The Orchard Morphism

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Résumé

We describe and prove uniqueness of a natural homomorphism (called the Orchard morphism) between some groups associated to finite sets.

1. Introduction

A natural way to plant two different species of trees in a row is to alternate them. Such a nice rule exists also in higher dimensions. In dimension 2 for instance, it provides a natural rule to plant (a finite number of) cherrytrees and plumtrees (up to transposition) at prescribed generic (no alignments of three trees) locations in an orchard. Figure 1 illustrates this: Our rule (given by Proposition 6.1 with $d = 2$, see also [2]) yields for the choosen nine positions three trees of one species and six trees of the remaining species.

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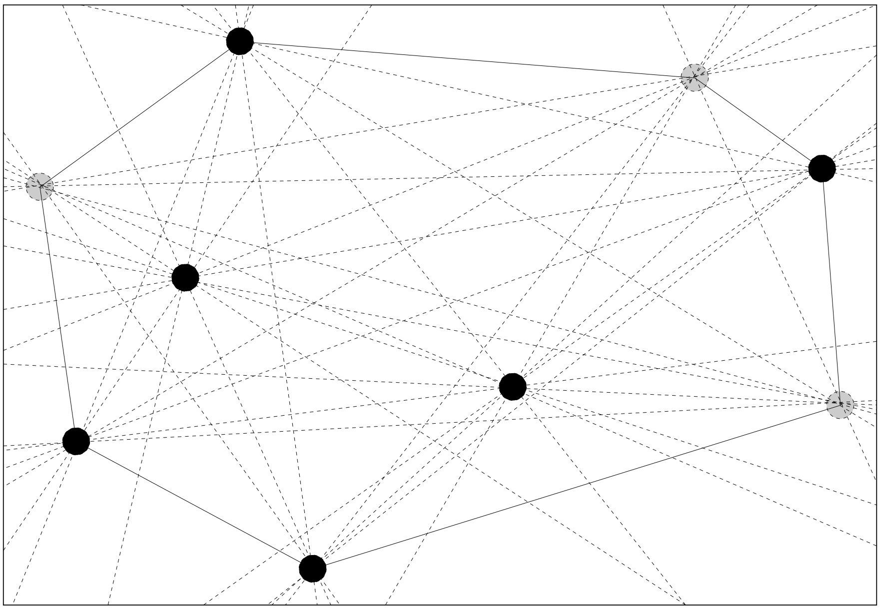

Figure 1: An orchard having 3 cherry- and 6 plumtrees in generic positions

More precisely, the content of this paper may be described as follows:

Let E be a set. We denote by

$$E^{(l)} = \{(x_1, \ldots, x_l) \in E^l \mid x_i \neq x_j \text{ for } 1 \leq i < j \leq l\}$$

the set of all sequences without repetitions of length l with values in E. Consider functions $\varphi : E^{(l)} \longrightarrow \{\pm 1\}$. The symmetric group $\mathrm{Sym}(\{1, \ldots, l\})$ acts on such functions by permutations of the l arguments. The subset of all functions on which this action is either trivial (symmetric functions) or multiplication by the signature homomorphism (antisymmetric functions) is a subgroup (with respect to the obvious product of functions). The main result of this paper is the existence of a unique non-trivial natural homomorphism (called the Orchard-morphism) from this group into the group of two-partitions of E (functions from E into $\{\pm 1\}$, defined up to multiplication by -1) where E is a finite set of cardinality $\sharp(E)$ (unicity fails however in the somewhat trivial case $l = \sharp(E) = 2$). Naturality means that this homomorphism is $\mathrm{Sym}(E)-$equivariant (with respect to the obvious actions by automorphisms of $\mathrm{Sym}(E)$ on both groups).

The result mentionned at the beginning concerning generic configurations of points in R^d follows then easily from the existence and properties of the Orchard morphism.

Finally, we deal also with the case where the finite set E is endowed with a natural fixed-point free involution $\iota : E \longrightarrow E$. We call such a set orientable and consider only structures on E which are invariant (perhaps up to a sign) under the involution ι.

2. Two-partitions

A *two-partition* is an unordered partition $\{A, B\}$ of a set $E = A \cup B$ into at most two disjoint subsets. Two-partitions are the same as equivalence relations having at most two classes. We will move freely between these two interpretations of two-partitions. The word "class" will hence often be used instead of "part of the two-partition" and a two-partition $\{A, B\}$ of E will generally be written as $A \cup B$ or $E = A \cup B$.

A two-partition $E = A \cup B$ can be given by a pair $\pm\alpha$ of opposite functions where

$$\alpha : E \longrightarrow \{\pm 1\}$$

is defined by $\alpha^{-1}(1) = A$ and $\alpha^{-1}(-1) = B$. The set $\mathcal{E}(E)$ of all such two-partitions is a vector space (of dimension $\sharp(E) - 1$ if E is finite) over the field $\mathbf{F}_2$ of two elements. The pair ± 1 of constant functions represents the identity and the group law $(\pm\alpha)(\pm\beta)$ is the obvious product $\pm\alpha\beta$ of functions. Set-theoretically, the product $(A_1 \cup A_2)(B_1 \cup B_2)$ of 2 two-partitions on a set E is given by $E = C_1 \cup C_2$ where $C_1 = (A_1 \cap B_1) \cup (A_2 \cap B_2)$ and $C_2 = (A_1 \cap B_2) \cup (A_2 \cap B_1)$.

Consider an unoriented (not necessarily finite) simple graph Γ with vertices V and unoriented edges $E \subset V^{(2)} = V \times V \setminus \{(v, v), v \in V\}$. Its adjacency matrix A is the symmetric matrix with rows and columns indexed by elements of V. All its entries are zero except $A_{v,w} = 1$ where $v \neq w$ are adjacent vertices of Γ (i.e. $\{u, v\}$ is an edge of Γ).

Our main tool in what follows is the following trivial and well-known observation:

Lemma 2.1. — *Let Γ be a simple graph with adjacency matrix A. Suppose that there exists a constant $\gamma \in \{0, 1\}$ such that*

$$A_{u,v} + A_{v,w} + A_{u,w} \equiv \gamma \pmod 2$$

for all triplets $(u, v, w) \in V^{(3)} \subset V^3$ of three distinct vertices.

Then either Γ or its complementary graph Γ^c (having adjacency matrix $A^c = J - I - A$ where J is the all one matrix and I the identity matrix) is a disjoint union of at most two complete graphs.

Proof. Up to replacing Γ by its complementary graph Γ^c we can suppose that $\gamma = 1$. This shows that given any three vertices of Γ, at least two of them are adjacent. The graph Γ contains hence at most two connected components. If a connected component of Γ is not a complete graph, then this component contains two vertices u, v at distance 2 implying that $A_{u,v} = 0$. Since u and v are at distance 2 they share a common neighbour w

for which we have $A_{u,w} = A_{v,w} = 1$. This yields a contradiction since $A_{u,v} + A_{u,w} + A_{v,w} = 2 \not\equiv \gamma \pmod 2$. $\square$

Given a set E we call a function $\sigma : E^{(2)} \longrightarrow \{\pm 1\}$ *symmetric* if $\sigma(x,y) = \sigma(y,x)$ for all $(x,y) \in E^{(2)}$.

Corollary 2.2. — *Any symmetric function*

$$\sigma : E^{(2)} \longrightarrow \{\pm 1\}$$

with

$$\sigma(a,b)\sigma(b,c)\sigma(a,c) = \gamma \in \{\pm 1\}$$

independent of $(a,b,c) \in E^{(3)} = \{(a,b,c) \in E^3, \mid a \neq b \neq c \neq a\}$ *gives rise to a two-partition of* E.

Proof. Consider the simple graph Γ with vertices E and adjacency matrix having coefficients $A_{x,x} = 0$ and $A_{x,y} = \frac{\sigma(x,y)+1}{2},\ x \neq y$.

The graph Γ satisfies the assumptions of Lemma 2.1 and consists hence, up to a sign change of σ (which replaces Γ by its complementary graph), of at most two non-empty complete graphs. The connected components of Γ define a two-partition on E. $\square$

Remark 2.3. — *The two-partition described by Corollary 2.2 can be constructed as follows: set* $\gamma = \sigma(a,b)\sigma(b,c)\sigma(a,c)$ *for* $a \neq b \neq c \neq a$ *and choose an element* $x_0 \in E$. *Up to multiplication by* -1 *the function* $\alpha : E \longrightarrow \{\pm 1\}$ *defined by* $\alpha(x_0) = 1$ *and* $\alpha(x) = \gamma\sigma(x, x_0),\ x \neq x_0$ *is then independent of the choice of the element* x_0 *and the classes of the associated two-partition are given by* $\alpha^{-1}(1)$ *and* $\alpha^{-1}(-1)$.

3. Symmetric and antisymmetric functions

A function $\varphi : E^{(l)} \longrightarrow \{\pm 1\}$ (where E is a set) is $l-$*symmetric* or *symmetric* if

$$\varphi(\ldots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \ldots) = \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_l)$$

for all $1 \leq i < l$ and $(x_1, \ldots, x_l) \in E^{(l)}$. We denote by $\mathcal{F}_+(E^{(l)})$ the set of all symmetric functions from $E^{(l)}$ into $\{\pm 1\}$.

Similarly, such a function $\varphi : E^{(l)} \longrightarrow \{\pm 1\}$ is $l-$*antisymmetric* or *antisymmetric* if

$$\varphi(\ldots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \ldots) = -\varphi(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_l)$$

for all $1 \leq i < l$ and $(x_1, \ldots, x_l) \in E^{(l)}$. We denote by $\mathcal{F}_-(E^{(l)})$ the set of all antisymmetric functions from $E^{(l)}$ into $\{\pm 1\}$.

The set $\mathcal{F}_\pm(E^{(l)}) = \mathcal{F}_+(E^{(l)}) \cup \mathcal{F}_-(E^{(l)})$ of all symmetric or antisymmetric functions from $E^{(l)}$ into $\{\pm 1\}$ is then a vector space (of dimension $\binom{|E|}{l} + 1$ for $1 < l \leq \sharp(E)$ and E finite) over $\mathbf{F}_2$. The identity element is given by the symmetric constant function $E^{(l)} \longrightarrow \{1\}$ and the group-law is the usual product of functions.

We define the *signature homomorphisme* $\text{sign} : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \{\pm 1\}$ by $\text{sign}(\varphi) = 1$ if $\varphi \in \mathcal{F}_+(E^{(l)})$ is symmetric and $\text{sign}(\varphi) = -1$ if $\varphi \in \mathcal{F}_-(E^{(l)})$ is antisymmetric. The set $\mathcal{F}_+(E^{(l)}) = \text{sign}^{-1}(1)$ of all symmetric functions on $E^{(l)}$ is of course a subgroup of $\mathcal{F}_\pm(E^{(l)})$ and the set $\mathcal{F}_-(E^{(l)}) = \text{sign}^{-1}(-1)$ of all antisymmetric functions on $E^{(l)}$ is a free $\mathcal{F}_+(E^{(l)})-$module.

4. The Orchard morphism

Given a finite set E, the aim of this section is to describe and construct the *Orchard morphism*

$$\rho : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \mathcal{E}(E) ,$$

a natural group homomorphism which factors through the quotient group $\mathcal{F}_\pm(E^{(l)})/\pm 1$ where ± 1 denote the obvious constant symmetric functions on $E^{(l)}$. Naturality means that ρ is equivariant with respect to the obvious actions of the symmetric group $\text{Sym}(E)$ on $\mathcal{F}_\pm(E^{(l)})$ and $\mathcal{E}(E)$.

Given a totally ordered set X we denote by $\binom{X}{k}$ the set of all strictly increasing sequences of length k in X.

For an arbitrary set X, we define $\binom{X}{k}$ by choosing first an arbitrary total order relation on X.

Given a function $\varphi \in \mathcal{F}_\pm(E^{(l)})$ where E is finite, we define $\sigma = \sigma_\varphi : E^{(2)} \longrightarrow \{\pm 1\}$ by setting

$$\sigma(y, z) = \prod_{(x_1,\dots,x_{l-1}) \in \binom{E\setminus\{y,z\}}{l-1}} \varphi(x_1, \dots, x_{l-1}, y)\ \varphi(x_1, \dots, x_{l-1}, z) .$$

PROPOSITION 4.1. — *The function σ is a well-defined symmetric function on $E^{(2)}$ such that*

$$\sigma(a, b)\sigma(b, c)\sigma(a, c) = (\text{sign}(\varphi))^{\binom{\sharp(E)-3}{l-2}}$$

for all $(a, b, c) \in E^{(3)}$ where $\text{sign}(\varphi)$ is the signature homomorphism sending $l-$symmetric functions on $E^{(l)}$ to 1 and $l-$antisymmetric functions to -1.

Proof. Since every sequence $(x_1, \dots, x_{l-1}) \in \binom{E\setminus\{y,z\}}{l-1}$ is involved twice in $\sigma(y, z)$, the value of $\sigma(y, z)$ is independent of the choice of a particular total order on $E \setminus \{y, z\}$. Symmetry ($\sigma(y, z) = \sigma(z, y)$) of σ is obvious.

Consider now first an element $(x_1,\dots,x_{l-1})\in\binom{E\setminus\{a,b,c\}}{l-1}$. Such an element contributes always a factor 1 to the product $\sigma(a,b)\sigma(b,c)\sigma(a,c)$. The product $\sigma(a,b)\sigma(b,c)\sigma(a,c)$ is hence equal to the the product over all elements $(x_1,\dots,x_{l-2})\in\binom{E\setminus\{a,b,c\}}{l-2}$ of factors of the form

$$\begin{array}{ll}\varphi(x_1,\dots,x_{l-2},c,a) & \varphi(x_1,\dots,x_{l-2},c,b)\\ \varphi(x_1,\dots,x_{l-2},a,b) & \varphi(x_1,\dots,x_{l-2},a,c)\\ \varphi(x_1,\dots,x_{l-2},b,a) & \varphi(x_1,\dots,x_{l-2},b,c)\end{array}$$

and each of these $\binom{\sharp(E)-3}{l-2}$ factors yields a contribution of $\mathrm{sign}(\varphi)$. $\Box$

By Proposition 4.1 the function σ_φ satisfies the conditions of Corollary 2.2 and gives hence rise to a two-partition $\rho(\varphi)\in\mathcal{E}(E)$. We call the application $\rho:\mathcal{F}_\pm(E^{(l)})\longrightarrow\mathcal{E}(E)$ defined in this way the *Orchard morphism*. Given an element $\varphi\in\mathcal{F}_\pm(E^{(l)})$ we call the two-partition $\rho(\varphi)$ the *Orchard-partition* of φ giving rise to the *Orchard-equivalence relation* having (at most 2) *Orchard classes*.

REMARK 4.2. — *(i) If $l=1$, the two-partition $\rho(\varphi)$ on E given by the Orchard morphism is the obvious one with classes $\varphi^{-1}(1)$ and $\varphi^{-1}(-1)$.*

(ii) Consider a $2-$symmetric function $\varphi\in\mathcal{F}_+(E^{(2)})$ satisfying the condition of Corollary 2.2. By Corollary 2.2 it gives hence rise to a two-partition on E. If E is finite, we get a second two-partition on E by considering the Orchard morphism $\rho(\varphi)$. An easy computation shows that these two-partitions coincide if $\sharp(E)$ is odd. If $\sharp(E)$ is even, the image of the Orchard morphism $\rho(\varphi)\in\mathcal{E}(E)$ is trivial for such a function φ.

Before stating the main result concerning the Orchard morphism, we recall the definition of equivariance: Let a group G act on two sets X and Y. An application $\psi:X\longrightarrow Y$ is *$G-$equivariant* if $\psi(g.x)=g.(\psi(x))$ for all $g\in G$ and $x\in X$. Given a set E, the symmetric group $\mathrm{Sym}(E)$ of all bijections of E acts in an obvious way on the groups groups $\mathcal{E}(E)$ and $\mathcal{F}_\pm(E^{(l)})$ and it is hence natural to study group homomorphisms from $\mathcal{F}_\pm(E^{(l)})$ into $\mathcal{E}(E)$ which are *natural*, i.e. $\mathrm{Sym}(E)-$equivariant.

THEOREM 4.3. — *For any finite set E and any natural integer $1\le l\le\sharp(E)$, the Orchard morphism*

$$\rho:\mathcal{F}_\pm(E^{(l)})\longrightarrow\mathcal{E}(E)$$

is the unique natural non-trivial (for $l<\sharp(E)$) group homomorphism. Moreover, ρ factors through the quotient group $\mathcal{F}^l_\pm(E)/\{\pm1\}$ (where, as always, ±1 denote the constant symmetric functions on $E^{(l)}$).

Remark 4.4. — *For $n = l = 2$ the unicity assertion of Theorem 4.3 fails: Defining $\rho' : \mathcal{F}_\pm(E^{(2)}) \longrightarrow \mathcal{E}(E)$ (where $E = \{1, 2\}$ has two elements) by $\rho'(\varphi) = (E = \{1, 2\})$ if $\varphi \in \mathcal{F}_+(E^{(2)})$ and $\rho'(\varphi) = (E = \{1\} \cup \{2\})$ if $\varphi \in \mathcal{F}_-(E^{(2)})$ defines a natural homomorphism distinct from the Orchard morphism (which is always trivial if $l = \sharp(E)$).*

This failure is due the fact that both two-partitions on the set $E = \{1, 2\}$ are Sym(E)-invariant. However, for finite sets E having more than 2 elements, only the trivial two-partition is Sym$(E)-$invariant.

A *flip* is a symmetric function $f_X \in \mathcal{F}_+(E^{(l)})$ such that $f^{-1}(-1) \subset E^{(l)}$ consists (up to permutation of its elements) of a unique sequence $X = (x_1, \dots, x_l) \in \binom{E}{l}$. We call the set $X = \{x_1, \dots, x_l\}$ the *flipset* of the flip f_X.

The set $\{f_X\}_{X \in \binom{E}{l}}$ of all flips is obviously a basis of the subspace $\mathcal{F}_+(E^{(l)})$ of symmetric functions on $E^{(l)}$.

Lemma 4.5. — *Given a flip $f_X \in \mathcal{F}_+(E^{(l)})$ and an arbitrary element $\varphi \in \mathcal{F}_\pm(E^{(l)})$ we have for $(a, b) \in E^{(2)}$*

$$\sigma_\varphi(a, b)\ \sigma_{(\varphi f_X)}(a, b) = -1$$

if and only if exactly one of the elements a, b belongs to X.

Proof. Every factor

$$\begin{aligned}
&\varphi(x_1, \dots, x_{l-1}, a)\varphi(x_1, \dots, x_{l-1}, b) \\
&\quad (\varphi\ f_X)(x_1, \dots, x_{l-1}, a)(\varphi\ f_X)(x_1, \dots, x_{l-1}, b) \\
&\qquad = f_X(x_1, \dots, x_{l-1}, a)\ f_X(x_1, \dots, x_{l-1}, b)
\end{aligned}$$

(with $(x_1, \dots, x_{l-1}) \in \binom{E \setminus \{a,b\}}{l-1})$) yields a contribution of 1 except if $X = \{x_1, \dots, x_{l-1}, a\}$ or $X = \{x_1, \dots, x_{l-1}, b\}$. This happens at most once and only if exactly one of the elements a, b belongs to X. □

Corollary 4.6. — *(i) The classes of the two-partition $\rho(f_X)$ associated to a flip $f_X \in \mathcal{F}_+(E^{(l)})$ are given by X and $E \setminus X$.*

(ii) If two functions $\varphi, \psi = \varphi\ f_X \in \mathcal{F}_\pm(E^{(l)})$ differ by a flip then the corresponding equivalence relations $\rho(\varphi)$ and $\rho(\psi) = \rho(\varphi f_X)$ differ exactly on the subsets $X \times (E \setminus X)$ and $(E \setminus X) \times X$ of $E \times E$.

Proof of Theorem 4.3. Since the set of all flips generates $\mathcal{F}_+(E^{(l)})$ and since $\mathcal{F}_-(E^{(l)})$ is a free $\mathcal{F}_+(E^{(l)})-$module, the Orchard morphism ρ behaves well under composition by assertion (ii) of Corollary 4.6. Since the equivalence relation associated to a constant function $\pm 1 \in \mathcal{F}_+(E^{(l)})$ is obviously trivial, ρ defines a group homomorphism from the quotient group $\mathcal{F}_+(E^{(l)})/\{\pm 1\}$ into $\mathcal{E}(E)$.

Equivariance of ρ with respect to $\mathrm{Sym}(E)$ is obvious.

We have yet to show that every other natural ($\mathrm{Sym}(E)-$equivariant) homomorphism $\rho' : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \mathcal{E}(E)$ is either trivial or coincides with the Orchard morphism ρ.

A flip f_X is clearly invariant under the subgroup $\mathrm{Sym}(X)\times\mathrm{Sym}(E\setminus X) \subset \mathrm{Sym}(E)$. If $\sharp(E) > 2$, any two-partition invariant under $\mathrm{Sym}(X) \times \mathrm{Sym}(E\setminus X)$ of E is either trivial or equal to $X \cup (E \setminus X)$. This implies that we have either $\rho'(f_X) = 1$ or $\rho'(f_X) = \rho(f_X)$ for any $\mathrm{Sym}(E)-$equivariant homomorphism $\rho' : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \mathcal{E}(E)$. Since $\mathrm{Sym}(E)$ acts transitively on the set of all flips, the first case implies triviality of ρ' restricted to $\mathcal{F}_+(E^{(l)})$ while we have $\rho' = \rho$ for the restriction onto $\mathcal{F}_+(E^{(l)})$ in the second case and this conclusion holds also for $\sharp(E) = 2$ as can easily be checked.

If ρ' restricted to $\mathcal{F}_+(E^{(l)})$ is trivial, the identity $\mathcal{F}_\pm(E^{(l)}) = \varphi\ \mathcal{F}_+(E^{(l)})$ for any $\varphi \in \mathcal{F}_-(E^{(l)})$ shows that ρ' restricted to $\mathcal{F}_-(E^{(l)})$ is constant and hence trivial for $\sharp(E) > 2$ by $\mathrm{Sym}(E)-$equivariance. For $\sharp(E) = 2$ and $l = 2$, this conclusion fails as shown by the example of Remark 4.4.

We might hence suppose that $\rho' = \rho$ on $\mathcal{F}_+(E^{(l)})$. Choose an antisymmetric function $\varphi \in \mathcal{F}_-(E^{(l)})$. If $n = \sharp(E)$ is odd, choose a cyclic permutation $\beta \in \mathrm{Sym}(E)$ (of maximal length n) of E and consider

$$\tilde{\varphi}(x_1, \dots, x_l) = \prod_{j=0}^{n-1} \varphi(\beta^j(x_1), \beta^j(x_2), \dots, \beta^j(x_l))$$

where $\beta^0(x) = x$ and $\beta^j(x) = \beta(\beta^{j-1}(x))$ for $x \in E$. The function $\tilde{\varphi} : E^{(l)} \longrightarrow \{\pm 1\}$ is antisymmetric on $E^{(l)}$ and invariant under the cyclic subgroup generated by $\beta \in \mathrm{Sym}(E)$. The corresponding two-partition $\rho'(\tilde{\varphi})$ is also invariant under the cyclic permutation β and hence trivial since $n = \sharp(E)$ is odd. The equality $\mathcal{F}_-(E^{(l)}) = \tilde{\varphi}\mathcal{F}_+(E^{(l)})$ implies now the result.

Suppose now $n = \sharp(E)$ even. Choose an element $z \in E$ and a cyclic permutation β of all $(n-1)$ elements of $E \setminus \{z\}$. Setting

$$\tilde{\varphi}(x_1, \dots, x_l) = \prod_{j=0}^{n-2} \varphi(\beta^j(x_1), \beta^j(x_2), \dots, \beta^j(x_l))$$

for a fixed element $\varphi \in \mathcal{F}_-(E^{(l)})$ and reasoning as above we see that $\rho'(\tilde{\varphi}) \in \mathcal{E}(E)$ is either trivial or corresponds to the two-partition $\{z\} \cup (E \setminus \{z\})$. This implies that the same conclusion holds for $\rho'(\tilde{\varphi})\rho(\tilde{\varphi})$ and the identity $\mathcal{F}_-(E^{(l)}) = \tilde{\varphi}\mathcal{F}_+(E^{(l)})$ shows that the product $\rho'(\varphi)\rho(\varphi) \in \mathcal{E}(E)$ is constant for $\varphi \in \mathcal{F}_-(E^{(l)})$. By $\mathrm{Sym}(E)-$equivariance this is only possible if $n = 2$ (cf. Remark 4.4) or if $\rho'(\varphi)\rho(\varphi)$ is trivial which establishes the Theorem. $\square$

4.1. An easy characterisation of ρ restricted to $\mathcal{F}_+(E^{(l)})$

In this subsection we give an elementary description of $\rho(\varphi)$ for $\varphi \in \mathcal{F}_+(E^{(l)})$ an $l-$symmetric function.

Given a finite set E and an $l-$symmetric function $\varphi \in \mathcal{F}_+(E^{(l)})$ we consider the function $\mu = \mu_\varphi : E \longrightarrow \{\pm 1\}$ defined by

$$\mu(x) = \prod_{(x_1,\dots,x_{l-1})\in\binom{E\setminus\{x\}}{l-1}} \varphi(x_1,\dots,x_{l-1},x)$$

.

Proposition 4.7. — *The two classes of the Orchard relation $\rho(\varphi)$ are given by $\mu_\varphi^{-1}(1)$ and $\mu_\varphi^{-1}(-1)$.*

Proof. The result clearly holds for the two $l-$symmetric constant functions. The Proposition follows now from the fact that μ_φ and $\mu_{\varphi f_X}$ differ exactly on X for a flip $f_X \in \mathcal{F}_+(E^{(l)})$. $\square$

Another proof can be given by remarking that μ defines a non-trivial $\mathrm{Sym}(E)-$equivariant homomorphism into $\mathcal{E}(E)$ which must hence be the Orchard homomorphism by unicity.

Remark 4.8. — *Setting*

$$\tilde{\mu}(x) = \prod_{(x_1,\dots,x_l)\in\binom{E\setminus\{x\}}{l}} \varphi(x_1,\dots,x_l)$$

we have $\tilde{\mu} = \mu$, up to a sign given by

$$\prod_{(x_1,\dots,x_l)\in\binom{E}{l}} \varphi(x_1,\dots,x_l)\ .$$

5. Invariants

An *invariant* is a map $I : \mathcal{F}_\pm(E^{(l)}) \longrightarrow \mathcal{R}$ into some set $\mathcal{R}$ which is $\mathrm{Sym}(E)-$equivariant with respect to the trivial action of $\mathrm{Sym}(E)$ on $\mathcal{R}$. Otherwise stated, an invariant is a constant map along $\mathrm{Sym}(E)-$orbits.

We describe in this section two ways to construct invariants using the Orchard morphism. There are of course many others.

5.1. Rooted binary trees

Given $\varphi \in \mathcal{F}_\pm(E^{(l)})$ where E is a finite set, we get a two-partition $E = E_1 \cup E_2$ of E by considering the Orchard map $\rho(\varphi) \in \mathcal{E}(E)$. Applying the Orchard morphism to the restriction of φ to E_1 and E_2 we get two-partitions $E_1 = E_{11} \cup E_{12}$ and $E_2 = E_{21} \cup E_{22}$. Iteration of this produces a rooted binary tree whose vertices are subsets of E. The root of this tree corresponds to the set E. It has two sons E_1 and E_2 (except if $\rho(\varphi)$ is trivial) etc. The leaves of this tree yield a partition of E into certain subsets A_i such that $\rho(\varphi_i)$ is trivial where φ_i is the restriction of φ onto $(A_i)^{(l)}$.

5.2. Euclideean lattices

Consider again $\varphi \in \mathcal{F}_\pm(E^{(l)})$ where E is a finite set. Given any subset $E' \subset E$, we get a two-partition on E' by restricting φ onto $(E')^{(l)}$. Encode such a two-partition by a pair of opposite vectors $\pm v \in \mathbf{Z}^{\sharp(E)}$ as follows: Coordinates v_x of v are indexed by elements of E and take the value $v_x = 0$ if $x \notin E'$ and $v_x \in \{\pm 1\}$ otherwise according to the two-partition associated to the restriction of φ to E'. Such vectors generate sublattices of the Euclidean vector space $\mathbf{R}^{\sharp(E)}$ by considering all subsets subject to some restrictions. All invariants (rank, determinant, minimal norm etc.) of such sublattices yield then invariants of φ.

A few interesting choices are perhaps as follows:

Consider the sublattice generated by all vectors associated to sets of the form $E \setminus \{x\},\ x \in E$ (all subsets of cardinality $\sharp(E) - 1$). One might also add the vector associated to E and/or the all 1 vector to this lattice.

One might also consider the unordered pair of sublattices $\Lambda_1,\ \Lambda_2$ (or their intersection $\Lambda_1 \cap \Lambda_2$) of the above lattice where Λ_i is generated by all vectors of the form $E \setminus \{x\}$ with $x \in E_i$ with $E = E_1 \cup E_2$ the two-partition $\rho(\varphi)$ of E, etc.

6. Generic configurations of points in $\mathbf{R}^d$

A finite set $\mathcal{P} = \{P_1, \dots, P_n\}$ of n points in the oriented real affine space $\mathbf{R}^d$ is a *generic configuration* if any subset of $k+1 \leq d+1$ points in $\mathcal{P}$ is affinely independent. Generic configurations of $n \leq d+1$ points in $\mathbf{R}^d$ are simply vertices of $(n-1)-$dimensional simplices. For $n \geq d+1$, genericity boils down to the fact that any set of $d+1$ points in $\mathcal{P}$ spans $\mathbf{R}^d$ affinely.

Two generic configurations $\mathcal{P}^1$ and $\mathcal{P}^2$ of $\mathbf{R}^d$ are *isomorphic* if there exists a bijection $\sigma : \mathcal{P}^1 \longrightarrow \mathcal{P}^2$ such that all pairs of corresponding $d-$dimensional simplices (with vertices $(P_{i_0}, \dots, P_{i_d}) \subset \mathcal{P}^1$ and $(\sigma(P_{i_0}), \dots, \sigma(P_{i_d})) \subset \mathcal{P}^2$)

have the same orientations (given for instance for the first simplex by the sign of the determinant of the $d\times d$ matrix with rows $P_{i_1}-P_{i_0},\dots,P_{i_d}-P_{i_0}$).

Two generic configurations $\mathcal{P}(-1)$ and $\mathcal{P}(+1)$ are *isotopic* if there exists a continuous path (with respect to the obvious topology on $\mathbf{R}^{dn}=\left(\mathbf{R}^d\right)^n$) of generic configurations $\mathcal{P}(t),\ t\in[-1,1]$, which joins them. Isotopic configurations are of course isomorphic. I ignore to what extend the converse holds.

Given a finite generic configuration $\mathcal{P}=\{P_1,\dots,P_n\}\subset\mathbf{R}^d$ we consider the $(d+1)-$antisymmetric function $\varphi:\mathcal{P}^{(d+1)}\longrightarrow\{\pm1\}$ defined by

$$\varphi(P_{i_0},\dots,P_{i_d})=1$$

if and only if

$$\det(P_{i_1}-P_{i_0},P_{i_2}-P_{i_0},\dots,P_{i_d}-P_{i_0})>0\ .$$

The Orchard morphism $\rho(\varphi)\in\mathcal{E}(\mathcal{P})$ (extended to be trivial if $n\leq d+1$) provides now a two-partition of the set $\mathcal{P}$.

The associated equivalence relation can be constructed geometrically as follows: Given two points $P,Q\in\mathbf{R}^d\setminus H$, call an affine hyperplane $H\subset\mathbf{R}^d$ *separating* if P,Q are not in the same connected component of $\mathbf{R}^d\setminus H$. For two points P,Q of a finite generic configuration $\mathcal{P}=\{P_1,\dots,P_n\}\subset\mathbf{R}^d$ we denote by $n(P,Q)$ the number of separating hyperplanes which are affinely spanned by d distinct elements in $\mathcal{P}\setminus\{P,Q\}$. This number $n(P,Q)$ depends obviously only of the isomorphism type of $\mathcal{P}$.

Proposition 6.1. — *The equivalence relation $\rho(\varphi)$ on a finite generic configuration $\mathcal{P}\subset\mathbf{R}^d$ is given by $P\sim Q$ if either $P=Q$ or if $n(P,Q)\equiv\binom{n-3}{d-1}\pmod 2$.*

Proof. Given two points $P,Q\in\mathcal{P}$ we have

$$\sigma(P,Q)=\prod_{(A_{i_1},\dots,A_{i_d})\in\binom{\mathcal{P}\setminus\{P,Q\}}{d}}\varphi(A_{i_1},\dots,A_{i_d},P)\varphi(A_{i_1},\dots,A_{i_d},Q)=(-1)^{\alpha(P,Q)}$$

where $\alpha(P,Q)$ denotes the number of subsets $(A_1,\dots,A_d)\in\mathcal{P}\setminus\{P,Q\}$ such that the two simplices with cyclically ordered vertices $(A_1,\dots,A_d,P)$ and $(A_1,\dots,A_d,,Q)$ have opposite orientations. This happens if and only if the affine hyperplane containing the points $A_1,\dots,A_d$ separates P from Q. We have hence $\alpha(P,Q)=n(P,Q)$ and

$$\sigma(P,Q)\sigma(Q,R)\sigma(P,R)=(-1)^{\binom{\sharp(\mathcal{P})-3}{(d+1)-2}}$$

and Proposition 6.1 follows from Remark 2.3. $\square$

A *geometric flip* is a continuous path

$$t \longmapsto \mathcal{P}(t) = (P_1(t), \ldots, P_n(t)) \in \left(\mathbf{R}^d\right)^n, \; t \in [-1, 1]$$

with $\mathcal{P}(t) = \{P_1(t), \ldots, P_n(t)\}$ generic except for $t = 0$ where there exists exactly one subset $\mathcal{F}(0) = (P_{i_0}(0), \ldots, P_{i_d}(0)) \subset \mathcal{P}(0)$, called the *flipset*, of $(d+1)$ points contained in an affine hyperplane spanned by any subset of d points in $\mathcal{F}(0)$. We require moreover that the simplices $(P_{i_0}(-1), \ldots, P_{i_d}(-1))$ and $(P_{i_0}(1), \ldots, P_{i_d}(1))$ carry opposite orientations. Geometrically this means that a point $P_{i_j}(t)$ crosses the hyperplane spanned by $\mathcal{F}(t) \setminus \{P_{i_j}(t)\}$ for $t = 0$.

It is easy to see that two generic configurations $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbf{R}^d$ having n points can be related by a continuous path involving at most a finite number of geometric flips.

The next result follows directly from the fact that configurations $\mathcal{P}(1)$ and $\mathcal{P}(-1)$ related by a geometric flip give rise to $(d+1)-$antisymmetric functions $\varphi_+, \varphi_- \in \mathcal{F}_-(\mathcal{P}^{(d+1)})$ which differ only by a flip:

Proposition 6.2. — *Let $\mathcal{P}(-1), \mathcal{P}(+1) \subset \mathbf{R}^d$ be two generic configurations related by a flip with respect to a subset $\mathcal{F}(t)$ of $(d+1)$ points.*

(i) If two distinct points $P(t), Q(t)$ are either both contained in $\mathcal{F}(t)$ or both contained in its complement $\mathcal{P}(t) \setminus \mathcal{F}(t)$ then we have

$$P(-1) \sim Q(-1) \text{ if and only if } P(+1) \sim Q(+1) \; .$$

(ii) For $P(t) \in \mathcal{F}(t)$ and $Q(t) \notin \mathcal{F}(t)$ we have

$$P(-1) \sim Q(-1) \text{ if and only if } P(+1) \not\sim Q(+1) \; .$$

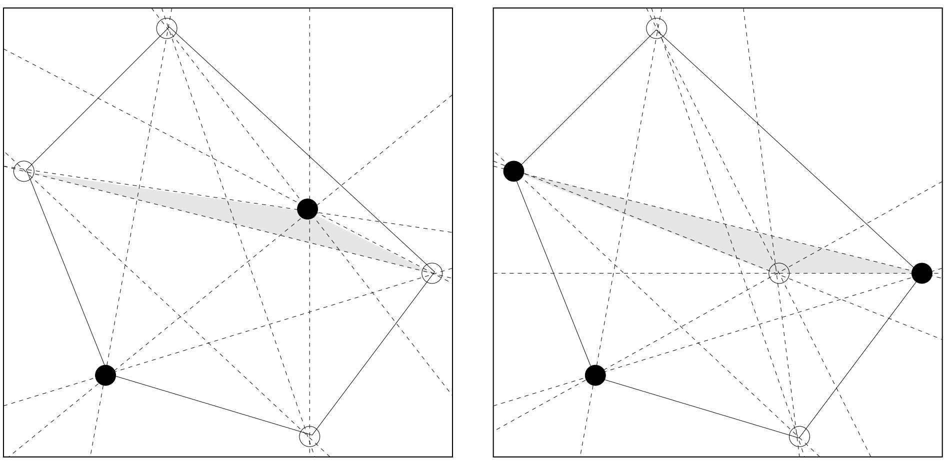

Figure 2: Two configurations of 6 points related by a geometric flip

Proposition 6.2 suggests also perhaps interesting problems concerning generic configurations: Call two generic configurations of n points in $\mathbf{R}^{2d+1}$ *orchard-equivalent* if they are related by flips whose flipsets have always exactly $(d+1)$ points in each class.

More generally, flips are of different types according to the number of points of each class involved in the corresponding flipset. A very special type of flips are the *monochromatic* ones, defined as involving only vertices of one class in their flipset.

Understanding isomorphism classes of generic configurations up to flips subject to some restrictions (e.g. only monochromatic flips or configurations up to orchard-equivalence in odd dimensions) might be interesting.

We close this section by discussing two further examples.

Example. Consider a configuration $\mathcal{P} \subset \mathbf{S}^2 \subset \mathbf{R}^3$ consisting of n points contained in the Euclideean unit sphere $\mathbf{S}^2$ and which are generic as a a subset of $\mathbf{R}^3$ in the above sense, i.e. 4 distinct points of $\mathcal{P}$ are never contained in a common affine plane of $\mathbf{R}^3$. A stereographic projection $\pi : \mathbf{S}^3 \setminus \{N\} \longrightarrow \mathbf{R}^2$ with respect to a point $N \in \mathbf{S}^2 \setminus \mathcal{P}$ sends the set $\mathcal{P} \subset \mathbf{S}^2$ into a set $\tilde{\mathcal{P}} = \pi(\mathcal{P}) \subset \mathbf{R}^2$ such that 4 points of $\tilde{\mathcal{P}}$ are never contained in a common Euclideean circle or line of $\mathbf{R}^2$. The Orchard relation on $\mathcal{P}$ can now be seen on $\tilde{\mathcal{P}}$ as follows: Given two distinct points $\tilde{P} \neq \tilde{Q} \in \tilde{\mathcal{P}}$ count the number $n(\tilde{P}, \tilde{Q})$ of circles or lines determined by 3 points in $\tilde{\mathcal{P}} \setminus \{\tilde{P}, \tilde{Q}\}$ which separate them. Two distinct points $P \neq Q \in \mathcal{P}$ are now Orchard-equivalent if and only if $n(\tilde{P}, \tilde{Q}) \equiv \binom{n-3}{2} \pmod 2$. This example can of course be generalised to finite generic configurations of points on the the $d-$dimensional unit sphere $\mathbf{S}^d \subset \mathbf{R}^{d+1}$ for $d \geq 2$.

Let $\mathcal{C}$ be a set of continuous real functions on $\mathbf{R}^k$. Suppose $\mathcal{C}$ is a $(d+1)-$dimensional vector space containing the constant functions. Call a set $\mathcal{P} \subset \mathbf{R}^k$ of n points $\mathcal{C}-$*generic* if for each subset $S = \{P_{i_1}, \dots, P_{i_d})$ of d distinct points in $\mathcal{P}$ if the set

$$I(S) = \{f \in \mathcal{C} \mid f(P_{i_j}) = 0,\ j = 0 \dots, d\}$$

is a $1-$dimensional affine line and all $\binom{n}{d}$ affine lines in $\mathcal{C}$ of this form are distinct.

Given $P, Q \in \mathcal{P}$, call a set $S = \{P_{i_1}, \dots, P_{i_d}\} \subset \mathcal{P} \setminus \{P, Q\}$ of d points as above $\mathcal{C}-$*separating* (or *separating* for short) if $f(P)f(Q) < 0$ for any $0 \neq f \in I(S)$ and denote by $n_{\mathcal{C}}(P, Q)$ the number of $\mathcal{C}-$separating subsets of $\mathcal{P}$.

Corollary 6.3. — *The relation $P \sim_{\mathcal{C}} Q$ if either $P = Q$ or*

$$n_{\mathcal{C}}(P, Q) \equiv \binom{n-3}{d-1} \pmod 2$$

defines an equivalence relation having at most two classes on a set $\mathcal{P} = \{P_1, \ldots, P_n\} \subset \mathbf{R}^k$ *of* n *points in* $\mathbf{R}^k$ *which are* $\mathcal{C}-$*generic.*

Examples. (i) Considering the $(d+1)-$dimensional vector space of all affine functions in $\mathbf{R}^d$, Corollary 1.6 boils down to Theorem 1.1.

(ii) Consider the vector space $\mathcal{C}$ of all polynomial functions $\mathbf{R}^2 \longrightarrow \mathbf{R}$ of degree at most 2. A finite subset $\mathcal{P} \subset \mathbf{R}^2$ is $\mathcal{C}-$generic if and only if every subset of five points in $\mathcal{P}$ defines a unique conic and all these conics are distinct.

(iii) Consider the vector space $\mathcal{C}$ of all polynomials of degree $< d$ in x together with the polynomials $\lambda y,\ \lambda \in \mathbf{R}$. A subset $\mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ with $x_1 < x_2 < \ldots, < x_n$ is $\mathcal{C}-$generic if all $\binom{n}{d}$ interpolation polynomials in x defined by d points of $\mathcal{P}$ are distinct.

7. Orientable sets

In the following sections we consider a set E together with a fixpoint-free involution $\iota : E \longrightarrow E$. We call ι an *orientation* and the pair (E, ι) an *orientable set*. The aim of the following sections is to define the Orchard morphism for finite orientable sets. In this case, all groups and homomorphisms are required to be also natural with respect to the involution ι.

Examples of orientable sets are for instance antipodal sets of points in $\mathbf{R}^d \setminus \{0\}$ or real Grassmannians endowed with orientations.

In the sequel we denote by $\pi : E \longrightarrow \overline{E} = E/\iota$ the quotient map $x \longrightarrow \overline{x} = \{x, \iota(x)\}$ onto the underlying (unoriented) quotient set. The set of all sections

$$\mathcal{S}(E, \iota) = \{s : \overline{E} \longrightarrow E | \pi \circ s(\overline{x}) = \overline{x}\ , \forall \overline{x} \in \overline{E}\}$$

is a free module over the group $(\pm 1)^{\overline{E}}$ of all functions $\overline{E} \longrightarrow \{\pm 1\}$ by setting $(\alpha s)(\overline{x}) = s(x)$ if $\alpha(\overline{x}) = 1$ and $(\alpha s)(\overline{x}) = \iota(s(x))$ otherwise where $\alpha : \overline{E} \longrightarrow \{\pm 1\}$ and $s \in \mathcal{S}(E, \iota)$. The quotient set $\mathcal{S}(E, \iota)/\{\pm 1\}$ corresponds to orientations defined up to global reversion (action of ι). We call an element of the quotient group $\mathcal{S}(E, \iota)/\{\pm 1\}$ a *semi-orientation* of the orientable set (E, ι).

Given an orientable set (E, ι), its automorphism group $\mathrm{Sym}(E, \iota)$ is the set of all $\iota-$equivariant permutations of E. Otherwise stated, a permutation $\pi : E \longrightarrow E$ belongs to $\mathrm{Sym}(E, \iota)$ if and only if $\pi(\iota(x)) = \iota(\pi(x))$ for all $x \in E$. As an abstract group, the group $\mathrm{Sym}(E, \iota)$ is easily seen to be isomorphic to the group of all isometries of the $e-$dimensional regular standard cube $[-1, 1]^e \subset \mathbf{R}^e$ where $2e = |E|$ is the cardinality of E. This group has $2^e\ e!$ elements and is the wreath product of $\mathrm{Sym}(\overline{E})$ with $\{\pm 1\}^e$. We have hence an obvious surjective homomorphisme $\mathrm{Sym}(E, \iota) \longrightarrow \mathrm{Sym}(\overline{E})$ with kernel $\{\pm 1\}^e$.

8. Two-sets of orientable sets

Given an orientable set (E, ι) it is natural to consider the set $\mathcal{E}(E, \iota)$ of all two-partitions of E which are invariant under ι. This set contains the subset $\mathcal{E}(E, \iota_+)$ consisting of all two-partitions factoring through π and inducing hence a two-partition on the quotient set $\overline{E}$. Otherwise stated, two elements x and $\iota(x)$ in an orbit under ι belong always to the same class. We call such a two-partition *even* since its classes are given $\alpha^{-1}(1)$ and $\alpha^{-1}(-1)$ where $\alpha : E \longrightarrow \{\pm 1\}$ is an even function with respect to the involution ι (it satisfies $\alpha(x) = \alpha(\iota(x))$ for all $x \in E$). Its complement $\mathcal{E}(E, \iota_-) = \mathcal{E}(E, \iota) \setminus \mathcal{E}(E, \iota_+)$, called the *odd* two-partitions, has equivalence classes defined as preimages of an odd function $\alpha : E \longrightarrow \{\pm 1\}$ satisfying $\alpha(x) = -\alpha(\iota(x))$ for all x. The set $\mathcal{E}(E, \iota_-)$ of all odd two-partitions on (E, ι) coincides with the set of semi-orientations of the orientable set (E, ι). Its elements are unordered pairs $\{s, \iota \circ s\}$ of complementary sections of the quotient map $\pi : E \longrightarrow \overline{E}$.

The set $\mathcal{E}(E, \iota) = \mathcal{E}(E, \iota_+) \cup \mathcal{E}(E, \iota_-)$ obtained by considering all even or odd two-partitions on the orientable set (E, ι) is a vector space (of dimension $\sharp(E)/2$ if E is finite) over $\mathbf{F}_2$. An element of $\mathcal{E}(E, \iota)$ is represented by $\pm\alpha$ where the function $\alpha : E \longrightarrow \{\pm 1\}$ is either even ($\alpha(\iota x) = \alpha(x)$ for all $x \in E$) or odd ($\alpha(\iota x) = -\alpha(x)$ for all $x \in E$) with respect to ι. The pair ± 1 of constant even functions represents the identity element and the group law is the usual product of (pairs of) functions. Given an element $\{\pm\alpha\} \in \mathcal{E}(E, \iota)$, we define a map parity : $\mathcal{E}(E, \iota) \longrightarrow \{\pm 1\}$, called the *parity homomorphism*, by setting parity$(\pm\alpha) = 1$ if $\{\pm\alpha\} \in \mathcal{E}(E, \iota_+)$ is a pair of even functions and parity$(\pm\alpha) = -1$ if α is an odd function.

Given an orientable set (E, ι), we define the set $(E, \iota)^{(l)}$ as the set of all sequences $(x_1, \ldots, x_l) \in E^l$ of length l such that $(\overline{x}_1, \ldots, \overline{x}_l) \in \overline{E}^{(l)}$. Otherwise stated, such a sequence satisfies $\{x_i, \iota(x_i)\} \neq \{x_j, \iota(x_j)\}$ for $i \neq j$.

Proposition 8.1. — *Any even (respectively odd) symmetric function*

$$\sigma : (E, \iota)^{(2)} \longrightarrow \{\pm 1\}$$

with

$$\sigma(a, b)\sigma(b, c)\sigma(a, c) = \gamma \in \{\pm 1\}$$

independent of $(a, b, c) \in (E, \iota)^{(3)}$ *gives rise to an even (respectively odd) two-partition on* (E, ι).

Proof. Results from Corollary 2.2 if σ is even.

For σ odd, choose a section $s : \overline{E} \longrightarrow E$ and define the two-partition in the obvious way on the section. This two-partition extends to a unique odd two-partition on (E, ι) which is independent of the choice of the section s. $\square$

Remark 8.2. — *The above equivalence relation can be constructed as follows: Choose a fixed base point $x_0 \in E$. Set $\alpha(x_0) = 1$ and $\alpha(\iota(x_0)) =$ parity(σ) where parity$(\sigma) = 1$ if σ is even and parity$(\sigma) = -1$ if σ is odd. For $y \notin \{x_0, \iota(x_0)\}$ we set $\alpha(y) = \gamma\, \sigma(x_0, y)$ with $\gamma \in \{\pm 1\}$ as in Proposition 8.1.*

9. Symmetric and antisymmetric functions on orientable sets

Recall that $(E, \iota)^{(l)}$ denotes the set of all sequences $(x_1, \ldots, x_l) \in E^l$ such that $(\overline{x_1}, \ldots, \overline{x_l}) \in \overline{E}^{(l)}$.

One defines $l-$symmetric (respectively $l-$antisymmetric) functions on $(E, \iota)^{(l)}$ in the obvious way as the subset of functions which are invariant (respectively which change sign) under transposition of two arguments.

A symmetric or antisymmetric function $\varphi : (E, \iota)^{(l)} \longmapsto \{\pm 1\}$ is *even* if

$$\varphi(x_1, x_2, \ldots, x_l) = \varphi(\iota(x_1), x_2, \ldots, x_l) = \varphi(x_1, \iota(x_2), x_3, \ldots, x_l) = \ldots\ .$$

We denote by $\mathcal{F}_\pm(E, \iota_+)^{(l)}$ the set of all even $l-$symmetric or $l-$antisymmetric functions. Notice that there exists an obvious bijection between $\mathcal{F}_\pm(E, \iota_+)^{(l)}$ and $\mathcal{F}_\pm(\overline{E}^{(l)})$.

Such a function is *odd* if

$$\varphi(x_1, x_2, \ldots, x_l) = -\varphi(\iota(x_1), x_2, \ldots, x_l) = -\varphi(x_1, \iota(x_2), x_3, \ldots, x_l) = \ldots\ .$$

The set of all odd symmetric or antisymmetric functions on $(E, \iota)^{(l)}$ will be denoted by $\mathcal{F}_\pm(E, \iota_-)^{(l)}$.

We denote by $\mathcal{F}_\pm(E, \iota)^{(l)} = \mathcal{F}_\pm(E, \iota_+)^{(l)} \cup \mathcal{F}_\pm(E, \iota_-)^{(l)}$ the set of all even or odd, $l-$symmetric or $l-$antisymmetric functions on the orientable set (E, ι). The set $\mathcal{F}_\pm(E, \iota)^{(l)}$ is of course a vector space (of dimension $\binom{\sharp(E)/2}{l}+2$ if E is finite) over $\mathbf{F}_2$. The set $\mathcal{F}_\pm(E, \iota_-)^{(l)}$ is a free $\mathcal{F}_\pm(E, \iota_+)^{(l)}-$module.

We define the signature and parity homomorphismes sign, parity: $\mathcal{F}_\pm(E, \iota)^{(l)} \to \{\pm 1\}$ by

$$\begin{array}{ll} \text{sign}(\varphi) = 1 \text{ if } \varphi \in \mathcal{F}_+(E, \iota)^{(l)}, & \text{sign}(\varphi) = -1 \text{ if } \varphi \in \mathcal{F}_-(E, \iota)^{(l)}\ , \\ \text{parity}(\varphi) = 1 \text{ if } \varphi \in \mathcal{F}_\pm(E, \iota_+)^{(l)}, & \text{parity}(\varphi) = -1 \text{ if } \varphi \in \mathcal{F}_\pm(E, \iota_-)^{(l)}\ . \end{array}$$

10. The Orchard morphism for finite orientable sets

Given $\varphi \in \mathcal{F}_\pm(E,\iota)^{(l)}$ where (E,ι) is a finite orientable set, we define $\sigma = \sigma_\varphi : (E,\iota)^{(2)} \longrightarrow \{\pm 1\}$ by setting

$$\sigma(y,z) = \prod_{(\overline{x}_1,\dots,\overline{x}_{l-1})\in\binom{\overline{E}\setminus\{\overline{y},\overline{z}\}}{l-1}} \varphi(x_1,\dots,x_{l-1},y)\ \varphi(x_1,\dots,x_{l-1},z)$$

where $x_1 = s(\overline{x}_1),\dots,x_{l-1} = s(\overline{x}_{l-1})$ are obtained using an arbitrary section $s : \overline{E} \longrightarrow E$ of the quotient map $\pi : E \longrightarrow \overline{E} = E/\iota$.

Proposition 10.1. — *Let $\varphi \in \mathcal{F}_\pm(E,\iota)^{(l)}$ be a function and define $\sigma = \sigma_\varphi$ as above.*

(i) The function σ is well defined, symmetric and satisfies the identity

$$\sigma(a,b)\sigma(b,c)\sigma(a,c) = (sign(\varphi))^{\binom{e-3}{l-2}}$$

for all $(a,b,c) \in (E,\iota)^{(3)}$ where $2e = |E| = 2\ |\overline{E}|$ is the cardinality of E.

(ii) If $\varphi \in \mathcal{F}_\pm(E,\iota_+)^{(l)}$ (i.e. φ even), then σ is even.

(iii) If $\varphi \in \mathcal{F}_\pm(E,\iota_-)^{(l)}$ (i.e. φ odd), then σ is even if $\binom{e-2}{l-1} \equiv 0 \pmod 2$ and odd otherwise.

Proof. Every element $(\overline{x}_1,\dots,\overline{x}_{l-1}) \in \binom{\overline{E}\setminus\{\overline{y},\overline{z}\}}{l-1}$ is involved twice in $\sigma(y,z)$ thus implying that the final value is independent of the choosen total order on $\overline{E} \setminus \{y,z\}$ and of the choosen section $s : \overline{E} \longrightarrow E$.

The definition of σ is obviously symmetric with respect to its arguments.

The proof of the identity $\sigma(a,b)\sigma(b,c)\sigma(a,c) = (\mathrm{sign}(\varphi))^{\binom{e-3}{l-2}}$ is exactly analogous to the corresponding proof in the non-orientable case.

Assertion (ii) is almost obvious since we have

$$\varphi(x_1,\dots,x_{l-1},u) = \varphi(x_1,\dots,x_{l-1},\iota(u))$$

for $u \in \{a,b\}$ and $\varphi \in \mathcal{F}_\pm(E^{(l)},\iota_+)$ even.

Assertion (iii) follows from the fact that

$$\varphi(x_1,\dots,x_{l-1},u) = -\varphi(x_1,\dots,x_{l-1},\iota(u))$$

for $u \in \{a,b\}$ and $\varphi \in \mathcal{F}_\pm(E^{(l)},\iota_-)$ odd and from the observation that the definition of $\sigma(a,b)$ involves $\binom{e-2}{l-1}$ such factors. □

The Orchard morphism $\rho : \mathcal{F}_\pm(E^{(l)},\iota) \longrightarrow \mathcal{E}(E,\iota)$ associates to a function $\varphi \in \mathcal{F}_\pm(E^{(l)},\iota)$ the two-partition in $\mathcal{E}(E,\iota)$ associated to $\sigma = \sigma_\varphi$ by Proposition 8.1.

Theorem 10.2. — *The oriented Orchard morphism is the unique non-trivial Sym$(E,\iota)-$equivariant homomorphism from $\mathcal{F}_\pm(E,\iota)^{(l)}$ into $\mathcal{E}(E,\iota)$ where (E,ι) is a finite orientable set containing at least 6 elements.*

Remark 10.3. — *If (E,ι) is an orientable set containing 4 elements $\pm a, \pm b$ (with ι given by $\iota(a) = -a$ and $\iota(b) = -b$), there exist several non-trivial natural homomorphisms $\mathcal{F}_\pm(E,\iota)^{(l)} \longrightarrow \mathcal{E}(E,\iota)$ for $l = 1, 2$.*

An example (distinct from the Orchard morphism) for $l = 1$ is given by $\rho'(\varphi) =$ trivial if $\varphi \in \mathcal{F}(E,\iota_+)^{(1)}$ and $\rho'(\varphi) = (E = \{\pm a\} \cup \{\pm b\})$ if $\varphi \in \mathcal{F}(E,\iota_-)^{(1)}$.

For $l = 2$, one can for instance extend the exotic homomorphism of the unoriented case (cf. Remark 4.2) in two ways by choosing an arbitrary even two-partition as the image $\rho'(\varphi)$ for $\varphi \in \mathcal{F}_+(E,\iota_-)^{(2)}$. The image $\rho'(\psi)$ for $\psi \in \mathcal{F}_-(E,\iota_-)^{(2)}$ is then the unique remaining two-partition (i.e. $\rho'(\varphi)\rho'(\psi) = \rho'(\theta)$ with $\theta \in \mathcal{F}_-(E,\iota_+)^{(2)}$ is the unique even non-trivial two-partition of (E,ι)).

Proof. The proof that ρ defines a homomorphism is as in the unoriented case.

The restriction of ρ to the subgroup $\mathcal{F}_\pm(E,\iota_+)^{(l)}$ consisting only of even functions coincides with the usual Orchard morphism $\mathcal{F}_\pm(\overline{E}) \longrightarrow \mathcal{E}(\overline{E})$ on $\overline{E}$ and the result holds hence for this restriction by Theorem 4.3.

We have hence to show unicity of the restriction to $\mathcal{F}_\pm(E,\iota_-)$ of such a homomorphisme ρ'. The identity $\mathcal{F}_\pm(E,\iota_-) = \varphi\mathcal{F}_\pm(E,\iota_+)$ for $\varphi \in \mathcal{F}_\pm(E,\iota_-)$ and Sym$(E,\iota)-$equivariance show that such a homomorphism with trivial restriction on $\mathcal{F}_\pm(E,\iota_+)$ is trivial.

We might hence suppose that $\rho' = \rho$ on $\mathcal{F}_\pm(E,\iota_+)$. We set $e = \sharp(\overline{E} = \sharp(E)/2$.

Consider now a section $s : \overline{E} \longrightarrow E$ and the unique symmetric odd function $\varphi \in \mathcal{F}_+(E,\iota_-)$ defined by

$$\varphi(s(\overline{x}_{i_1}), \dots, s(\overline{x}_{i_l})) = 1$$

for all $(\overline{x}_{i_1}, \dots, \overline{x}_{i_l}) \in \overline{E}^{(l)}$. Sym$(E,\iota)-$equivariance of ρ' implies that $\rho'(\varphi)$ is either trivial or the semi-orientation associated to the section s. Choose now an element $\overline{x} \in \overline{E}$ and consider the corresponding function $\tilde{\varphi}$ associated as above to the section $\tilde{s}$ which coincides with s on $\overline{E} \setminus \{\overline{x}\}$ and sends $\overline{x}$ to $\iota(s(\overline{x}))$. The functions φ and $\tilde{\varphi}$ differ by the product of all $\binom{e-1}{l-1}$ flips with flipsets $\{\overline{x}, \overline{y}_1, \dots, \overline{y}_{l-1}\}$ where $(\overline{y}_1, \dots, \overline{y}_{l-1}) \in \binom{\overline{E}\setminus\{\overline{x}\}}{l-1}$. An element $\overline{y} \in \overline{E} \setminus \{\overline{x}\}$ is involved in $\binom{e-2}{l-2}$ such flipsets and $\overline{x}$ is involved in $\binom{e-1}{l-1} = \binom{e-2}{l-2} + \binom{e-2}{l-1}$ such flipsets. This shows that $\rho'(\varphi) = \rho'(\tilde{\varphi})$ if $\binom{e-2}{l-1} \equiv 0$

(mod 2) and $\mathrm{Sym}(E,\iota)-$equivariance forces $\rho'(\varphi)$ to be even. It coincides hence with the Orchard morphism.

If $\binom{e-2}{l-1} \equiv 1 \pmod 2$, the two-partitions $\rho'(\varphi)$ and $\rho'(\tilde{\varphi})$ differ exactly on $\pi^{-1}(\overline{x})$ and $\mathrm{Sym}(E,\iota)-$equivariance forces $\rho'(\varphi)$ to be the semi-orientation of $\mathcal{E}(E,\iota)$ associated to the section s. $\square$

11. Geometric examples

In this section we discuss a few orientable sets arising from geometric configurations: finite generic antipodal configurations of points (or generic configurations of lines through the origin) in $\mathbf{R}^d$ and generic configurations of the real projective space $\mathbf{R}P^d$.

A finite *antipodal set* of $\mathbf{R}^d$ is a finite subset $\mathcal{P} \subset \mathbf{R} \setminus \{0\}$ invariant under the involution $x \longmapsto \iota(x) = -x$. We call such a set *generic* if the linear span of any subset $\{\pm x_1, \dots, \pm x_k\} \subset \mathcal{P}$ is k for $k \leq d$. We get then an element $\varphi \in \mathcal{F}_-(E,\iota)^{(d)}$ by considering the sign $\in \{\pm 1\}$ of

$$\det(x_1, \dots, x_d)$$

for $(x_1, \dots, x_d) \in (\mathcal{P}, \iota)^{(d)}$ (where $\det(x_1, \dots, x_d)$ denotes the non-zero determinant of the $d \times d$ matrix with rows $x_1, \dots, x_d$).

Applying the oriented Orchard morphism ρ of the preceeeding section to φ we get a two-partition $\rho(\varphi) \in \mathcal{E}(\mathcal{P},\iota)$. Obviously, $\rho(\varphi)$ remains the same by rescaling each pair $\pm x \in \mathcal{P}$ by some strictly positive constant $\lambda_x \in \mathbf{R}_{>0}$.

We may hence rescale such an antipodal set in order to lie on the Euclideean sphere $\mathbf{S}^{d-1} = \{x \in \mathbf{R}^d \mid \quad \| x \| = 1\} \subset \mathbf{R}^d$. Similarly, we might interprete $\mathcal{P}$ as a set $\mathcal{L}$ of lines (defined by opposite pairs $\pm x \in \mathcal{P}$). The Orchard morphism $\rho(\varphi)$ endows then such a generic finite set of lines either with a two-partition (in the case where $\binom{\sharp(\mathcal{L})-2}{d-1} \equiv 0 \pmod 2$) or with a semi-orientation (if $\binom{\sharp(\mathcal{L})-2}{d-1} \equiv 1 \pmod 2$).

A finite subset $\mathcal{P} \subset \mathbf{R}P^d$ of the real projective space is generic if its completed preimage $\mathcal{L} = \overline{\pi^{-1}(\mathcal{P})} \subset \mathbf{R}^{d+1}$ is a finite set of generic lines in $\mathbf{R}^{d+1}$. If $\binom{\sharp(\mathcal{P})-2}{d} \equiv 0 \pmod 2$ we get a two-partition on such a set $\mathcal{P}$ by applying the Orchard morphism to $\mathcal{L}$.

In the case where the Orchard morphism endows $\mathcal{L}$ with a semi-orientation, we get also an interesting structure on $\mathcal{P}$ as follows:

Any pair $P, Q \in \mathcal{P}$ of distinct points defines two connected components on $L_{P,Q} \setminus \{P,Q\}$ where $L_{P,Q} \subset \mathbf{R}P^d$ denotes the projective line containing P and Q. One of these connected components is now selected by a semi-orientation on $\mathcal{L}$ by choosing the connected component of $L_{P,Q} \setminus \{P,Q\}$

whose preimage in $\mathbf{S}^d \subset \mathbf{R}^{d+1} = \pi^{-1}(\mathbf{R}P^d) \cup \{0\}$ joins elements of $\pi^{-1}(\mathcal{P})$ which are in the same class. We get in this way an immersion of the complete graph $K_{\mathcal{P}}$ with vertices $\mathcal{P}$ into the projective space $\mathcal{R}P^d$. It is straightforward to show that this immersion is homologically trivial: each cycle of $K_{\mathcal{P}}$ is immerged in a contractible way into $\mathbf{R}P^d$.

REMARK 11.1. — *A preliminary version of this paper (cf. [1]) contained also a section concerning simple arrangements of (pseudo)lines in the projective plane. The corresponding invariants (two-partitions and semi-orientations) are however not based on the Orchard-morphism but use only Proposition 8.1. They are hence not directly related to the topic of this text and will perhaps be discussed elsewhere.*

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Bibliographie

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