COMPLEXITY OF HOMOGENEOUS SPACES AND GROWTH OF MULTIPLICITIES

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Abstract. The complexity of a homogeneous space G/H under a reductive group G is by definition the codimension of general orbits in G/H of a Borel subgroup $B \subseteq G$. We give a representation-theoretic interpretation of this number as the exponent of growth for multiplicities of simple G-modules in the spaces of sections of homogeneous line bundles on G/H. For this, we show that these multiplicities are bounded from above by the dimensions of certain Demazure modules. This estimate for multiplicities is uniform, i.e., it depends not on G/H, but only on its complexity.

1. Introduction

Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of characteristic 0, and $H \subseteq G$ a closed subgroup. Consider the homogeneous space G/H. Choose a Borel subgroup $B \subseteq G$. By lower semicontinuity, general B-orbits in G/H have maximal dimension. The minimal (=typical) codimension $c = c(G/H)$ of B-orbits is called the *complexity* of G/H. (Clearly, it does not depend on the choice of B since all Borel subgroups are conjugate.) By the Rosenlicht theorem [VP, 2.3], $c(G/H)$ equals the transcendence degree of $\Bbbk(G/H)^B$ over $\Bbbk$.

This numerical invariant plays an important rôle in the geometry of G/H. For instance, the class of homogeneous spaces of complexity zero, called *spherical* spaces, is particularly nice [Kn1], [Bri2], [Vin]. It includes many classical spaces, all symmetric spaces, etc. Also the equivariant embedding theory of G/H depends crucially on its complexity, see [LV], [Tim].

In this note, we describe $c(G/H)$ in terms of representation theory related to G/H.

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Let $\Lambda = \Lambda(B)$ be the weight lattice of B, and $\Lambda_+ \subseteq \Lambda$ be the set of dominant weights. By $V(\lambda)$ denote the simple G-module of highest weight $\lambda \in \Lambda_+$. For any rational G-module M, let $\operatorname{mult}_\lambda M$ denote the multiplicity of $V(\lambda)$ in M.

It turns out that $c(G/H)$ characterizes the growth of multiplicities in spaces of global sections of G-line bundles over G/H. Here is our main result:

Theorem 1. *The complexity $c(G/H)$ is the minimal integer c such that $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}) = O(|\lambda|^c)$ over all $\lambda \in \Lambda_+$ and all G-line bundles $\mathcal{L} \to G/H$, where $|\cdot|$ is any fixed norm on the vector space spanned by Λ. Moreover, this estimate for multiplicities is uniform over all $H \subseteq G$ such that $c(G/H) = c$. If G/H is quasiaffine, then it suffices to consider only* $\operatorname{mult}_\lambda \Bbbk[G/H]$.

A weaker version of Theorem 1 under some restrictive conditions (multiplicities in $\Bbbk[G/H]$ for quasiaffine G/H provided that $\Bbbk[G/H]$ is finitely generated; no uniform estimate) appeared in [AP]. The relation between complexity and growth of multiplicities is known for quite a time, see partial results in [Pa1, 1.1], [Pa2, 2.4], [Bri2, 1.3].

We prove Theorem 1 in Section 2. The idea is to embed the "space of multiplicity" in the dual of a certain Demazure submodule in $V(\lambda)$, associated with an element of length c in the Weyl group (Lemma 2).

In Section 3, we justify the term "complexity" by providing a much more precise information on multiplicities on homogeneous spaces of complexity ≤ 1. Actually, the spherical case is well known [VK] and is included in the text only for convenience of the reader. In the case of complexity 1, a formula similar to ours for multiplicities in $\Bbbk[G/H]$ provided that it is a finitely generated unique factorization domain, and G/H is quasiaffine, was obtained in [Pa1, 1.2], see also [Pa2, 2.4.19].

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Notation.

- The character lattice of an algebraic group H is denoted by $\Lambda(H)$ and is written additively.
- By M^H we denote the set of H-fixed points in the set M acted on by a group H. If M is a vector space, and H acts linearly, then $M_\chi = M_\chi^{(H)}$ is the H-eigenspace of eigenweight $\chi \in \Lambda(H)$.
- Throughout the paper, G is a connected reductive group, and $B \subseteq G$ a fixed Borel subgroup. The semigroup Λ_+ of dominant weights is considered relative to B. We fix an opposite Borel subgroup B^-, a maximal torus $T = B \cap B^-$, and consider the Weyl group W of G relative to T.

- By λ^* denote the highest weight of the dual G-module to the simple G-module $V(\lambda)$ of highest weight $\lambda \in \Lambda_+$.

2. Upper bound for multiplicities

We begin with basic facts about line bundles on homogeneous spaces. Every line bundle $\mathcal{L} \to G/H$ admits a G-linearization, i.e., a fiberwise linear G-action compatible with the projection onto the base, if we possibly replace G by a finite cover [KKLV, §2]. (Alternatively, one may replace $\mathcal{L}$ by its sufficiently big tensor power.) Every G-line bundle is isomorphic to a homogeneous bundle $\mathcal{L}(\chi) = \mathcal{L}_{G/H}(\chi) = G \times^H \Bbbk_\chi$, where H acts on the fiber $\Bbbk_\chi \simeq \Bbbk$ via a character $\chi \in \Lambda(H)$. The bundle $\mathcal{L}(\chi)$ is trivial (regardless the G-linearization) iff χ is the restriction of a character of G.

The space of global sections $H^0(G/H, \mathcal{L}(\chi)) \simeq \Bbbk[G]^{(H)}_{-\chi}$ is a rational G-module, where H acts on G by right translations, and G by left translations. By the Frobenius reciprocity [Jan, I.3.3–3.4], we have $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim V(\lambda^*)^{(H)}_{-\chi}$.

In particular, $H^0(G/B, \mathcal{L}(-\lambda)) = V(\lambda^*)$ whenever $\lambda \in \Lambda_+$, and 0, otherwise (the Borel–Weil theorem).

Observe that for any rational G-module M we have $\operatorname{mult}_\lambda M = \dim M^{(B)}_\lambda$, $\forall \lambda \in \Lambda_+$. In particular,

$$\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim \Bbbk[G]^{(B\times H)}_{(\lambda,-\chi)}$$

The nonzero spaces $\Bbbk[G]^{(B\times H)}_{(\lambda,-\chi)}$ represent complete linear systems of B-stable divisors on G/H, i.e., linear systems of pairwise rationally equivalent B-stable divisors which cannot be enlarged by adding new B-stable effective divisors. The respective biweights (λ, χ) form a subsemigroup $\Sigma = \Sigma(G/H) \subseteq \Lambda(B \times H)$. Two biweights $(\lambda, \chi), (\lambda', \chi') \in \Sigma$ determine the same linear system on G/H iff (λ, χ) differs from (λ', χ') by a twist of G-linearization, i.e., by $(\varepsilon|_B, -\varepsilon|_H)$, $\varepsilon \in \Lambda(G)$.

We need a useful result, essentially due to Brion. Replacing H by a conjugate, we may assume that $\dim B(eH)$ is maximal among all B-orbits, i.e., $\operatorname{codim} B(eH) = c = c(G/H)$.

Lemma 1 (cf. [Bri1, 2.1])**.** *There exists a sequence of minimal parabolics $P_1, \dots, P_c \supset B$ such that $\overline{P_c \cdots P_1(eH)} = G/H$. The decomposition $w = s_1 \cdots s_c$, where $s_i \in W$ are the simple reflections corresponding to P_i, is reduced, and $D_w = \overline{BwB} = P_1 \cdots P_c$ is a "Schubert subvariety" in G of dimension $c + \dim B$.*

Proof. If $c > 0$, then $\overline{B(eH)}$ is not G-stable, whence it is not stabilized by some minimal parabolic $P_1 \supset B$. Since $P_1/B \simeq \mathbb{P}^1$, the natural map $P_1 \times^B B(eH) \to G/H$ is generically finite, and $\operatorname{codim} P_1(eH) = c - 1$. Continuing in the same way, we construct a sequence of minimal parabolics $P_1, \dots, P_c \supset B$ such that $\overline{P_c \cdots P_1(eH)} = G/H$, i.e.,

$P_c\cdots P_1 H$ is dense in G. The map $P_c\cdots P_1\times^B B(eH)\to G/H$ is generically finite, hence $\dim P_c\cdots P_1 = c+\dim B$, which yields all the remaining assertions. □

Remark 1. The "Schubert subvarieties" D_w, $w\in W$, form a monoid w.r.t. the multiplication of sets in G, called the Richardson–Springer monoid. It is generated by the D_s, $s\in W$ a simple reflection, and is naturally identified with W, defining relations being $s^2=s$ and braid relations for W. The action of the Richardson–Springer monoid on the set of B-stable subvarieties (appearing implicitly in Lemma 1) is studied in [Kn2].

Let $v_\lambda\in V(\lambda)$ be a highest weight vector. The B-submodule $V_w(\lambda)\subseteq V(\lambda)$ generated by wv_λ, $w\in W$, is called a *Demazure module*. In the notation of Lemma 1, we have $V_w(\lambda)=\langle D_w v_\lambda\rangle\simeq H^0(S_w,\mathcal{L}_{G/B}(-\lambda))^*$, where $S_w=D_w/B$ is a Schubert subvariety in G/B of dimension c. Indeed, the restriction map $V(\lambda^*)=H^0(G/B,\mathcal{L}(-\lambda))\to H^0(S_w,\mathcal{L}(-\lambda))$ is surjective [Jan, II.14.15, e)], and $D_w v_\lambda$ is the affine cone over the image of S_w under the map $G/B\to\mathbb{P}(V(\lambda))$.

Lemma 2. *In the notation of Lemma 1,* $\operatorname{mult}_\lambda H^0(G/H,\mathcal{L}(\chi))\le\dim V_w(\lambda)$.

Proof. We show that the pairing between $V(\lambda^*)$ and $V(\lambda)$ provides an embedding $V(\lambda^*)^{(H)}_{-\chi}\hookrightarrow V_w(\lambda)^*$. Otherwise, if $v^*\in V(\lambda^*)^{(H)}_{-\chi}$ vanishes on $V_w(\lambda)$, then it vanishes on $D_w v_\lambda$, i.e., $\langle D_w^{-1}v^*\rangle=\langle Gv^*\rangle=V(\lambda^*)$ vanishes at v_λ, a contradiction. □

Remark 2. A similar idea was used in [Pa2, 2.4.18] to obtain an upper bound for multiplicities in coordinate algebras of homogeneous spaces of complexity 1.

Remark 3. The assertion of Lemma 2 can be refined and viewed in a more geometric context as follows. Consider the natural B-equivariant proper map $\varphi: D_w^{-1}/B\cap H\simeq D_w^{-1}\times^B B(eH)\to G/H$, which is generically finite by construction. For $\mathcal{L}=\mathcal{L}_{G/H}(\chi)$ we have $\varphi^*\mathcal{L}=\mathcal{L}_{D_w^{-1}/B\cap H}(\chi|_{B\cap H})$, and $\varphi^*: H^0(G/H,\mathcal{L})\hookrightarrow H^0(D_w^{-1}/B\cap H,\varphi^*\mathcal{L})$ gives rise to

$$\begin{aligned}H^0(G/H,\mathcal{L})^{(B)}_\lambda &\hookrightarrow H^0(D_w^{-1}/B\cap H,\varphi^*\mathcal{L})^{(B)}_\lambda=\Bbbk[D_w^{-1}]^{(B\times B\cap H)}_{(\lambda,-\chi)}\\ &\simeq\Bbbk[D_w]^{(B\cap H\times B)}_{(-\chi,\lambda)}=H^0(S_w,\mathcal{L}(-\lambda))^{(B\cap H)}_{-\chi}=V_w(\lambda)^{*(B\cap H)}_{-\chi}\end{aligned}$$

I am indebted to M. Brion for this remark.

Lemma 2 applies to obtaining upper bounds for multiplicities in branching to reductive subgroups, cf. [AP, Thm. 2].

Corollary. *If $L\subseteq G$ is a connected reductive subgroup, then*

$$\operatorname{mult}_\mu\operatorname{res}^G_L V(\lambda)\le\dim V_w(\lambda^*)$$

for any two dominant weights λ, μ of G, L, respectively, where $w \in W$ is provided by Lemma 1 for H equal to a Borel subgroup of L. Similarly,

$$\operatorname{length} \operatorname{res}^G_L V(\lambda) \le \dim V_w(\lambda^*)$$

where $w \in W$ corresponds to H equal to a maximal unipotent subgroup of L. (Here length *is the number of simple factors in an L-module.)*

Proof. Just note that

$$\operatorname{mult}_\mu \operatorname{res}^G_L V(\lambda) = \dim V(\lambda)^{(H)}_\mu = \operatorname{mult}_{\lambda^*} H^0(G/H, \mathcal{L}(-\mu))$$

in the first case, and $\operatorname{length} \operatorname{res}^G_L V(\lambda) = \dim V(\lambda)^H = \operatorname{mult}_{\lambda^*} \Bbbk[G/H]$ in the second case, and then apply Lemma 2. □

Example 1. Let $P \supseteq B^-$ be a parabolic subgroup with the Levi decomposition $P = L \rightthreetimes P_{\mathrm{u}}$, $L \supseteq T$. By w_L denote the longest element in the Weyl group of L, and consider the decomposition $w_G = w_L w^L$. Let H be the unipotent radical of $B^- \cap L$. Then we may take $w = w^L$ and obtain $\operatorname{length} \operatorname{res}^G_L V(\lambda) \le \dim V_{w^L}(\lambda^*)$.

Proof of Theorem 1. Recall the character formula for Demazure modules [Jan, II.14.18, b)]:

$$\operatorname{ch}_T V_w(\lambda) = \frac{1 - e^{-\alpha_1} s_1}{1 - e^{-\alpha_1}} \cdots \frac{1 - e^{-\alpha_c} s_c}{1 - e^{-\alpha_c}}\ e^\lambda$$

where e^μ is the monomial in the group algebra $\mathbb{Z}[\Lambda]$ corresponding to $\mu \in \Lambda$, α_i are the simple roots defining P_i, and $s_i \in W$ are the respective simple reflections acting on $\mathbb{Z}[\Lambda]$ in a natural way. One easily computes

$$\frac{1 - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}\ e^\mu = e^\mu(1 + e^{-\alpha_i} + \dots + e^{-\langle \mu, \alpha_i^\vee \rangle \alpha_i}), \qquad \forall i,\ \forall \mu \in \Lambda,$$

where $\alpha_i^\vee$ is the respective simple coroot. It is then easy to deduce that $\dim V_w(\lambda) = O(|\lambda|^c)$, and Lemma 2 yields the desired estimate in Theorem 1.

Alternatively, observing that S_w is a projective variety of dimension c, one may deduce that $\dim H^0(S_w, \mathcal{L}(-\lambda))$ grows no faster than $|\lambda|^c$ as follows.

Without loss of generality we may assume G to be semisimple and simply connected. Let $\omega_1, \dots, \omega_l$ be the fundamental weights of G. Put $X = \mathbb{P}_{S_w}(\mathcal{L}(\omega_1) \oplus \dots \oplus \mathcal{L}(\omega_l))$, a projective space bundle over S_w with fiber $\mathbb{P}^{l-1}$. Let $\mathcal{O}_X(1)$ be the antitautological line bundle over X, and $\pi : X \to S_w$ the projection map. Then

$$\pi_* \mathcal{O}_X(k) = \bigoplus_{k_1 + \dots + k_l = k,\ k_j \ge 0} \mathcal{L}(-k_1\omega_1 - \dots - k_l\omega_l)$$

$$R^i \pi_* \mathcal{O}_X(k) = 0, \qquad \forall i > 0$$

Now $R_w = \bigoplus_{k \ge 0} H^0(X, \mathcal{O}(k)) = \bigoplus_{\lambda \in \Lambda_+} H^0(S_w, \mathcal{L}(-\lambda))$ is a Λ-graded algebra, and the quotient field of R_w is a monogenic transcendental extension of $\Bbbk(X)$, hence it has transcendence degree $n = c + l$. Furthermore, R_w is finitely generated. Indeed, R_w is a quotient of the G-algebra $R = \bigoplus_{\lambda\in\Lambda_+} H^0(G/B, \mathcal{L}(-\lambda))$. The multihomogeneous components of R are the simple G-modules $V(\lambda^*)$, hence R is generated by its components with $\lambda = \omega_1, \dots, \omega_l$. (The spectrum of R_w is the so called *multicone* over the Schubert variety S_w, studied in [KR]. For instance, this multicone has rational singularities.) Finally, a general property of finitely generated multigraded algebras implies that $\dim H^0(S_w, \mathcal{L}(-\lambda)) = O(|\lambda|^{n-\operatorname{rk}\Lambda}) = O(|\lambda|^c)$.

This estimate is uniform in H, because there are finitely many choices for w. It remains to show that the exponent c cannot be made smaller.

Let $f_1, \dots, f_c$ be a transcendence base of $\Bbbk(G/H)^B$. There exists a G-line bundle $\mathcal{L}$ and B-eigenvectors $\sigma_0, \dots, \sigma_c \in H^0(G/H, \mathcal{L})$ of the same weight λ such that $f_i = \sigma_i/\sigma_0$, $\forall i = 1, \dots, c$. (Indeed, $\mathcal{L}$ and σ_0 may be determined by a sufficiently big B-stable effective divisor majorizing the poles of all f_i.)

Consider the graded algebra $R = \bigoplus_{n\geq 0} R_n$, $R_n = H^0(G/H, \mathcal{L}^{\otimes n})^{(B)}_{n\lambda}$. Clearly, $\sigma_0, \dots, \sigma_c$ are algebraically independent in R, whence

$$\operatorname{mult}_{n\lambda} H^0(G/H, \mathcal{L}^{\otimes n}) = \dim R_n \geq \binom{n+c}{c} \sim n^c$$

This proves our claim.

Finally, if G/H is quasiaffine, then there even exist $\sigma_0, \dots, \sigma_c \in \Bbbk[G/H]$ with the same properties. □

Remark 4. For a given H, the estimate of the multiplicity by $\dim V_w(\lambda)$ may be not sharp. However, it is natural to ask whether it is a sharp uniform estimate over all homogeneous spaces with given complexity. More precisely, we formulate the following

Question. Given an element $w \in W$ of length c, does there exist a subgroup $H \subseteq G$ such that $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim V_w(\lambda)$ for sufficiently general $(\lambda, \chi) \in \Sigma(G/H)$?

Example 2. In the notation of Example 1, put $H = P_{\mathrm{u}}$. Then always $\chi = 0$, so that $H^0(G/H, \mathcal{L}(\chi)) = \Bbbk[G/H]$. We may take $w = w_L$. Then $V(\lambda^*)^H$ is a simple L-module of lowest weight $-\lambda$, and $V_w(\lambda)$ is the dual L-module of highest weight λ. It follows that $\operatorname{mult}_\lambda \Bbbk[G/H] = \dim V_w(\lambda)$.

3. Case of small complexity

Homogeneous spaces of complexity ≤ 1 are distinguished among all homogeneous spaces by their nice behaviour. For instance, they have a well developed equivariant embedding theory [LV, 8–9], [Tim, 2–5]. There are also more explicit formulæ for multiplicities in this case.

Theorem 2. *In the above notation,*

(1) *If* $c(G/H) = 0$, *then* $\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) \le 1$, $\forall(\lambda, \chi) \in \Sigma$.
(2) *If* $c(G/H) = 1$, *then there exists a pair* $(\lambda_0, \chi_0) \in \Sigma$, *unique up to a shift by* $(\varepsilon|_B, -\varepsilon|_H)$, $\varepsilon \in \Lambda(G)$, *such that*

$$\operatorname{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = n + 1$$

where n *is the maximal integer such that* $(\lambda, \chi) - n(\lambda_0, \chi_0) \in \Sigma(G/H)$.

Proof. The assertion is well known in the case $c = 0$, and we prove it just to keep the exposition self-contained. Assuming the contrary yields two non-proportional B-eigenvectors $\sigma_0, \sigma_1 \in H^0(G/H, \mathcal{L}(\chi))$ of the same weight λ. Hence $f = \sigma_1/\sigma_0 \in \Bbbk(G/H)^B$, $f \ne \text{const}$, a contradiction.

In the case $c = 1$, we have $\Bbbk(G/H)^B \simeq \Bbbk(\mathbb{P}^1)$ by the Lüroth theorem. Consider the respective rational map $\pi : G/H \dashrightarrow \mathbb{P}^1$, whose general fibers are (the closures of) general B-orbits. By a standard argument, π is given by two B-eigenvectors $\sigma_0, \sigma_1 \in H^0(G/H, \mathcal{L}(\chi_0))$ of the same weight λ_0 for a certain $(\lambda_0, \chi_0) \in \Sigma$. Moreover, σ_0, σ_1 are algebraically independent, and each $f \in \Bbbk(G/H)^B$ can be represented as a homogeneous rational fraction in σ_0, σ_1 of degree 0.

Now put $(\mu, \tau) = (\lambda, \chi) - n(\lambda_0, \chi_0)$, fix $\sigma_\mu \in H^0(G/H, \mathcal{L}(\tau))^{(B)}_\mu$, and take any $\sigma_\lambda \in H^0(G/H, \mathcal{L}(\chi))^{(B)}_\lambda$. Then $f = \sigma_\lambda/\sigma_0^n\sigma_\mu \in \Bbbk(G/H)^B$, whence $f = F_1/F_0$ for some m-forms F_0, F_1 in σ_0, σ_1. We may assume the fraction to be reduced and decompose $F_1 = L_1 \cdots L_m$, $F_0 = M_1 \cdots M_m$, as products of linear forms, with all L_i distinct from all M_j. Then $\sigma_\lambda M_1 \cdots M_m = \sigma_\mu \sigma_0^n L_1 \cdots L_m$.

Being fibers of π, the divisors of σ_0, L_i, M_j on G/H either coincide or have no common components. By the maximality of n, the divisor of σ_μ does not majorize any one of M_j. Therefore $M_1 = \cdots = M_m = \sigma_0$, $m \le n$, and $\sigma_\lambda/\sigma_\mu$ is an n-form in σ_0, σ_1. The assertion follows. $\square$

Example 3. Let $G = \mathrm{SL}_3$, $H = T$ (the diagonal torus), B be the upper-triangular subgroup. The space G/H can be regarded as the space of ordered triangles in $\mathbb{P}^2$, i.e.,

$$G/H \simeq \{p = (p_1, p_2, p_3) \mid p_i \ne p_j\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$$

Let $\ell_i \subset \mathbb{P}^2$ denote the line joining p_j and p_k, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. By p_0 denote the B-fixed point in $\mathbb{P}^2$, and by ℓ_0 the B-stable line.

There are the following B-stable prime divisors on G/H:

$$D_i = \{p \mid p_i \in \ell_0\} = \operatorname{div} g_{3i}, \qquad \lambda_i = \omega_2,\ \chi_i = -\varepsilon_i$$

$$D'_i = \{p \mid p_0 \in \ell_i\} = \operatorname{div} \Delta_i, \quad \Delta_i = \begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{vmatrix}, \qquad \lambda'_i = \omega_1,\ \chi'_i = \varepsilon_i$$

$$D_t = \overline{B \cdot p(t)} = \operatorname{div}(g_{32}\Delta_2 + t g_{33}\Delta_3), \qquad \lambda_0 = \omega_1 + \omega_2,\ \chi_0 = 0$$

where $i = 1, 2, 3$, $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the vertices of the triangle $p(t)$ are: $p_1(t) = (0 : 0 : 1)$, $p_2(t) = (0 : 1 : 1)$, $p_3(t) = (1 : t : 1)$. Here g_{ij} are matrix entries of $g \in G$, and H-semiinvariant polynomials in g_{ij} are regarded as sections of G-line bundles on G/H. We also indicate their biweights $(\lambda, \chi) \in \Sigma$, denoting by ω_i the fundamental weights, and by ε_i the diagonal matrix entries of H. Observe that $g_{31}\Delta_1 + g_{32}\Delta_2 + g_{33}\Delta_3 = 0$.

It follows that $c(\mathrm{SL}_3 / T) = 1$. Now it is an easy combinatorial exercise to deduce from Theorem 2(2) that $\operatorname{mult}_\lambda H^0(\mathrm{SL}_3 / T, \mathcal{L}(\chi)) = n+1$, where

$$n = \frac{k_1 + k_2}{2} - \frac{1}{6} \sum_{i=1}^{3} |k_1 - k_2 + 2l_i - l_j - l_k|$$

whenever $(\lambda, \chi) \in \Sigma$, $\lambda = k_1\omega_1 + k_2\omega_2$, $\chi = l_1\varepsilon_1 + l_2\varepsilon_2 + l_3\varepsilon_3$; and $(\lambda, \chi) \in \Sigma$ whenever $k_1 - k_2 \equiv l_1 + l_2 + l_3 \pmod 3$ and $n \geq 0$.

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