ON PROJECTIONS OF SYZYGIES AND GONALITY OF CURVES

Marian Aprodu [1]

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0. Introduction.

If X is a (complex, smooth, connected) projective curve, and L a globally generated, non-special line bundle on X, and $x \in X$ is a point, the projection of syzygies (in degree p) centered in x is a map between Koszul cohomology groups:

$$K_{p+1,1}(X,L) \xrightarrow{\mathrm{pr}_x} K_{p,1}(X,L-x),$$

defined in a natural way, for any non-negative integer p. Introduced for the first time by Ehbauer (see [Ehb]), these maps turned out to be an useful tool for comparing vanishing of Koszul cohomology of a curve when changing the embedding; see, for instance, [Ehb] and [A1]. One of their fundamental properties is that the induced map

$$K_{p+1,1}(X,L) \xrightarrow{\oplus \mathrm{pr}_x} \bigoplus_{x\in X} K_{p,1}(X,L-x)$$

is injective for any p. In other words, any non-zero element from $K_{p+1,1}(X,L)$ survives when we project it from some point of X (see [A1]). In turn, there is no particular reason for having injectivity of *all* the projection maps pr_x at one time, as some non-zero element of $K_{p+1,1}(X,L)$ might vanish when we project it from some particular point. This phenomenon is actually controlled by the Koszul cohomology group $K_{p+1,1}(X,L,W_x)$, which coincides to $\mathrm{Ker}(\mathrm{pr}_x)$ (cf. [A1]), where W_x denotes $H^0(X,L-x)$ inside $H^0(X,L)$.

The aim of the present paper is to prove two general results (Theorem 3 in Section 1, and Theorem 8 in Section 3) whose consequences suggest that injectivity of *all* the projection maps occurs in close relation with the geometry of the curve. We conjecture that gonality of any curve (by a *curve* we shall always mean a complex, smooth, connected, projective curve) can be read off injectivity of projection maps. For formulating a precise statement, we introduce the following invariant.

Definition 1. Let X be a non-hyperelliptic curve of genus g, and L be a line bundle on X of degree at least $2g$. We set

$$m_L := \max\{d, \text{ such that } d \le h^0(X,L) \text{ and } K_{h^0(L)-d,1}(X,L,W_x) = 0 \text{ for all } x \in X\}.$$

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It is not hard to see that this invariant is well-defined, and strictly lower than $h^0(X,L)$.

The statement we propose is a *weak Green-Lazarsfeld gonality conjecture*, in the sense that the Green-Lazarsfeld gonality conjecture implies our conjecture:

Conjecture 2. *Let X be a d-gonal curve of genus g, and L be a line bundle of degree at least $3g$. Then $m_L = d$.*

We will verify in the next section (see Corollary 5) that inequality $d \geq m_L$ always holds if the degree of L is large enough, and prove later the converse for generic curves of odd genus (cf. Corollary 11 in Section 3). In view of the main results of [AV], Conjecture 2 is also true for generic curves of even genus, as well as for generic d-gonal curves of genus $g < 3d$. Green's conjecture on syzygies of canonical curves will appear once again in the picture, reinforcing our belief that Green's conjecture and the gonality conjecture are intimately related to each other.

1. A short digression on Green-Lazarsfeld's non-vanishing Theorem.

In this section, we present a version of the Green-Lazarsfeld non-vanishing Theorem [GL1], and we give an ad-hoc proof of it. This result represents one of the main motivations for having formulated Conjecture 2.

We begin by recalling very briefly the definition of Koszul cohomology groups (cf. [Gr1]). If X is a compact complex manifold, $L \in \mathrm{Pic}(X)$ is a line bundle, $\mathcal{F}$ is a coherent sheaf, and $W \subset H^0(X,L)$, denote by $K_{p,q}(X,\mathcal{F},L,W)$ the Koszul cohomology groups of the graded SW-module $\bigoplus_{q\in\mathbf{Z}} H^0(X,\mathcal{F}\otimes qL)$ (following Green, we use the additive notation for the group law in $\mathrm{Pic}(X)$). By definition, $K_{p,q}(X,\mathcal{F},L,W)$ is the cohomology at the middle of the Koszul complex

$$\bigwedge^{p+1} W\otimes H^0(X,\mathcal{F}\otimes(q-1)L) \to \bigwedge^{p} W\otimes H^0(X,\mathcal{F}\otimes qL) \to \bigwedge^{p-1} W\otimes H^0(X,\mathcal{F}\otimes(q+1)L).$$

Adopting Green's notation, if $W = H^0(X,L)$, we drop it and write $K_{p,q}(X,\mathcal{F},L)$, if $\mathcal{F} \cong \mathcal{O}_X$, we suppress it, and write $K_{p,q}(X,L,W)$ instead; the notation $K_{p,q}(X,L)$ corresponds to the choice $W = H^0(X,L)$ and $\mathcal{F} \cong \mathcal{O}_X$.

We the notation above, we have the following (compare to [GL1]):

Theorem 3. *Let X be a compact complex manifold, L be a line bundle on X, and D be an effective divisor such that $h^0(X,\mathcal{O}_X(D)) \geq 2$, and $h^0(X,L-D) \geq h^0(X,\mathcal{O}_X(D))$. Then $K_{h^0(L-D)-1,1}(X,L,W) \neq 0$ for any intermediate vector subspace $H^0(X,L-D) \subset W \subset H^0(X,L)$.*

Proof. We start by considering the Koszul differential:

$$\bigwedge^{h^0(L-D)} H^0(X,L-D)\otimes H^0(X,\mathcal{O}_X(D)) \xrightarrow{\delta} \bigwedge^{h^0(L-D)-1} H^0(X,L-D)\otimes H^0(X,L).$$

Its kernel obviously equals to $K_{h^0(L-D),0}(X,D,L-D)$. By means of Green's vanishing Theorem ([Gr1] Theorem 3.a.1), and using the hypothesis $h^0(X,L-D) \geq h^0(X,\mathcal{O}_X(D))$, we see that $K_{h^0(L-D),0}(X,D,L-D) = 0$, which means δ is injective.

Let $\alpha \in H^0(X, \mathcal{O}_X(D))$ be a section vanishing along D, and $\beta \in H^0(X, \mathcal{O}_X(D))$ another non-zero section which is not a multiple of α. The space $H^0(X, L-D)$ canonically embedds into $H^0(X, L)$ via the multiplication by α (we keep this embedding fixed).

If $t \in \bigwedge^{h^0(L-D)} H^0(X, L-D)$ is a non-zero element, then $\delta(t \otimes \beta)$ defines a class in $K_{h^0(L-D)-1,1}(X, L, H^0(L-D))$. Since α and β are linearly independent, this class cannot be zero, as it doesn't belong to the image of $\bigwedge^{h^0(L-D)} H^0(X, L-D)$ inside $\bigwedge^{h^0(L-D)-1} H^0(X, L-D) \otimes H^0(X, L)$ (use the injectivity of δ). Therefore, $K_{h^0(L-D)-1,1}(X, L, H^0(L-D)) \neq 0$. For any intermediate vector subspace $H^0(X, L-D) \subset W \subset H^0(X, L)$, Green's spectral sequence for base change [Gr2] Proposition 1.b.1 induces natural injective maps $K_{h^0(L-D)-1,1}(X, L, H^0(L-D)) \subset K_{h^0(L-D)-1,1}(X, L, W)$. Thus $K_{h^0(L-D)-1,1}(X, L, W) \neq 0$.

An immediate consequence of Theorem 3 is the following.

Corollary 4. *Let X be a curve, $d \geq 2$ an integer, and D be an effective divisor of degree d moving in a pencil, i.e. $h^0(X, \mathcal{O}_X(D)) = 2$. If $L-D$ is non-special, and has at least two linearly independent sections, then $K_{h^0(L)-d-1,1}(X, L, H^0(L-x)) \neq 0$ for any $x \in X$.*

In particular, we can apply Corollary 4 for a minimal pencil on X to obtain the following:

Corollary 5. *On a d-gonal curve X of genus g the inequality $d \geq m_L$ holds for any line bundle L of degree at least $2g+d$.*

One half (the easiest) of Conjecture 2 is thus solved.

2. Projection maps between Koszul cohomology groups.

We present next Ehbauer's projection maps from a point of view which is slightly different from those of [Ehb], [A1], [A2]. For the sake of simplicity, the exposition of the facts in the sequel considers only the case of curves. Nevertheless, the reader should be aware that all the arguments that follow can be easily adapted to higher-dimensional varieties.

So, let X be a curve, L a line bundle on X, and $W \subset H^0(X, L)$ of dimension at least two. The Euler sequence on $\mathbf{P}W^*$ pulls back to an Euler type sequence on X:

$$0 \to M_W \to W \otimes \mathcal{O}_X \to L.$$

In higher exterior degrees, we obtain an exact sequence, for any positive p:

$$0 \to \bigwedge^{p+1} M_W \to \bigwedge^{p+1} W \otimes \mathcal{O}_X \to \bigwedge^{p} M_W \otimes L,$$

and further

$$0 \to \bigwedge^{p+1} M_W \otimes \mathcal{F} \otimes (q-1)L \to \bigwedge^{p+1} W \otimes \mathcal{F} \otimes (q-1)L \to \bigwedge^{p} M_W \otimes \mathcal{F} \otimes qL.$$

It is well-known (see for example [Gr3] or [V1]) that the exact sequence of global sections associated to the latter exact sequence computes the Koszul cohomology, i.e.

$$K_{p,q}(X,\mathcal{F},L,W) \cong \frac{H^0\left(\bigwedge^p M_W \otimes \mathcal{F} \otimes qL\right)}{\operatorname{Im}\left(\bigwedge^{p+1} W \otimes H^0(\mathcal{F}\otimes(q-1)L) \to H^0\left(\bigwedge^p M_W \otimes \mathcal{F} \otimes qL\right)\right)}.$$

We make a short break to mention that important things happen if W is base-point-free. In this case, $K_{p,q}(X,\mathcal{F},L,W)$ is moreover isomorphic to

$$\operatorname{Ker}\left(H^1\left(\bigwedge^{p+1} M_W \otimes \mathcal{F}\otimes(q-1)L\right) \to \bigwedge^{p+1} W \otimes H^1(\mathcal{F}\otimes(q-1)L)\right).$$

This fact is the main argument for having a *duality theorem* (see [Gr3], and also [Gr1] Corollary 2.c.10) in the context of a base-point-free sublinear system:

Theorem 6. (Green's duality theorem for curves) *If W is base-point-free inside $H^0(X,L)$, then*

$$K_{p,q}(X,\mathcal{F},L,W) \cong K_{\dim(W)-2-p,2-q}(X,K_X\otimes\mathcal{F}^*,L,W)^*,$$

for all integers p and q.

The projection maps between Koszul cohomology groups arise naturally in this context. We suppose $x \in X$ is a point which is not a base point of L. It corresponds to a short exact sequence

$$0 \to W_x \to H^0(X,L) \to \mathbf{C}_x \to 0,$$

where $W_x = H^0(X, L-x)$. From the restricted Euler sequences corresponding to L, and $L-x$ respectively, we obtain an exact sequence:

$$0 \to M_{L-x} \to M_L \to \mathcal{O}_X(-x),$$

and further

$$0 \to \bigwedge^{p+1} M_{L-x} \otimes L \to \bigwedge^{p+1} M_L \otimes L \to \bigwedge^{p} M_{L-x} \otimes (L-x),$$

for any positive integer p. With the previous notation, M_{L-x} coincides to M_{W_x}. The exact sequence of global sections, together with the natural sequence

$$0 \to \bigwedge^{p+2} W_x \to \bigwedge^{p+2} H^0(X,L) \to \bigwedge^{p+1} W_x \to 0,$$

induce an exact sequence:

$$0 \to K_{p+1,1}(X,L,W_x) \to K_{p+1,1}(X,L) \stackrel{\mathrm{pr}_x}{\to} K_{p,1}(X,L-x),$$

where pr_x is the projection map centered in x we were talking about at the beginning.

In a similar way, if $y \in X$ is not a base point of $L - x$, we have a projection map centered in y

$$K_{p+1,1}(X, L, H^0(L - x)) \to K_{p,1}(X, L - y, H^0(L - x - y))$$

whose kernel equals $K_{p+1,1}(X, L, H^0(L - x - y))$.

Several other projection maps of same flavour can be obtained by playing with various Euler sequences. A fundamental property which they bear in commun is the *projection principle*: any non-zero element in a Koszul cohomology group survives when we project it from a generic point of X (see [A1] Lemma 2.5, and also [Ehb]). In particular, for the latter projection map, the same arguments as in [A1] Lemma 4.1 yield to the following:

Lemma 7. *Suppose L is non-special, and $x \in X$ is not a base point of L such that $K_{p,1}(X, L, H^0(L - x)) = 0$ for a positive integer $p \geq 1$. Then $K_{p+e,1}(X, L + E, H^0(L - x + E)) = 0$ for any effective divisor E of degree e on X.*

3. Proof of Conjecture 2 for generic curves of odd genus.

The main result of this section is the following.

Theorem 8. *Let X be a curve of odd genus $g = 2k + 1$, with $k \geq 1$, and maximal gonality $k + 2$. Then*

$$K_{k,1}(X, K_X + x + y, H^0(K_X)) = 0$$

for any two points $x, y \in X$.

Proof. The curve X being non-hyperelliptic, we know that $h^0(X, \mathcal{O}_X(x + y)) = 1$, and a simple analysis of the corresponding Koszul complexes yields to a natural identification:

$$K_{k,1}(X, K_X + x + y, H^0(K_X)) = K_{k,1}(X, x + y, K_X).$$

Green's duality theorem gives rise to an isomorphism $K_{k-j+1,j}(X, x + y, K_X) \cong K_{k+j-2,3-j}(X, -x-y, K_X)^*$ for any integer j. In particular, we obtain $K_{k-j+1,j}(X, x+y, K_X) = 0$ for $j \geq 3$. Therefore, as $K_{k-j+1,j}(X, x + y, K_X) = 0$ for $j \leq 0$, the non-vanishing Koszul groups among $K_{k-j+1,j}(X, x + y, K_X)$ could only be obtained for $j = 1$ or $j = 2$.

We compute first the dimension of $K_{k-1,2}(X, x + y, K_X)$. By Green's duality Theorem it equals the dimension of $K_{k,1}(X, -x - y, K_X)$.

At this end, we need an elementary Lemma, which holds in the most general context.

Lemma 9. *Let X be a compact complex manifold, and $D \neq 0$ be an effective divisor. Then, for any $L \in \mathrm{Pic}(X)$, any integer p, the kernel of the canonical map $K_{p,1}(X, -D, L) \to K_{p,1}(X, L)$ is isomorphic to $\bigwedge^{p+1} H^0(L - D)$.*

Proof. Let $V = H^0L$, and consider the graded SV-modules $B' = \bigoplus H^0(qL - D)$, $B = \bigoplus H^0(qL)$, and $C = B/B'$, where the inclusion of B' in B is given by the multiplication with the non-zero section of $\mathcal{O}_X(D)$ vanishing along D. Obviously, $B'_0 = 0$, and $C_0 \cong \mathbf{C}$. The long cohomology sequence for syzygies (cf. [Gr1], Corollary 1.d.4) yields an exact sequence:

$$0 \to \mathrm{Ker}\left(\bigwedge^{p+1} V \to \bigwedge^{p} V \otimes C_1\right) \to K_{p,1}(X, -D, L) \to K_{p,1}(X, L) \to ...$$

The aim is to prove that

$$\mathrm{Ker}\left(\bigwedge^{p+1} V \to \bigwedge^{p} V \otimes C_1\right) \cong \bigwedge^{p+1} H^0(L - D).$$

For this, choose a basis $\{w_1, ..., w_N\} \subset V$, such that $\{w_1, ..., w_s\}$ is a basis of $H^0(L - D)$, and pick an element $\alpha = \sum_{1 \le i_1 < ... < i_{p+1} \le N} \alpha_{i_1 ... i_{p+1}} w_{i_1} \wedge ... \wedge w_{i_{p+1}} \in \bigwedge^{p+1} V$. It belongs to $\mathrm{Ker}\left(\bigwedge^{p+1} V \to \bigwedge^p V \otimes C_1\right)$ if and only if the following relations are satisfied, for any $1 \le k_1 < ... < k_p \le N$:

$$\sum_{k \notin \{k_1, ..., k_p\}} (-1)^{\#\{k_i < k\}} \alpha_{k_1 ... k ... k_p} w_k \in H^0(L - D).$$

In particular, for any $1 \le k_1 < ... < k_p \le N$, and $k > s$, $\alpha_{k_1 ... k ... k_p} = 0$, in other words all $\alpha_{i_1 ... i_{p+1}}$ with $i_{p+1} > s$ vanish, so α belongs to $\bigwedge^{p+1} H^0(L - D)$.

We return to the proof of Theorem 8. Thanks to Voisin's solution to Green's conjecture for generic curves of odd genus, and to the main result of [HR], we know that $K_{k,1}(X, K_X)$ vanishes. Lemma 9, applied for $D = x + y$, and $L = K_X$ gives us an isomorphism $\bigwedge^{k+1} H^0(K_X - x - y) \cong K_{k,1}(X, -x - y, K_X)$. Therefore, the dimension of $K_{k,1}(X, -x - y, K_X)$, and thus of $K_{k-1,2}(X, x + y, K_X)$, equals the binomial coefficient $\binom{2k-1}{k+1}$ (use once again the hypothesis that X is non-hyperelliptic). What is left from the proof is a combinatorial computation. Analysing the Koszul complex which computes $K_{k,1}(X, x + y, K_X)$, one can prove:

Lemma 10. *The Euler characteristic of the complex*

$$0 \to \bigwedge^{k+1} H^0(K_X) \otimes H^0(\mathcal{O}_X(x + y)) \to \bigwedge^{k} H^0(K_X) \otimes H^0(K_X + x + y) \to$$

$$\to ... \to \bigwedge^{k-j+1} H^0(K_X) \otimes H^0(jK_X + x + y) \to ...$$

equals the binomial coefficient $\binom{2k-1}{k+1}$.

The proof of this Lemma is left to the reader. By the facts we exposed above, Lemma 10 concludes the proof of Theorem 8.

Gathering together Corollary 5, Lemma 7 and Theorem 8, and observing that $H^0(X, K_X) \cong H^0(X, K_X + x)$, for any point $x \in X$, we are lead to:

Corollary 11. *Conjecture 2 holds for any curve of odd genus and maximal gonality.*

The case of elliptic curves follows directly from Green's $K_{p,1}$-Theorem [Gr1].

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AUTHOR'S ADDRESSES:

Institute of Mathematics "Simion Stoilow", Romanian Academy, P.O. Box 1-764, RO-70700 Bucharest, Romania
email: Marian.Aprodu@imar.ro
&
Université de Grenoble 1, Laboratoire de Mathématiques, UMR 5582 du CNRS, Institut Fourier, BP 74, F-38402 Saint Martin d'Hères Cedex, France
email: aprodu@mozart.ujf-grenoble.fr