ implies that the rotation number changes between neighboring tori. An invariant torus $\mathbb{T}_c$ for which the twist vanishes is called a twistless torus, for short. To check the non-degeneracy of the frequency map in an example requires tools from complex analysis, see e.g. Horozov's work on the spherical pendulum [10]. General results in the neighborhood of critical points of the energy momentum map are due to Knörrer [13] and Nguyên Tiên Zung [24]. We are not aware of general results about the iso-energetic non-degeneracy condition; the spherical pendulum has again been treated by Horozov [11].

The twist condition is only a sufficient condition for persistence, and KAM theorems with weaker conditions exist, see [17]. However, the perturbed dynamics is quite unusual when the unperturbed twistless torus is resonant, i.e. when it has a rational rotation number. A resonant torus with twist breaks into a Poincaré-Birkhoff island chain, see e.g. [15, 18, 2]. A twistless resonant torus instead breaks into two island chains, and near the collision of these chains interesting dynamics occurs on so called meandering invariant curves, see [12, 4, 19, 8] and the references therein. Our goal is to show that such dynamics occurs near a loxodromic equilibrium of a generically perturbed integrable system. More precisely we will prove the following

**Theorem 1.** *In every integrable Hamiltonian system with two degrees of freedom and a focus-focus singularity with loxodromic eigenvalues there exists a regular torus with vanishing twist for each value of the energy close to the critical one. For this energy all other tori close enough to the singular fibre have non-vanishing twist.*

This theorem might look surprising when compared to a result by Nguyen Tien Zung [24]: He showed that near a focus-focus point the Kolmogorov non-degeneracy condition is always satisfied, *i.e.* on almost all tori the frequency map is non-degenerate. Zung did not provide a way to find the tori for which the non-degeneracy condition fails; but, as we show in section 4, our techniques improve his theorem by showing that the frequency map is non-degenerate for *all* regular tori close enough to the singular fiber.

It is well known that the Kolmogorov condition and the Twist condition are independent, and our results show that whenever there is a focus-focus point the twist-condition is violated while the frequency map is non-degenerate.

## 2   Rotation Number

Let the critical value of the equilibrium be $(h, l) = (0, 0)$. For simplicity, we assume that the corresponding singular fibre $F^{-1}(0)$ contains only a single critical point $m$. We then need to assume that the component of $F^{-1}(0)$ containing $m$ is compact. Such a singularity is called of focus-focus type. Since we are interested in tori close to the singular component of $F^{-1}(0)$ we may disregard other connected components. Then $(0, 0)$ is an isolated critical value in the image of the energy-momentum map $F$ restricted to those tori. Additional structure appears near the loxodromic equilibrium by

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the assumption that the system be integrable. Eliasson showed [9] that the momenta $J_1$ and $J_2$ of the quadratic normal form are the components of a momentum map $J = (J_1, J_2)$ of the full system near the focus-focus point in some symplectic coordinates. This means that we may assume that both $H$ and $L$ (and not only their quadratic parts) are functions of $J_1$ and $J_2$ near $m$:

$$H = \Phi \circ J \qquad \text{and} \qquad L = \Psi \circ J, . \tag{5}$$

where $g = (\Phi, \Psi)$ is a local diffeomorphism of $\mathbb{R}^2$ near the origin. Recall that (5) is valid only near the focus-focus point $m$. However this gives a way of extending the momentum map $J$ to a whole neighborhood of the singular fiber simply by enforcing $J = g^{-1}(H, L)$.

Since $H$ is loxodromic the coefficient $\alpha$ in (3) is non-zero hence $\partial_1 \Phi(0) \neq 0$. Then by the implicit function theorem the additional integral $L$ can be chosen equal to $J_2$.

Observe that $J_2$ has a $2\pi$ periodic flow near $m$, and this still holds in a neighborhood of the singular fiber, as was shown in [21]. Therefore $J_2$ is a generator of an $S^1$ action on the neighboring tori and hence is equal to one action, say $I_2$.

Instead of using the energy-momentum map $F$ it is advantageous to use the momentum map $J$ instead, and whenever the Hamiltonian enters the calculation to use $\Phi$.

In [22] it is shown using only the momentum map $J$ without the Hamiltonian (or, equivalently, with Hamiltonian equal to $J_1$) that the $2\pi$ periodic flow $\mathscr{X}_{I_1}$ of the first action is given by

$$2\pi \mathscr{X}_{I_1} = \tau_1 \mathscr{X}_{J_1} + \tau_2 \mathscr{X}_{J_2} \tag{6}$$

where the periods $\tau_i$, $i = 1, 2$ satisfy

$$\begin{aligned} \tau_1(j) &= \sigma_1(j) - \Re(\ln \zeta), \\ \tau_2(j) &= \sigma_2(j) + \Im(\ln \zeta), \end{aligned} \tag{7}$$

and $\sigma_i(j)$, $i = 1, 2$, are smooth and single-valued functions near the origin. Here the point $j = (j_1, j_2)$ in the image of the momentum map $J$ is identified with the complex number $\zeta = j_1 + \mathrm{i}j_2$. The smooth contribution comes from the dynamics far away from the focus-focus point, while the singular contribution is obtained from the flows of $J_1$ and $J_2$ near the focus-focus point, see [22] for the details. The points $(\tau_1, \tau_2)$ and $(0, 2\pi)$ form a basis of the period lattice of the foliation. Using this result we obtain

**Lemma 2.** *The rotation number $W(j)$ near a focus-focus point is*

$$2\pi W(j) = -A(j)\Re(\ln\zeta) - \Im(\ln\zeta) + \sigma(j) \qquad (8)$$

*where $A(j)$ is smooth and determined from the Hamiltonian $H = \Phi \circ J$ by*

$$A(j) = \frac{\partial_2 \Phi}{\partial_1 \Phi}(j), \quad A(0) = \frac{\omega}{\alpha}, \qquad (9)$$

*and $\sigma(j)$ is a smooth (and single-valued) function near the origin.*

*Proof.* The Hamiltonian vector field $\mathscr{X}_H$ is obtained from (5) as

$$\mathscr{X}_H = \partial_1 \Phi \mathscr{X}_{J_1} + \partial_2 \Phi \mathscr{X}_{J_2}. \qquad (10)$$

Now eliminating $\mathscr{X}_{J_1}$ with the aid of (6) and using $J_2 = I_2$ gives

$$\mathscr{X}_H = \frac{2\pi}{\tau_1}\partial_1 \Phi \mathscr{X}_{I_1} + \left(\partial_2 \Phi - \frac{\tau_2}{\tau_1}\partial_1 \Phi\right)\mathscr{X}_{I_2}. \qquad (11)$$

Comparing with (4) the frequencies $\omega_j$ can be read off, and the rotation number as a function of $j$ is

$$W(j) = \frac{\omega_2(j)}{\omega_1(j)} = \frac{1}{2\pi}(\tau_1(j)A(j) - \tau_2(j)). \qquad (12)$$

Finally using (7) gives the result with $\sigma = A\sigma_1 - \sigma_2$. From (3) we see that $A(0) = \omega/\alpha$. $\qquad\qquad\square$

The fact that $\Im(\ln\zeta)/2\pi$ is multivalued and increases by 1 upon completing a cycle $\zeta = \exp(i\phi)$ is a manifestation of the fact that a simple focus-focus point has Monodromy with index 1, see [5, 14, 25, 21].

The functions $\tau_i$ that determines the period lattice depend on the foliation alone, while $A(j)$ is determined by the Hamiltonian alone. In order to understand the behaviour of $W(j)$ near the origin we now prove the following

**Theorem 3.** *There is a local diffeomorphism of $\mathbb{R}^2$ near the origin which is $C^1$ at the origin and $C^\infty$ elsewhere which maps the level sets of $W$ to the integral curves of the equation $\dot{\zeta} = -\bar{\lambda}\zeta$. In particular these curves are spirals determined by the eigenvalue $\lambda$ of the focus-focus point.*

*Proof.* We use the notation of lemma 2. The strategy is to change the coordinates such that the level sets of $W$ in the new coordinates satisfy a

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linear differential equation. Let $\varphi$ be the change of variables defined in polar coordinates $\zeta = j_1 + \mathrm{i}j_2 = \rho e^{\mathrm{i}\theta}$ by

$$\begin{pmatrix} \rho \\ \theta \end{pmatrix} \to \begin{pmatrix} \tilde{\rho} = \rho^{A(j)/A(0)} \exp(-\sigma(j)/A(0)) \\ \tilde{\theta} = \theta \end{pmatrix}.$$

Writing $A(j) = A(0) + \langle j, \tilde{A}(j) \rangle$, where $\tilde{A}$ is smooth, we have $\varphi(j) = f(j)j$, where $f : \mathbb{R}^2 \to \mathbb{R}$ locally near the origin is given by

$$f(j) = \rho^{\langle j, \tilde{A}(j) \rangle/A(0)} \exp(-\sigma(j)/A(0)).$$

We see that $f$ is continuous at the origin, with $f(0) = \exp(-\sigma(0)/A(0))$, and that the partial derivatives of $f$ are of order $\ln \rho$ at the origin. Hence $\varphi$ is $C^1$ at the origin, with Jacobian

$$\mathrm{d}\varphi(0) = f(0)\mathrm{Id} = \exp(-\sigma(0)/A(0))\mathrm{Id}.$$

Under this diffeomorphism, the rotation number is simplified and we only need to study the level sets of

$$2\pi W \circ \varphi^{-1}(\tilde{j}) = (-A(0)\Re(\ln \tilde{\zeta}) - \Im(\ln \tilde{\zeta})).$$

Forget now the tildes. Viewing $W$ as a local Hamiltonian for the standard canonical structure $\mathrm{d}j_2 \wedge \mathrm{d}j_1$, we compute the levels sets of $W$ as the integral curves of the associated Hamiltonian vector field: $\frac{\mathrm{d}}{\mathrm{d}s}(j_1, j_2) = (\partial_2 W, -\partial_1 W)$. Rescaling the time $s$ by $2\pi|\zeta|^2\alpha$ and rewriting the dynamical system in complex form gives the equation $\dot{\zeta} = -\bar{\lambda}\zeta$.

From Lemma 2 we know that $W(j)$ diverges when $j$ approaches the origin. These statements are compatible because different branches of the complex logarithm are involved. After each complete turn of the spiral the rotation number jumps by one. Alternatively one could view the rotation number as globally defined on the Riemann surface of the complex logarithm. $\qquad\square$

Recall that as coordinates in $\mathbb{R}^2$ we may use $j = (j_1, j_2)$ or $c = (h, l)$, and that they are related by $l = j_2$ and $h = \Phi(j)$. Therefore the spirals in the image of the momentum map $J$ will be mapped to spirals in the image of the momentum map $F$. Near the origin the transformation between the two is the linear map $(h, l) = (j_1\alpha + j_2\omega, j_2)$.

# 3 Vanishing Twist

In the isoenergetic problem fixing the energy $H = h$ gives a smooth curve $\mathscr{C}_h$ in the image of the momentum map. The twist condition is obtained

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from $W(j)$ by differentiating along this curve. Since $\partial_1 \Phi(0) = \alpha \neq 0$ we may apply the implicit function theorem and the curve $\mathscr{C}_h$ is the graph of a function $j_1(j_2)$. The slope of this curve near the origin is given by implicit differentiation of $\Phi(j_1, j_2) = h$ which yields $\partial j_1 / \partial j_2 = -\partial_2 \Phi / \partial_1 \Phi = -A(j)$. Therefore the twist is found to be

$$\mathcal{T}(j) = \left. \frac{\partial W(j)}{\partial j_2} \right|_{\mathscr{C}_h} = \partial_1 W \frac{\partial j_1}{\partial j_2} + \partial_2 W = -A(j)\partial_1 W + \partial_2 W \,. \tag{13}$$

Expanding the remaining derivatives gives

$$2\pi \mathcal{T}(j) = -A^2 \partial_1 \tau_1 + 2A \partial_2 \tau_1 - \partial_2 \tau_2 - \tau_1 A \partial_1 A + \tau_1 \partial_2 A \tag{14}$$

where $\partial_1 \tau_2 = \partial_2 \tau_1$ has been used, which follows from the fact that both periods can be obtained by differentiating a single action function, see [22]. The main contribution near the origin comes from differentiating the complex logarithm in (7), which produces terms proportional to $1/|j|^2$. Therefore we now define $\tilde{\mathcal{T}}(j) = 2\pi |j|^2 \mathcal{T}(j)$ and have the following

**Lemma 4.** $\tilde{\mathcal{T}}$ *is a $C^1$ map at the origin that satisfies*

$$\tilde{\mathcal{T}}(0) = 0, \qquad \partial_1 \tilde{\mathcal{T}}(0) = A^2(0) - 1, \quad \partial_2 \tilde{\mathcal{T}}(0) = -2A(0) \tag{15}$$

*Proof.* Using (7) where $\sigma_i$ is smooth we find that the last two terms in (14), $\tau_1 A \partial_1 A$ and $\tau_1 \partial_2 A$, are both of the form $f(j) + g(j) \ln |j|$ with smooth $f$ and $g$, while the remaining first three terms are of the form $f(j) + g(j)/|j|^2$ for some smooth $f$ and $g$. Since $|j|^2 \ln |j|$ is of class $C^1$ at the origin the first statement follows. A simple computation gives

$$\tilde{\mathcal{T}}(j) = A^2 j_1 - 2A j_2 - j_1 + O(|j|^2 \ln |j|) \,, \tag{16}$$

and the result follows. $\qquad \square$

Since the derivatives of $\tilde{\mathcal{T}}$ cannot both vanish at the origin, $\mathcal{T}^{-1}(0)$ is a $C^1$ curve through the origin. By Lemma 4 the equation for the tangent of this curve at the origin is

$$(A(0)^2 - 1)j_1 = 2A(0)j_2 \tag{17}$$

Using $h = j_1 \alpha + j_2 \omega$ at the origin to express this tangent in $(h, j_2)$ gives

$$h = \omega \frac{\omega^2 + \alpha^2}{\omega^2 - \alpha^2} j_2 \,. \tag{18}$$

When $\omega \neq 0$ the curve $\tilde{\mathcal{T}}(0)$ therefore transversally intersects the curves $\{h = \text{const}\}$ near the origin, and we have proven Theorem 1. Note that the

condition $\omega \neq 0$ is only needed in the last step. Therefore also for $\omega = 0$ twistless tori exist near the origin, but transversality is lost. Therefore twistless tori might exist only for positive or only for negative values of $h$, but possibly not for both.

Different choices of actions are possible. The semi-global $S^1$ action $L = J_2 = I_2$ is unique up to sign $\varepsilon = \pm 1$, but the other action $I_1$ can be changed, $\tilde{I}_1 = \varepsilon I_1 + k I_2$, $k \in \mathbb{Z}$, so that new and old actions are related by a unimodular transformation. The frequencies $\omega$ change into $\tilde{\omega}_1 = \varepsilon \omega_1$ and $\tilde{\omega}_2 = \varepsilon \omega_2 - k \omega_1$, so that the rotation number transforms as $\tilde{W} = W - k\varepsilon$. Therefore a different choice of actions amounts to a different choice of sheet of the Riemann surface of the complex logarithm. This argument also shows that the vanishing twist is independent of the choice of actions.

When there are more than one critical point in the singular leave similar results can be derived, based on modified formulas (7) whose derivation is sketched in [22].

As an application of the above results one can consider the integrable normal form of the Hamiltonian Hopf bifurcation, see [20, 6] and the references therein. In [7] the rotation number in the compact case was computed explicitly in terms of elliptic integrals. Expansion of these integrals at the focus-focus point gave the above results (8) and (18) for the first time in this special case. Pictures of the spirals of constant $W$ can also be found in [7]. But the method employed could not deal with the case in which the normal form has a non-compact singular leave. And even with a compact singular fibre in a more complicated system this approach might lead to hyperelliptic integrals, and their expansion at the singular point would be quite difficult. A prominent example of an integrable system with a focus-focus point is the spherical pendulum, see e.g. [3]. In this case the eigenvalues are degenerate because $\omega = 0$, and the spiral of Theorem 3 degenerates into a star.

## 4    Kolmogorov Condition

Near a focus-focus point we can improve Zung's theorem [24].

**Theorem 5.** *On regular tori close to the focus-focus singular fiber, the Jacobian determinant of the frequency map admits the following asymptotic expansion (recall that $\tau_1$ is of order $\ln |j|$):*

$$\det \frac{\partial \omega}{\partial I} = -\left(\frac{2\pi\alpha}{|j|\tau_1^2}\right)^2 + O(\frac{1}{|j|\tau_1^3}). \tag{19}$$

*In particular this shows that the Kolmogorov condition is uniformly satisfied for every torus near the singular fibre of a simple focus-focus point.*

*Proof.* First notice that the Jacobian of the frequency map does not depend on the choice of action variables and hence does not care about monodromy; in our case, this means that it does not depend on the determination of the complex logarithm in (7). Thus the leading order of this Jacobian can easily be calculated. We need to compute

$$\frac{\partial \omega}{\partial I} = \frac{\partial \omega}{\partial J} \frac{\partial J}{\partial I}.$$

From (6) we know that by definition

$$2\pi \frac{\partial I_1}{\partial J_1} = \tau_1, \quad 2\pi \frac{\partial I_1}{\partial J_2} = \tau_2.$$

Since $J_2 = I_2$ we easily find that $\det \partial I / \partial J = \tau_1/2\pi$. The frequencies can be read off from (11); they are

$$\omega_1 = \frac{2\pi}{\tau_1} \Phi_1, \quad \omega_2 = \Phi_2 - \frac{\tau_2}{\tau_1} \Phi_1.$$

The subscripts of $\Phi$ denote partial derivatives. Using that $\tau_1$ is of order $\ln|j|$ and that the first derivatives of $\tau_i$ have all leading order $1/|j|$, one easily computes:

$$
\begin{aligned}
\partial_1 \omega_1 &= -2\pi \Phi_1 \frac{\tau_{11}}{\tau_1^2} + O(1/\tau_1) & = O\left(\frac{1}{|j|\tau_1^2}\right) \\
\partial_2 \omega_1 &= -2\pi \Phi_1 \frac{\tau_{12}}{\tau_1^2} + O(1/\tau_1) & = O\left(\frac{1}{|j|\tau_1^2}\right) \\
\partial_1 \omega_2 &= -\frac{(\tau_1 \tau_{21} - \tau_2 \tau_{11})}{\tau_1^2} \Phi_1 + O(1) & = O\left(\frac{1}{|j|\tau_1}\right) \\
\partial_2 \omega_2 &= -\frac{(\tau_1 \tau_{22} - \tau_2 \tau_{12})}{\tau_1^2} \Phi_1 + O(1) & = O\left(\frac{1}{|j|\tau_1}\right)
\end{aligned}
\tag{20}
$$

Here the second index of $\tau$ denotes a partial derivative. At leading order $\tau_1$ can be replaced by $\ln|j|$. The leading order of the determinant $\det \partial \omega / \partial J$ is obtained by taking the determinant of the matrix in which only the terms that are given explicitly in (20) are kept, provided one shows *a posteriori* that the result has the expected order of $\frac{1}{|j|^2 \tau_1^3} = \frac{1}{|j|^2 \ln^3 |j|}$. The result is

$$\det \frac{\partial \omega}{\partial J} = 2\pi (\tau_{11} \tau_{22} - \tau_{12}^2) \frac{\Phi_1^2}{\tau_1^3} + O\left(\frac{1}{|j|\tau_1^2}\right).$$

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Using the form of $\tau_1$ and $\tau_2$ we find

$$\det \frac{\partial \tau}{\partial J} = \tau_{11}\tau_{22} - \tau_{12}^2 = -\frac{1}{|j|^2} + O(1/|j|)$$

so that

$$\det \frac{\partial \omega}{\partial J} = -2\pi \frac{\alpha^2}{|j|^2 \tau_1^3} + O(\frac{1}{|j|\tau_1^2}) = -2\pi \frac{\alpha^2}{|j|^2 \ln^3 |j|} + O(\frac{1}{|j|^2 \ln^4 |j|}) \,.$$

Since by hypothesis $\alpha \neq 0$ the leading term has indeed the required order. Returning to the true actions $I$ the final result is

$$\det \frac{\partial \omega}{\partial I} = -\left(\frac{2\pi\alpha}{|j|\tau_1^2}\right)^2 + O(\frac{1}{|j|\tau_1^3}) \,,$$

thereby proving the theorem. $\qquad\square$

*Remark.* In the process of finishing this paper we became aware of a preprint by Rink [16] in which the determinant of the Jacobian of the frequency map was calculated in a very similar way.

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