GENERALIZED DANIELEWSKI SURFACES

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ABSTRACT. In this paper we study a class of normal affine surfaces with $\mathbb{C}_+$-action which contains in particular the Danielewski surfaces in $\mathbb{A}^3$ given by the equations $x^n z = P(y)$, where P is a nonconstant polynomial with simple roots. We call them generalized Danielewski surfaces. Using the ideas of [5] we describe these surfaces as fiber bundles over an affine line with an r-fold origin. Then we give some explicit embeddings of such surfaces into an affine space. Finally we characterize those surfaces with a trivial Makar-Limanov invariant.

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Introduction

The Makar-Limanov $ML(V)$ of a normal affine surface $V = \mathrm{Spec}(B)$ is defined as the intersection in B of all the invariant rings of $\mathbb{C}_+$-action on V. In view of the well known correspondence between algebraic $\mathbb{C}_+$-action on V and locally nilpotent derivation of the $\mathbb{C}$-algebra B, $ML(V)$ is the intersection of all kernels of locally nilpotent derivations of B. We say that V has a trivial Makar-Limanov invariant (or that V is an *ML-surface*) if $ML(V) = \mathbb{C}$. In [9] and [10] Makar-Limanov calculated $ML(V)$ for all Danielewski surfaces $V = V_{P,n}$, given by the equations $x^n z = P(y)$ in $\mathbb{A}^3_{\mathbb{C}}$ ([3]). In particular, he established the following.

Theorem. *A Danielewski surface $V = V_{P,n}$ has a trivial Makar-Limanov invariant if and only if $n = 1$ and $\deg(P) \geq 1$.*

In the same papers, Makar-Limanov also calculates the automorphism group $\mathrm{Aut}(V_{P,n})$ of a Danielewski surface $V_{P,n}$. In case $n = 1$, the symmetry between x and z implies that the group $\mathrm{Aut}(V_{P,1})$ is 'big'. More generally, if a normal affine surface V has a trivial Makar-Limanov invariant then it follows from [7] (see also [1] and [4]) that the group $\mathrm{Aut}(V)$ has a Zariski dense orbit with finite complement.

In this paper we introduce a class of normal affine surfaces which generalizes the nonsingular Danielewski surfaces.

Definition. A normal affine surface $V = \mathrm{Spec}(B)$ with a $\mathbb{C}_+$-action is called a generalized Danielewski surface (a *GDS* for short) if the following conditions hold.

1) The algebra $B^{\mathbb{C}_+}$ of invariant functions is a polynomial ring in one variable.

2) The quotient morphism $q : V \to Z = \mathrm{Spec}\left(B^{\mathbb{C}_+}\right) \simeq \mathbb{A}^1_{\mathbb{C}}$ has reduced fibers.

3) There exists a closed point $z_0 \in Z$ such that $q^{-1}(Z \setminus \{z_0\})$ is equivariantly isomorphic to $(Z \setminus \{z_0\}) \times \mathbb{A}^1_{\mathbb{C}}$ with $\mathbb{C}_+$ acting by translations on the second factor.

The ordinary nonsingular Danielewski surfaces satisfy this definition. Indeed, such a surface $V = V_{P,n}$ admits a nontrivial algebraic $\mathbb{C}_+$-action defined by the triangular locally nilpotent $\mathbb{C}[x]$-derivation of $\mathbb{C}[x,y,z]$

$$\partial := x^n \partial_y + P'(y)\,\partial_z \tag{0.1}$$

which annihilates $x^n z - P(y)$. The inclusion of algebras $\mathbb{C}[x] = \mathbb{C}[V]^{\mathbb{C}_+} = \mathrm{Ker}(\partial) \subset \mathbb{C}[V]$ yields a quotient morphism $q : V \to Z = \mathrm{Spec}\left(\mathbb{C}[V]^{\mathbb{C}_+}\right) \simeq \mathbb{A}^1_{\mathbb{C}}$. Over $Z_* = Z \setminus \{0\}$ the morphism

$$\begin{array}{ccc} Z_* \times \mathbb{A}^1 & \to & V \\ (x,y) & \mapsto & \left(x, y, x^{-1}P(y)\right) \end{array}$$

restricts to an equivariant isomorphism $q^{-1}(Z_*) \simeq Z_* \times \mathbb{A}^1$, where $\mathbb{C}_+$ acts by translations on the second factor. If $\deg(P) = 1$ then $V_{P,n}$ is isomorphic to $\mathbb{A}^2$. Otherwise the fiber $F_0 = q^{-1}(0)$ is reduced, reducible, consisting of $\deg(P) \geq 2$ components.

The paper is divided as follows. After recalling in section 1 some generalities on $\mathbb{C}_+$-actions on normal affine surfaces, we first describe in section 2, using the ideas of [5], a *GDS* V as a fiber bundle $V \xrightarrow{\pi} X$ over an affine line X with an r-fold origin. In order to understand morphisms between *GDS*'s we introduce in section 3 a particular class of affine modifications

[8] which we call fibered modifications. This leads to the following description of morphisms between GDS's (see theorem 3.8).

Theorem 0.1. *Every equivariant morphism $\beta : V \to \tilde{V}$ between two GDS's V and $\tilde{V}$ factors into a sequence of equivariant fibered modifications followed by an equivariant open embedding.*

In section 4 we explain how to construct a GDS from the data consisting of a rooted tree Γ with a weight function w. This is just a function which assign a complex number $w(e)$ to every node e of Γ. The GDS's $V_{\Gamma,w}$ constructed by our procedure come naturally embedded in an affine space $\mathbb{A}^m_{\mathbb{C}}$. Moreover these GDS's $V_{\Gamma,w} \subset \mathbb{A}^m_{\mathbb{C}}$ admit canonical $\mathbb{C}_+$-actions induced by certain explicit $\mathbb{C}_+$-actions on the ambient space $\mathbb{A}^m_{\mathbb{C}}$.

Then in section 5 we prove the following theorem (see theorem).

Theorem 0.2. *For every GDS V there exists a weighted rooted tree $\Gamma_w = \Gamma_w(V)$ such that V is is isomorphic to $V_{\Gamma,w}$.*

Finally in section 5 we determine which GDS's have a trivial Makar-Limanov invariant. Theorem 6.3 gives a criterion for that in terms of the fiber bundle $\pi : V \to X$. Then, using the constructions of sections 3 and 4, we compute equivariant affine embeddings of a GDS with a trivial Makar-Limanov invariant. We obtain the following description (theorem 6.12).

We let $P_1, \ldots, P_n \in \mathbb{C}[T]$ be a sequence of polynomials with simple roots, one of these roots, say $\lambda_{i,1}$, $1 \leq i \leq n$, being distinguished. For every $1 \leq i \leq n$ we let

$$\tilde{P}_i(T) = (T - \lambda_{i,1})^{-1} P_i(T),$$

and we consider the affine variety

$$V_{P_1,\ldots,P_n} \subset \mathbb{A}^{n+2}_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}[X_0, X_1, \ldots, X_{n+1}]\right)$$

with equations

$$\begin{cases} X_0 X_{l+1} & = \displaystyle\prod_{i=1}^{l-1} \tilde{P}_i(X_i) P_l(X_l) \quad \text{for } 1 \leq l \leq n \\ (X_{k-1} - \lambda_{k-1,1}) X_{l+1} & = X_k \displaystyle\prod_{i=k}^{l-1} \tilde{P}_i(X_i) P_l(X_l) \quad \text{for } 2 \leq k \leq l \leq n \end{cases}$$

where, by convention, $\displaystyle\prod_{i=k}^{l-1} \tilde{P}_i(X_i) = 1$ if $k > l-1$.

Theorem 0.3. *A normal affine surface V is a GDS with a trivial Makar-Limanov invariant if and only if there exist $n = n(V) \geq 1$ nonzero polynomials $P_1, \ldots, P_n$ as above such that $V \simeq V_{P_1,\ldots,P_n}$.*

Actually there exists a canonical $\mathbb{C}_+$-action α on $\mathbb{A}^{n+2}_{\mathbb{C}}$ which leaves $V_{P_1,\ldots,P_n}$ invariant so that the isomorphism of the theorem is equivariant. Moreover there also exists a second canonical $\mathbb{C}_+$-action α' on $\mathbb{A}^{n+2}_{\mathbb{C}}$, which leaves $V_{P_1,\ldots,P_n}$, such that the general orbits of α and α' on $V_{P_1,\ldots,P_n}$ do not coincide.

1. Fibrations and $\mathbb{C}_+$-actions on normal affine surfaces

Here we recall some generalities on $\mathbb{C}_+$-actions in the form appropriate to our needs. The reader may also consult [2] for the properties of affine rulings on a normal rational surface.

Definition 1.1. We let Z be a connected integral prescheme over $\mathbb{C}$, with function field $K = \mathbb{C}(Z)$. We let V be a prescheme over Z with a surjective structural morphism $q : V \to Z$.

1) We say that $q : V \to Z$ is an $\mathbb{A}^1$-bundle if there exists a Zariski open covering of Z by affine open subsets Z_i, $i \in I$ such that $V_i = V \times_Z Z_i$ is isomorphic to $\mathbb{A}^1_{Z_i} = Z_i \times \mathbb{A}^1$.

2) We say that $q : V \to Z$ is an $\mathbb{A}^1$-fibration if the generic fiber $V_\eta = V \times_Z \mathrm{Spec}(K)$ of q is isomorphic to $\mathbb{A}^1_K$.

We assume further that V admits a nontrivial action of the additive group-scheme $\mathbb{G}_{a,Z}$, *i.e.* $\mathbb{C}_+ = \left(\mathbb{A}^1_{\mathbb{C}}, +\right)$ acts on V by Z-automorphisms.

3) We say that $q : V \to Z$ is a $\mathbb{G}_a$-bundle if there exist an affine open covering $Z = \bigcup_{i \in I} \mathrm{Spec}(A_i)$ and regular functions $f_i \in A_i \setminus \{0\}$ such that $V_i = V \times_Z \mathrm{Spec}(A_i)$ is equivariantly isomorphic to $\mathbb{A}^1_{A_i} = \mathrm{Spec}(A_i[U_i])$ where $\mathbb{G}_a(A_i) = \mathrm{Spec}(A_i[T])$ acts by the A_i-algebra morphism

$$\begin{array}{ccc} A_i[U_i] & \to & A_i[U_i] \otimes_{A_i} A_i[T] \\ U_i & \mapsto & U_i \otimes 1 + 1 \otimes f_i T \end{array}. \tag{1.1}$$

4) We say that $q : V \to Z$ is a $\mathbb{G}_a$-fibration if $p_2 : V_\eta = V \times_Z \mathrm{Spec}(K) \to \mathrm{Spec}(K)$ is a $\mathbb{G}_a$-bundle.

1.2. If V is a normal affine surface then a surjective morphism $q : V \to Z$ to a nonsingular integral affine Z curve is an $\mathbb{A}^1$-fibration if and only if there exists an affine open subset $U \subset Z$ such that $V \times_Z U$ is isomorphic to $\mathbb{A}^1_U$. This implies in particular that the general fiber of an $\mathbb{A}^1$-fibration $q : V \to Z$ is isomorphic to the affine line $\mathbb{A}^1_{\mathbb{C}}$. A fiber $F_z = q^{-1}(z)$, over a closed point $z \in Z$, which is either singular, reducible or non-reduced is called *degenerate*. It is known (see [11, Lemma 1.4.2 and 1.4.4 p. 196]) that a degenerate fiber of q is a disjoint union of affine rational curves with one place at infinity. Thus a connected component of a degenerate fiber of q is isomorphic to $\mathbb{A}^1$ provided it is nonsingular.

Given a connected normal affine surface $V = \mathrm{Spec}(B)$ with a nontrivial $\mathbb{C}_+$-action α, it is known (see *e.g.* [5, Lemma 1.1]) that the algebra of invariants $A = B^{\mathbb{C}_+}$ is finitely generated and that the natural quotient morphism $q : V \to Z = \mathrm{Spec}(A)$ is a $\mathbb{G}_a$-fibration over a nonsingular curve. We say that $q : V \to Z$ is the *quotient $\mathbb{G}_a$-fibration associated to the action α on V*. The following lemma is well-known (see e.g. [12] and [5, lemma 1.1]).

Lemma 1.3. *If V is a normal affine surface then the following conditions are equivalent.*

1) There exists an $\mathbb{A}^1$-fibration $q : V \to Z$ over a nonsingular affine curve Z.

3) There exists a nontrivial $\mathbb{C}_+$-action on V with associated quotient $\mathbb{G}_a$-fibration $q : V \to Z$.

Definition 1.4. For a normal affine surface $V = \mathrm{Spec}(B)$ with a nontrivial $\mathbb{C}_+$-action α given by a locally nilpotent derivation $\partial \in LND(V)$ and an invariant subscheme C with defining ideal $I_C \subset B$, the fixed point order $\mu = \mu(\alpha, C)$ along C is defined as the maximal $n \in \mathbb{N}$ such that $\partial(B) \subset I_C^n B$.

Note that this definition is equivalent to definition 1.3 of [5] which evokes the vector field associated to the $\mathbb{C}_+$-action. The following lemma describes the structure of V in a neighbourhood of a reduced, connected fiber of q.

Lemma 1.5. (see [5, lemma 1.2]) *We let $q : V = Spec\,(B) \to Z = Spec\,(A)$ be the quotient $\mathbb{G}_a$-fibration associated to a nontrivial $\mathbb{C}_+$-action α on a normal affine surface V. We let $z \in Z$ be a closed point and $h \in \mathcal{O}_{Z,z}$ be a generator of the maximal ideal $\mathfrak{m}_{Z,z}$.*

If the fiber F_z is reduced and connected then there exists an affine neighbourhood $U \subset Z$ of z, such that $h \in \mathcal{O}_Z(U)$and $V \times_Z U$ is equivariantly isomorphic to $U \times \mathbb{A}^1$with $\mathbb{C}_+$ acting by

$$t \star (z,u) := \left(z, u + th^{\mu(\alpha,F_z)}(z)\right).$$

If V is a normal affine surface then it follows from lemma 1.3 that for every $\mathbb{A}^1$-bundle $q : V \to Z$ over a nonsingular affine curve Z there exists a nontrivial $\mathbb{C}_+$-action on V such that $q : V \to Z$ is its associated quotient $\mathbb{G}_a$-fibration. By virtue of lemma 1.5 above every such $\mathbb{G}_a$-fibration is in fact a $\mathbb{G}_a$-bundle. Thus the following lemma holds.

Lemma 1.6. *For a normal affine surface V the following conditions are equivalent.*

(1) There exists an $\mathbb{A}^1$-bundle $q : V \to Z$ over a nonsingular affine curve Z.

(2) There exists a nontrivial $\mathbb{G}_{a,Z}$-action on V such that $q : V \to Z$ is a $\mathbb{G}_a$-bundle with respect to this action.

1.7. If $q : V \to Z$ is a $\mathbb{G}_a$-bundle over an affine curve Z then, by definition there exists an open covering $Z = \bigcup_{i \in I} Z_i$ by open subsets $Z_i = \mathrm{Spec}\,(A_i)$ and regular function $f_i \in A_i$ such that $V_i = V \times_Z Z_i$ is equivariantly isomorphic to $\mathbb{A}^1_{Z_i}$ with $\mathbb{C}_+$ acting by $t \star (z,u) = (z, u + tf_i(z))$. Letting $Z_{ij} = Z_i \cap Z_j$ it is clear that f_i and f_j must coincide on Z_{ij}. Thus the collection $(Z_i, f_i)_{i \in I}$ is a local data for an effective Cartier divisor on Z.

Lemma 1.8. *If Pic Z is trivial then every $\mathbb{A}^1$-bundle is trivial.*

Proof. We let $q : V \to Z$ be an $\mathbb{A}^1$-bundle. It follows from lemma 1.6 that we can find a $\mathbb{G}_{a,Z}$-action αon V such that $q : V \to Z$ is a $\mathbb{G}_a$-fibration. Then the discussion above tells us that such $\mathbb{G}_a$-fibration corresponds to an effective Cartier divisor on Z, with local data $(Z_i, f_i)_{i \in I}$, whence to an invertible sheaf $\mathcal{L} \in \mathrm{Pic}\, Z$. As Pic Z is trivial we can find $f \in \mathbb{C}[Z]$ such that $f_i = f|_{Z_i}$. Letting ∂ be the locally nilpotent $\mathbb{C}[Z]$-derivation of which $\mathbb{C}[V]$ which corresponds to α, this means that $\partial(\mathbb{C}[V]) \subset f \cdot \mathbb{C}[V]$. Thus $\tilde{\partial} = f^{-1}\partial$ is again locally nilpotent, corresponding to a free $\mathbb{G}_{a,Z}$-action $\tilde{\alpha}$ on V. Then $q : V \to Z$ is a principal $\mathbb{G}_{a,Z}$-bundle, whence is trivial as Z is affine. □

2. Generalized Danielewski Surfaces

Definition and examples of GDS**.** In what follows a base scheme $Z = \mathrm{Spec}\,(A) \simeq \mathbb{A}^1_{\mathbb{C}}$ and a closed point $z_0 \in Z$ are fixed. We let $h \in A$ be a generator of the maximal ideal $\mathfrak{m}_{Z,z_0} \subset \mathcal{O}_{Z,z_0}$ which does not vanish on $Z_* := Z \setminus \{z_0\} \simeq \mathrm{Spec}\,(A_h)$. In particular h is a generator of A as a $\mathbb{C}$-algebra.

We introduce the following generalization of nonsingular Danielewski Surfaces (compare with definition) .

Definition 2.1. By a *generalized Danielewski surface over Z* (GDS, in brief) we mean a normal affine surface $V \xrightarrow{q} Z$ whose structure morphism q is an $\mathbb{A}^1$-fibration, with reduced fibers, which restricts to an $\mathbb{A}^1$-bundle over Z_*.

2.2. It follows from lemma 1.3 that a GDS over $Z \simeq \mathbb{A}^1_{\mathbb{C}}$ can be equivalently defined as a normal affine surface which admits a nontrivial $\mathbb{C}_+$-action α whose associated quotient $\mathbb{G}_a$-fibration $q : V \to Z$ has reduced fibers and at most one degenerate fiber $F_{z_0} = q^{-1}(z_0)$. Notice that a GDS V comes equipped with a nontrivial $\mathbb{G}_{a,Z}$-action α which restricts to a free action on $V_* = V \times_Z Z_*$. Indeed such an action exists by lemma 1.8, as $Z_* \simeq A^1_* = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ is factorial. We will use the notation (V, α) or (V, α, q) whenever it is necessary to underline a particular $\mathbb{G}_{a,Z}$-action α as above.

Definition 2.3. We denote by $GDS_{/Z}$ the category whose objects are GDS's (V, α) over Z and whose morphisms are equivariant Z-morphisms.

2.4. Clearly $GDS_{/Z}$ is a subcategory of the category $\mathrm{Sch}_{/Z}$ of schemes over Z. If (V, α) and (V', α') are 2 elements of $\mathrm{Ob}\left(GDS_{/Z}\right)$ then we will denote a $Z-$morphism between V and V by $\beta : V \to V'$, whereas the notation $\beta : (V, \alpha) \to (V', \alpha')$ will mean that β is equivariant, that is $\beta \in \mathrm{Mor}\left(GDS_{/Z}\right)$.

2.5. Given $(V, \alpha, q) \in \mathrm{Ob}\left(GDS_{/Z}\right)$, we have $\mu(\alpha, F_z) = 0$ for every fiber $F_z = q^{-1}(z)$, $z \in Z_*$, as α restricts to a free action on V_*. We let $\mu(\alpha) := \mu(\alpha, F_{z_0})$ be the fixed point order along the fiber F_{z_0} of q. Letting $C_1, \ldots, C_r$ be the connected components of F_{z_0} we have $\mu(\alpha) = \min_{1 \leq i \leq r} \{\mu(\alpha, C_i)\}$.

Example 2.6. Using the quotient $\mathbb{G}_a$-fibration $q : V_{P,n} \to Z := \mathrm{Spec}(\mathbb{C}[x])$ associated to the locally nilpotent derivations $\partial_{P,n}$ of (0.1) we see that a nonsingular Danielewski surface is a GDS over Z.

By slightly modifying an example of [1] we obtain the following example of a GDS which is not an ordinary Danielewski surface.

Example 2.7. We let $V \subset \mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, u])$ be the nonsingular surface with equations

$$\left\{ \begin{array}{rcl} xz & = & y(y-1) \\ yu & = & z^2 \\ xu & = & (y-1)z \end{array} \right. .$$

The first projection $pr_x : \mathbb{A}^4 \to \mathbb{A}^1$, $(x, y, z, u) \mapsto x$ restricts to an $\mathbb{A}^1$-fibration $q_1 : V \to Z_1 = \mathrm{Spec}(\mathbb{C}[x])$ with reduced fibers. For every $z \in Z_* := Z_1 \setminus \{0\}$ the fiber $F_z = q^{-1}(z)$ is connected, whence V is a GDS by corollary 1.8. The fiber F_0 is degenerate since it has two connected components, namely $C_1 = \{x = 0;\ y = 1\} \cap V$ and $C_2 = \{x = 0;\ y = 0;\ z = 0\} \cap V$. The triangular $\mathbb{C}[x]$-derivation

$$\partial_1 = x^2 \partial_y + x(2y - 1)\partial_z + \left(xz + 2y^2 - 3y + 1\right)\partial_u$$

of $\mathbb{C}[x, y, z, u]$ annihilates the ideal I_V of V and thus induces a nontrivial $\mathbb{C}_+$-action α_1 on V with associated quotient $\mathbb{G}_a$-fibration $q_1 : V_1 \to Z_1$.

Similarly the triangular $\mathbb{C}[u]$-derivation

$$\partial_2 := \left((y-1)u + 2z^2\right)\partial_x + 2uz\partial_y + u^2 \partial_z$$

of $\mathbb{C}[x, y, z, u]$ induces a nontrivial $\mathbb{C}_+$-action α_2 on V whose associated quotient $\mathbb{G}_a$-fibration $q_2 : V \to Z_2 \simeq \mathbb{A}^1$ coincides with the restriction to V of the projection $pr_u : \mathbb{A}^4 \to \mathbb{A}^1$, $(x, y, z, u) \mapsto u$.

Since the general orbits of α_1 and α_2 do not coincide it follows that V has a trivial Makar-Limanov invariant. By [9] and [10], a nonsingular Danielewski Surface has a trivial Makar-Limanov invariant if and only if it is of the form $V_{P,1}$, that is, isomorphic to the hypersurface $xz = P(y) \subset \mathbb{A}^3$, where P is a nonconstant polynomial with simple roots. In this situation, it follows from [9] that, up to the action of $Aut(V_{P,1})$ on $LND(V_{P,1})$, every locally nilpotent derivation $\delta \in LND(V_{P,1})$ is conjugated to one of the form

$$\Delta_R = R(x)\left(x\partial_y + P'(y)\partial_z\right)$$

for some polynomial $R \in \mathbb{C}[x]$. In particular the quotient $\mathbb{G}_a$-fibration associated to every $\mathbb{C}_+$-action on $V_{P,1}$ has reduced fibers. In our situation, we observe that the unique degenerate fiber $q_2^{-1}(0)$ of the $\mathbb{G}_a^1$-fibration $q_2 : V \to Z_2$ associated to α_2 is not reduced, whence V is not a Danielewski surface.

The Fieseler presentation of a ***GDS*****.**
If the fiber $F_{z_0} = q^{-1}(z_0)$ of $(V, \alpha, q) \in \mathrm{Ob}\left(GDS_{/Z}\right)$ is not degenerate then it follows from lemmas 1.8 and 1.5 that V is equivariantly Z-isomorphic to $\mathbb{A}^1_Z$ with $\mathbb{G}_{a,Z}$ acting by

$$t \star (z, u) = \left(z, u + th^{\mu(\alpha)}(z)\right).$$

If F_{z_0} is degenerate then we let $C_1, \ldots, C_r$, $r \geq 2$, be the connected components of F_{z_0}. We denote by $\mu_i = \mu(\alpha, C_i)$, $1 \leq i \leq r$ the fixed point order along C_i. By lemmas 1.5 and 1.8 again we know that V_* is equivariantly Z-isomorphic to $\mathbb{A}^1_{Z_*} \simeq Z_* \times \mathbb{A}^1$ with $\mathbb{C}_+$ acting by translation on the second factor (equivalently $q|_{V_*} : V_* \to Z_*$ is the structure morphism of a $\mathbb{C}_+$-principal bundle) .

2.8. Following [5] we let $X \overset{p}{\to} Z$ be the prescheme obtained from Z by replacing the point z_0 by as many points x_i, $1 \leq i \leq r$, as there are connected components in F_{z_0}. More precisely X is obtained by gluing r copies $p_i : X_i \overset{\sim}{\to} Z$ of Z by the identity along $X_{i,*} := p_i^{-1}(Z_*)$. As $Z \simeq \mathbb{A}^1_{\mathbb{C}}$ it follows that X is isomorphic to the affine line with an r-fold origin $\{x_1, \ldots, x_r\}$. We obtain a factorization

$$q : V \overset{\pi}{\to} X \overset{p}{\to} Z$$

such that $\pi : V \to X$ is a $\mathbb{G}_a$-bundle. More precisely, as $X_i \simeq \mathbb{A}^1_{\mathbb{C}}$ we conclude again from lemmas 1.8 and 1.5 that $V_i := V \times_X X_i$ is equivariantly X_i-isomorphic to $\mathbb{A}^1_{X_i}$ where $\mathbb{C}_+$ acts by $t \star (x, u) := (x, u + t(h^{\mu_i} \circ p)(x))$. Letting $\tau_i : V_i \overset{\sim}{\to} \mathbb{A}^1_{X_i}$, $1 \leq i \leq r$, be a set of equivariant trivializations, we finally arrive at an equivariant Z-isomorphism

$$\tau : (V, \alpha, q) \overset{\sim}{\to} \bigsqcup_{i=1}^{r} X_i \times \mathbb{A}^1_{\mathbb{C}} / \sim \tag{2.1}$$

with the identification $\sim$ given by the transition isomorphisms $\phi_{ij} = \tau_j \circ \tau_i^{-1}$ over $X_{ij} = X_i \cap X_j \simeq Z_*$. We can identify X_i and $X_{i,*} \cup \{x_i\}$ so that $C_i = \tau^{-1}\left(\{x_i\} \times \mathbb{A}^1_{\mathbb{C}}\right)$.

Remark 2.9. A consequence of the description above is that a GDS is always an irreducible, nonsingular affine surface.

Definition 2.10. Given $(V,\alpha)\in \mathrm{Ob}\left(GDS_{/Z}\right)$, an equivariant Z-isomorphism

$$\tau:(V,\alpha)\xrightarrow{\sim}\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$$

as above is called a *Fieseler presentation of* (V,α).

2.11. Clearly the isomorphisms $\phi_{ij}:X_{ij}\times\mathbb{A}^1_{\mathbb{C}}\xrightarrow{\sim}X_{ij}\times\mathbb{A}^1_{\mathbb{C}}$ are of the form

$$(x,u)\mapsto\left(x,h^{\mu_j-\mu_i}\left(p\left(x\right)\right)u+g_{ij}\left(p\left(x\right)\right)\right)\tag{2.2}$$

for some regular functions $(g_{ij})_{1\leq i,j\leq r}\in A_h=\mathbb{C}\left[Z_*\right]$ which satisfy the cocycle relations

$$g_{ii}=0\qquad g_{ik}=h^{\mu_k-\mu_j}g_{ij}+g_{jk}\tag{2.3}$$

for every $1\leq i,j,k\leq r$. We say that the $(g_{ij})_{1\leq i,j\leq r}$ are the *transition functions* associated to the given Fieseler presentation of (V,α).

Any other equivariant trivialization $\tau_i':V_i\xrightarrow{\sim}\mathbb{A}^1_{X_i}$ differs from τ_i by an X_i-automorphism θ_i of $\mathbb{A}^1_{X_i}$ which is thus of the form $(x,u)\mapsto(x,a_i\left(p\left(x\right)\right)u+b_i\left(p\left(x\right)\right))$ for some regular functions $a_i\in A^*$ and $b_i\in A$. Therefore, for any other Fieseler presentation

$$\tau':(V,\alpha)\xrightarrow{\sim}\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$$

with associated transition functions $\left(g'_{ij}\right)_{1\leq i,j\leq r}\in A_h$, the relations

$$a_i=a_j\quad\text{and}\quad g'_{ij}-a_jg_{ij}=b_j-h^{\mu_j-\mu_i}b_i,\tag{2.4}$$

$1\leq i,j\leq r$, must hold in A^* and A_h respectively.

Conversely the following proposition tells us which collections of transition functions $(g_{ij})_{1\leq i,j\leq r}\in A_h$ correspond to GDS's.

Proposition 2.12. ([5, Proposition 1.4]) *We let μ_i, $1\leq i\leq r$ be natural numbers and we let $(g_{ij})_{1\leq i,j\leq r}\in A_h$ be a collection of transition functions which satisfy the cocycle relations (2.3). Then the Z-prescheme*

$$\tilde{V}=\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$$

associated to $(g_{ij})_{1\leq i,j\leq r}$ by mean of identification (2.2) is affine if and only if

$$n_{ij}=-ord_{z_0}\left(g_{ij}\right)>0\ \textit{for every}\ i\neq j,\ 1\leq i,j\leq r.$$

The index of a ***GDS*****.**

2.13. For every $n \in \mathbb{N}$ we let $\mathbb{A}^1_Z(n)$ be the triple $\left(\mathbb{A}^1_Z, \alpha(n), pr_Z\right)$, where $\alpha(n)$ denotes the $\mathbb{C}_+$-action $t \star (z,u) := (z, u + th^n(z))$ on $\mathbb{A}^1_Z$. Clearly $\mathbb{A}^1_Z(n)$ is an object of $GDS_{/Z}$ and it follows from lemma 1.5 that there is no other $\mathbb{C}_+$-action α on $\mathbb{A}^1_Z$ such that $\left(\mathbb{A}^1_Z, \alpha, pr_Z\right)$ belongs to $Ob\left(GDS_{/Z}\right)$. Obviously there exists a morphism $\mathbb{A}^1(m) \to \mathbb{A}^1(n)$ in $GDS_{/Z}$ if and only if $n \geq m$ and, given some coordinates z and u on $\mathbb{A}^1_Z$, every such morphism is of the form

$$\theta_{m,n} : (z,u) \mapsto \left(z, a(z)\, h^{n-m}(z)\, u + b(z)\right) \tag{2.5}$$

for some regular functions $a \in A^*$ and $b \in A$. If $(V,\alpha) \in Ob\left(GDS_{/Z}\right)$ is isomorphic to $\mathbb{A}^1_Z$ then there exists a unique integer $n_V(\alpha)$ such that (V,α) is equivariantly isomorphic to $\mathbb{A}^1_Z(n_V(\alpha))$.

2.14. Given $(V,\alpha) \in Ob\left(GDS_{/Z}\right)$ non isomorphic to $\mathbb{A}^1_Z$ we consider a Fieseler presentation $\tau : (V,\alpha) \xrightarrow{\sim} \bigsqcup_{i=1}^{r} X_i \times \mathbb{A}^1_{\mathbb{C}}/\sim$ of (V,α), with associated transition functions $(g_{ij})_{1 \leq i,j \leq r} \in A_h$. We let

$$n_V(\alpha) := \max_{1 \leq i,j \leq r} \{n_{ij} + \mu_j\} = \max_{1 \leq i,j \leq r} \{n_{ji} + \mu_i\}, \tag{2.6}$$

where $\mu_i = \mu\left(\alpha, \{x_i\} \times \mathbb{A}^1\right)$ and $n_{ij} = -ord_{z_0}(g_{ij})$. As $V \not\simeq \mathbb{A}^1_Z$ it follows from proposition 2.12 that $n_{ij} > 0$ for every $i \neq j$, $1 \leq i,j \leq r$, whence $n_V(\alpha) \in \mathbb{Z}_{>0}$. We deduce from the relation (2.4) that $n_V(\alpha)$ does not depend on the choice of a particular Fieseler presentation of (V,α). The integer $n_V = n_V(\alpha) - \mu(\alpha)$ is called the *index* of the GDS (V,α). This is an invariant of the $\mathbb{A}^1$-bundle $\pi : V \to X$.

Lemma 2.15. *Given* $(V,\alpha) \in Ob\left(GDS_{/Z}\right)$*, there exists a morphism* $\beta : (V,\alpha) \to \mathbb{A}^1_Z(n)$ *if and only if* $n \geq n_V(\alpha)$*.*

Proof. If $V \simeq \mathbb{A}^1_Z$ then β is of the form $\theta_{n(V,\alpha),n}$ and the lemma is proved. We now suppose that $V \not\simeq \mathbb{A}^1_Z$ and we suppose that $\beta : (V,\alpha) \to \mathbb{A}^1_Z(n)$ is a morphism. We let $\tau : (V,\alpha) \xrightarrow{\sim} \bigsqcup_{i=1}^{r} X_i \times \mathbb{A}^1_{\mathbb{C}}/\sim$ be a Fieseler presentation of (V,α), with associated transition functions $(g_{ij})_{1 \leq i,j \leq r} \in A_h$. Letting $\mu_i = \mu\left(\alpha, \{x_i\} \times \mathbb{A}^1\right)$, the $\mathbb{C}_+$-action α restricts to $V_i = V \times_X X_i$ in such a way that

$$(p \times I_d) \circ \tau_i : (V_i, \alpha|_{V_i}) \xrightarrow{\tau_i} \left(\mathbb{A}^1_{X_i}(\mu_i)\right) \xrightarrow{p \times I_d} \mathbb{A}^1_Z(\mu_i)$$

is an isomorphism for every $1 \leq i \leq r$. Thus $n \geq \mu_i$ for every $1 \leq i \leq r$ and β induces a morphism

$$\tilde{\beta}_i := \beta \circ \tau_i^{-1} : \mathbb{A}^1_{X_i}(\mu_i) \to \mathbb{A}^1_Z(n), (x,u) \mapsto \left(p(x), a_i(p(x))\, h^{n-\mu_i} u + b_i(p(x))\right),$$

for some regular functions $a_i \in A^*$ and $b_i \in A$. Moreover, for every $1 \leq i,j \leq r$ the compatibility relations

$$a_i = a_j \quad \text{and} \quad b_i = a_j h^{n-\mu_j} g_{ij} + b_j \tag{2.7}$$

hold in A^* and A_h respectively. Since b_i and b_j are regular functions on Z it follows that $n-\mu_j-n_{ij}\geq 0$ for every $1\leq i,j\leq r$. Thus $n\geq n_V(\alpha)=\max_{1\leq i,j\leq r}\{n_{ij}+\mu_j\}$.

Conversely, given any $n\geq n_V(\alpha)$ we can choose some regular functions $b_i\in A$ such that the second relation of (2.7) holds for every $1\leq i,j\leq r$. Then

$$\begin{array}{rcl}\tilde{\beta}:\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim & \rightarrow & \mathbb{A}^1_Z(n)\\ X_i\times\mathbb{A}^1\ni(x,u) & \mapsto & \left(p(x),h(p(x))^{n-\mu_i}u+b_i(p(x))\right)\end{array}$$

is an equivariant morphism, whence $\beta=\tilde{\beta}\circ\tau:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n)$ is a morphism in $GDS_{/Z}$. □

Given 2 morphisms $\beta:(V,\alpha)\rightarrow(V',\alpha')$ and $\beta':(V',\alpha')\rightarrow\mathbb{A}^1_Z(n(V',\alpha'))$ in $GDS_{/Z}$ it is clear that $\beta'\circ\beta:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n(V',\alpha'))$ is a morphism too. This implies the following corollary.

Corollary 2.16. *If $\beta:(V,\alpha)\rightarrow(V',\alpha')$ in a morphism in $GDS_{/Z}$ then*

$$n_{V'}(\alpha')\geq n_V(\alpha).$$

Remark 2.17. If $V\not\simeq\mathbb{A}^1_Z$ then the proof of lemma 2.15 shows that for every morphism $\beta:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n)$ the image $\beta(F_{z_0})\subset\{z_0\}\times\mathbb{A}^1_{\mathbb{C}}\subset\mathbb{A}^1_Z$ is a finite set. Moreover this set contains more than one element if and only if $n=n_V(\alpha)$.

3. Generalized Danielewski surfaces and affine modifications

For every $(V,\alpha)\in Ob\left(GDS_{/Z}\right)$ the definition implies that $V_*=V\times_Z Z_*$ is equivariantly isomorphic to $\mathbb{A}^1_{Z_*}\simeq Z_*\times\mathbb{A}^1$ with $\mathbb{C}_+$ acting by translation on the second factor. Being equivariant, every morphism $\beta:(V,\alpha)\rightarrow\left(\tilde{V},\tilde{\alpha}\right)$ must restrict to an isomorphism over Z_*. Thus every morphism $\beta\in\operatorname{Mor}\left(GDS_{/Z}\right)$ is birational, whence an affine modification.

In what follows we will use freely the results of the theory of affine modification, referring the reader to [8] for complete proofs. We should mention however that we will frequently use the following geometrical definition of an affine modification.

Definition 3.1. Suppose we are given an affine scheme $X=\operatorname{Spec}(A)$, an effective principal divisor $D=div(f)$ and subscheme $Y\subset V$ such that $Y\subset Supp(D)$. We let $I_Y\subset A$ be the defining ideal of Y. By the affine modification of X along D with center at Y (or, shortly, with the locus (Y,D)) we mean the affine modification of A with the locus (I_Y,f) in the sense of [8, Definition 1.1].

A consequence of lemma 2.15 is that every GDS (V,α,q) is an equivariant affine modification of the affine plane $\mathbb{A}^1_Z(n)$, $n\geq n(V,\alpha)$ along a divisor D supported on $\{z_0\}\times\mathbb{A}^1_{\mathbb{C}}$ and with center at a subscheme $Y\subset D$. In general such a modification is quite complicated to understand. However we will show that a morphism $\beta:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n)$ can be decomposed into a succession of simple affine modifications.

Definition 3.2. An affine modification $\sigma:V\rightarrow\mathbb{A}^2_{\mathbb{C}}=\operatorname{Spec}(\mathbb{C}[x,y])$ along the principal divisor $D_{x_0}=\operatorname{div}(x-x_0)$ with center at a reduced subscheme $Y\subset D$ is called a *simple fibered modification* (or, shortly, an SFM).

Remark 3.3. In suitable coordinates every simple fibered modification is an affine modification of $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[x,y\right]\right)$ with the locus $(I,f) = \left(\left(x, S\left(x,y\right)\right), x\right)$, where $S\left(x,y\right)$ is a nonconstant polynomial such that $P\left(y\right) = S\left(0,y\right)$ is either 0 or a polynomial with simple roots. If $P=0$ then $Y=D$, whence $\sigma : V \rightarrow \mathbb{A}^2_{\mathbb{C}}$ is a trivial affine modification . Otherwise $P \neq 0$ and V is isomorphic to the Danielewski surface $V_{P,1}$. Indeed we can write $S\left(x,y\right) = P\left(y\right) + xR\left(x,y\right)$ for a polynomial $R\left(x,y\right) \in \mathbb{C}\left[x,y\right]$. Since then $I = \left(x, P\left(y\right)\right)$we conclude that V embeds in $\mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[x,y,z\right]\right)$ as the affine surface with equation $xz - P\left(y\right) = 0$. The morphism $\sigma : V \rightarrow \mathbb{A}^2$ is induced by the projection $\mathbb{A}^3_{\mathbb{C}} \rightarrow \mathbb{A}^2_{\mathbb{C}}$, $(x,y,z) \mapsto (x,y)$. We shall mention that if the divisor $D = (f=0)$ is not reduced then description above does not hold. Indeed if $f = x^n$, $n \geq 2$ then by a result of Mauser-Jauslin and Freudenburg [6] the polynomials $S_1\left(x,y\right) = P\left(y\right)$ and $S_2\left(x,y\right) = P\left(y\right) + x$ give two non equivalent embeddings of the Danielewski surface $V_{P,n}$ in $\mathbb{A}^3_{\mathbb{C}}$.

Example 3.4. To illustrate how simple fibered modifications appear naturally as morphisms between generalized Danielewski surfaces we consider the surface $V \subset \mathbb{A}^4_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[x,y,z,u\right]\right)$ with equations

$$\left\{ \begin{array}{ccc} xz & = & y\left(y-1\right) \\ yu & = & zQ\left(z\right) \\ xu & = & \left(y-1\right)Q\left(z\right) \end{array} \right. ,$$

where $Q\left(z\right) = \prod_{i=1}^{r}\left(z - z_i\right) \in \mathbb{C}\left[z\right]$ is a nonconstant polynomial with simple roots. Similarly as in example 2.7 we see that the restriction to V of the projection $\mathbb{A}^4_{\mathbb{C}} \rightarrow \mathbb{A}^1_{\mathbb{C}}$, $(x,y,z,u) \mapsto x$ is an $\mathbb{A}^1$-fibration $q : V \rightarrow Z = \mathrm{Spec}\left(\mathbb{C}\left[x\right]\right)$ with reduced fibers. The degenerate fiber $F_0 = q^{-1}\left(0\right)$ has $r+1$ connected components, namely $C_1 := \{x=0;\ y=1\} \cap V$ and $C_{2,i} := \{x=0;\ y=0;\ z=z_i\} \cap V$, $1 \leq i \leq r$. Thus V is a GDS over Z . Letting V' the Danielewski Surface with equation $xz = y\left(y-1\right)$ in $\mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[x,y,z\right]\right)$ we see that the restriction to V of the projection $\mathbb{A}^4_{\mathbb{C}} \rightarrow \mathbb{A}^3_{\mathbb{C}}$, $(x,y,z,u) \mapsto (x,y,z)$ induces a morphism $\beta : V \rightarrow V'$.

We can cover V' by the open subsets $V'_1 := \left(\left(y \neq 0\right) \cup \left(x \neq 0\right)\right) \cap V'$ and $V'_2 := \left(y \neq 1\right) \cup \left(x \neq 0\right) \cap V'$. These 2 open subsets are both isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ with coordinates $x, t_0 = \left(y-1\right)/x$ and $x, t_1 = y/x = z/\left(y-1\right)$ respectively. Clearly $\beta : V \rightarrow V'$ restricts to a trivial affine modification $\beta^{-1}\left(V'_1\right) \xrightarrow{\sim} V'_1$. On V'_2, $z = t_1\left(xt_1 + 1\right)$ so that $\beta^{-1}\left(V'_2\right) \subset V$ is given by the equation $xu - \left(1 + xt_1\right)Q\left(t_1\left(xt_1+1\right)\right) = 0$ in $\mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[x,t_1,u\right]\right)$. We notice that in this coordinates $\beta : \beta^{-1}\left(V'_2\right) \rightarrow V'_2$ is just the restriction of the projection $\mathbb{A}^3_{\mathbb{C}} \rightarrow \mathbb{A}^2_{\mathbb{C}}$, $(x,t_1,u) \mapsto (x,t_1)$. By mean of Taylor's formula we can write $\left(1+xt_1\right)Q\left(t_1\left(xt_1+1\right)\right) = Q\left(t_1\right) + xR\left(x\right)$ and hence, the automorphism $(x,t_1,u) \mapsto (x,t_1,u - R\left(x\right))$ restricts to an isomorphism $V \simeq V_{Q,1}$. Thus $\beta : \beta^{-1}\left(V'_2\right) \rightarrow V'_2$ is a simple fibered modification along the divisor $D = (x=0)$ with center at the reduced subscheme $V\left(x, Q\left(t_1\right)\right) \subset D$.

The previous example leads us naturally to the following definition.

Definition 3.5. An affine modification $\sigma : W \rightarrow V$ of a nonsingular affine surface V is called a *fibered modification* if there exists an open covering $V = \bigcup_{i \in I} V_i$ such that $V_i \simeq \mathbb{A}^2_{\mathbb{C}}$ and

$$\sigma|_{\sigma^{-1}\left(V_i\right)} : \sigma^{-1}\left(V_i\right) \rightarrow V_i$$

is an SFM.

Definition 3.6. We say that a fibered modification $\sigma : V' \to V$ of $(V, \alpha) \in Ob\left(GDS_{/Z}\right)$ is equivariant if there exists a nontrivial $\mathbb{C}_+$-action α' on V' such that $(V', \alpha') \in Ob\left(GDS_{/Z}\right)$ and $\sigma \in \mathrm{Mor}\left(GDS_{/Z}\right)$.

Example 3.7. We let $\sigma : V \to \mathbb{A}^1_Z(n)$ be the simple fibered modification of $\mathbb{A}^1_Z = \mathrm{Spec}\left(\mathbb{C}[z][u]\right)$ along the divisor $D = \mathrm{div}(z)$ with center at a reduced proper subscheme $Y \subsetneq D$. Similarly as in remark 3.3 we can find a polynomial a polynomial with simple roots $P \in \mathbb{C}[u]$ such that V is isomorphic to the hypersurface of $\mathbb{A}^2_Z = \mathrm{Spec}\left(\mathbb{C}[z,u,v]\right)$with the equation $zv - P(u) = 0$. The restriction to V of the projection $pr_Z : \mathbb{A}^2_Z \to Z$ is an $\mathbb{A}^1$-fibration $q : V \to Z$ with reduced fibers. If $n = 0$ then the center $I = (z, P(u) + R(z,u))$ of σ is not $\mathbb{C}_+$-invariant. Hence there is no nontrivial $\mathbb{G}_{a,Z}$-action α on V such $\sigma : (V, \alpha) \to \mathbb{A}^1_Z(0)$ is equivariant. Otherwise $n \geq 1$ and the action $\alpha(n)$ on $\mathbb{A}^1_Z$ corresponds to the locally nilpotent $\mathbb{C}[z]$-derivation of $\mathbb{C}[z,u]$ given by

$$\partial(z) = 0 \; ; \; \partial(u) = z^n.$$

For every $m \in \mathbb{N}$ the triangular $\mathbb{C}[z]$-derivation ∂_m of $\mathbb{C}[z,u,v]$ given by

$$\partial_m(z) = 0 \; ; \; \partial_m(u) = z^{m+1} \; ; \; \partial_m(v) = z^m P'(u)$$

annihilates the ideal $I_V \subset \mathbb{C}[z,u,v]$ of V, whence induces a nontrivial $\mathbb{C}_+$-action on V. Clearly $\sigma : (V, \alpha) \to \mathbb{A}^1_Z(n)$ is equivariant if and only if α is induced by the derivation ∂_{n-1}. Moreover we observe that $\mu\left(\alpha, q^{-1}(0)\right) = n - 1$.

We will prove the following theorem.

Theorem 3.8. *Every morphism $\beta : (V, \alpha) \to \left(\tilde{V}, \tilde{\alpha}\right)$ in $GDS_{/Z}$ factors (in $GDS_{/Z}$) into a sequence of equivariant fibered modifications followed by an equivariant open embedding.*

Proof. Let us assume for a moment that we have already proved the assertion of the theorem for the particular morphisms $\beta : (V, \alpha) \to \mathbb{A}^1_Z(n)$, where $n \geq n_V(\alpha)$.

We can thus suppose that $\left(\tilde{V}, \tilde{\alpha}\right)$ is not isomorphic to $\mathbb{A}^1_Z\left(n_V(\tilde{\alpha})\right)$. We let

$$\tilde{\tau} : \left(\tilde{V}, \tilde{\alpha}\right) \xrightarrow{\sim} \bigsqcup_{i \in I} X_i \times \mathbb{A}^1_{\mathbb{C}} / \sim$$

be a Fieseler presentation of $\left(\tilde{V}, \tilde{\alpha}\right)$. Letting $I_0 := \left\{ i \in I, \; (\tilde{\tau} \circ \beta)^{-1}\left(\{x_i\} \times \mathbb{A}^1_{\mathbb{C}}\right) \neq \emptyset \right\}$ we see that

$$V_0 = \tilde{V} \times_X \bigcup_{i \in I_0} X_i = \tilde{\tau}^{-1}\left(\bigsqcup_{i \in I_0} X_i \times \mathbb{A}^1_{\mathbb{C}} / \sim\right)$$

is a $\mathbb{C}_+$-invariant open open subset of $\tilde{V}$. Then $(V_0, \alpha_0) := (V_0, \tilde{\alpha}|_{V_0})$ is a GDS and β factors as

$$\beta : (V, \alpha) \xrightarrow{\phi} (V_0, \alpha_0) \xrightarrow{i} \left(\tilde{V}, \tilde{\alpha}\right),$$

where $(V_0, \alpha_0) \xrightarrow{i} (V', \alpha')$ is the equivariant open embedding induced by the open inclusion $V_0 \subset \tilde{V}$. For every $i \in I_0$ we let $V_i = (\tilde{\tau} \circ \phi)^{-1}\left(X_i \times \mathbb{A}^1_{\mathbb{C}}\right)$. Then, as ϕ is equivariant, $(V_i, \alpha|_{V_i})$

is a GDS and β restricts to a morphism

$$\beta_i : (V_i, \alpha|_{V_i}) \stackrel{\phi\circ\tilde{\tau}}{\longrightarrow} \mathbb{A}^1_{X_i}\left(\alpha_0\left(\{x_i\}\times\mathbb{A}^1_{\mathbb{C}}\right)\right) \simeq \mathbb{A}^1_Z\left(\alpha_0\left(\{x_i\}\times\mathbb{A}^1_{\mathbb{C}}\right)\right).$$

By our assumption every such morphism can be factored as a succession of equivariant fibered modification and hence, the theorem is proved. □

To complete the proof of theorem 3.8 it sufficient to show that every morphism $\beta : (V,\alpha) \to \mathbb{A}^1_Z(n)$ factors as a succession of equivariant fibered modifications. However we will prove the following stronger result.

Proposition 3.9. *If $\beta : (V,\alpha) \to \mathbb{A}^1_Z(n)$ is a morphism in $GDS_{/Z}$ then there exist $m = n-\mu(\alpha)$ GDS (V_k,α_k), $1\leq k\leq m$ such that β factors as*

$$\beta : (V,\alpha) = (V_m,\alpha_m) \stackrel{\sigma_m}{\to} (V_{m-1},\alpha_{m-1}) \stackrel{\sigma_{m-1}}{\to} \ldots \stackrel{\sigma_2}{\to} (V_1,\alpha_1) \stackrel{\sigma_1}{\to} (V_0,\alpha_0) = \mathbb{A}^1_Z(n),$$

where for every $1\leq k\leq m$, $\sigma : (V_k,\alpha_k)\to(V_{k-1},\alpha_{k-1})$ is an equivariant fibered modification.

Proof. A morphism $\mathbb{A}^1_Z(n-1)\to\mathbb{A}^1_Z(n)$ is a simple Danielewski modification as it is given by $(z,u')\mapsto\left(z,h^{n-n'}(z)\,u'\right)$ in suitable coordinates. It is thus sufficient to prove the assertion for a morphism $\beta : (V,\alpha)\to\mathbb{A}^1_Z(n_V(\alpha))$. If $(V,\alpha)\simeq\mathbb{A}^1_Z(n_V(\alpha))$ then β is an isomorphism and the proposition is proved.

We now suppose that $(V,\alpha)\not\simeq\mathbb{A}^1_Z(n_V(\alpha))$. It follows from proposition 2.12 and lemma 2.15 that $n_V(\alpha)>0$. The remark 2.17 tells us the image $\beta(F_{z_0})$ is a finite subset of $\{z_0\}\times\mathbb{A}^1_{\mathbb{C}}$ which contains at least 2 elements. We let $\sigma_1 : V_1\to V_0$ be the simple fibered modification of $V_0=\mathbb{A}^1_Z$ along the divisor $D=\{z_0\}\times\mathbb{A}^1_{\mathbb{C}}$ with center at the unique reduced affine subscheme $Y\subset D$ supported on $\beta(F_{z_0})\subset V_0$. We let $q_1 : V_1\to Z$ be the $\mathbb{A}^1$-fibration which lifts the morphism $pr_Z : \mathbb{A}^1_Z\to Z$. As $n_V(\alpha)>0$ it follows from example 3.7 that there exists a unique $\mathbb{G}_{a,Z}$-action α_1 on V_1 lifting $\alpha(n_V(\alpha))$ such that $(V_1,\alpha_1,q_1)\in\mathrm{Ob}\left(GDS_{/Z}\right)$ and $\beta_1 : (V_1,\alpha_1)\to\mathbb{A}^1_Z(n_V(\alpha))$ is an equivariant morphism. Moreover $\mu\left(\alpha_1,q_1^{-1}(z_0)\right)=n_V(\alpha)-1$.

By virtue of the universal property of affine modifications (see [8, Proposition 2.1]) the morphism $\beta : (V,\alpha)\to\mathbb{A}^1_Z(n)$ lifts to a morphism $\phi_1 : (V,\alpha)\to(V_1,\alpha_1)$. If

$$\tau_1 : (V_1,\alpha_1)\stackrel{\sim}{\to}\bigsqcup_{i=1}^{r} X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$$

is a Fieseler presentation of (V_1,α_1) then, for every $1\leq i\leq r$, the $\mathbb{C}_+$-invariant open subset $V_{1,i}=\tau_1^{-1}\left(X_i\times\mathbb{A}^1_{\mathbb{C}}\right)$ of V_1 is equivariantly isomorphic to $\mathbb{A}^1_Z(n_V(\alpha)-1)$.

We come across the following alternative : either $\phi_1 : \left(\phi_1^{-1}(V_{1,i}),\alpha\right)\to(V_{1,i},\alpha_1)$ is an isomorphism or we may apply the same procedure as previously. By doing this where ever it is possible we obtain an equivariant fibered modification $\sigma_2 : (V_2,\alpha_2,q_2)\to(V_1,\alpha_1,q_1)$ such that $\phi_1 : (V,\alpha)\to(V_1,\alpha_1)$ factors through a morphism $\phi_2 : (V,\alpha)\to(V_2,\alpha_2)$. Moreover we have $\mu\left(\alpha_2,q_2^{-1}(z_0)\right)=\mu\left(\alpha_1,q_1^{-1}(z_0)\right)-1$.

Now the proof can be completed by induction since after exactly $n_V=n_V(\alpha)-\mu(\alpha)$ steps this procedure becomes stationnary. □

Definition 3.10. Given a morphism $\beta : (V,\alpha)\to\left(\tilde{V},\tilde{\alpha}\right)$ and a factorisation

$$\beta : (V,\alpha) = (V_m,\alpha_m) \stackrel{\sigma_m}{\to} (V_{m-1},\alpha_{m-1}) \stackrel{\sigma_{m-1}}{\to} \ldots \stackrel{\sigma_2}{\to} (V_1,\alpha_1) \stackrel{\sigma_1}{\to} (V_0,\alpha_0) \stackrel{i}{\hookrightarrow} \left(\tilde{V},\tilde{\alpha}\right)$$

as in theorem 3.8, we call the integer m *the length of the factorisation.* We say that a factorisation is *minimal* if its length is minimal among all such factorisations.

3.11. It is clear from the construction above that every factorisation of a morphism $\beta : (V,\alpha) \to \mathbb{A}^1_Z(n)$ has length at least $n - \mu(\alpha)$. During the proof of theorem 3.9 we have constructed a particular factorisation

$$\beta : (V,\alpha) = (V_m,\alpha_m) \overset{\sigma_m}{\to} (V_{m-1},\alpha_{m-1}) \overset{\sigma_{m-1}}{\to} \ldots \overset{\sigma_2}{\to} (V_1,\alpha_1) \overset{\sigma_1}{\to} (V_0,\alpha_0) = \mathbb{A}^1_Z(n_V(\alpha))$$

of a morphism $\beta : (V,\alpha) \to \mathbb{A}^1_Z(n_V(\alpha))$. We recall that if $(V,\alpha) \not\simeq \mathbb{A}^1_Z(n_V(\alpha))$ then this factorisation is obtained by performing simple fibered modification where ever it is possible each step $1 \leq k \leq m-1$. This is clearly a minimal factorisation, whence $m = n_V(\alpha) - \mu(\alpha) = n_V$.

Remark 3.12. We let $\tau : (V,\alpha,q) \overset{\sim}{\to} \bigsqcup_{i=1}^{r} X_i \times \mathbb{A}^1_{\mathbb{C}}/\sim$ be a Fieseler presentation of (V,α) with associated transition functions $(g_{ij})_{1\leq i,j\leq r} \in A_h$. As usual we let $n_{ij} = -ord_{z_0}(g_{ij}) > 0$ and $\mu_i = \mu(\alpha, \{x_i\} \times \mathbb{A}^1)$. The morphisms $\beta_k := \sigma_{k+1} \circ \ldots \circ \sigma_m : V = V_m \to V_k$ enjoy the following properties we will need later on.

1) The images $\beta_k(\tau^{-1}(\{x_i\} \times \mathbb{A}^1_{\mathbb{C}}))$ and $\beta(\tau^{-1}(\{x_i\} \times \mathbb{A}^1_{\mathbb{C}}))$ are disjoint if and only if

$$n_V(\alpha) - \mu_j - n_{ij} \leq k$$

and equal iff $n_V(\alpha) - \mu_j - n_{ij} > k$.

2) The image $\beta_k(\tau^{-1}(\{x_i\} \times \mathbb{A}^1_{\mathbb{C}}))$ is a curve iff $k \geq n_V(\alpha) - \mu_i$.

4. Generalized Danielewski surface associated to a weighted rooted tree

In this part we explain how to construct a GDS from a weighted rooted tree.

Preliminaries on rooted trees.

4.1. We recall that a tree is a connected acyclic graph Γ. We let $N(\Gamma)$ be the set of *nodes* (or vertices) of Γ, whereas $E(\Gamma)$ denotes the set of *edges* of Γ. A *rooted tree* is a tree Γ together with a distinguished node $e_0 \in V(\Gamma)$ which we call the *root* of Γ. The nodes to which e_0 is connected by one edge are called *children* of e_0 and conversely e_0 is called the *parent* of those nodes. Each of the children of e_0, in turn, may have one, many or no children. However every node has always exactly one parent (except of course the root node e_0). The set of children of a given node $e \in N(\Gamma)$ is denoted by $\mathrm{Child}_\Gamma(e)$ (or simply $\mathrm{Child}(e)$), whereas $\mathrm{Par}_\Gamma(e)$ (or simply $\mathrm{Par}(e)$) denotes the parent of e. A node with no children is referred to as a *leaf* and we denote by $\mathrm{Leaf}(\Gamma)$ the sets of those nodes. The *degree* $\deg_\Gamma(e)$(or simply $\deg(e)$) of a node $e \in N(\Gamma)$ is the number of its children.

We say that a node e' is an *ancestor* of a given node e, and we write $e \preceq e'$, if e' belongs to the shortest path in Γ from e to the root e_0. We will also use the notation $e \prec e'$ when e' is an ancestor of e distinct from e itself. We denote by $\mathrm{Anc}_\Gamma(e)$ (or simply $\mathrm{Anc}(e)$) the set of ancestors of e. Given two nodes e and e' the *first common ancestor* of e and e' is the unique minimal element of the set $\mathrm{Anc}(e) \cap \mathrm{Anc}(e') \subset N(\Gamma)$ for the partial order $\preceq$on $N(\Gamma)$.

For every $l \in \mathbb{N}$ we let $N_l(\Gamma) \subset N(\Gamma)$ be the set of nodes e which are at distance l from the root (*i.e* nodes who have exactly l edges lying between them and the root). We say that the nodes in $N_l(\Gamma)$ are at *level* l. The *height* $H(\Gamma)$ of Γ is the maximal distance between the nodes and the root.

Definition 4.2. Given a tree Γ rooted in e_0 and a node $e \in N(\Gamma)$ we define Γ^e as the subtree of Γ with nodes $N(\Gamma^e) = \{f \in N(\Gamma),\ e \in \mathrm{Anc}_\Gamma(f)\}$ and with edges $E(\Gamma^e)$ consisting of all the edges in $E(\Gamma)$ which connect two given nodes in $N(\Gamma^e)$. We call Γ^e the *maximal subtree of* Γ *rooted in* e.

Definition 4.3. Given a tree Γ rooted in $e_0 \in N(\Gamma)$, a *fine weight function* on Γ (with values in $\mathbb{C}$) is a function $w : N(\Gamma) \to \mathbb{C}$ such that $w(e) \neq w(e')$ whenever e and e' share the same parent. A rooted tree Γ together with a fine weight function $w : N(\Gamma) \to \mathbb{C}$ is called a *weighted rooted tree* and is denoted by Γ_w.

For every $e \in N(\Gamma)$ we let Γ_w^e be the maximal subtree of Γ rooted in e, together with the fine weight function induced by $w : N(\Gamma) \to \mathbb{C}$.

Affine scheme associated to a weighted rooted tree.

In this section we explain how to construct an affine scheme $V_{\Gamma,w}$ from the data consisting of a weighted rooted tree Γ_w.

Definition 4.4. Given a tree Γ rooted in $e_0 \in N(\Gamma)$ we let W be the the vector space over $\mathbb{C}$ with basis $(X_e)_{e\in N(\Gamma)}$ and we let $\mathbb{C}[N(\Gamma)]$ be the symmetric algebra of W. We denote by I_S the ideal of $\mathbb{C}[N(\Gamma)]$ generated by the polynomials $S_{e,e'} = X_e - X_{e'}$ whenever e and e' share the same parent and we let $\mathbb{C}[\Gamma] = \mathbb{C}[N(\Gamma)]/I_S$.

4.5. By definition $\mathbb{C}[N(\Gamma)]$ is a polynomial ring in $\mathrm{Card}(N(\Gamma))$ variables. Using the same symbol to denote the variable $X_e \in \mathbb{C}[N(\Gamma)]$ and its image in $\mathbb{C}[\Gamma]$ we see that $X_e = X_{e'}$ in $\mathbb{C}[\Gamma]$ if and only if e and e' share the same parent, whence $\mathbb{C}[\Gamma]$ is a polynomial ring in

$$d(\Gamma) = 1 + \sum_{i=1}^{H(\Gamma)} \mathrm{Card}\left(N_{i-1}(\Gamma) \setminus \mathrm{Leaf}(\Gamma)\right)$$

variables.

Given a new symbol X_0 we let $A = \mathbb{C}[X_0]$ and $A[\Gamma] = A \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]$. We let $Z = \mathrm{Spec}(A) \simeq \mathbb{A}^1_{\mathbb{C}}$ and $z_0 \in Z$ be the closed point $(X_0) \in \mathrm{Spec}(A)$.

Definition 4.6. For every node $e \in N(\Gamma)\setminus\mathrm{Leaf}(\Gamma)$ we define the *sibling polynomial* $S_e(\Gamma_w) \in A[\Gamma]$ of e by the formula

$$S_e(\Gamma_w) = \prod_{e' \in \mathrm{Child}(e)} \left(X_e - w(e')\right) \in \mathbb{C}[X_e] \subset A[\Gamma].$$

As w is a fine weight function $S_e(\Gamma_w)$ is a polynomial of degree $\deg(e) \geq 1$ with simple roots. For every $f \in \mathrm{Child}(e)$ we let $S_e^{(f)}(\Gamma_w) = (X_e - w(f))^{-1} S_e(\Gamma_w)$. Similarly, if g is a child of e disctint from f then we let $S_e^{(f,f')}(\Gamma_w) = (X_e - w(g))^{-1}(X_e - w(f))^{-1} S_e(\Gamma_w)$.

Definition 4.7. For every $e \in N(\Gamma)$ the *root polynomial* $R_e(\Gamma_w) \in A[\Gamma]$ of the weighted subtree Γ_w^e (with respect to Γ_w) is defined by the recursive formula

$$R_e(\Gamma_w) := \begin{cases} 1 & \text{if } e = e_0 \\ S_{\mathrm{Par}(e)}^{(e)}\left(X_{\mathrm{Par}(e)}\right) R_{\mathrm{Par}(e)}(\Gamma_w) & \text{otherwise} \end{cases}.$$

4.8. Letting $\mathrm{Anc}(e) = \{e_{i+1} = e, e_i, \ldots, e_0\}$, it is clear that $R_e(\Gamma_w) \in \mathbb{C}[X_{e_0}, \ldots, X_{e_i}] \subset A[\Gamma]$. Moreover if $e' \in N(\Gamma)$ is an ancestor of e then $R_e(\Gamma_w) = R_{e'}(\Gamma_w) R_e\left(\Gamma_w^{e'}\right)$ as Γ_w^e is the maximal weighted subtree of $\Gamma_w^{e'}$ rooted in e.

4.9. For every $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ we let $Q_e(\Gamma_w) = S_e(\Gamma_w) R_e(\Gamma_w) \in A[\Gamma]$. We again notice that if $e' \in N(\Gamma)$ is an ancestor of e then

$$Q_e(\Gamma_w) = R_{e'}(\Gamma_w) Q_e\left(\Gamma_w^{e'}\right) \in A[\Gamma].$$

For every $e_1 \prec e_2 \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ we let $f_1 \in \mathrm{Child}(e_1)$ be the unique child of e_1 such that $e_2 \in N\left(\Gamma^{f_1}\right)$ and we let $f_2 \in \mathrm{Child}(e_2)$. Then we let

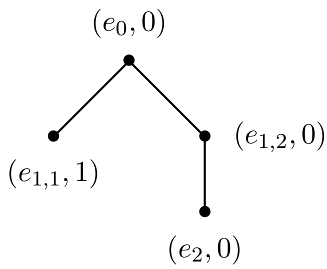
$$\tilde{\Delta}_{e_1,e_2}(\Gamma_w) = (X_{e_1} - w(f_1)) X_{f_2} - X_{f_1} Q_{e_2}\left(\Gamma_w^{f_1}\right) \in A[\Gamma]$$

and

$$\tilde{\Delta}_{0,e_2}((\Gamma_w)) = X_0 X_{f_2} - Q_{e_2}(\Gamma_w) \in A[\Gamma].$$

Definition 4.10. Given a weighted rooted tree Γ_w we let $I_{\Gamma,w} \subset A[\Gamma]$ be the ideal generated by the polynomials $\tilde{\Delta}_{0,e}(\Gamma_w)$, $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ and $\tilde{\Delta}_{e',e}(\Gamma_w)$, $e \prec e' \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$. Letting $B_{\Gamma,w} = A[\Gamma]/I_{\Gamma,w}$, we call $V_{\Gamma,w} = \mathrm{Spec}(B_{\Gamma,w}) \subset \mathbb{A}_Z^{d(\Gamma)}$ the *affine scheme associated to the weighted rooted tree* Γ_w.

Example 4.11. We consider the following weighted rooted tree Γ_w



Then $A[\Gamma] = \mathbb{C}[X_0, X_{e_0}, X_{e_1}, X_{e_2}]$ and $V_{\Gamma,w} \subset \mathbb{A}_Z^3$ is given by the equations

$$X_0 X_{e_1} - X_{e_0}(X_{e_0} - 1) = 0;\ X_0 X_{e_2} - (X_{e_1} - 1) X_{e_1} = 0;\ X_{e_0} X_{e_2} - X_{e_1}^2 = 0.$$

We recognize the *GDS* V of example 2.7.

Remark 4.12. We shall mention that the ideal $I_{\Gamma,w}\subset A[\Gamma]$ can be defined using a matrix as follows. For every $1\leq l\leq H(\Gamma)-1$ and every $e\in N_l(\Gamma)\setminus\mathrm{Leaf}(\Gamma)$ we let

$$M(e)=\begin{pmatrix} Q_e(\Gamma_w)\\ X_f\end{pmatrix}\in M_{2,1}(A[\Gamma]),$$

where $f\in\mathrm{Child}(e)$. If $N_l(\Gamma)\setminus\mathrm{Leaf}(\Gamma)=\{e_1,\ldots,e_{r_l}\}$ then we let

$$M_{\Gamma,w}(l)=(M(e_1),\ldots,M(e_{r_l}))\in M_{2,r_l}(A[\Gamma])$$

and

$$M(\Gamma_w)=(M_0,M_\Gamma(1),\ldots,M_\Gamma(H(\Gamma)-1))\in M_{2,d(\Gamma)}(A[\Gamma]),$$

where

$$M_0=\begin{pmatrix} X_0\\ 1\end{pmatrix}\in M_{2,1}(A[\Gamma]).$$

Then it is clear that

$$\tilde{\Delta}_{0,e}=\det\begin{pmatrix} X_0 & Q_e(\Gamma_w)\\ 1 & X_f\end{pmatrix}$$

for every $e\in N(\Gamma)\setminus\mathrm{Leaf}(\Gamma)$.

If $e_1,e_2\in N(\Gamma)\setminus\mathrm{Lea}(\Gamma)$ are two distinct nodes with first common ancestor $e_{12}\in N(\Gamma)$ then either $e_1\prec e_2$ (or $e_2\prec e_1$) or there exists two nodes $g_i\in N_1(\Gamma^{e_{12}})$ such that $e_i\in N(\Gamma^{g_i})$, $i=1,2$. In the first case we can suppose that $e_1\prec e_2$ and we let $g_1\in N_1(\Gamma^{e_2})$ be the unique node such that $e_1\in N(\Gamma^{g_1})$. Then $Q_{e_1}(\Gamma_w)=R_{e_2}(\Gamma_w)\,S_{e_2}^{(g_1)}Q_{e_1}\left(\Gamma_w^{(g_1)}\right)$. In the second case we see that $Q_{e_i}(\Gamma_w)=R_{e_{12}}(\Gamma_w)\,S_{e_{12}}^{(g_i)}(\Gamma_w)\,Q_{e_i}(\Gamma_w^{g_i})$. Thus the greatest common divisor of $Q_{e_1}(\Gamma_w)$and $Q_{e_2}(\Gamma_w)$ is the polynomial $G_{e_1,e_2}\in A[\Gamma]$ defined by

$$G_{e_1,e_2}=\begin{cases} R_{e_{12}}(\Gamma_w)\,S_{e_{12}}^{(g_1)}(\Gamma_w) & \text{if } e_1\prec e_2\\ R_{e_{12}}(\Gamma_w)\,S_{e_{12}}^{(g_1,g_2)}(\Gamma_w) & \text{otherwise}\end{cases}.$$

If $e_1\prec e_2\in N(\Gamma)\setminus\mathrm{Leaf}(\Gamma)$ then it is clear that

$$\tilde{\Delta}_{e_1,e_2}(\Gamma_w)=G_{e_1,e_2}^{-1}\det\begin{pmatrix} Q_{e_1}(\Gamma_w) & Q_{e_2}(\Gamma_w)\\ X_{g_1} & X_{f_2}\end{pmatrix},$$

where $f_2\in\mathrm{Child}(e_2)$. Thus the generators of $I_{\Gamma,w}$ can be obtained from certain 2×2 minors of the matrix $M(\Gamma_w)$ by *simplifying the common factors*.

We notice that if e_1 is not an ancestor of e_2 then

$$G_{e_1,e_2}^{-1}\det\begin{pmatrix} Q_{e_1}(\Gamma_w) & Q_{e_2}(\Gamma_w)\\ X_{f_1} & X_{f_2}\end{pmatrix}=Q_{e_1}(\Gamma_w^{g_1})\,\tilde{\Delta}_{e_{12},e_2}(\Gamma_w)-Q_{e_2}(\Gamma_w^{g_2})\,\tilde{\Delta}_{e_{12},e_1}(\Gamma_w)$$

so that we can even define $I_{\Gamma,w}$ as the ideal of $A[\Gamma]$ generated by all the simplified 2×2 minors of $M(\Gamma_w)$.

First properties of the schemes $V_{\Gamma,w}$.

This section is devoted to the proof of the following theorem.

Theorem 4.13. *For any weighted rooted tree Γ_w, the closed affine subscheme $V_{\Gamma,w}\subset\mathbb{A}_Z^{d(\Gamma)}$ is a GDS over Z.*

The proof is given in 4.14-4.19 below. To begin with, we will prove that $V_{\Gamma,w}$ admits an $\mathbb{A}^1$-fibration $q_{\Gamma,w};V_{\Gamma,w}\to Z$ with reduced fibers.

Lemma 4.14. *The inclusion $A \to B_{\Gamma,w}$ defines an $\mathbb{A}^1$-fibration $q_{\Gamma,w} : V_{\Gamma,w} \to Z = Spec\,(A)$ which restricts to the trivial $\mathbb{A}^1$-bundle over $Z_* = Spec\,(A_{X_0})$.*

Proof. For every $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ and $f \in \mathrm{Child}(e)$ we deduce from the polynomials $\tilde{\Delta}_{0,e}(\Gamma_w) \in I_{\Gamma,w}$ that $X_f = X_0^{-1} Q_e(\Gamma_w)$ in $B_{\Gamma,w} \otimes_A A_{X_0}$. As $Q_e(\Gamma_w)$ only involves the variables $X_{e'}$, $e' \in \mathrm{Anc}(e)$ we conclude that $B_{\Gamma,w} \otimes_A A_{X_0} \simeq A_{X_0}[X_{e_0}]$. Equivalently $V_{\Gamma,w} \times_Z Z_* \simeq \mathbb{A}^1_{Z_*}$ and hence, $q_{\Gamma,w} : V_{\Gamma,w} \to Z$ is an $\mathbb{A}^1$-fibration which restricts to the trivial $\mathbb{A}^1$-bundle over Z_*. □

In order to describe the fiber $F_{\Gamma,w} = V_{\Gamma,w} \times_Z \mathrm{Spec}\,(A/(X_0)) = q_{\Gamma,w}^{-1}(z_0)$ we need the following auxiliary lemma.

Lemma 4.15. *If $g \in N_1(\Gamma)$ then the A-algebra*

$$B_{\Gamma,w}(0,g) := A[\Gamma] / (X_0, X_{e_0} - w(g), I_{\Gamma,w})$$

is isomorphic to $A[\Gamma^g] / (X_0, I_{\Gamma^g,w})$.

Proof. It is clear that for every $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ the polynomial $\tilde{\Delta}_{0,e}(\Gamma_w) \in I_{\Gamma,w}$ reduces to $Q_e(\Gamma_w)$ modulo X_0. If $e \neq e_0$ then there exists a unique node $e_1 \in N_1(\Gamma)$ such that $e \in N(\Gamma^{e_1})$. Since then $Q_e(\Gamma_w) = S_{e_0}^{(e_1)}(X_{e_0})\, Q_e(\Gamma_w^{e_1})$ and $Q_e(\Gamma_w^{e_1})$ does not contain the variable X_{e_0}we conclude that

$$Q_e(\Gamma_w) \mod (X_{e_0} - w(g)) = \begin{cases} S_{e_0}^{(e_1)}(w(g))\, Q_e(\Gamma_w^{e_1}) & \text{if } e_1 = g \\ 0 & \text{otherwise} \end{cases}.$$

If $e \neq e_1$ then there exists a unique node $e_2 \in \mathrm{Child}(e_1)$ such that $e \in N(\Gamma_w^{e_2})$. If $e_1 \neq g$ and $f \in \mathrm{Child}(e)$ then $\tilde{\Delta}_{e_1,e}$ reduces to $(w(g) - w(e_1)) X_f - X_{e_1} Q_e(\Gamma_w^{e_2})$ modulo $(X_{e_0} - w(g))$ and $w(e_1) \neq w(g)$ as w is a fine weight function.

Thus, starting with the case $e = e_2$ we conclude by induction that every variable X_f, $f \in N(\Gamma) \setminus (\{e_0\} \cup N(\Gamma^g))$ can be expressed in $B_{\Gamma,w}(0,g)$ as a polynomial in X_g. Letting I_0 be the ideal of $A[\Gamma^g]$ generated by the polynomials $\tilde{\Delta}_{e_1,e_2}(\Gamma_w^g) = \tilde{\Delta}_{e_1,e_2}(\Gamma_w)$, $e_1 \prec e_2 \in N(\Gamma^g) \setminus (\mathrm{Leaf}(\Gamma^g))$ and $S_{e_0}^{(g)}(w(g))\, Q_e(\Gamma_w^g)$, $e \in N(\Gamma^g) \setminus (\mathrm{Leaf}(\Gamma^g))$, we conclude that

$$B_{\Gamma,w}(0,g) \simeq (\mathbb{C}[X_{e_0}] \otimes_{\mathbb{C}} A[\Gamma^g]) / (X_{e_0} - w(g), X_0, I_0) \simeq A[\Gamma^g] / (X_0, I_0)$$

as $\tilde{\Delta}_{e_1,e_2}(\Gamma_w^g)$ does not contain the variables X_0 and X_{e_0}. As $S_{e_0}^{(g)}(w(g)) \in \mathbb{C}^*$ the ideals $(X_0, I_{\Gamma^g,w})$ and (X_0, I_0) of $A[\Gamma^g]$ coincide and hence, $B_{\Gamma,w}(0,g) \simeq A[\Gamma^g] / (X_0, I_{\Gamma^g,w})$. □

Lemma 4.16. *The fiber $F_{\Gamma,w}$ is reduced and its connected components are in bijection with the leaves of Γ. More precisely if $Leaf(\Gamma) = \{e_1, \ldots, e_r\}$ then $F_{\Gamma,w}$ is the disjoint union of the curves $C_{e_j} \simeq \mathbb{A}^1_{\mathbb{C}}$ with defining ideals*

$$I_{\Gamma,w}(e_j) = \left(I_{\Gamma,w}, X_0, \left(X_{e_{j,i}} - w(e_{j,i+1})\right)_{1 \leq i \leq k_j}\right) \subset A[\Gamma],$$

where $Anc(e_j) = \left\{e_{j,k_j+1} = e_j, e_{j,k_j} \ldots, e_{j,1}, e_{j,0} = e_0\right\}$. Moreover X_{e_j} restricts to a coordinate function on C_{e_j}.

Proof. If $H(\Gamma)=0$ then $N(\Gamma)=\{e_0\}$, $V_{\Gamma,w}=\mathrm{Spec}\left(A\left[X_{e_0}\right]\right)\simeq\mathbb{A}^1_Z$ and $F_{\Gamma,w}\simeq\mathrm{Spec}\left(\mathbb{C}\left[X_{e_0}\right]\right)$ is reduced.

We now suppose that the assertion of the lemma is true for every weighted rooted tree of height $H<H(\Gamma_w)$. As $\tilde{\Delta}_{0,e_0}$ reduces to the polynomial with simple roots $Q_{e_0}(\Gamma_w)=S_{e_0}(\Gamma_w)$ modulo X_0 we conclude that

$$F_{\Gamma,w}\simeq\mathrm{Spec}\left(\prod_{g\in\mathrm{Child}(e_0)}B_{\Gamma,w}(0,g)\right).$$

Thus $F_{\Gamma,w}$ is reduced if and only if $F_{0,g}=\mathrm{Spec}\left(B_{\Gamma,w}(0,g)\right)$ is reduced for every $g\in N_1(\Gamma)$.

Letting $q_{\Gamma^g,w}:V_{\Gamma^g,w}:\longrightarrow Z$ be the $\mathbb{A}^1$-fibration of lemma 4.14 on $V_{\Gamma^g,w}$ we deduce from lemma 4.15 that $F_{0,g}$ is isomorphic to the fiber $F_{\Gamma^g,w}\subset V_{\Gamma^g,w}$ of $q_{\Gamma^g,w}$. Since Γ^g has height $H(\Gamma)-1$ we conclude from our induction hypothesis that $F_{0,g}$ is reduced, whence $F_{\Gamma,w}$ is reduced. The precise description of the connected components of $F_{\Gamma,w}$ then follows easily by induction. □

4.17. To achieve the proof of theorem 4.13 it sufficient to check that $V_{\Gamma,w}$ is a connected normal affine surface. Since $V_*=V_{\Gamma,w}\times_Z Z_*$ is isomorphic to $\mathrm{Spec}\left(A_{X_0}\left[X_{e_0}\right]\right)$ by virtue of lemma 4.14 we conclude that any section of $q_{\Gamma,w}$ over Z_* is uniquely determined by a Laurent polynomial $\sigma(X_0)\in A_{X_0}=\mathbb{C}\left[X_0,X_0^{-1}\right]$. Then for every $e\in N(\Gamma)$ the restriction of the variable $X_e\in A[\Gamma]$ to the image in V_* of the section σ defines a new Laurent polynomial $X_e(\sigma)\in A_{X_0}$.

Lemma 4.18. *We let $e=e_{i+1}\in Leaf(\Gamma)$, $Anc_\Gamma(e)=\{e_{i+1},\ldots,e_1,e_0\}$ and $\lambda\in\mathbb{C}$. Then for every $N\geq i+1$ there exists a section $\sigma(X_0)=\sum_{i=0}^{N}a_iX_0^i\in A$ of $q_{\Gamma,w}$ over Z_* such that the following assertions hold.*

1) For every $0\leq k\leq i$, $X_{e_k}(\sigma)=w(e_{k+1})+X_0P_{e_k}(X_0)$, where $P_{e_k}(X_0)=\sum_{i=1}^{N_k}b_{k,i}X_0^{i-1}\in A$.
2) If $k+i\leq N$ then $b_{k,i}\in\mathbb{C}\left[a_0,\ldots a_{k+i-1}\right]\left[a_{k+i}\right]$ is a polynomial of degree 1 in a_{k+i}.
3) $X_{e_{i+1}}(\sigma)=\lambda+X_0P_{e_{i+1}}(X_0)$, where $P_{e_{i+1}}(X_0)\in A$.
3) For every $e'\in N(\Gamma)\setminus Anc(e)$, $X_e(\sigma)\in A$.

Proof. Letting X_{e_0} be coordinate function on $V_*\simeq\mathrm{Spec}\left(A_{X_0}\left[X_{e_0}\right]\right)$ we see that $X_{e_0}(\sigma)=a_0+X_0\sum_{i=1}^{N}a_iX_0^{i-1}$. Thus assertions (1) and (2) hold provided we let $a_0=w(e_1)$.

As $\tilde{\Delta}_{0,e_0}(\Gamma_w)\in I_{\Gamma,w}$ we conclude that on V_*

$$\begin{aligned}X_{e_1}(\sigma)&=X_0^{-1}S_{e_0}\left(X_{e_0}(\sigma)\right)=S_{e_0}^{(e_1)}\left(X_{e_0}(\sigma)\right)\left(\sum_{i=1}^{N}a_iX_0^{i-1}\right)\\&=a_1S_{e_0}^{(e_1)}\left(w(e_1)\right)+X_0\sum_{i=1}^{N_1}b_{1,i}X_0^{i-1}\end{aligned}$$

by virtue of Taylor's Formula. Letting $P_{e_1}=\sum_{i=1}^{N_1} b_{1,i}X_0^{i-1}$ we see that $b_{1,i}\in\mathbb{C}\left[a_0,a_1,\ldots a_i\right]\left[a_{1+i}\right]$ is a linear polynomial in a_{i+1} for every $1+i\leq N$, as $\left(S_{e_0}^{(e_1)}\right)^{(l)}\left(w\left(e_1\right)\right)\neq 0$ for every $1\leq l\leq \deg_\Gamma\left(e_0\right)-1$. Finally, as $S_{e_0}^{(e_1)}\left(w\left(e_1\right)\right)\neq 0$ we can find $a_1\in\mathbb{C}$ such that $a_1S_{e_0}^{(e_1)}\left(w\left(e_1\right)\right)=w\left(e_2\right)$.

Let us assume that $a_0,\ldots,a_{k_0}$ have been chosen in such a way that

$$X_{e_k}\left(\sigma\right)=w\left(e_{k+1}\right)+X_0P_{e_k},$$

where $P_{e_k}=\sum_{i=1}^{N_k} b_{k,i}X_0^{i-1}\in A$ satisfies (1) and (2) for every $0\leq k\leq k_0\leq i$. Then the polynomial $\tilde{\Delta}_{0,e_{k_0}}\left(\Gamma_w\right)\in I_{\Gamma,w}$ tells us that, over V_*

$$X_{e_{k_0+1}}\left(\sigma\right)=P_{e_{k_0}}\prod_{j=0}^{k_0}S_{e_j}^{(e_{j+1})}\left(w\left(e_{j+1}\right)+X_0P_{e_j}\right).$$

Then we can again use Taylor's Formula to conclude that

$$X_{e_{k_0+1}}\left(\sigma\left(X_0\right)\right)=b_{k_0,1}\prod_{j=0}^{k_0}S_{e_j}^{(e_{j+1})}\left(w\left(e_{j+1}\right)\right)+X_0\sum_{i=1}^{N_{k_0+1}}b_{k_0+1,i}X_0^{i-1},$$

where, for every $k_0+i+1\leq N$, $b_{k_0+1,i}\in\mathbb{C}\left[b_{k_0,i},\ldots,b_{k_0,1},\left(b_{k,j}\right)_{k,j}\right]\left[b_{k_0,i+1}\right]$, $k+j<k_0+i+1$ is a linear polynomial in $b_{k_0,i+1}$. By induction hypothesis $b_{k_0,i+1}$ is linear polynomial in a_{k_0+i+1} whereas $b_{k,j}$ only contains the coefficients a_l for $l\leq k+j<k_0+i+1$. Thus $b_{k_0+1,i}$ is a linear polynomial in a_{k_0+i+1} for every $k_0+i+1\leq N$ and hence, $P_{e_{k_0+1}}=\sum_{i=1}^{N_{k_0+1}}b_{k_0+1,i}X_0^{i-1}$ satisfies (2). As $b_{k_0,1}=ca_{k_0+1}+d$, $c\in\mathbb{C}^*$, and $\prod_{j=0}^{k_0}S_{e_j}^{(e_{j+1})}\left(w\left(e_{j+1}\right)\right)\neq 0$ we can choose a_{k_0+1} in such a way that

$$\left(ca_{k_0+1}+d\right)\prod_{j=0}^{k_0}S_{e_j}^{(e_{j+1})}\left(w\left(e_{j+1}\right)\right)=\begin{cases} w_{e_{k_0+2}} & \text{if } k_0<i\\ \lambda & \text{if } k_0=i\end{cases}.$$

This proves (1) and (2) and (3). We also notice that the coefficients $a_0\left(v\right),\ldots,a_i\left(v\right)$ are in fact uniquely determined by connected component C_e of $F_{\Gamma,w}$ which contains v.

To prove (4) we observe that, over V_*, $X_{e'}\left(\sigma\right)=X_0^{-1}Q_{\mathrm{Par}_\Gamma(e')}\left(\Gamma_w\right)\left(\sigma\right)$ for every $e'\in N\left(\Gamma\right)\setminus\mathrm{Anc}_\Gamma\left(e\right)$. As $e'\notin\mathrm{Anc}\left(e\right)$ there exist a unique k, $0\leq k\leq i$ such that e_k is the first common ancestor of e_{i+1} and e'. Then $\left(X_{e_k}\left(\sigma\right)-w\left(e_{k+1}\right)\right)=X_0P_{e_k}$ divides $Q_{\mathrm{Par}_\Gamma(e')}\left(\sigma\right)$, whence $X_e\left(\sigma\right)\in A$. □

Proposition 4.19. *The scheme $V_{\Gamma,w}$ is a nonsingular connected affine surface.*

Proof. Lemma 4.18 tells us that for every closed point $v\in F_{\Gamma,w}$ there exists a section σ of $q_{\Gamma,w}$ over Z_* whose closure in $V_{\Gamma,w}$ meets $F_{\Gamma,w}$ in v. Thus none of the connected components of $F_{\Gamma,w}$ is an irreducible component of $V_{\Gamma,w}$. As $V_*=V_{\Gamma,w}\times_Z Z_*\simeq\mathbb{A}^1_{Z_*}$ we conclude that $V_{\Gamma,w}$ is irreducible and that $\dim_v V_{\Gamma,w}=2$ for every closed point $v\in V_{\Gamma,w}$.

We let $J_{\Gamma,w}$ be the Jacobian matrix of the system of equations $\tilde{\Delta}_{0,e}\left(\Gamma_w\right)=0$, $e\in N\left(\Gamma\right)\setminus\mathrm{Leaf}\left(\Gamma\right)$, and $\tilde{\Delta}_{e',e}\left(\Gamma_w\right)=0$, $e'\prec e\in N\left(\Gamma\right)\setminus\mathrm{Leaf}\left(\Gamma\right)$. As $\dim\left(A\left[\Gamma\right]\right)=d\left(\Gamma\right)+1$, it is sufficient to prove that $J_{\Gamma,w}$ has rank $d(\Gamma)-1$ at every closed point $v \in V_{\Gamma,w}$. Since $V_* \simeq \mathbb{A}^1_{Z_*}$ we may restrict our attention to the closed points $v \in F_{\Gamma,w}$. Due to lemma 4.16, we may suppose that v belongs to the component $C_{e_{i+1}} \subset V_{\Gamma,w}$ with equations $X_0 = 0$; $X_{e_j} = w(e_{j+1}), 1 \leq j \leq i$, where $e_{i+1} \in \mathrm{Leaf}(\Gamma)$ and $\mathrm{Anc}(e_{i+1}) = \{e_{i+1}, e_i \ldots, e_1, e_0\}$. For every integer j, $0 \leq j \leq i$, we consider the set

$$K_j = N(\Gamma^{e_j}) \setminus (\{e_j\} \cup N(\Gamma^{e_{j+1}}) \cup \mathrm{Leaf}(\Gamma^{e_j})) \subset N(\Gamma).$$

If $K_j \neq \emptyset$ then for every $e \in K_j$ there exists a unique $e' \in N_1(\Gamma^{e_j})$, $e' \neq e_{j+1}$, such that $e \in N\left(\Gamma^{e'}\right)$. If $f \in \mathrm{Child}(e)$ then $Q_e\left(\Gamma^{e'}_w\right)$ does not contains the variable $X_f \in A[\Gamma]$, whence

$$\frac{\partial \tilde{\Delta}_{e_j,e}(\Gamma_w)}{\partial X_f}(v) = w(e_{j+1}) - w(e') \neq 0$$

as w is a fine weight function. This implies that the rank of $J_{\Gamma,w}$ at v is at least $d(\Gamma) - i - 2$.

For every $0 \leq j \leq i$, $Q_{e_j}(\Gamma_w) = R_{e_j}(\Gamma_w) S_{e_j}\left(X_{e_j}\right)$ and $R_{e_j}(\Gamma_w)$ does not contains the variable X_{e_j}. Moreover $R_{e_j}(v) = \prod_{j=0}^{i} S_{e_j}^{(e_{j+1})}(w(e_{j+1})) \neq 0$ by construction. As S_{e_j} is a polynomial with simple roots, we conclude that

$$\frac{\partial \tilde{\Delta}_{0,e_j}(\Gamma_w)}{\partial X_{e_j}}(v) = R_{e_j}(v) \frac{\partial S_{e_j}}{\partial X_{e_j}}(v) \neq 0.$$

Thus $J_{\Gamma,w}$ has rank at least $(d(\Gamma) - i - 2) + i + 1 = d(\Gamma) - 1$ at v, whence $\mathcal{O}_{V_{\Gamma,w},v}$ is a regular local ring. Since this holds for every closed point $v \in V_{\Gamma,w}$ we finally conclude that $V_{\Gamma,w}$ is a nonsingular affine surface. □

Corollary 4.20. *For every $e \in$ Leaf(Γ) the open subset $V_e = V_* \cup C_e$ of $V_{\Gamma,w}$ isomorphic to $\mathbb{A}^1_Z$.*

Proof. It follows from lemma 4.18 that for every closed point $v \in C_e$ there exists a section $\sigma = \sum_{k=0}^{N} a_k X_0^k$of $q_{\Gamma,w}$ over Z_* whose closure in $V_{\Gamma,w}$ meets $F_{\Gamma,w}$ in v. More precisely, letting $\mathrm{Anc}(e) = \{e_{i+1} = e, \ldots, e_1, e_0\}$, we must choose $N \geq i+1$ and the coefficients $a_k = a_k(C_e)$, $0 \leq k \leq i$, in such a way that $X_{e_k}(\sigma) = w(e_{k+1}) + X_0 P_{e_k}(X_0) \in A$ for every $0 \leq k \leq i$. Then

$$X_{e_{i+1}}(\sigma) = \alpha a_{i+1} + \beta + X_0 P_{e_{i+1}}(X_0)$$

where $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and $P_{e_{i+1}}(X_0) \in A$. Letting u be a new symbol we consider the polynomial

$$\sigma_u(X_0) = \sum_{k=0}^{i} a_k(v) X_0^k + u X_0^{i+1} \in A[u]$$

and we define an A-morphism $\tau^* : B_{\tilde{\Gamma},w} \to A[u]$ by $\tau^*\left(\bar{X}_e\right) = \bar{X}_e(\sigma_u)$ for every $e \in N(\Gamma)$. The corresponding morphism $\tau : \mathbb{A}^1_Z \to V_{\tilde{\Gamma},w}$ restricts to an isomorphism on V_e. Indeed for every closed point $z \in Z_*$ we know that X_{e_0} is a coordinate function on the fiber $F_z = q^{-1}_{\Gamma,w}(z)$. For every such closed point z we see immediately that $X_{e_0}(\sigma_u)$ restricts on F_z to a polynomial of degree 1 in u. Similarly $X_{e_{i+1}}(\sigma_u) = \alpha u + \beta + X_0 P_{e_{i+1}}(X_0)$ restricts on C_e to a polynomial of degree 1 in u. Since $X_{e_{i+1}}$ is a coordinate function on $C_e \simeq \mathbb{A}^1_{\mathbb{C}}$ (see lemma 4.16) it follows that τ restricts to a bijective Z-morphism $\tau_e : \mathbb{A}^1_Z \to V_e$. Since V_e is nonsingular we conclude that τ_e is an isomorphism by virtue of Zariski Main Theorem. □

$\mathbb{C}_+$-actions on the *GDS*'s $V_{\Gamma,w}$.

In this small section we study $\mathbb{G}_{a,Z}$-actions on the *GDS* $V_{\Gamma,w} \overset{q_{\Gamma,w}}{\to} Z$ associated to a weighted rooted tree. The following proposition tells us that $V_{\Gamma,w}$ admits canonical $\mathbb{G}_{a,Z}$-actions which comes from the restriction of some $\mathbb{G}_{a,Z}$-actions on $\mathbb{A}^{d(\Gamma)}_Z$.

Proposition 4.21. *For every weighted rooted tree Γ_w and every $m \geq H(\Gamma)$ the A-derivation $\partial_{\Gamma,w,m}$ of $A[\Gamma] \otimes_A A_{X_0}$ defined by*

$$\partial_{\Gamma,w,m}(X_0) = 0, \qquad \partial_{\Gamma,w,m}(X_{e_0}) = X_0^m,$$
$$\partial_{\Gamma,w,m}(X_e) = X_0^{-1}\partial_{\Gamma,w,m}\left(Q_{Par(e)}(\Gamma_w)\right) \quad \forall e \in N(\Gamma) \setminus \{e_0\}$$

is a triangular derivation of $A[\Gamma]$ which induces a nontrivial $\mathbb{G}_{a,Z}$-action $\alpha_{\Gamma,w}(m)$ on $V_{\Gamma,w} \overset{q_{\Gamma,w}}{\to} Z$.

Proof. As $\partial_{\Gamma,w,m}(X_{e_0}) \in (X_0^m)\, A[\Gamma]$ it follows that $\partial_{\Gamma,w,m}(X_e) \in \left(X_0^{m-l}\right) A[\Gamma]$ for every $e \in N_l(\Gamma)$, $1 \leq l \leq H(\Gamma)$. As $m \geq H(\Gamma)$ we conclude that $\partial_{\Gamma,w,m}$ is a A-derivation of $A[\Gamma]$. If $e \in N(\Gamma)$ and $\mathrm{Anc}(e) = \{e_{i+1} = e, \ldots, e_1, e_0\}$, then by construction the polynomial $Q_{\mathrm{Par}(e)}(\Gamma_w)$ contains only monomials in the variables X_{e_j}, $0 \leq j \leq i$ so that $\partial_{\Gamma,w,m}(X_e) \in A[X_{e_0}, \ldots, X_{e_i}]$. Hence $\partial_{\Gamma,w,m}$ is a triangular A-derivation of $A[\Gamma]$ which defines a nontrivial $\mathbb{G}_{a,Z}$-action on $\mathbb{A}^{d(\Gamma)}_Z = \mathrm{Spec}(A[\Gamma])$.

It is clear that $\partial_{\Gamma,w,m}$ annihilates the polynomials $\tilde{\Delta}_{0,e}(\Gamma_w)$, $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$. Given $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ we let $\mathrm{Anc}(e) = \{e_i = e, \ldots, e_1, e_0\}$ and we let $e_{i+1} \in N(\Gamma)$ be a child of e_i. Then a short computation shows that

$$\partial_{\Gamma,w,m}\left(\tilde{\Delta}_{e_0,e_i}\right) = X_0^{m-1}\tilde{\Delta}_{0,e_i} - \left(X_0^{-1}\partial_{\Gamma,w,m}\left(Q_{e_i}\left(\Gamma_w^{e_1}\right)\right)\right)\tilde{\Delta}_{0,e_0}$$

whereas

$$\partial_{\Gamma,w,m}\left(\tilde{\Delta}_{e_j,e_i}\right) = \left(X_0^{-2}\partial_{\Gamma,w,m}Q_{e_{j-1}}(\Gamma_w)\right)\tilde{\Delta}_{0,e_i} - \left(X_0^{-1}\partial_{\Gamma,w,m}Q_{e_i}\left(\Gamma_w^{e_{j+1}}\right)\right)\tilde{\Delta}_{0,e_j}$$

for every $1 \leq j \leq i-1$. Thus $\partial_{\Gamma,w,m}\left(\tilde{\Delta}_{e_j,e_i}\right) \in I_{\Gamma,w}$ for every $0 \leq j \leq i-1$ and hence $\partial_{\Gamma,w,m}(I_{\Gamma,w}) \subset I_{\Gamma,w}$. Therefore $\partial_{\Gamma,w,m}$ induces a locally nilpotent A-derivation on $B_{\Gamma,w} = A[\Gamma]/I_{\Gamma,w}$ with corresponding $\mathbb{G}_{a,Z}$-action $\alpha_{\Gamma,w}(m)$ on $V_{\Gamma,w}$. □

Lemma 4.22. *For every $m \geq H(\Gamma)$ the $\mathbb{G}_{a,Z}$-action $\alpha_{\Gamma,w}(m)$ on $V_{\Gamma,w}$ restricts to a free action on $V_* = V_{\Gamma,w} \times_Z Z_*$ whereas $\mu(\alpha_{\Gamma,w}(m), C_e) = m - l$ for every connected components C_e of $F_{\Gamma,w}$ corresponding to a node $e \in$ Leaf$(\Gamma) \cap N_l(\Gamma)$.*

Proof. As $V_* \simeq \mathrm{Spec}(A[X_{e_0}])$ and $\partial_{\Gamma,w,m}(X_{e_0}) = X_0^m$ it follows that $\alpha_{\Gamma,w}(m)$ restricts to a free $\mathbb{C}_+$-action on V_*. Letting $\mathrm{Anc}(e) = \{e_{i+1} = e, \ldots, e_1, e_0\}$, the ideal $\tilde{I}_{C_e}$of C_e in $B_{\Gamma,w}$ is generated by the image of the ideal $I_{C_e} = (X_0, X_{e_0} - w(e_1), \ldots, X_{e_i} - w(e_{i+1}), I_{\Gamma,w}) \subset A[\Gamma]$ in $B_{\Gamma,w}$ by virtue of lemma 4.16. By definition of $\alpha_{\Gamma,w}(m)$ we see that

$$\partial_{\Gamma,w,m}(X_0, X_{e_0} - w(e_1), \ldots, X_{e_i} - w(e_{i+1})) \subset I_{C_e}^{m-i}A[\Gamma].$$

For every $e' \in N(\Gamma) \setminus \mathrm{Anc}(e)$ we let $e_k \in \mathrm{Anc}(e)$ be the first common ancestor of e_{i+1} and e' and we let $\mathrm{Anc}_\Gamma(e') = \left\{e'_{n+1}, \ldots e'_l = e_k, e'_{l-1} = e_{k-1}, \ldots, e'_1 = e_1, e'_0 = e_0\right\}$. Then

$$Q_{e'_j}(\Gamma_w) = R_{e_k}(\Gamma_w) S_{e_k}^{(e'_{l+1})} Q_{e'_j}\left(\Gamma_w^{e'_{l+1}}\right) \in (X_{e_k} - w(e_{k+1})) A[\Gamma]$$

for every $l+1 \leq j \leq n$, which implies that $\partial_{\Gamma,w,m}\left(X_{e'_{j+1}}\right) \in I_{C_e}^{m-k-1} A[\Gamma]$ for every $l \leq j \leq n-1$. Finally as

$$\partial_{\Gamma,w,m}\left(X_{e_{i+1}}\right) = X_0^{-1} \partial_{\Gamma,w,m}(Q_{e_i}(\Gamma_w)) \in I_{C_e}^{m-i-1} \setminus I_{C_e}^{m-i-2}$$

we conclude that that $\partial_{\Gamma,w,m}(B_{\Gamma,w}) \subset \tilde{I}_{C_e}^{m-i-1} \setminus \tilde{I}_{C_e}^{m-i-2}$, which means exactly that

$$\mu(\alpha_{\Gamma,w}(m), C_e) = m - i - 1.$$

□

Corollary 4.23. *For every $e \in N_l(\Gamma) \cap \mathit{Leaf}(\Gamma)$ and every $m \geq H(\Gamma)$ the $\mathbb{G}_{a,Z}$-action $\alpha_{\Gamma,w}(m)$ on V restricts to the $\mathbb{C}_+$-action $t \star (z,u) = \left(z, u + tX_0^{m-l}(z)\, u\right)$ on $V_e \simeq \mathit{Spec}(A[u])$.*

Fibered modifications of a GDS $V_{\Gamma,w}$**.**

Given an equivariant morphism $\beta : (V_{\Gamma,w}, \alpha_{\Gamma,w}(m)) \to \mathbb{A}^1_Z(\alpha_{\Gamma,w})$ in $\mathrm{Mor}\left(GDS_{/Z}\right)$ we know from theorem 3.8 that β can be factored in $GDS_{/Z}$ by a succession of fibered modifications. It appears that this factorisation is strongly related to the structure of the tree Γ_w. We first introduce the following operation on a rooted tree.

Definition 4.24. Given a tree Γ rooted in e_0 and a node $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ such that $\mathrm{Child}_\Gamma(e) \subset \mathrm{Leaf}(\Gamma)$ we let $\tilde{\Gamma}$ be the unique rooted subtree of Γ such that $N\left(\tilde{\Gamma}\right) = N(\Gamma) \setminus \mathrm{Child}_\Gamma(e)$. We say that $\tilde{\Gamma}$ is obtained from Γ by *cutting the leaves at e*.

Proposition 4.25. *Given a weighted rooted tree Γ_w and a node $g \in N(\Gamma)$ such that $\mathit{Child}_\Gamma(g) \subset \mathit{Leaf}(\Gamma)$ we let $\tilde{\Gamma}_w$ be the tree obtained from Γ by cutting the leaves at g, together with the fine weight function induced by the restriction of w. The following assertions hold.*

1) $A[\Gamma] = A\left[\tilde{\Gamma}\right] \otimes_{\mathbb{C}} \mathbb{C}[X_f]$, where $f \in \mathit{Child}(g)$.

2) Letting $B_{\tilde{\Gamma},w} = A\left[\tilde{\Gamma}\right]/I_{\tilde{\Gamma},w}$ there exists an isomorphism of A-algebra

$$B_{\Gamma,w} \simeq B'_{\tilde{\Gamma},w} := B_{\tilde{\Gamma},w}\left[\frac{\tilde{Q}_g(\Gamma_w)}{X_0}\right],$$

where $\tilde{Q}_g(\Gamma_w)$ denotes the image in $B_{\tilde{\Gamma},w}$ of $Q_g(\Gamma_w) \in A[\Gamma]$.

Proof. The first assertion is obvious. For the second one we recall [8, definition 2.1] that if R is a integral domain, $I \subset R$ is an ideal, $f \in I$ and $J \subset R$ is a prime ideal then the strict transform of J in $R[I/f]$ is defined by

$$J^{st} = \left\{r \in R[I/f],\ f^k r \in J \text{ for some } k \in \mathbb{N}\right\}.$$

This is a prime ideal of $R[I/f]$ containing J.

In our situation we let $R = A\left[\tilde{\Gamma}\right]$, $I = (X_0, Q_g(\Gamma_w))$, $f = X_0$ and $J = I_{\tilde{\Gamma},w}$. We notice that $J = I_{\tilde{\Gamma},w}$ is a prime ideal of $A\left[\tilde{\Gamma}\right]$ as $V_{\tilde{\Gamma},w}$ is an irreducible affine surface by proposition 4.19. Letting $\tilde{Q}_g(\Gamma_w)$ be the image of $Q_g(\Gamma_w)$ in $B_{\tilde{\Gamma},w}$ it follows from [8, proposition 2.2] that

$$B'_{\tilde{\Gamma},w} \simeq A\left[\tilde{\Gamma}\right]\left[Q_g(\Gamma_w)/X_0\right]/J^{st} \simeq A[\Gamma]/\left(X_0X_f - Q_g(\Gamma_w), J^{st}\right).$$

Since $J = I_{\tilde{\Gamma},w}$ is prime, J^{st} is also prime and hence $V'_{\tilde{\Gamma},w} = \mathrm{Spec}\left(B'_{\tilde{\Gamma},w}\right)$ is an irreducible affine surface. It is clear that $I_{\tilde{\Gamma},w} \subset \left(\tilde{\Delta}_{0,g}, J^{st}\right)A[\Gamma]$. Moreover, letting $\mathrm{Anc}(g) = \{e_{i+1} = g, e_i, \ldots, e_0\}$ we see that for every $0 \leq j \leq i$

$$X_0\tilde{\Delta}_{e_j,g}(\Gamma_w) = \left(X_{e_j} - w(e_{j+1})\right)\tilde{\Delta}_{0,g}(\Gamma_w) - Q_g\left(\Gamma_w^{e_{j+1}}\right)\tilde{\Delta}_{0,e_{j+1}}(\Gamma_w)$$

belongs to $\left(\tilde{\Delta}_{0,g}(\Gamma_w), J^{st}\right)$ and hence, $I_{\Gamma,w} \subset \left(\tilde{\Delta}_{0,g}(\Gamma_w), J^{st}\right)$. Therefore $V'_{\tilde{\Gamma},w}$ is an irreducible closed 2-dimensional subscheme of $V_{\Gamma,w}$, which implies that $V'_{\tilde{\Gamma},w} = V_{\Gamma,w}$ as $V_{\Gamma,w}$ is itself irreducible due to proposition 4.19. Thus $B_{\Gamma,w} \simeq B'_{\tilde{\Gamma},w} = B_{\tilde{\Gamma},w}\left[\tilde{Q}_g(\Gamma_w)/X_0\right]$. □

Remark 4.26. If A is an integral domain and $I = (f_1, \ldots, f_r) \subset A$ is an ideal then the affine modification of A with the locus (I, f_1) is isomorphic to $A[It](1 - f_1t)$. If $f_1, \ldots, f_r$ form a regular sequence in A then it is well knwon that the Rees algebra $A[It]$ is isomorphic to the symmetric algebra $\mathrm{Sym}_A(I)$ which is in turn defined as $A[X_1, \ldots, X_r]/J$, where J is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} f_1 & \ldots & f_r \\ X_1 & \ldots & X_r \end{pmatrix}.$$

In our situation $A = B_{\tilde{\Gamma},w}$ and I is the image in $B_{\tilde{\Gamma},w}$ of the ideal $(X_0, Q_e(\Gamma_w)) \subset A\left[\tilde{\Gamma}\right]$. Although X_0 and $Q_e(\Gamma_w)$ form a regular sequence in $A\left[\tilde{\Gamma}\right]$ we see that the image $\tilde{Q}_e(\Gamma_w)$ in $B_{\tilde{\Gamma},w}$ is a zero divisor modulo X_0. Indeed as $\tilde{\Delta}_{0,\mathrm{Par}(e)} = X_0X_e - Q_{\mathrm{Par}(e)}\left(\tilde{\Gamma}_w\right) = 0$ in $B_{\tilde{\Gamma,w}}$ we see that $Q_e(\Gamma_w) = \left(X_{\mathrm{Par}(e)} - w(e)\right)^{-1} Q_e\left(\tilde{\Gamma}_w\right) S_e(\Gamma_w)$ is annihilated by $\left(X_{\mathrm{Par}(e)} - w(e)\right)$ in $B_{\tilde{\Gamma},w}/(X_0)$. This explains why the polynomials $\tilde{\Delta}_{e,e'}(\Gamma_w)$ were added to the obvious ones $\tilde{\Delta}_{0,e}(\Gamma_w)$, $e \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$, to define the ideal $I_{\Gamma,w}$.

Proposition 4.27. *The morphism $\beta : V_{\Gamma,w} \to V_{\tilde{\Gamma},w}$ induced by the inclusion $B_{\tilde{\Gamma},w} \subset B_{\Gamma,w} = B_{\tilde{\Gamma},w}\left[\tilde{Q}_g/X_0\right]$ is a fibered modification.*

Proof. Letting $\mathrm{Anc}(g) = \{e_{i+1} = e, e_i, \ldots, e_0\} \subset N(\Gamma)$ we see immediately that $\beta : V_{\Gamma,w} \to V_{\tilde{\Gamma},w}$ restricts to an isomorphism outside the connected component $C_g \subset F_{\tilde{\Gamma},w}$, whereas $\beta^{-1}(C_g)$ is the union of the connected components $C_{f_j} \subset F_{\Gamma,w}$, where $\mathrm{Child}_\Gamma(g) = \left\{f_1, \ldots, f_{\deg_\Gamma(g)}\right\}$. It follows from lemma 4.20 that

$$\tilde{V}_e = \left(V_{\tilde{\Gamma},w} \times_Z Z_*\right) \cup C_e$$

is an open subset of $V_{\tilde{\Gamma},w}$ isomorphic to $\mathbb{A}^1_Z$. Thus $\beta : V_{\Gamma,w} \to V_{\tilde{\Gamma},w}$ will be a fibered modification provided the restriction $\beta : \beta^{-1}\left(\tilde{V}_g\right) \to \tilde{V}_g \subset V_{\tilde{\Gamma},w}$ is a simple fibered modification (see definitions 3.2 and 3.5).

It follows from the proof of lemma 4.20 that there exists a polynomial

$$\sigma_u(X_0) = \sum_{k=0}^{i} a_k(v) X_0^k + uX_0^{i+1} \in A[u]$$

such that the isomorphism $\tilde{V}_g \simeq \mathbb{A}^1_Z$ is induced by the A-morphism $\tau_g^* : B_{\tilde{\Gamma},w} \to A[u]$, $X_e \mapsto X_e(\sigma_u)$, $e \in N\left(\tilde{\Gamma}\right)$. As $A[\Gamma] = A\left[\tilde{\Gamma}\right] \otimes_A \mathbb{C}[X_f]$, $f \in \mathrm{Child}_\Gamma(g)$, we can extend τ_g^* to a morphism $\bar{\tau}_g^* B_{\Gamma,w} \to A_{X_0}[u]$ by letting

$$\bar{\tau}_g^*(X_f) = X_f(\sigma_u) = X_0^{-1} Q_g(\Gamma_w)(\sigma_u) \in A_{X_0}[u].$$

Due to lemmas 4.18 and 4.20 we can write

$$X_{e_k}(\sigma_u) = w(e_{k+1}) + X_0 P_{e_k}(X_0, u)$$

for every $0 \le k \le i$, whereas

$$X_g(\sigma_u) = (cu+d) \prod_{j=0}^{i} S_{e_j}^{(e_{j+1})}(w(e_{j+1})) + X_0 P_g(X_0),$$

where $c \in \mathbb{C}^*$. Using Taylor's Formula we have

$$Q_g(\Gamma_w)(\sigma_u) = R_g(\Gamma_w)(\sigma_u) S_g(X_g(\sigma_u)) = P(u) + X_0 R(X_0, u)$$

where $\lambda \in \mathbb{C}^*$ and $P(u) = S_g\left(\prod_{j=0}^{i} S_{e_j}^{(e_{j+1})}(w(e_{j+1}))(cu+d)\right) \in \mathbb{C}[u]$ is a polynomial with simple roots. Thus $X_f(\sigma_u) = X_0^{-1} P(u) + R(X_0, u) \in A_{X_0}[u]$. This implies that $\beta^{-1}\left(\tilde{V}_g\right)$ is isomorphic to the surface $W \subset \mathbb{A}^2_Z = \mathrm{Spec}(A[u,v])$ with equation $X_0 v - P(u) - X_0 R(X_0, u) = 0$ which is in turn isomorphic to the Danielewski surface $V_{P,1}$. Thus $\beta : \beta^{-1}\left(\tilde{V}_e\right) \to \tilde{V}_e$ is a simple fibered modification and hence, $\beta : V_{\Gamma,w} \to V_{\tilde{\Gamma},w}$ is a fibered modification. □

We notice that the fibered modification $\beta : V_{\Gamma,w} \to V_{\tilde{\Gamma},w}$ coincide with the restriction to $V_{\Gamma,w}$ of the projection $\mathbb{A}_Z^{d(\Gamma)} \to \mathbb{A}_Z^{d(\tilde{\Gamma})}$ induced by the inclusion $A\left[\tilde{\Gamma}\right] \subset A[\Gamma]$. Moreover, for every $m \ge H(\Gamma)$, β equivariant when we let $\mathbb{G}_{a,Z}$ acts on $V_{\Gamma,w}$ by $\alpha_{\Gamma,w}(m)$ and on $V_{\tilde{\Gamma},w}$ by $\alpha_{\tilde{\Gamma},w}(m)$. If $g \in N(\Gamma) \setminus \mathrm{Leaf}(\Gamma)$ and $\mathrm{Child}_\Gamma(g) \subset \mathrm{Leaf}(\Gamma)$ then g becomes a leaf in the tree $\tilde{\Gamma}$ obtained by cutting the leaves of Γ_w at g. Starting with the nodes in Γ_w whose childrens are leaves of Γ_w we can iterate the construction until $N\left(\tilde{\Gamma}_w\right) = \{e_0\}$. Thus the following theorem holds.

Theorem 4.28. *For every $m \ge H(\Gamma)$, the birational equivariant morphism*

$$\beta : (V_{\Gamma,w}, \alpha_{\Gamma,w}(m)) \to \mathbb{A}^1_Z(m)$$

induced by the inclusion $A[X_{e_0}] \subset A[\Gamma]$ factors equivariantly as a succession of fibered modifications obtained by successively cutting the leaves of Γ_w.

Remark 4.29. Remind that the equations which defines the embedding of $V_{\Gamma,w}$ in $\mathbb{A}_Z^{d(\Gamma)}$ can be obtained from all the 2×2 minors of the matrix $M(\Gamma_w)$ of 4.12 by simplifying the common factors. Using this description we see that the operation consisting of cutting the leaves at a node $g\in N(\Gamma)$ coincide we the deletion of the column

$$M(g)=\left(\begin{array}{c} Q_g(\Gamma_w) \\ X_f \end{array}\right)$$

of $M(\Gamma_w)$ corresponding to the node g.

5. Embeddings of generalized Danielewski surfaces

We shall prove the following theorem.

Theorem 5.1. *For every $(V,\alpha)\in Ob\left(GDS_{/Z}\right)$ there exists a weighted rooted tree $\Gamma_w = \Gamma_w(V)$ such that (V,α) is equivariantly isomorphic to $(V_{\Gamma,w},\alpha_{\Gamma,w}(n_V(\alpha)))$.*

The proof is given in 5.2-5.13 below. We begin with the construction of a rooted tree naturally associated to a GDS (V,α).

5.2. We recall that every GDS $(V,\alpha)\not\simeq \mathbb{A}_Z^1(n_V(\alpha))$ admits a Fieseler presentation

$$\tau:(V,\alpha)\overset{\sim}{\to}\bigsqcup_{i=1}^{r} X_i\times\mathbb{A}_{\mathbb{C}}^1/\sim$$

with the identification $\sim$ given by

$$X_i\times\mathbb{A}^1\ni(x,u)\sim\left(x',u'\right)\in X_j\times\mathbb{A}^1 \;\Leftrightarrow\; x'=x \text{ and } u'=h^{\mu_j-\mu_i}(p(x))\,u+g_{ij}(p(x)),$$

where $\mu_i=\mu\left(\alpha,\{x_i\}\times\mathbb{A}_{\mathbb{C}}^1\right)$ and the transition functions $(g_{ij})_{1\leq i,j\leq r}\in A_h$ satisfy the cocycle relations

$$g_{ii}=0 \quad\text{and}\quad g_{ik}=h^{\mu_k-\mu_j}g_{ij}+g_{jk}$$

for every $1\leq i,j,k\leq r$. Moreover $n_{ij}=-ord_{z_0}(g_{ij})>1$ as $r\geq 2$ (see proposition 2.12).

We construct a tree $\Gamma(\tau)$ by the following procedure. For every every $i\in I=\{1,\ldots,r\}$ we let $\Gamma(i)$ be the linear chain $\Gamma(i)=[e_{i,0},e_{i,1},\ldots,e_{i,r_i}]$, where $r_i=n_V(\alpha)-\mu_i$. We let $\Gamma(\tau)$ be the graph obtained from the disjoint union of the trees $\Gamma(i)$, $i\in I$, by identifying two nodes $e_{i,k}$ and $e_{j,k}$ whenever $n_V(\alpha)-\mu_j-n_{ij}>k-1$. If two nodes $e_{i,k}$ and $e_{j,k}$ are identified then the nodes $e_{i,l}$ and $e_{j,l}$, $0\leq l\leq k$, are also identified and hence, $\Gamma(\tau)$ is a tree. As $n_V(\alpha)-\mu_j-n_{ij}\geq 0$ by definition of $n_V(\alpha)$ (see 2.14) we conclude that the nodes $e_{0,i}$, $i\in I$, are all identified to a unique node $e_0\in N(\Gamma(\tau))$ which we consider as the root of $\Gamma(\tau)$. In contrast the nodes e_{i,r_i}, $i\in I$, are not identified with any other node e_{j,r_i}. Indeed otherwise $n_{ji}=n_{ij}-\mu_i+\mu_j<1$ in contradiction with proposition 2.12. By construction the leaves of $\Gamma(\tau)$ are in bijection with the connected component of the fiber $F_{z_0}=q^{-1}(z_0)$. We also notice that the path from e_{i,r_i} to e_0 in $\Gamma(\tau)$ consists of $n_V(\alpha)-\mu_i$ edges and it joins the path from e_{j,r_j} to e_0 after exactly $n_{ji}\geq 1$ edges. We will see later on that this tree is closely related to the dual graph of the boundary divisor in a minimal equivariant completion of (V,α).

5.3. By construction $\Gamma(\tau)$ depends on the order $n_{ij}=-ord_{z_0}(g_{ij})$ of the transition functions associated to the Fieseler presentation $\tau:(V,\alpha)\overset{\sim}{\to}\bigsqcup_{i=1}^{r} X_i\times\mathbb{A}_{\mathbb{C}}^1/\sim$ of (V,α). Due to the relation (2.4) we known that if $\tau':(V,\alpha)\overset{\sim}{\to}\bigsqcup_{i=1}^{r} X_i\times\mathbb{A}_{\mathbb{C}}^1/\sim$ is an other Fieseler presentation of

(V,α) corresponding to transition functions $\left(g'_{ij}\right)_{1\leq i,j\leq r}$ then

$$n_{ij}=-ord_{z_0}\left(g_{ij}\right)=-ord_{z_0}\left(g'_{ij}\right)=n'_{ij}.$$

This implies that $\Gamma(\tau)$ depends only on (V,α). As $n_V(\alpha)=n_V+\mu(\alpha)$ by definition of the index n_V of V (see 2.14) we finally conclude that $\Gamma(\tau)$ depends only on the $\mathbb{A}^1$-bundle $\pi:V\rightarrow X$ (see 2.8) and not on the particular choice of a $\mathbb{G}_{a,Z}$-action α on V. We will consequently denote $\Gamma(\tau)$ as $\Gamma(V)$.

Notation 5.4. By construction a node $e\in\mathrm{Leaf}(\Gamma(V))$ corresponds to a connected component $C(e)=\left\{x_{i(e)}\right\}\times\mathbb{A}^1_{\mathbb{C}}$ of F_{z_0}. More generally for every node $e\in N(\Gamma(V))$ we let

$$C(e)=\bigcup_{f\in\mathrm{Leaf}(\Gamma^e(V))}C(f)\subset F_{z_0},$$

where $\Gamma^e(V)$ denotes the maximal subtree of $\Gamma(V)$ rooted in e. We observe that $C(e_0)=F_{z_0}$

Definition 5.5. We say that two rooted trees Γ_1 and Γ_2 of same height H are isomorphic if the following recursive condition holds.

1) If $H=1$ then $\mathrm{Card}(N_1(\Gamma_1))=\mathrm{Card}(N_1(\Gamma_2))$.

2) If $H\geq 1$ then $\mathrm{Card}(N_1(\Gamma_1))=\mathrm{Card}(N_1(\Gamma_2))$ and for every $e_1\in N_1(\Gamma_1)$ there exists $e_2\in N_1(\Gamma_2)$ such that $\Gamma_1^{e_1}$ is isomorphic to $\Gamma_2^{e_2}$.

The following lemma tells us the the tree $\Gamma(V)$ is an invariant of a *GDS* V.

Lemma 5.6. *If two GDS's V and V' are Z-isomorphic then the trees $\Gamma(V)$ and $\Gamma(V')$ are isomorphic.*

Proof. We let $q:V\overset{\pi}{\rightarrow}X\overset{p}{\rightarrow}Z$ and $q':V'\overset{\pi'}{\rightarrow}X'\overset{p'}{\rightarrow}Z$ be the factorizations obtained in 2.8. Any Z-isomorphism $\chi:V\rightarrow V'$ induces a Z-isomorphism $\bar{\chi}:X\rightarrow X'$. As X and X' are both isomorphic to the affine line with an r-fold origin we see $\bar{\chi}$ coincides with a permutation of the origins $\{x_1,\ldots,x_r\}$ of X. We may choose two $\mathbb{G}_{a,Z}$-action α on V and α' on V' in such a way that $\chi:V\rightarrow V'$ is equivariant. We let $\tau:(V,\alpha)\overset{\sim}{\rightarrow}\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$ and $\tau':\left(V',\alpha'\right)\overset{\sim}{\rightarrow}\bigsqcup_{i=1}^{r}X'_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$ be two Fieseler presentations of (V,α) and (V',α') respectively. Then for every $1\leq i\leq r$ there exists $k=k(i)$, $1\leq k\leq r$ such that χ induces an equivariant Z-isomorphism $X_i\times\mathbb{A}^1_{\mathbb{C}}\overset{\sim}{\rightarrow}X'_k\times\mathbb{A}^1_{\mathbb{C}}$. Thus $\mu_i=\mu\left(\alpha,\{x_i\}\times\mathbb{A}^1_{\mathbb{C}}\right)=\mu\left(\alpha',\{x'_k\}\times\mathbb{A}^1_{\mathbb{C}}\right)=\mu'_k$. Letting $(g_{ij})_{1\leq i,j\leq r}\in A_h$ and $\left(g'_{ij}\right)_{1\leq i,j\leq r}\in A_h$ be the corresponding transition functions we conclude that for every $1\leq i,j\leq r$, $i\neq j$, there exists $k\neq l$, $1\leq k,l\leq r$ such that

$$n_{ij}=-ord_{z_0}\left(g_{ij}\right)=-ord_{z_0}\left(g'_{kl}\right)=n'_{kl}.$$

Thus $n_V(\alpha)-\mu_j-n_{ij}=n_{V'}(\alpha)-\mu'_l-n'_{kl}$ and hence, $\Gamma(V)$ and $\Gamma(V')$ are isomorphic. $\square$

5.7. We will now construct a fine weight function on the tree $\Gamma(V)$. If $(V,\alpha)\simeq\mathbb{A}^1_Z(n_V(\alpha))$ then $\Gamma(V)$ consist of a unique node e_0 to which we can assign any complex number $w(e_0)$. Otherwise we let $\beta:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n_V(\alpha))$ be a morphism. In a Fieseler presentation $\tau:(V,\alpha)\overset{\sim}{\rightarrow}\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$ of (V,α), β corresponds to a morphism

$$\tilde{\beta}=\beta\circ\tau^{-1}:\overset{\sim}{\rightarrow}\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1/\sim\rightarrow\mathbb{A}^1_Z.$$

Letting $(g_{ij})_{1\leq i,j\leq r}\in A_h$ be the transition functions associated to this presentation it follows from lemma 2.15 that $\tilde{\beta}$ is uniquely determined by the choice of regular functions $b_i\in A, 1\leq i\leq r$, satisfying the relation

$$b_i=h^{n_V(\alpha)-\mu_j}g_{ij}+b_j \tag{5.1}$$

for every $1\leq i,j\leq r$. We write $b_i=\sum_{k=0}^{N_i}a_{i,k}h^k$ for every $1\leq i\leq r$ and we consider again the linear chain $\Gamma(i)=[e_{i,0},e_{i,1},\ldots,e_{i,r_i}]$, where $r_i=n_V(\alpha)-\mu_i$. Then we let $w_{\beta,\tau}(e_0)$ be any complex number whereas $w_{\beta,\tau,i}(e_{i,k})=a_{i,k-1}\in\mathbb{C}$ for every $1\leq k\leq r_i$. The relation (5.1) implies that $b_i-b_j=\left(h^{n_V(\alpha)-\mu_j-n_{ij}}\right)D_{ij}(h)$ where $D_{ij}\in A\simeq\mathbb{C}[h]$ is a nonconstant polynomial. Hence $w_{\beta,\tau,i}(e_{i,k})=w_{\beta,\tau,j}(e_{j,k})$ for every $k-1<n_V(\alpha)-\mu_j-n_{ij}$. It then follows from the definition of $\Gamma(V)$ that the weight functions $w_{\beta,\tau,i}$, $i\in I$, patch to define a weight function $w_{\beta,\tau}$ on $\Gamma(V)$. If $k_0-1=n_V(\alpha)-\mu_j-n_{ij}$ then $w_i(e_{i,k_0})\neq w_j(e_{j,k_0})$ and hence $w_{\beta,\tau}$ is a fine weight function on $\Gamma(V)$.

If $\tau':(V,\alpha)\overset{\sim}{\rightarrow}\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$ is an other Fieseler presentation of (V,α) with associated transition functions $\left(g'_{ij}\right)_{1\leq i,j\leq r}\in A_h$ then it follows from relation (2.4) that there exists $c\in A^*=\mathbb{C}^*$ and some regular functions $d_i\in A$ such that

$$g_{ij}=cg'_{ij}+d_j-ch^{\mu_j-\mu_i}d_i$$

for every $i\neq j, 1\leq i,j\leq r$. Thus $\beta\circ(\tau')^{-1}$ is uniqueley determined by $b'_1=b_1+h^{n_V(\alpha)-\mu_1}d_1$ and $b'_i=b'_1-h^{n_V(\alpha)-\mu_j}g_{1i}$ for every $2\leq i\leq r$. As $b_i-b'_i\in\left(h^{n_V(\alpha)-\mu_i}\right)A$ we conclude that the weight function $w_{\beta,\tau}$ does not depend on a particular choice of a Fieseler presentation. This natural fine weight function associated to a morphism $\beta:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n_V(\alpha))$ will be denoted w_β.

5.8. In the sequel it will be convenient to use a less natural fine weight function $\tilde{w}_\beta$ which we derive from w_β as follows. We let $\tilde{w}_\beta(e_0)=1$ and $\tilde{w}_\beta(e)=w_\beta(e)$ for every node $e=e_l(1)\in N_1(\Gamma(V))$. If $e=e_{l_{k+1}}(k+1,l_1,\ldots,l_k)\in N_{k+1}(\Gamma(V))$ is a node at level $k+1\geq 2$ then there exists a unique node $e'=e_{l_{k-1}}(k-1,l_1,\ldots,l_{k-2})\in N_{k-1}(\Gamma(V))$ such that $e\in\Gamma^{e'}(V)$ and we let

$$\tilde{w}_\beta(e)=w_\beta(e)\left(\prod_{f\in\mathrm{Child}(e')\setminus\mathrm{Par}(e)}(\tilde{w}_\beta(f)-\tilde{w}_\beta(e))\right)\tilde{w}_\beta(e').$$

We see by induction that If e_1 and e_2 share the same parent in $\Gamma(V)$ then $\tilde{w}_\beta(e_1)=\lambda w_\beta(e_1)$ and $\tilde{w}_\beta(e_2)=\lambda w_\beta(e_2)$, where $\lambda\in\mathbb{C}^*$. Hence $\tilde{w}_\beta$ is a fine weight function on $\Gamma(V)$.

Lemma 5.9. *We let $\Gamma_{\tilde{w}_\beta}(V)$ be the weighted rooted tree associated to a morphism $\beta:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n_V(\alpha))$. Then for every node $e\in N(\Gamma(V))$ there exists a regular function g_e on (V,α) such that the following assertions hold.*

1) If $e\in Leaf(\Gamma(V))$ then g_e is a coordinate function on $C_e\subset F_{z_0}$.

2) If $e\in N(\Gamma(V))\setminus Leaf(\Gamma(V))$ then for every $f\in Child(e)$ g_e is constant on $C(f)\subset F_{z_0}$ with value $\tilde{w}_\beta(f)$.

Proof. If $(V,\alpha)\simeq\mathbb{A}^1_Z(n_V(\alpha))$ then β is an isomorphism. In this case the tree $\Gamma(V)$ consist of a unique node e_0. Letting $\mathbb{A}^1_Z=\mathrm{Spec}(A[u])$ we see that $g_{e_0}=u\circ\beta$ satisfies (1).

- If $(V,\alpha)\not\simeq\mathbb{A}^1_Z(n_V(\alpha))$ then we let $b_i\in A$, $1\leq i\leq r$ be the regular functions defining $\tilde{\beta}=\beta\circ\tau$ in a Fieseler presentation $\tau:(V,\alpha)\xrightarrow{\sim}\bigsqcup_{i=1}^{r}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$ of (V,α). Since $\tilde{\beta}$ is given by

$$\begin{array}{rcl}\tilde{\beta}:\bigsqcup_{i\in I}X_i\times\mathbb{A}^1_{\mathbb{C}}/\sim & \rightarrow & \mathbb{A}^1_Z(n_V(\alpha))\\ X_i\times\mathbb{A}^1_{\mathbb{C}}\ni(x,u) & \mapsto & \left(p(x),h(p(x))^{n_V(\alpha)-\mu_i}u+b_i(p(x))\right)\end{array}$$

we conclude that the regular function $g_{e_0}\in\mathbb{C}[V]$ defined by

$$g_{e_0}(v)=h(p(x))^{n_V(\alpha)-\mu_i}u+b_i(p(x))$$

if $\tau(v)=(x,u)\in X_i\times\mathbb{A}^1_{\mathbb{C}}$ satisfy (2). Moreover g_{e_0} restricts to a coordinate function on every fiber $F_z=q^{-1}(z)$ for $z\in Z_*$.

- As $g_{e_0}(C(e_l(1)))=\tilde{w}_\beta(e_l(1))$, $1\leq l\leq r(1)$, we see that letting

$$S_{e_0}(T_{e_0})=\prod_{l=1}^{r(1)}(T_{e_0}-\tilde{w}_\beta(e_l(1)))\in\mathbb{C}[T_{e_0}],$$

the rational function

$$g_1(v)=h(q(v))^{-1}S_{e_0}(g_{e_0}(v))$$

on V is in fact a regular function. If $e_l(1)$ is a leaf of $\Gamma(V)$ then $\mu_l=n_V(\alpha)-1$ by definition of $\Gamma(V)$ and g_1 restricts to a coordinate on $C(e_l(1))$. Otherwise it follows from Taylor's Formula that for every $f\in\mathrm{Child}(e_l(1))$ g_1 is constant on $C(f)\subset F_0$ with value $\tilde{w}_\beta(f)$. Thus $g_{e_l(1)}=g_1\in\mathbb{C}[V]$ satisfies (1) and (2) for every $1\leq l\leq r(1)$.

- For every $1\leq l\leq r(1)$ we consider the maximal subtree $\Gamma^{e_l(1)}(V)$ of $\Gamma(V)$ rooted in $e_l(1)$. If $e_l(1)$ is not a leaf of $\Gamma(V)$ then we let $\mathrm{Child}(e_l(1))=\left\{e_1(2,l),\ldots,e_{r(2,l)}(2,l)\right\}$,

$$R_{e_l(1)}(T_{e_0})=S_{e_0}^{(e_l(1))}(T_{e_0})=\prod_{\substack{i=1\\ i\neq l}}^{r(1)}(T_{e_0}-\tilde{w}_\beta(e_i(1)))\in\mathbb{C}[T_{e_0}]$$

and

$$S_{e_l(1)}\left(T_{e_l(1)}\right)=\prod_{i=1}^{r(2,i)}\left(T_{e_l(1)}-\tilde{w}_\beta(e_i(2,l))\right)\in\mathbb{C}\left[T_{e_l(1)}\right].$$

Then

$$g_{2,l}(v)=h(q(v))^{-1}R_{e_l(1)}(g_{e_0}(v))\,S_{e_l(1)}\left(g_{e_l(1)}\right)$$

is a regular function on V and we conclude similarly as above that for every $e_i(2,l)\in\mathrm{Child}(e_l(1))$, $g_{e_i(2,l)}=g_{2,l}\in\mathbb{C}[V]$ satisfies (1) and (2).

- Now the proof can be completed by induction as follows. Suppose that $g_e \in \mathbb{C}[V]$, where $e = e_{l_k}(k, l_1, \ldots, l_{k-1})$ has been constructed by the above procedure. We let $e' = e_{l_{k-1}}(k-1, l_1, \ldots, l_{k-2})$,

$$R_e\left(T_{e_0}, \ldots, T_{e_{l_{k-2}}(k-1,l_1,\ldots,l_{k-3})}, T_{e'}\right) = S_{e'}^{(e)}(T_{e'})\, R_{e'}\left(T_{e_0}, \ldots, T_{e_{l_{k-2}}(k-1,l_1,\ldots,l_{k-3})}\right)$$

and

$$S_e(T_e) = \prod_{f \in \mathrm{Child}(e)} (T_e - \tilde{w}_\beta(f)) \in \mathbb{C}[T_e].$$

Then for every $f_{l_{k+1}} = e_{l_{k+1}}(k+1, l_1, \ldots, l_k) \in \mathrm{Child}(e)$

$$g_{f_{l_{k+1}}}(v) = h(q(v))^{-1} R_e\left(g_{e_0}(v), g_{e_{l_1}(1)}(v), \ldots, g_{e_{l_{k-1}}(k-1,l_1,\ldots,l_{k-1})}(v)\right) S_e(g_e(v))$$

is a regular function on V which satisfies (1) and (2). □

Corollary 5.10. *If $e, e' \in N(\Gamma(V))$ share the same parent then we can choose $g_{e'} = g_e \in \mathbb{C}[V]$ in lemma 5.9.*

5.11. If $(V, \alpha) \in \mathrm{Ob}\left(GDS_{/Z}\right)$ then we consider $\mathbb{C}[V]$ as an A-algebra by mean of the quotient morphism $q : V \to Z = \mathrm{Spec}(A)$. Given a morphism $\beta : (V, \alpha) \to \mathbb{A}^1_Z(n_V(\alpha))$ we let $\phi_\beta^* : A \otimes_{\mathbb{C}} \mathbb{C}[\Gamma(V)] \to \mathbb{C}[V]$ be the homomorphism of A-algebra sending X_e to g_e for every $e \in N(\Gamma(V))$. By virtue of corollary 5.10 we can choose $g_{e'} = g_e$ whenever e and e' share the same parent in $\Gamma(V)$. By making this choice we see that ϕ_β^* factors through an homomorphism of A-algebra $\overline{\phi_\beta^*} : A[\Gamma(V)] \to \mathbb{C}[V]$ (see 4.4).

Proposition 5.12. *For every morphism $\beta : (V, \alpha) \to \mathbb{A}^1(n_V(\alpha))$, the Z-morphism $\bar{\phi}_\beta : V \to \mathbb{A}_Z^{d(\Gamma(V))}$ corresponding to $\overline{\phi_\beta^*} : A[\Gamma(V)] \to \mathbb{C}[V]$ is a closed embedding which induces an isomorphism $\psi_\beta : V \xrightarrow{\sim} V_{\Gamma(V), \tilde{w}_\beta}$.*

Proof. If $(V, \alpha) \simeq \mathbb{A}^1_Z(n_V(\alpha))$ then $\Gamma(V)$ consists of the unique node e_0. In this situation $V_{\Gamma(V), w} \simeq \mathbb{A}^1_Z = \mathrm{Spec}(A[X_{e_0}])$ for every weight function w (see the proof of lemma 4.16). As $g_{e_0} \in \mathbb{C}[V]$ restricts to a coordinate function on every fiber F_z, $z \in Z$, we see that $\psi_\beta = \bar{\phi}_\beta$ is an isomorphism.

We now suppose that $(V, \alpha) \simeq \mathbb{A}^1_Z(\alpha)$. If $e_1, e_2 \in N(\Gamma(V)) \setminus \mathrm{Leaf}(\Gamma(V))$ are two disctint nodes with first common ancestor $e_{12} \in N(\Gamma(V))$ then $g_{e_{12}} \in \mathbb{C}[V]$ takes distinct values on the curves $C(e_1) \subset F_{z_0}$ and $C(e_2) \subset F_{z_0}$. Thus $X_{e_{12}} \in A[\Gamma(V)]$ separates $\bar{\phi}_\beta(C(e_1))$ and $\bar{\phi}_\beta(C(e_2))$. In particular if e_1 and e_2 are leaves of $\Gamma(V)$ which share the parent then $X_{e_{12}}\left(\bar{\phi}_\beta(C(e_1))\right) \neq X_{e_{12}}\left(\bar{\phi}_\beta(C(e_2))\right)$. If $f \in \mathrm{Leaf}(\Gamma(V))$ then $C(f) \simeq \mathbb{A}^1_{\mathbb{C}}$ is a connected component of F_{z_0} and X_f restricts to a coordinate function on $\bar{\phi}_\beta(C(f)) \subset \bar{\phi}_\beta(F_{z_0})$. Finally X_{e_0} restricts to a coordinate function on the image by $\bar{\phi}$ of every fiber $F_z = q^{-1}(z)$, $z \in Z_*$. This implies that $\bar{\phi}_\beta : V \to \mathbb{A}_Z^{d(\Gamma(V))}$ is a closed embedding.

Due to the definition of the regular functions g_e, $e \in N(\Gamma(V))$, and the weight function $\tilde{w}_\beta$ it is clear that $\bar{\phi}_\beta(V) \subset V_{\Gamma(V), \tilde{w}_\beta}$. It is also clear that $\bar{\phi}_\beta(V \times_Z Z_*) \simeq V_{\Gamma(V), \tilde{w}_\beta} \times_Z Z_* \simeq \mathbb{A}^1_{Z_*}$ as g_{e_0} restricts to a coordinate function on every fiber $F_z \subset V$, $z \in Z_*$. We observe that F_{z_0} and $F_{\Gamma(V), w}$ have the same number of connected components. Moreover for every $f \in \mathrm{Leaf}(\Gamma(V))$

the coordinate X_f on $\mathbb{A}_Z^{d(\Gamma(V))}$ restricts to a coordinate function on

$$\bar{\phi}_\beta\left(C\left(f\right)\right) \subset F_{\Gamma(V),w} = \operatorname{Spec}\left(\prod_{f\in \operatorname{Leaf}(\Gamma(V))} \mathbb{C}\left[X_f\right]\right)$$

(see lemma 4.16) and hence, $\bar{\phi}_\beta\left(F_{z_0}\right) = F_{\Gamma(V),w}$. Thus $\bar{\phi}_\beta : V \to \mathbb{A}_Z^{d(\Gamma(V))}$ induces a bijective morphism $\psi_\beta : V \to V_{\Gamma(V),\tilde{w}_\beta}$ which is an isomorphism by virtue of Zariski Main Theorem as $V_{\Gamma(V),\tilde{w}_\beta}$ is nonsingular (see proposition 4.19). □

The following lemma is then an immediate consequence of lemma 4.22.

Lemma 5.13. *If* $(V,\alpha) \in Ob\left(GDS_{/Z}\right)$ *then*

$$\psi_\beta : (V,\alpha) \stackrel{\sim}{\to} \left(V_{\Gamma(V),\tilde{w}_\beta}, \alpha_{\Gamma(V),\tilde{w}_\beta}\left(n_V\left(\alpha\right)\right)\right)$$

is an equivariant isomorphism.

5.14. By proposition 3.9 every morphism $\beta : (V,\alpha) \to \mathbb{A}_Z^1\left(n_V\left(\alpha\right)\right)$ in $GDS_{/Z}$ factors as a succession of equivariant fibered modifications. On the other hand, it follows from theorem 4.28 that for every weighted rooted tree Γ_w the equivariant morphism $\beta' : \left(V_{\Gamma,w_\beta}, \alpha_{\Gamma,w}\left(m\right)\right) \to \mathbb{A}_Z^1\left(\alpha_{\Gamma,w_\beta}\left(m\right)\right)$ defined by the inclusion $A\left[X_{e_0}\right] \subset B_{\Gamma,w_\beta}$ factors as a succession of equivariant fibered modification obtained by cutting the leaves of Γ (see definition 4.24). Moreover by proposition 4.27 every such equivariant fibered modification $\sigma : V_{\tilde{\Gamma},w_\beta} \to V_{\Gamma,w_\beta}$ is induced by the inclusion $A\left[\tilde{\Gamma}\right] \subset A\left[\Gamma\right]$.

Given a morphism $\beta : (V,\alpha) \to \mathbb{A}_Z^1\left(\alpha\right)$, we let

$$\beta : (V,\alpha) = (V_m,\alpha_m) \stackrel{\sigma_m}{\to} (V_{m-1},\alpha_{m-1}) \stackrel{\sigma_{m-1}}{\to} \dots \stackrel{\sigma_2}{\to} (V_1,\alpha_1) \stackrel{\sigma_1}{\to} (V_0,\alpha_0) = \mathbb{A}_Z^1\left(n_V\left(\alpha\right)\right)$$

be the minimal factorisation of obtained in proposition 3.9. For every $1 \leq i \leq m$ we let $\beta_i = \sigma_1 \circ \cdots \circ \sigma_i : (V_i,\alpha_i) \to \mathbb{A}_Z^1\left(n_V\left(\alpha\right)\right)$ and we let

$$\psi_{\beta_i} : (V_i,\alpha_i) \to \left(V_{\Gamma(V_i),\tilde{w}_{\beta_i}}, \alpha_{\Gamma(V_i),\tilde{w}_{\beta_i}}\left(n_V\left(\alpha\right)\right)\right)$$

be the corresponding equivariant isomorphism constructed above. The way we constructed $\Gamma\left(V\right)$ and the remark 3.12 imply immediately the following theorem.

Theorem 5.15. *For every* $1 \leq i \leq m-1$ *the following assertion holds.*

1) The tree $\Gamma\left(V_i\right)$ *is obtained from the tree* $\Gamma\left(V_{i+1}\right)$ *by deleting all the leaves at level* $H\left(\Gamma\left(V_i\right)\right)$.

2) The weight function $\tilde{w}_{\beta_i}$ *coincide with the restriction of* $\tilde{w}_{\beta_{i+1}}$ *to* $N\left(\Gamma\left(V_i\right)\right) \subset N\left(\Gamma\left(V_{i+1}\right)\right)$.

3) Letting $\tilde{\sigma}_{i+1} : V_{\Gamma(V_{i+1}),\tilde{w}_{\beta_{i+1}}} \to V_{\Gamma(V_i),\tilde{w}_{\beta_i}}$ *be the fibered modification induced by the inclusion* $A\left[\Gamma\left(V_i\right)\right] \subset A\left[\Gamma\left(V_{i+1}\right)\right]$*, the following diagram is commutative (in* $GDS_{/Z}$*)*

$$\begin{array}{ccc} (V_{i+1},\alpha_{i+1}) & \stackrel{\sigma_{i+1}}{\to} & (V_i,\alpha_i) \\ \psi_{\beta_{i+1}} \downarrow & & \downarrow \psi_{\beta_i} \\ \left(V_{\Gamma(V_{i+1}),\tilde{w}_{\beta_{i+1}}}, \alpha_{\Gamma(V_{i+1}),\tilde{w}_{\beta_{i+1}}}\left(n_V\left(\alpha\right)\right)\right) & \stackrel{\tilde{\sigma}_{i+1}}{\to} & \left(V_{\Gamma(V_i),\tilde{w}_{\beta_i}}, \alpha_{\Gamma(V_i),\tilde{w}_{\beta_i}}\left(n_V\left(\alpha\right)\right)\right) \end{array}.$$

Remark 5.16. Since every factorisation of a morphism $\beta:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n_V(\alpha))$ as a succession of equivariant fibered modification can be obtained from a minimal one, we conclude that the embedding $\bar{\phi}_\beta:V\rightarrow\mathbb{A}^{d(\Gamma(V))}_Z$ is compatible with any factorisation of β.

It is clear that given two morphisms $\beta_1,\beta_2:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n_V(\alpha))$ there exists a unique Z-automorphism χ_{12} of $\mathbb{A}^1_Z$ such that $\beta_2=\chi_{12}\circ\beta_1$. In particular $\beta_2(F_{z_0})=\chi_{12}(\beta_1(F_{z_0}))$. If $(V,\alpha)\not\simeq\mathbb{A}^1_Z(n_V(\alpha))$ then $\beta_i(F_{z_0})\subset\{z_0\}\times\mathbb{A}^1_{\mathbb{C}}\subset\mathbb{A}^1_Z$, $i=1,2$, is a finite set with a least two elements (see 2.17). We let $\sigma_i:V_i\rightarrow\mathbb{A}^1_Z$, $i=1,2$, be the equivariant simple fibered modification along the divisor $(h=0)\subset\mathbb{A}^1_Z$ with center as the unique reduced closed subscheme Y_i supported on $\beta_i(F_{z_0})$. By the universal property of affine modifications there exists a unique equivariant Z-morphism $\beta_{i,1}:V\rightarrow V_i$ such that β_i factors as $\sigma_i\circ\beta_{i,1}$, $i=1,2$. It follows from [8, Corollary 3.2] that χ_{12} lifts in a unique way to an equivariant isomorphism $\tilde{\chi}_{12}:V_1\stackrel{\sim}{\rightarrow}V_2$ such that $\beta_{2,1}=\tilde{\chi}_{12}\circ\beta_{1,1}$. This procedure can be continued by induction and hence, the following corollary holds.

Corollary 5.17. *If $\beta_1,\beta_2:(V,\alpha)\rightarrow\mathbb{A}^1_Z(n_V(\alpha))$ are two morphisms then there exists a unique Z-automorphism χ_{12} of $\mathbb{A}^{d(\Gamma(V))}_Z$ such that $\bar{\phi}_{\beta_2}=\chi_{12}\circ\bar{\phi}_{\beta_1}:(V,\alpha)\rightarrow\mathbb{A}^{d(\Gamma(V))}_Z$.*

6. *GDS*'s with a trivial Makar-Limanov invariant

In this section we determine which *GDS* have a trivial Makar-Limanov invariant. It is well known that a normal affine surface V is an is an ML-surface if and only if it admits 2 nontrivial $\mathbb{C}_+$-action whose general orbits do not coincide. Given two such actions α_1 and α_2, we let $q:V\rightarrow Z$ be the $\mathbb{G}_a$-fibration associated to α_1. Then $q:V\rightarrow Z$ has at most one degenerate fiber (see e.g. [4, Proposition 2.15]) and Z is isomorphic to the affine line $\mathbb{A}^1$. Indeed Z is an irreducible nonsingular affine curve which contains the image by q of a general orbit of α_2. A general orbit of a α_2 being isomorphic to the affine line $\mathbb{A}^1_{\mathbb{C}}$ we conclude that $Z\simeq\mathbb{A}^1_{\mathbb{C}}$. Thus an ML-surface V is a *GDS* if and only if it admits a nontrivial $\mathbb{C}_+$-action whose associated quotient $\mathbb{G}_a$-fibration $q:V\rightarrow Z$ has reduced fibers.

Example 6.1. Every nonsingular Danielewski surface $V_{P,1}$ has a trivial Makar-Limanov invariant. This is also the case of the surface V of example 2.7. The latter example shows that, in contrast with the case of Danielewski surfaces (see [9]), a *GDS* with a trivial Makar-Limanov, may admit a nontrivial $\mathbb{C}_+$-action such that the degenerate fiber of the associated quotient $\mathbb{G}_a$-fibration is not reduced.

Definition 6.2. A rooted tree Γ is called a *comb* if for every node $e\in N(\Gamma)\setminus\mathrm{Leaf}(\Gamma)$ of degree $\deg(e)>1$ all but possibly one of the children of e are leaves of Γ.

The following theorem characterizes ML-surfaces among *GDS*'s.

Theorem 6.3. *For $(V,\alpha)\in Ob\left(GDS_{/Z}\right)$ non isomorphic to $\mathbb{A}^1_Z(n_V(\alpha))$ the following assertions are equivalent.*

1) V is an ML-surface.

2) The graph $\Gamma(V)$ of 5.2 is a comb.

3) If $\tau : (V, \alpha) \simeq \bigsqcup_{i=1}^{r} X_i \times \mathbb{A}^1_{\mathbb{C}} / \sim$ *is a Fieseler presentation of* (V, α) *such* $\mu\left(\alpha, \{x_i\} \times \mathbb{A}^1_{\mathbb{C}}\right) \geq \mu\left(\alpha, \{x_j\} \times \mathbb{A}^1_{\mathbb{C}}\right)$ *for every* $1 \leq i \leq j \leq r$ *then*

$$n_{ji} = -ord_{z_0}(g_{ij}) = 1$$

for every $1 \leq i < j \leq r$.

Our proof is based on the following theorem.

Theorem 6.4. ([4, proposition 2.10 and theorem 2.16]) *A normal affine surface* $V \not\simeq \mathbb{C}^* \times \mathbb{A}^1$ *is an* ML*-surface if and only if for every minimal completion* $\bar{V}$ *of* V *the dual graph of the boundary divisor* $B := \bar{V} \setminus V$ *is a linear chain.*

For the convenience of the reader we recall that a minimal completion of V (in the sense of [4]) is an open embedding $V \hookrightarrow \bar{V}$ of V into a normal projective surface $\bar{V}$ such that the following conditions hold.
(0) $\bar{V}$ is nonsingular along $B = \bar{V} \setminus V$.
(1) B is an SNC-divisor.
(2) Every irreducible component of B is a nonsingular rational curve.
(3) B does not contains (-1)-curves which meet at most two other component transversally in a single point.

In [5, theorem 2.1] Fieseler gives a general algorithm which produces a completion of any $\mathbb{C}_+$-surface with quotient $\mathbb{G}_a$-fibration $\tilde{q} : \tilde{V} \rightarrow \tilde{Z}$. Using the factorisation of a morphism $\beta : (V, \alpha) \rightarrow \mathbb{A}^1_Z(n_V(\alpha))$ into fibered modifications we give below a construction which coresponds to the one of [5, theorem 2.1] in case that $\tilde{q} : \tilde{V} \rightarrow \tilde{Z}$ has reduced fibers.

6.5. We recall [8, Definition 1.3] that given a scheme $V = \mathrm{Spec}(A)$, an ideal $I \subset A$ and an element $f \in I$ which is not a zero divisor in A, the proper transform D_f^{pr} of the divisor D_f in $\bar{V}' = Bl_I V = \mathrm{Proj}_A A[It]$ is the set of prime ideals $\mathfrak{p} \in Z$ such that $ft \in \mathfrak{p}$. We let $\sigma : V' \rightarrow V$ be the affine modification with the locus (I, f), whereas $\bar{\sigma} : \bar{V}' = Bl_I(V) \rightarrow V$ denotes the corresponding blow-up morphism. Then V' is isomorphic to $\bar{V}' \setminus Supp\left(D_f^{pr}\right)$ (see [8, Lemma 1.2]). In general D_f^{pr} is different from the usual strict transform D_f' of D_f in $\bar{V}'$ which is defined as the closure in $\bar{V}'$ of the preimage $\bar{\sigma}^{-1}(D_f \setminus V(I))$.

Lemma 6.6. *If* $\sigma : V' \rightarrow \mathbb{A}^2_{\mathbb{C}} = Spec(\mathbb{C}[x, y])$ *is a simple fibered modification with the locus* $(I, f) = ((x, P(y)), x)$*, then the strict transform* $D_x' \subset \bar{V}'$ *and the proper transform* $D_x^{pr} \subset \bar{V}'$ *coincide.*

Proof. Since $(x, P(y))$ is a regular sequence in $\mathbb{C}[x, y]$ the Rees algebra

$$\mathbb{C}[x, y][It] = \mathbb{C}[x, y][xt, P(y)t]$$

is isomorphic to $\mathbb{C}[x, y][u, v] / (xv - P(y)u)$ via the map $u \mapsto xt$, $v \mapsto P(y)t$. The curve D_x^{pr} is then given in $\bar{V}'$ by the equation $\{xt = u = 0\}$, and hence corresponds exactly to the strict transform D_x' of the line $D_x = \{x = 0\} \subset \mathbb{A}^2$. □

Given a fibered modification $\beta : W \to V$ of an affine surface V we let $\bar{\beta} : \bar{W} \to V$ be the corresponding blow-up morphism. By definition of a fibered modification there exists an open covering $V = \bigcup_{k\in K} V_k$ by open subsets $V_k \simeq \mathbb{A}^2 = \operatorname{Spec}(\mathbb{C}[x_k, y_k])$ such that

$$\beta|_{\beta^{-1}(V_k)} : W_k := \beta^{-1}(V_k) \to V_k$$

is an SDM with the locus $(I_k, f_k) = ((x_k, P_k(y_k)), x_k)$. Over every V_k the blow-up $\bar{\beta} : \bar{W} \to V$ restricts to the blow-up morphism $Bl_{I_k}V_k \to V_k$. We conclude from lemma 6.6 that W is isomorphic to the complement in $\bar{W}$ of the strict transforms of the lines $\{x_k = 0\} \subset V_k$.

6.7. Given $(V, \alpha) \in Ob(GDS_{/Z})$ and a morphism $\beta : (V, \alpha) \to V_0 = \mathbb{A}^1_Z(n_V(\alpha))$, we let $\bar{V}_0$ be the $\mathbb{P}^1$-bundle $\bar{q}_0 : \bar{V}_0 := \mathbb{P}^1_{\bar{Z}} \to \bar{Z}$, where $\bar{Z} \simeq \mathbb{P}^1_{\mathbb{C}}$ denotes a nonsingular projective model of $Z \simeq \mathbb{A}^1_{\mathbb{C}}$. We let F_0 be the fiber $\bar{q}_0^{-1}(z_0)$ and F_∞ be the fiber $\bar{q}_0^{-1}(\infty)$, where $\infty := \bar{Z} \setminus Z$. Letting $S \subset \bar{V}_0$ be a section of $\bar{q}_0$, we embed V_0 equivariantly in $\bar{V}_0$ as the complement of the ample divisor $F_\infty \cup S$. If (V, α) is isomorphic to $\mathbb{A}^1_Z(n_V(\alpha))$ then $\bar{V}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a minimal completion of V as $(F_\infty^2) = (S^2) = 0$.

Otherwise we let

$$\beta : (V, \alpha) = (V_m, \alpha_m) \stackrel{\beta_m}{\to} (V_m, \alpha_m) \stackrel{\beta_{m-1}}{\to} \dots \stackrel{\beta_2}{\to} (V_1, \alpha_1) \stackrel{\beta_1}{\to} (V_0, \alpha_0) = \mathbb{A}^1_Z(n_V(\alpha))$$

be a factorisation of β by fibered modifications as in proposition 3.9. For every $1 \leq k \leq m$ we let $\bar{\beta}_k : \bar{V}_k \to \bar{V}_{k-1}$ be the blow-up morphism which corresponds to $\beta_k : V_k \to V_{k-1}$. Thus, for every $0 \leq k \leq m$ we see that $\bar{V}_k$ is a completion of V_k by an SNC-divisor B_k. Moreover B_{k+1} is the union of the strict transforms of B_k and the divisor of the fibered modification $\beta_{k+1} : V_{k+1} \to V_k$.

6.8. We let

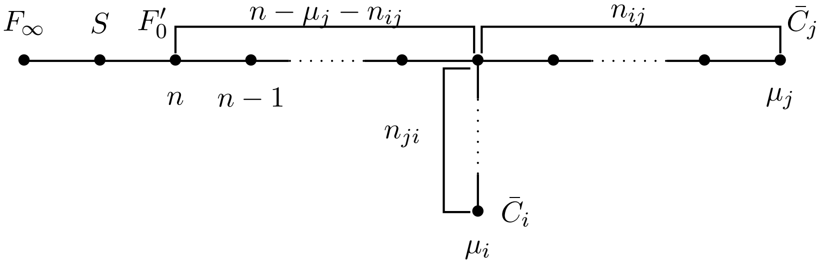
$$\beta : (V, \alpha, q) = V_m \stackrel{\beta_m}{\to} V_{m-1} \stackrel{\beta_{m-1}}{\to} \dots \stackrel{\beta_2}{\to} V_1 \stackrel{\beta_1}{\to} V_0,$$

where $m = n_V$ be the particular minimal factorisation of 3.11. We let $\bar{V} := \bar{V}_m$ and we denote by $\phi : V \hookrightarrow \bar{V}$ the corresponding closed embedding. We let by $\bar{C}_i \subset \bar{V}$, $1 \leq i \leq r$, be the closure of the connected component C_i of the degenerate fiber F_{z_0} of q. We let $\bar{\beta} := \bar{\beta}_1 \circ \cdots \circ \bar{\beta}_m : \bar{V} \to \bar{V}_0$ and we denote by E_j, $j \in J$, the irreducible components of $\bar{\beta}^{-1}(F_0)$ different from the strict transform F_0' of F_0 and the $\bar{C}_i$, $1 \leq i \leq r$. Finally we let

$$B := \bar{V} \setminus V = F'_\infty \cup S' \cup F'_0 \cup \left(\bigcup_{j \in J} E_j \right). \tag{6.1}$$

As $S' \simeq \bar{Z} \simeq \mathbb{P}^1_{\mathbb{C}}$ is rational it follows that every irreducible component of B is a rational curve. By construction $\left(E_j^2\right) \leq -2$ for every $j \in J$. Since $\left(F_\infty'^2\right) = \left(S'^2\right) = 0$ it follows that $\bar{V}$ is a minimal completion of V provided $\left(F_0'^2\right) \neq -1$. The center of the first blow-up $\bar{\beta}_1 : \bar{V}_1 \to \bar{V}_0$ is $\beta(F_{z_0})$ which contains at least 2 elements by virtue of remark 2.17. Therefore $\left(F_0'^2\right) \leq -2$ and so $\bar{V}$ is a minimal completion of V by the divisor B, as desired. This also implies that for every $1 \leq k \leq m-1$ the projective surface $\bar{V}_k$ is a minimal completion of V_k by the SNC-divisor $B_k = \bar{V}_k \setminus V_k$.

6.9. The dual graph $\bar{\Gamma}$ of $B\cup\bar{C}_1\cup\ldots\cup\bar{C}_r$ is a tree emanating from the vertex corresponding to F'_{∞} and having the vertices corresponding to the curves $\bar{C}_i$ as terminal points. Using the properties of our particular factorisation (see 3.11) we see that the path from $\bar{C}_i$ to F'_0 in $\bar{\Gamma}$ consists of $n-\mu_i$ edges and, after exactly n_{ji} edges it joins the path from $\bar{C}_j$ to F'_0.



This implies in particular that the dual graph Γ of $(B\setminus(S'\cup F'_{\infty}))\cup\bar{C}_1\cup\ldots\cup\bar{C}_r$ is isomorphic to the tree $\Gamma(V)$ constructed in 5.2.

Since we have assumed that $\mu_i\geq\mu_j$ for every $1\leq i\leq j\leq r$ we conclude that the dual graph $\Gamma(B)$ of B is a linear chain if and only if $n_{ji}=1$ for all $1\leq i<j\leq r$. It then follows from theorem 6.4 that V is an ML-surface iff $n_{ji}=1$ for all $1\leq i<j\leq r$, which proves the equivalence (1) $\Leftrightarrow$ (3) of theorem 6.3. As $n_{ij}=1$ for all $1\leq i<j\leq r$ if and only if $\Gamma\simeq\Gamma(V)$ is a comb, the equivalence (2) $\Leftrightarrow$ (3) is also proved.

As a consequence of the construction above we obtain the following proposition.

Proposition 6.10. *Given an ML-surface $(V,\alpha)\in Ob\left(GDS_{/Z}\right)$ non isomorphic to $\mathbb{A}^1_Z(n_V(\alpha))$ and a morphism $\beta:(V,\alpha)\to\mathbb{A}^1_Z(n_V(\alpha))$ we let*

$$\beta:V=V_n\stackrel{\beta_n}{\to}V_{n-1}\stackrel{\beta_{n-1}}{\to}\cdots\stackrel{\beta_2}{\to}V_1\stackrel{\beta_1}{\to}V_0=\mathbb{A}^1_Z(n_V(\alpha)),$$

where $n=n_V$, be a minimal factorisation as in proposition 3.9. Then the following assertions hold.

(1) For every $1\leq k\leq n$, (V_k,α_k) is an ML-surface.

(2) For every $1\leq k\leq n$ the morphism $\beta_k:(V_k,\alpha_k)\to(V_{k-1},\alpha_{k-1})$ restricts to an isomorphism over all but exactly one open subset $X^{k-1}_i\times\mathbb{A}^1_{\mathbb{C}}$ of a given Fieseler presentation of (V_{k-1},α_{k-1}).

Proof. We let $\bar{V}_0,\ldots,\bar{V}_n$ be the minimal completions of $V_0,\ldots,V_n$ constructed above.

(1) If $(V,\alpha)=(V_n,\alpha_n)$ is an ML-surface then the dual graph of $B_n=\bar{V}_n\setminus V_n$ is a linear chain. Thus for every $1\leq k\leq n$ the dual graph of $B_k=\bar{V}_k\setminus V_k$ is a linear chain, owing to the fact B_n is obtained by successively adding some irreducible components to the B_k. Hence (V_k,α_k) is an ML-surface by theorem 6.4.

(2) Given a Fieseler presentation $\tau:(V_{k-1},\alpha_k)\stackrel{\sim}{\to}\bigsqcup_{i\in I}X^{k-1}_i\times\mathbb{A}^1_{\mathbb{C}}/\sim$ it follows from the construction of proposition 3.9 that

$$(\beta_k\circ\tau)^{-1}\left(X^{k-1}_i\times\mathbb{A}^1_{\mathbb{C}}\right)\stackrel{\beta_k\circ\tau}{\longrightarrow}X^{k-1}_i\times\mathbb{A}^1_{\mathbb{C}}$$

is either an isomorphism or a simple fibered modification of $X_i^{k-1}\times\mathbb{A}^1_{\mathbb{C}}\simeq\mathbb{A}^2_{\mathbb{C}}$.Since the factorisation is minimal it follows that there exists at least an $i\in I$ over which $\beta_k\circ\tau$ is not an isomorphism. If there exists $j\in I$, $j\neq i$ such that

$$(\beta_k\circ\tau)^{-1}\left(X_i^{k-1}\times\mathbb{A}^1_{\mathbb{C}}\right)\stackrel{\beta_k\circ\tau}{\longrightarrow}X_i^{k-1}\times\mathbb{A}^1_{\mathbb{C}}$$

is not an isomorphism then the dual graph of B_{k+1} is not a linear chain, a contradiction with theorem 6.4 as V_{k+1} is an ML-surface. This proves (2). $\square$

6.11. Using the fact that the tree $\Gamma(V)$ associated to a GDS (V,α) with a trivial Makar-Limanov invariant is a comb we can compute equivariant closed embeddings of (V,α) in $\mathbb{A}_V^{d(\Gamma)}$ as in theorem 5.1.

If $\Gamma_w(V)$ is a comb of height $n\geq 1$ then $d(\Gamma)=n+1$ and we can find n nonzero polynomials with simple roots

$$P_i(T):=(T-\lambda_{i,1})\tilde{P}_i(T)=\prod_{j=1}^{r_i}(T-\lambda_{i,j})\in\mathbb{C}[T]$$

such that the scheme $V_{\Gamma(V),w}\subset\mathbb{A}_Z^{d(\Gamma)}=\mathbb{A}_Z^{n+1}$ of definition 4.10 is given by the equations

$$\left\{\begin{array}{lcll} X_0X_{l+1}-\left(\prod_{i=1}^{l-1}\tilde{P}_i(X_i)\right)P_l(X_l) & = & 0 & \text{for } 1\leq l\leq n\\ (X_{k-1}-\lambda_{k-1,1})X_{l+1}-X_k\left(\prod_{i=k}^{l-1}\tilde{P}_i(X_i)\right)P_l(X_l) & = & 0 & \text{for } 2\leq k\leq l\leq n\end{array}\right. ,$$

where, by convention, $\prod_{i=k}^{l-1}\tilde{P}_i(X_i)=1$ if $k>l-1$. In this situation we will denote $V_{\Gamma(V),w}$ as $V_{P_1,\ldots,P_n}$.

The following theorem is then an immediate consequence of theorem 5.1.

Theorem 6.12. *If $(V,\alpha)\in Ob\left(GDS_{/Z}\right)$ is an ML-surface then either $(V,\alpha)\simeq\mathbb{A}^1_Z(n_V(\alpha))$ or there exist $n=n_V$ nonzero polynomials with simple roots*

$$P_i(X_i):=\prod_{j=1}^{r_i}(X_i-\lambda_{i,j})=(X_i-\lambda_{i,1})\tilde{P}_i(X_i),$$

$1\leq i\leq n$, and an equivariant Z-isomorphism

$$\phi:(V,\alpha)\stackrel{\sim}{\to}\left(V_{P_1,\ldots,P_n},\alpha_{\Gamma(V),w}(n_V(\alpha))\right).$$

We know that for every weighted rooted tree Γ_w and every $m\geq H(\Gamma)$ the $\mathbb{G}_{a,Z}$-action $\alpha_{\Gamma,w}(m)$ on $V_{\Gamma,w}\subset\mathbb{A}_Z^{d(\Gamma)}$ is induced by a $\mathbb{G}_{a,Z}$-action on $\mathbb{A}_Z^{d(\Gamma)}$. If $\Gamma=\Gamma(V)$ is a comb then there exists some other natural $\mathbb{C}_+$-actions $V_{P_1,\ldots,P_n}$ coming from the restriction of such actions on $\mathbb{A}_Z^{d(\Gamma)}=\mathbb{A}_Z^{n+1}$.

Lemma 6.13. *For every $n \geq 1$ the projection $pr_{n+1} : \mathbb{A}^{n+1}_Z \to \mathbb{A}^1$, $(x_0, x_1, \ldots, x_{n+1}) \mapsto x_{n+1}$ restricts to an $\mathbb{A}^1$-fibration $\rho : V_{P_1,\ldots,P_n} \to Z' = Spec\,(\mathbb{C}\,[X_{n+1}])$ on $V_{P_1,\ldots,P_n}$. Moreover the general fibers of $q_{\Gamma,w} : V_{\Gamma,w} = V_{P_1,\ldots,P_n} \to Z$ are distinct.*

Proof. It is clear that $\rho : V_{P_1,\ldots,P_n} \to Z'$ is surjective. We see from the equations

$$X_0 X_{n+1} = \left(\prod_{i=1}^{n-1} \tilde{P}_i\,(X_i) \right) P_n\,(X_n)$$

and

$$(X_{k-1} - \lambda_{k-1,1})\, X_{n+1} - X_k \left(\prod_{i=k}^{n-1} \tilde{P}_i\,(X_i) \right) P_n\,(X_n) \;\; 2 \leq k \leq n$$

that, $V_{P_1,\ldots,P_n} \times_{Z'} \mathrm{Spec}\left(\mathbb{C}\left[X_{n+1}, X_{n+1}^{-1}\right]\right)$ is isomorphic to $\mathbb{A}^1_{Z'_*}$ with coordinates X_n and X_{n+1}. Thus $\rho : V_{P_1,\ldots,P_n} \to Z'$ is an $\mathbb{A}^1$-fibration. Clearly the general fibers of $q_{\Gamma,w} : V_{P_1,\ldots,P_n} \to Z$ and $\rho : V_{P_1,\ldots,P_n} \to Z'$ do not coincide. □

Similarly as in proposition 4.21 we can find a $\mathbb{G}_{a,Z'}$-action α' on $\mathbb{A}^{n+1}_Z \simeq \mathbb{A}^{n+1}_{Z'}$ such that α' restricts to a nontrivial $\mathbb{G}_{a,Z'}$-action on $V_{P_1,\ldots,P_n}$. Hence $V_{P_1,\ldots,P_n}$ comes equipped with two natural $\mathbb{C}_+$-actions whose general orbits do not coincide.

We end this paper by the following corollary of theorem 6.12.

Corollary 6.14. *For an ML-surface $(V, \alpha) \in Ob\left(GDS_{/Z}\right)$ the following assertions are equivalent.*
1) V is a Danielewski Surface.
2) The canonical sheaf ω_V of V is trivial.
3) There exists a free $\mathbb{G}_{a,Z}$-action on V.

Proof. This is a consequence of [1] but we will give an alternative proof. If (V, α) is isomorphic to $\mathbb{A}^1_Z\,(n_V\,(\alpha))$ then there is nothing to prove. We now suppose that $(V, \alpha) \not\simeq \mathbb{A}^1_Z\,(n_V\,(\alpha))$ and we let $\tau : (V, \alpha) \xrightarrow{\sim} \bigsqcup_{i=1}^r X_i \times \mathbb{A}^1_{\mathbb{C}} / \sim$ be a Fieseler presentation of (V, α) with associated transition functions $(g_{ij})_{1 \leq i \leq j} \in A_h$.

- As a Danielewski Surface V is isomorphic to a nonsingular hypersurface of $\mathbb{A}^3$ it follows from the adjunction formula that ω_V is trivial. Thus (1)⇒(2).

- As usual we let $\mu_i = \mu\left(\alpha, \{x_i\} \times \mathbb{A}^1\right)$. Remind that the identification $\sim$ above is given by

$$(X_i \setminus \{x_i\}) \times \mathbb{A}^1_{\mathbb{C}} \ni (x, u) \overset{\phi_{ij}}{\mapsto} \left(x, \left(h^{\mu_j - \mu_i} \circ p\right)(x)\, u + g_{ij}\,(p\,(x))\right) \in (X_j \setminus \{x_j\}) \times \mathbb{A}^1_{\mathbb{C}}.$$

If ω_V is trivial then there exists a nowhere vanishing holomorphic 2-form ω such that $\omega_V = \omega \cdot \mathcal{O}_V$. It is clear that for every $1 \leq i \leq r$, $\omega_i := \omega|_{X_i \times \mathbb{A}}$ is of the form $a_i dx \wedge du$ for $a_1 \in \mathbb{C}^*$. Moreover the identification $\sim$ implies that $\omega_i = \phi^*_{ij}\,(\omega_j) = a_j\,(h^{\mu_j - \mu_i} \circ p)\, dx \wedge du$ for every $1 \leq i, j \leq r$. Thus $a_i = a_j\,(h^{\mu_j - \mu_i} \circ p)$, which implies that $\mu_i = \mu_j = \mu\,(\alpha)$ for every $1 \leq i, j \leq r$. If α is gvien by a locally nilpotent A-derivation ∂ of $\mathbb{C}\,[V]$ then $\tilde{\partial} = h^{-\mu(\alpha)} \partial$ is again locally nilpotent and corresponds to a free $\mathbb{G}_{a,Z}$-action on V. Thus (2)⇒(3).

- If there exists a free $\mathbb{G}_{a,Z}$-action α on V then $\mu_i = \mu\left(\alpha, \{x_i\} \times \mathbb{A}^1\right) = 0 = \mu\,(\alpha)$ for every $1 \leq i \leq r$. As V is an ML-surface it follows from theorem 6.3 that

$$n_{ij} = -ord_{z_0}\,(g_{ij}) = -ord_{z_0}\,(g_{ji}) = n_{ji} = 1.$$

Thus $n_V(\alpha) = 1 + \max_{1\leq i,j\leq r} \{\mu_i\} = 1$, whence $n_V = n_V(\alpha) - \mu(\alpha) = 1$. By virtue of theorem 6.12 there exists a nonconstant polynomial with simple roots P such that V is isomorphic to $V_{P,1}$ which is a Danielewski Surface.Thus (3)⇒(1) □

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