SEVERI VARIETIES AND THEIR VARIETIES OF REDUCTIONS

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ABSTRACT. We study the varieties of reductions associated to the four Severi varieties, the first example of which is the Fano threefold of index 2 and degree 5 studied by Mukai and others. We prove that they are smooth but very special linear sections of Grassmann varieties, and rational Fano manifolds of dimension $3a$ and index $a+1$, for $a = 1, 2, 4, 8$. We study their maximal linear spaces and prove that through the general point pass exactly three of them, a result we relate to Cartan's triality principle. We also prove that they are compactifications of affine spaces.

CONTENTS

[1]Mots clés : algèbre de Jordan, algèbre normée, compactification, schéma de Hilbert, trialité, variété de Fano, variété de Calabi-Yau, variété de Severi. Classification A.M.S. : 14J45, 14M17, 14N05, 32M15.

1. Introduction

1.1. **Preliminary: Reductions of Quadrics.** Let $\mathbb{P}^n = \mathbb{P}(V_{n+1})$, $V_{n+1} = \mathbb{C}^{n+1}$ be the complex projective n-space, and let $\hat{\mathbb{P}}^n = \mathbf{P}(\hat{V}_{n+1})$, $\hat{V}_{n+1} = Hom(V_{n+1}, \mathbb{C})$, the dual projective n-space. Let Q be a smooth quadric in $\mathbb{P}^n$. A *non-singular reduction* of Q is any n-simplex $\Delta \subset \mathbb{P}^n$ (equivalently, any set of $n+1$ independent point in $\hat{\mathbb{P}}^n$), such that in homogeneous coordinates $(x) = (x_1 : \cdots : x_{n+1})$, defining the $(n-1)$-faces $\Delta_i = (x_i = 0)$, $i = 1, ..., n+1$ of Δ, the quadratic form $Q(x)$ defining the quadric $Q = (Q(x) = 0)$ becomes diagonal. The hyperplanes $h_i = (x_i = 0)$, spanned on the $(n-1)$-faces Δ_i, $i = 1, ..., n+1$ of Δ , are the principal axes of Q defined by the reduction Δ.

The $(n+1)$ principal axes h_i defined by a (nonsingular) reduction Δ of Q are also the vertices of the dual simplex $\hat{\Delta} \subset \hat{\mathbb{P}}^n$ to Δ; and the fact that Δ is a reduction of Q means that the point $Q \in \mathbb{P}(Sym^2\ \hat{V}_{n+1})$ lies in the projective n-space $\Pi^n = \Pi^n(\Delta)$ spanned on the Veronese images ε_i of the axes h_i, $i = 1, ..., n+1$. That is:

$(*)$ *The family $Red^o(Q)$ of non-singular reductions of the smooth quadric $Q \subset \mathbb{P}^n$ is isomorphic to the family of $(n+1)$-secant n-spaces to the Veronese n-fold $v_2(\hat{\mathbb{P}}^n)$ passing thourgh Q.*

1.2. **Reductions in Jordan algebras.** We shall see that the projectivized simple Jordan algebras are the natural projective representation spaces where one can define analogs of reductions in a way similar to the case of quadrics.

On the one hand, the observation is that the space $\mathbb{P}(W) = \mathbb{P}(Sym^2\ \hat{V}_{n+1})$ is an irreducible projective representation space of the group $G = SL_{n+1}$. Moreover $\mathbb{P}(Sym^2\ \hat{V}_{n+1})$ is a prehomogeneous projective space of SL_{n+1}, i.e. the group $G = SL_{n+1}$ acts transitively over an open subset of this space – the set $\mathbb{P}(Sym^2\ \hat{V}_{n+1}) - Det$ of quadrics of rank $n+1$. The last identifies the varieties of reductions of any two quadrics of rank $n+1$.

On the other hand, the Veronese variety $v_2(\hat{\mathbb{P}}^n)$, which is the closed orbit of the projective action ρ of SL_{n+1} on $\mathbb{P}(Sym^2\ \hat{V}_{n+1})$, is isomorphic to the projective n-space. This makes it possible to define the reductions as simplices with faces defined by intersections of coordinate hyperplanes in $\mathbb{P}^n$.

In the above context, it is natural to look for smooth projective varieties that are, in some way, analogs of the projective space. The most natural possible analogs of the projective space are the Severi varieties $\mathbb{AP}^2$ – the Veronese models of the projective planes over the 4 complexified composition algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and the varieties of Scorza – the projective n-spaces $\mathbb{AP}^n$, $n \geq 3$ over $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ [26, 31].

The ambient projective spaces of all the varieties $\mathbb{AP}^n$ – of Severi, as well of Scorza – are projectivized prehomogeneous spaces. The representation spaces supporting the varieties $\mathbb{AP}^n$, $n = 2, a = 1, 2, 4, 8$ and $n \geq 3, a = 1, 2, 4$ are exactly the spaces of all the simple complex Jordan algebras $\mathbb{P}(\mathcal{J}_{n+1}(\mathbb{A}))$ of rank $n \geq 3$. In particular, for $\mathbb{A} = \mathbb{R} = \mathbf{C}$ one obtains again the Jordan algebra $\mathcal{J}_{n+1}(\mathbb{R}) \cong Sym^2\ \mathbf{C}^{n+1}$ of symmetric matrices of order $n+1$.

1.3. **The principal results and structure of the paper.** In this paper we treat the case $n = 3$, i.e. the question about the description of the varieties of reductions in the four projectivized simple complex Jordan algebras of order 3.

The four complex composition algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are the complexifications of the four normed division algebras $\mathbf{A} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$ – the reals, the complexes, the quaternions, and the octonions. As complex vector spaces these algebras have correspondingly dimensions $a = \dim_{\mathbf{C}} \mathbb{A} = 1, 2, 4, 8$; in particular $\mathbb{R} = \mathbf{C}$.

For any $a = 1, 2, 4, 8$, the Jordan algebra $\mathcal{J}_3(\mathbb{A})$ is the complex vector space of $\mathbb{A}$-Hermitian matrices of order 3. Its projectivization contains three types of matrices, depending on the *rank* (which can be defined properly even over the octonions). In particular, the (projectivization of the) set of rank one matrices is the Severi variety X_a, the projective $\mathbb{A}$-plane, a homogeneous variety of dimension $2a$.

For a point $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})$ defined by a rank three matrix, a non-singular reduction of w is a 3-secant plane to X_a through w – see **1.1**$(*)$. The projection from the fixed point w sends the quasiprojective set Y_a^o of non-singular reductions of w isomorphically to the family of 3-secant lines to the projected Severi variety $\overline{X}_a$ inside the projective space $\mathbb{P}\mathcal{J}_3(\mathbb{A})_o = \mathbb{P}^{3a+1}$ of traceless matrices; and the projective closure Y_a of Y_a^o in the Grassmannian $G(2, \mathcal{J}_3(\mathbb{A})_o) = G(2, 3a+2)$ is *the variety of reductions* of w, our main object of study.

In the first section we relate the family Y_a^0 of simple reduction planes with another series of homogeneous varieties X^a. These varieties appear in the $\mathbb{C}$-column of the geometric Freudenthal square explored in [17] (while the Severi varieties are those of the $\mathbb{C}$-line). Specifically, we note that the choice of w gives an embedding in these varieties X^a of a copy X^a_w of a variety from the $\mathbb{R}$-column of the magic square. Then, we prove that the choice of a reduction plane gives an embedding in X^a_w of what we call a *triality variety* Z_ε, a variety from the $\underline{0}$-column of the magic square. We give an interpretation of these varieties as zero-set of sections of homogeneous vector bundles whose spaces of global sections are precisely the Jordan algebras $\mathcal{J}_3(\mathbb{A})$. This is another example of the fascinating geometry related to Freudenthal's magic square, the new feature here being, while we usually understand the square line by line, the geometry of the first three columns is deeply interwoven with that of the first two lines.

In the second section, we focus on the beautiful geometry of the completion Y_a of Y_a^0, the variety of reductions. This subvariety of $G(2,\mathcal{J}_3(\mathbb{A})_0)$ has dimension $3a$, and is endowed with a natural action of the automorphism group $SO_3(\mathbb{A}) := \mathrm{Aut}(\mathcal{J}_3(\mathbb{A}))$ of the Jordan algebra. We prove that Y_a has four $SO_3(\mathbb{A})$-orbits, which we describe explicitely: they have codimension 0, 1, 2 and 4 (Proposition 3.2). We prove that Y_a can be defined as a linear section of the ambient Grassmannian $G(2,\mathcal{J}_3(\mathbb{A})_0)$, in its Plücker embedding (Proposition 3.1). For $a>1$, this section is non transverse, not even proper. Nevertheless, we prove that Y_a is a smooth subvariety of $G(2,\mathcal{J}_3(\mathbb{A})_0)$ (Theorem 3.11). This unexpected phenomenon is related to the presence in Y_a of large linear spaces: namely, Y_a is covered by a family of $\mathbb{P}^a$'s parametrized by the Severi variety X_a. We prove that the stabilizer of a generic point of Y_a is the semi-direct product of a triality group by a symmetric group $\mathfrak{S}_4$ (Propositions 3.8). Moreover, exactly three $\mathbb{P}^a$'s pass through that point, which are permuted by the symmetric group $\mathfrak{S}_3$, obtained as the quotient of $\mathfrak{S}_4$ by the normal subgroup of permutations given by products of two disjoint transpositions. This leads to a very nice geometric picture of the Lie algebra isomorphism

$$\mathfrak{so}_3(\mathbb{A}) = \mathfrak{t}(\mathbb{A}) \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3$$

(see **2.7.1** for the notations), which was used in [18] in a very different context. This geometric occurence of triality completes the picture given by E. Cartan in his paper on isoparametric families of hypersurfaces, the first geometric appearance of the exceptional group F_4 [5].

Using the geometry of linear subspaces on Y_a, we prove that the point-line incidence variety Z_a over Y_a is the blow-up of the projected Severi variety $\overline{X_a}$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ (Proposition 3.16). An easy consequence is that Y_a is a smooth Fano manifold of index $a+1$ (and dimension $3a$) with a cyclic Picard group. We also compute its Betti numbers, and its degree with respect to the Plücker embedding. Finally, we use the group action to prove that Y_a is a minimal compactification of $\mathbb{C}^{3a}$. More precisely, the maximally degenerate hyperplane sections of Y_a are parametrized by its closed orbit (which identifies with the space of special lines on the hyperplane section X_a^0 of the Severi variety), and their complements in Y_a are affine cells (Theorem 3.22). Remember that the only minimal projective compactification of $\mathbb{C}^2$ is the projective plane, and that there exists only four types of such compactifications of $\mathbb{C}^3$. Several people asked for the classification of minimal compactifications of $\mathbb{C}^n$, but very few explicit examples seem to be known. Our varieties of reductions give a series of such examples.

The case $a=1$ is classical: Y_1 is a transverse intersection of the Grassmannian $G(2,5)\subset\mathbb{P}^9$ with a codimension three linear subspace. This is the Fano threefold of degree 5, studied in particular by Mukai [25]. The fact that Y_1 is a compactification of $\mathbb{C}^3$ was discovered by Furushima (see [11] and the references therein). We realized that the second variety of reductions Y_2

also appears in the litterature, in a slightly disguised form: it is called in [30] the variety of determinantal nets of quadrics. In our interpretation, it is rather the space of abelian planes in $\mathfrak{sl}_3$, and is therefore closely connected with the commuting variety of $\mathfrak{sl}_3$. We show in the last section that it is the image of one of the two extremal contractions of the punctual Hilbert scheme $\mathrm{Hilb}^3\mathbb{P}^2$ (the other one is the Hilbert-Chow morphism, whose image is of course singular). Since Y_2 is a Fano manifold of index 3 and dimension 6, a generic codimension three linear section is a smooth Calabi-Yau threefold. We conclude the paper by a computation of the Euler number of this Calabi-Yau manifold, showing that its general deformation is not induced by a deformation of the section.

2. Reductions in the Jordan algebras $\mathcal{J}_3(\mathbb{A})$

2.1. **The four special complex Jordan algebras of order 3.**
Let $\mathbf{A} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$, the reals, the complexes, the quaternions and the octonions, be the four real division algebras. As real vector spaces, they have dimensions correspondingly $a = 1, 2, 4, 8$. They are endowed with non degenerate quadratic forms q such that $q(xy) = q(x)q(y)$. We denote by $q(x, y)$ the associated scalar product, so that $q(x) = q(x, x)$. By orthogonal symmetry with respect to the unit element 1, a vector $x \in \mathbf{A}$ is transformed into its conjugate $\bar{x} \in \mathbf{A}$, which is such that $x\bar{x} = \bar{x}x = q(x)1$. The real and imaginary parts of x are defined by the identities $x = Re(x)1 + Im(x)$, $\bar{x} = Re(x)1 - Im(x)$.

For any $a = 1, 2, 4, 8$, let let $\mathbb{A} = \mathbf{A} \otimes_{\mathbf{R}} \mathbf{C}$ be the complex composition algebra with multiplication $(x_1 \otimes c_1)(x_2 \otimes c_2) = x_1x_2 \otimes c_1c_2$ and conjugation $\overline{x \otimes c} = \bar{x} \otimes c$. In particular $\mathbb{R} \cong \mathbf{C}$, $\mathbb{C} \cong \mathbf{C} \oplus \mathbf{C}$, and $\mathbb{H} \cong M_2(\mathbf{C})$, the algebra of complex matrices of order 2.

For any $a = 1, 2, 4, 8$ the space

$$\mathcal{J}_3(\mathbb{A}) = \left\{ \begin{pmatrix} c_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & c_2 & x_1 \\ x_2 & \bar{x}_1 & c_3 \end{pmatrix} : c_i \in \mathbf{C}, x_i \in \mathbb{A} \right\} \cong \mathbf{C}^{3a+3}$$

of $\mathbb{A}$-Hermitian matrices of order 3, together with the Jordan multiplication $A \circ B = \frac{1}{2}(AB + BA)$ is a Jordan algebra, i.e. $(\mathcal{J}_3(\mathbb{A}), \circ)$ is commutative and the equality

$$(A \circ B) \circ (A \circ A) = A \circ (B \circ (A \circ A))$$

holds for any $A, B \in \mathcal{J}_3(\mathbb{A})$.

2.2. **The Severi varieties and reductions.** On $\mathcal{J}_3(\mathbb{A})$ there is a well defined determinant det, a cubic form which can be defined in terms of the trace of a matrix and its second and third powers, by the formula which is usual in $M_3(\mathbf{C})$.

For $a = 1, 2, 4, 8$ the subgroup of $GL_{\mathbf{C}}(\mathcal{J}_3(\mathbb{A}))$ of complex-linear transformations preserving the determinant, is the product of its center by the derived group $SL_3(\mathbb{A})$. This semi-simple group is isomorphic correspondingly to $SL_3, SL_3 \times SL_3, SL_6, E_6$; and in fact the action $\rho_a : SL_3(\mathbb{A}) \times \mathcal{J}_3(\mathbb{A}) \to \mathcal{J}_3(\mathbb{A})$ is an irreducible representation of $SL_3(\mathbb{A})$ in the complex vector space

$\mathcal{J}_3(\mathbb{A}) = \mathbf{C}^{3a+3}$. More precisely, let $A_2, A_2 + A_2, A_5, E_6$ be correspondingly the Dynkin diagrams of $SL_3(\mathbb{A})$, $a = 1, 2, 4, 8$. Then, in the notation of [Bourbaki], we may consider that:

– The representation ρ_1 is defined by the weight $2\omega_1$, i.e. ρ_1 is the 2-nd symmetric power of the standard representation of SL_3; in particular the Jordan algebra $\mathcal{J}_3(\mathbb{R})$ is isomorphic to the algebra $Sym^2\ \mathbf{C}^3$ of symmetric complex matrices of order 3.

– The representation ρ_2 is defined by any pair of weights (ω_i', ω_j''), $1 \leq i, j \leq 2$ of the two copies (A_2', A_2'') of A_2 in the diagram of $SL_3 \times SL_3$; in particular $\mathcal{J}_3(\mathbb{C})$ is isomorphic to the algebra $\otimes^2\ \mathbf{C}^3$ of matrices of order 3.

– The representation ρ_4 is defined by ω_2 or by ω_4 , i.e. ρ_4 is the second or the fourth alternative power of the standard representation of SL_6; in particular $\mathcal{J}_3(\mathbb{C})$ is isomorphic to the algebra $\wedge^2\ \mathbf{C}^6$ of antisymmetric matrices of order 6.

– The representation ρ_8 is defined by any of the weights ω_1 or ω_6 of E_6.

Denote by ρ_a also the projectivized action

$$\rho_a : SL_3(\mathbb{A}) \times \mathbb{P}\mathcal{J}_3(\mathbb{A}) \to \mathbb{P}\mathcal{J}_3(\mathbb{A})$$

on the projective complex space $\mathbb{P}\mathcal{J}_3(\mathbb{A}) = \mathbb{P}^{3a+2}$. The following is well-known, see e.g. [28, 17]:

Lemma 2.1. *The projective action of ρ_a of $SL_3(\mathbb{A})$ splits $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ into a union of 3 orbits*

$$\mathbb{P}\mathcal{J}_3(\mathbb{A}) = (\mathbb{P}\mathcal{J}_3(\mathbb{A}) - \mathbb{D}_3(\mathbb{A}))\ \cup\ (\mathbb{D}_3(\mathbb{A}) - \mathbb{AP}^2)\ \cup\ \mathbb{AP}^2,$$

where $\mathbb{D}_3(\mathbb{A}) = (det = 0)$ and $\mathbb{AP}^2$ are correspondingly the determinantal cubic hypersurface and the locus of Jordan $\mathbb{A}$-matrices of order 3 and of rank 1. Moreover $\mathbb{D}_3(\mathbb{A}) = Sec(\mathbb{AP}^2)$ = the union of all the secant lines to $\mathbb{AP}^2$; and $\mathbb{AP}^2 = Sing\ \mathbb{D}_3(\mathbb{A})$.

The *rank* of a matrix with coefficients in $\mathbb{O}$ is a rather delicate notion. In this paper, it will suffice to consider that by definition, these three orbits in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ consist in matrices of rank 3, 2 and 1 respectively.

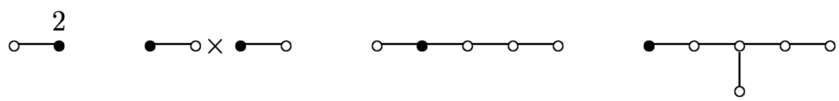
The varieties $\mathbb{AP}^2 = \mathbb{RP}^2, \mathbb{CP}^2, \mathbb{HP}^2$ and $\mathbb{OP}^2$ can be interpreted as the four complex projective $\mathbb{A}$-planes, see [26]. From another point of view they are also all the four Severi varieties [26, 31, 6]. They can be described very explicitely:

Lemma 2.2. *The Severi variety $X_a \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ is defined by the equations $X^2 = \mathrm{trace}(X)X$, which generate its ideal. Its intersection with the affine subspace of $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ on which the first diagonal coefficient is non zero is*

$$X_a \cap \{c_1 \neq 0\} = \left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & \bar{x}x & \bar{x}y \\ \bar{y} & \bar{y}x & \bar{y}y \end{pmatrix}, \quad x, y, \in \mathbb{A} \right\} \cong \mathbf{C}^{2a}.$$

The Severi varieties really show up the geometry of projective planes. They are covered by a family of projective $\mathbb{A}$-lines $\mathbb{AP}^1 \cong \mathbb{Q}^a$, quadrics of dimension a, and two (generic) such $\mathbb{A}$-lines intersect at a unique point.

In the interpretation $\mathcal{J}_3(\mathbb{R}) = Sym^2\ \mathbf{C}$, the $\mathbb{R}$-plane $\mathbb{RP}^2$ is the Veronese image $v_2(\mathbb{P}^2)$ of the complex projective plane, and the hypersurface $\mathbb{D}_3(\mathbb{R})$ is the symmetric determinant cubic in $\mathbb{P}(Sym^2\ \mathbf{C})$. For $\mathcal{J}_3(\mathbb{R}) \cong \otimes^2\mathbf{C^3}$, the $\mathbb{C}$-plane $\mathbb{CP}^2$ is the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$, and $\mathbb{D}_3(\mathbb{C})$ is the symmetric determinant cubic in $\mathbb{P}(\otimes^2\ \mathbf{C})$. For $\mathcal{J}_3(\mathbb{H}) \cong \wedge^2\ \mathbf{C}^6$, the complex quaternionic plane $\mathbb{HP}^2$ is the grassmannian $G(2,6)$, and $\mathbb{D}_3(\mathbb{H})$ is the pfaffian cubic $Pf \subset \mathbb{P}(\wedge^2\ \mathbf{C}^6)$. At the end, the complex octonionic plane, or the complex Cayley plane $\mathbb{OP}^2 \subset \mathbb{P}(\mathcal{J}_3(\mathbb{O})) = \mathbb{P}^{26}$ is a smooth Fano 16-fold of degree 78 with $Pic\ \mathbb{OP}^2 = \mathbb{Z}H$, H being the hyperplane section, and $K_{\mathbb{OP}^2} = -12H$.



Weighted Dynkin diagrams of the Severi varieties X_a

To the four Jordan algebras one can attach the algebra $\mathcal{J}_3(\underline{0}) \cong \mathbf{C}^3$ of complex diagonal matrices of order 3, coming from the algebra $\underline{0} = (0)$. Then the determinant hypersurface $\mathbb{D}(\underline{0}) \subset \mathcal{J}_3(\underline{0}) = \mathbb{P}^2$ is, of course, the coordinate triangle $\Delta \subset \mathbb{P}^2$, and the $\underline{0}$-plane (or the 0-th Severi variety) $\underline{0}\mathbb{P}^2 = \vee^3\ \mathbb{P}^0$ is the triple of vertices of Δ.

The five Severi varieties fill in the 2-nd line of the extended Freudenthal square, see [LM]:

	$\underline{0}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	
$\mathbb{R}$	$\emptyset$	$v_4(\mathbb{P}^1)$	$\mathbb{P}(T_{\mathbb{P}^2})$	$IG(2,6)$	$\mathbb{OP}^2_0$	*section of Severi*
$\mathbb{C}$	$\vee^3\mathbb{P}^o$	$v_2(\mathbb{P}^2)$	$\times^2\mathbb{P}^2$	$G(2,6)$	$\mathbb{OP}^2$	*Severi*
$\mathbb{H}$	$\times^3\mathbb{P}^1$	$LG(3,6)$	$G(3,6)$	...	...	*Legendre*
$\mathbb{O}$	D_4^{ad}	F_4^{ad}	E_6^{ad}	...	...	*adjoint*

Here we denoted by $IG(2,6)$ (resp. $LG(3,6)$) the isotropic (resp. lagrangian) grassmannian of isotropic 2-planes (resp. 3-planes) in $\mathbb{C}^6$ with respect to a symplectic form. The adjoint varieties are the closed orbits in the projectivizations of the adjoint representations of the simple Lie algebras.

As we will see below, the $\underline{0}$-th column will play an important role in the description of the varieties of reductions in the Jordan algebras $\mathcal{J}_3(\mathbb{A}), a = 1,2,4,8$.

Definition 2.3. (Reductions and reduction planes) *Let $a = 1,2,4,8$, and let $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o = \mathbb{P}\mathcal{J}_3(\mathbb{A}) - \mathbb{D}_3(\mathbb{A})$ be a projective rank 3 Jordan matrix of order 3. Call a non-singular reduction of w any simply 3-secant plane $\mathbb{P}^2 \subset \mathbb{P}^{3a+2} = \mathbb{P}\mathcal{J}_3(\mathbb{A})$ to $\mathbb{AP}^2$ which passes through the point w.*

2.3. **The four varieties of reductions $Y_a \subset G(2, \mathcal{J}_3(\mathbb{A})_0)$.** Let $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o = \mathbb{P}\mathcal{J}_3(\mathbb{A}) - \mathbb{D}_3(\mathbb{A})$ be as above, and let

$$\pi_w : \mathbb{P}\mathcal{J}_3(\mathbb{A}) \dashrightarrow \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$$

be the rational projection from w. One can identify the base space $\mathbb{P}\mathcal{J}_3(\mathbb{A})_w$ of π_w with the polar hyperplane $\mathbb{P}^{3a+1}_w \subset \mathbb{P}^{3a+2}$ to the rank 3 point w, defined has follows: let $\det(X,Y,Z)$ be the polarisation of the determinant, i.e. the unique symmetric trilinear form on $\mathcal{J}_3(\mathbb{A})$ such that $\det(X) = \det(X, X, X)$. Then the polar hyperplane to w is defined by the equation $\det(W, W, X) = 0$, where $W \in \mathcal{J}_3(\mathbb{A})$ is any representative of w.

Since the point w doesn't lie in the secant variety $Sec(\mathbb{AP}^2) = \mathbb{D}_3(\mathbb{A})$ (see Lemma 2.1), then the projection π_w sends: **(a)** the Severi variety $X_a = \mathbb{AP}^2$ isomorphically to its image $\overline{X}_a \subset \mathbb{P}^{3a+1}_w$; **(b)** any non-singular reduction $\mathbb{P}^2$ of w to a line l which is simply 3-secant to $\bar{X}_a$.

Inversely, any line $l \subset \mathbb{P}^{3a+1}_w$ which is simply 3-secant to $\overline{X_a}$ is a projection from w of a unique plane $\mathbb{P}^2 \in Y^o_w$. Therefore π_w embeds the set of reductions of w as a subset $Y^o_{a,w}$ of the grassmannian $G(2, \mathcal{J}_3(\mathbb{A})_w)$ of lines in $\mathbb{P}^{3a+1}_w = \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$, which yields the following

Definition 2.4. *For the point $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o = \mathbb{P}\mathcal{J}_3(\mathbb{A}) - \mathbb{D}_3(\mathbb{A})$, define the variety $Y_{a,w}$ to be the closure of $Y^o_{a,w}$ in the grassmannian $G(2, \mathcal{J}_3(\mathbb{A})_w) = G(2, 3a+2)$.*

Since the group $SL_3(\mathbb{A})$ acts transitively on the points $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$, as well as on the Severi variety $\mathbb{AP}^2$ then all the varieties $Y_{a,w}$, $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ are projectively equivalent, by the induced action of $SL_3(\mathbb{A})$ on $G(2, \mathcal{J}_3(\mathbb{A}))$, to the same variety Y_a; and we let

$$Y_a := Y_{a,I}$$

where I is the projectivized unit matrix $diag(1,1,1) \in \mathcal{J}_3(\mathbb{A})$. Note that $\mathcal{J}_3(\mathbb{A})_I$ coincides with $\mathcal{J}_3(\mathbb{A})_0$, the space of traceless matrices in $\mathcal{J}_3(\mathbb{A})$.

2.4. **The $\mathbb{C}$-column.** Denote by $X^a = \mathbb{P}(T^2_{\mathbb{P}}),\ \times^2\mathbb{P}^2, LG(3,6),\ E^{ad}_6,\ a = 1,2,4,8$ the four varieties of the $\mathbb{C}$-column of the Freudenthal square. By the preceding, for any $a = 1,2,4,8$ the choice of the Jordan algebra $\mathcal{J}_3(\mathbb{A})$ defines uniquely the group $SL_3(\mathbb{A})$, and the variety $X_a = \mathbb{AP}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ in the $\mathbb{C}$-line of the Freudenthal square. Similarly, $\mathcal{J}_3(\mathbb{A})$ defines uniquely the variety X^a in the $\mathbb{C}$-column as the closed orbit of the projective representation ρ^a of the group $SL_3(\mathbb{A})$, as follows; as above we use the notation of weights from [3]:

– ρ^1 is the adjoint representation of SL_3, defined by the weight $\omega_1 + \omega_2$ of A_2.

– ρ^2 is defined by any pair of weights (ω'_i, ω''_j), $1 \le i,j \le 2$ of the two copies (A'_2, A''_2) of A_2 in the diagram of $SL_3 \times SL_3$; in particular $\mathcal{J}_3(\mathbb{C})$ is isomorphic to the algebra $\otimes^2\ \mathbf{C}^3$ of matrices of order 3.

– ρ^4 is defined by the weight ω_3 of A_5 giving the 3-rd alternative power of the standard representation of SL_6;

– ρ^8 is the adjoint representation of the group E_6 given by the weight ω_2 of the diagram E_6.

2.5. **Points in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ and isotropic varieties from the $\mathbb{R}$-column.**

2.5.1. *Points and isotropic groups.* Let ρ_a be the action $SL_3(\mathbb{A})$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$. For a point $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ define $SO_3(\mathbb{A})_w \subset SL_3(\mathbb{A})$ to be the connected component in the isotropy subgroup of w. Since $\mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ is a ρ_a-orbit of $SL_3(\mathbb{A})$, then the group $SL_3(\mathbb{A})$ permutes the set

$$\{SO_3(\mathbb{A})_w \subset SL_3(\mathbb{A}), w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o\}$$

of all these copies this way identifying any of them with the subgroup $SO_3(\mathbb{A}) = SO_3(\mathbb{A})_I \subset SL_3(\mathbb{A})$ preserving the projective unit matrix $I = diag(1,1,1)$. By [7], Proposition 3.2, this group $SO_3(\mathbb{A})$ coincides with the automorphism group $\mathrm{Aut}\mathcal{J}_3(\mathbb{A})$ of the Jordan algebra $\mathcal{J}_3(\mathbb{A})$. It preserves not only the determinant, but also the linear form $\mathrm{trace}(X) = \det(I,I,X)$ and the quadratic form $Q(X) = \mathrm{trace}(X^2) = \det(I,I,X)^2 - 2\det(I,X,X)$.

The following result is well-known, see e.g. [1]:

Lemma 2.5. *The action of $SO_3(\mathbb{A})$ on the Severi variety X_a has exactly two orbits, the hyperplane section $X_a^0 = X_a \cap \mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ and its complement.*

For $a = 1,2,4,8$, $SO_3(\mathbb{A})$ is correspondingly SO_3, SL_3, Sp_6, F_4, see e.g. [17]. We denote by $SO_{3,w}$, $SL_{3,w}$, $Sp_{6,w}$, $F_{4,w}$ the copy of $SO_3(\mathbb{A})_w$ in $SL_3(\mathbb{A})_w = SL_3$, $SL_3 \times SL_3$, SL_6, E_6 defined by $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$.

For a fixed, $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$, the projective representation ρ^a of $SL_3(\mathbb{A})$ induces a projective representation $\rho^a|_w$ of the subgroup $SO_3(\mathbb{A})$. The representation $\rho^a|_w$ is already reducible; in fact one has, in terms of highest weights:

– for $a = 1$, $\rho^1|_w = 4\omega_1 \oplus 2\omega_1$, and we let $\rho^1_w = 4\omega_1$.
– for $a = 2$, $\rho^2|_w = 2\omega_1 \oplus \omega_1$, and we let $\rho^2_w = 2\omega_1$.
– for $a = 4$, $\rho^4|_w = \omega_3 \oplus \omega_1$ and we let $\rho^4_w = \omega_3$.
– for $a = 8$, $\rho^8|_w = \omega_1 \oplus \omega_4$, and we let $\rho^8_w = \omega_1$.

For any $a = 1,2,4,8$, the subrepresentation ρ^a_w of $SO_3(\mathbb{A})_w$ is irreducible; and the choice of $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})$ defines uniquely the projective representation subspace $\mathbb{P}(V^a_w)$ of ρ^a_w, in the space $\mathbb{P}(V^a)$ of ρ^a. For $a = 1,2,4,8$ denote these subspaces correspondingly by

$$\mathbb{P}(Sym^4_w\mathbf{C}^2) \subset \mathbb{P}(sl_3), \quad \mathbb{P}(Sym^2_w\mathbf{C}^3) \subset \mathbb{P}(\otimes^2\ \mathbf{C}^3),$$
$$\mathbb{P}(\wedge^{(3)}_w\mathbf{C}^6) \subset \mathbb{P}(\wedge^3_w\mathbf{C}^6), \qquad \mathbb{P}(\mathfrak{f}_{4,w}) \subset \mathbb{P}(\mathfrak{e}_6).$$

In particular $dim\ \mathbb{P}(V^a_w) = 4,5,13,51$ in the space $\mathbb{P}(V^a)$ of dimension $7,8,19,77$.

Definition 2.6. (Isotropic spaces and isotropic varieties) *For $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$, $a = 1,2,4,8$ we call the subspace $V^a_w \subset V^a$ (respectively the projective subspace $\mathbb{P}(V^a_w) \subset \mathbb{P}(V^a)$) the isotropic subspace of w (respectively the isotropic projective space of w). We call the closed $SO_3(\mathbb{A})_w$-orbit $X^a_w \subset \mathbb{P}(V^a_w)$ the isotropic variety of w.*

For $a = 1,2,4,8$, denote the isotropic subvariety $X^a_w \subset X^a$ by

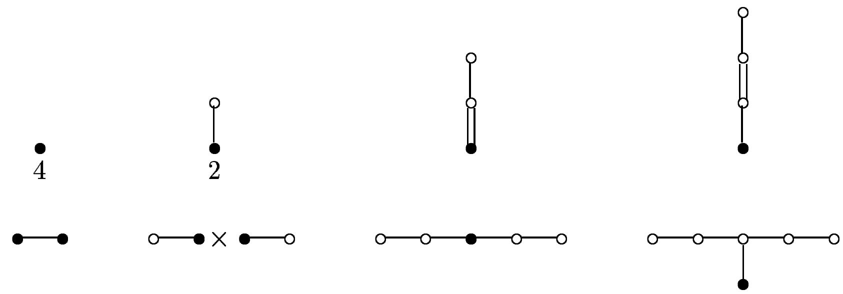
$$v_4(\mathbb{P}^1)_w \subset \mathbb{P}(T^2_{\mathbb{P}}), \quad v_2(\mathbb{P}^2)_w \subset \times^2\mathbb{P}^2, \quad LG_w(3,6) \subset G(3,6) \quad \text{and } F^{ad}_{4,w} \subset E^{ad}_6.$$

Now the following is direct:

Lemma 2.7. *Let $a = 1,2,4,8$. Then any $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ defines uniquely the isotropic subvariety $X^a_w \subset X^a$ in the $\mathbb{R}$-column of the Freudenthal square as the intersection*

$$X^a_w = X^a \cap \mathbb{P}(V^a_w)$$

of $X^a \subset \mathbb{P}(V^a)$ with the isotropy projective subspace $\mathbb{P}(V^a_w)$ of w.



Weighted Dynkin diagrams of the varieties X^a and X^a_w

The weighted Dynkin diagrams of the varieties X_a^w are obtained by *folding* those of the varieties X_a.

2.5.2. **Isotropic varieties as zero-sets.** In this section we let $a > 1$. It turns out that the isotropy subvarieties $X^a_w \subset X^a$ are zero-sets of sections of a homogeneous vector bundle $\mathcal{E}^a$ on X^a. Recall that an irreducible homogeneous vector bundle on a homogeneous variety $X = G/P$ is determined by the highest weight of the corresponding P-module, which can be encoded in a weighted Dynkin diagram. To get the weighted Dynkin diagram of our vector bundle on X^a, we just superimpose the weighted diagram of the Severi variety X_a, to that of X^a.

Since the weighted Dynkin diagram of X^a has a twofold symmetry, while that of X_a does not, we obtain in fact *two* vector bundles $\mathcal{E}^a$ and $\mathcal{E}^{a\prime}$ on X^a, which can be deduced one from the other through the action of an outer involutive automorphism of X^a.

Proposition 2.8. *The vector bundles $\mathcal{E}^a$ and $\mathcal{E}^{a\prime}$ on X^a are generated by their global sections, and their spaces of global sections are isomorphic to the Jordan algebra $\mathcal{J}_3(\mathbb{A})$ and its dual $\mathcal{J}_3(\mathbb{A})^*$. Their ranks and determinants are as follows:*

$$\begin{array}{ll} rank\ \mathcal{E}^8 = 6, & det\ \mathcal{E}^8 = \mathcal{O}(3), \\ rank\ \mathcal{E}^4 = 3, & det\ \mathcal{E}^4 = \mathcal{O}(2), \\ rank\ \mathcal{E}^2 = 2, & det\ \mathcal{E}^2 = \mathcal{O}(1,2). \end{array}$$

Proof. The first assertion is an immediate consequence of the Borel-Weil theorem. The rank of $\mathcal{E}^a$ (and $\mathcal{E}^{a\prime}$) can be read off its weighted Dynkin diagram, since the P-module that defines $\mathcal{E}^a$ is encoded in the weighted diagram which is obtained after deleting the black nodes that define X^a. To compute the determinant, we need to list the weights of this P-module, which are just the images of the highest weight by the Weyl group of P (indeed, theses modules are minuscule, as we can see case by case). Taking the sum of these weights, we get the weight of the determinant. □

More explicitly, the vector bundles $\mathcal{E}^a$ and $\mathcal{E}^{a\prime}$ on X^a can be described as follows. On $X^2 = \mathbb{P}^2 \times \mathbb{P}^2$, we have the pull-backs $\mathcal{O}(1)$ and $\mathcal{O}(1)'$ of the hyperplane line bundles on the two copies of $\mathbb{P}^2$, and the pull-backs T and T' of the rank two tautological bundles; then $\mathcal{E}^2$ and $\mathcal{E}^{2\prime}$ are the bundles $Hom(T, \mathcal{O}(1)')$ and $Hom(T', \mathcal{O}(1))$. On $X^4 = G(3,6)$, let T and Q denote the tautological and quotient vector bundles, both of rank three; then $\mathcal{E}^4$ and $\mathcal{E}^{4\prime}$ are the bundles $\wedge^2 Q$ and $\wedge^2 T^*$.

The case of $X^8 = E_6^{ad}$ is slightly more subtle. Recall that the adjoint variety E_6^{ad} is the closed orbit in $\mathbb{P}\mathfrak{e}_6$. If $X \in \mathfrak{e}_6$ defines a point of the adjoint variety, consider its action $d\rho_8(X)$ on $\mathcal{J}_3(\mathbb{O})$ (see **2.2**). We claim that $d\rho_8(X)$ has rank six. To check this, we use the fact that $\mathfrak{e}_6$ contains a copy of $\mathfrak{so}_8$ whose action on $\mathcal{J}_3(\mathbb{O})$ can be described very explicitely. Recall that the *infinitesimal triality principle* asserts that for any $g = g_1 \in \mathfrak{so}_8$, there exists uniquely defined operators $g_2, g_3 \in \mathfrak{so}_8$ such that $g_2(xy) = xg_1(y) + g_3(x)y$ for all $x, y \in \mathbb{O}$. Then the action of g on $\mathcal{J}_3(\mathbb{O})$ is given by the formula [16, 13, 20]

$$d\rho_8(g)\begin{pmatrix} c_1 & x_3 & x_2 \\ \bar{x}_3 & c_2 & x_1 \\ \bar{x}_2 & \bar{x}_1 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & g_3(x_3) & g_2(x_2) \\ \overline{g_3(x_3)} & 0 & g_1(x_1) \\ \overline{g_2(x_2)} & \overline{g_1(x_1)} & 0 \end{pmatrix}.$$

When g belongs to the adjoint variety of $\mathfrak{so}_8$ (which is naturally contained in that of $\mathfrak{e}_6$), g_1, g_2 and g_3 have minimal rank, that is rank two, so clearly $d\rho_8(X)$ has rank six. Therefore we get two vector bundles whose fibers at $X \in E_6^{ad}$ are $\mathcal{J}_3(\mathbb{O})/Ker\ d\rho_8(X)$ and $(Im\ d\rho_8(X))^*$, respectively. These bundles are homogeneous of rank six, and respectively quotients of the trivial bundles with fibers $\mathcal{J}_3(\mathbb{O})$ and $\mathcal{J}_3(\mathbb{O})^*$: they are our bundles $\mathcal{E}^8$ and $\mathcal{E}^{8\prime}$.

Distinguishing the two bundles $\mathcal{E}^a$ and $\mathcal{E}^{a\prime}$ is really a matter of convention. Our choice will be such that the space of global sections of the vector bundle $\mathcal{E}^a$ is the Jordan algebra $\mathcal{J}_3(\mathbb{A})$, rather than its dual.

Since for a section $w \in H^0(X^a, \mathcal{E}^a)^o = \mathcal{J}_3(\mathbb{A})^o$, and $c \in \mathbb{C}^*$, the zero-sets $Z(w)$ and $Z(cw)$ coïncide, one can regard equivalently the elements $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})$ as *projective sections* of $\mathcal{E}^a$, and their zero-sets $Z(w) \subset X^a$.

Proposition 2.9. *Let $a > 1$. Then for any projective section*

$$w \in \mathbb{P}(H^0(X^a, \mathcal{E}^a)^o) = \mathbb{P}\mathcal{J}_3(\mathbb{A})^o,$$

the zero-set $Z(w) \subset X^a$ coincides with the isotropy subvariety $X^a_w \subset X^a$.

Proof. We treat the case $a = 8$, the other ones are simpler. Since $\mathbb{P}\mathcal{J}_3(\mathbb{O})^0$ is an orbit of E_6, we may suppose that $w = I$, the identity of $\mathcal{J}_3(\mathbb{O})$. By the description we have just given of $\mathcal{E}^8$, a point $X \in E_6^{ad}$ belongs to the zero-set $Z(I)$ if and only if $I \in Ker\ d\rho_8(X)$, which means that X belongs to the isotropy Lie algebra of I. But recall that the (connected component of the identity in the) isotropy group of I is the automorphism group $\mathrm{Aut}\mathcal{J}_3(\mathbb{O}) = F_4$, hence the isotropy Lie algebra is $\mathfrak{f}_4$. We conclude that $Z(I) = E_6^{ad} \cap \mathbb{P}\mathfrak{f}_4 = F_4^{ad}$. □

2.6. **The 3-secant Lemma.** Let $G(2, \mathcal{J}_3(\mathbb{A}))$ be the grassmannian of lines in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$, and let $\Delta(\mathbb{A}) \subset G(2, \mathcal{J}_3(\mathbb{A}))$ be the subset of lines $L \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ which are not simply 3-secant to the determinant hypersurface $\mathbb{D}_3(\mathbb{A})$. Clearly $\Delta(\mathbb{A})$ is a hypersurface in $G(2, \mathcal{J}_3(\mathbb{A}))$, so $\Delta(\mathbb{A}) \in |\mathcal{O}_{G(2,\mathcal{J}_3(\mathbb{A}))}(d)|$ for some d in the Plücker polarization of $G(2, \mathcal{J}_3(\mathbb{A}))$; and we shall see that $d = 6$. Indeed, d is the number of intersection points of $\Delta(\mathbb{A})$ with the general line $\Lambda \subset G(2, \mathcal{J}_3(\mathbb{A}))$. Such a general line Λ is a plane pencil of lines in a general plane $\mathbb{P}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ passing through a fixed general point $w \in \mathbf{P}^2$. The plane

$\mathbf{P}^2$ intersects $\mathbb{D}_3(\mathbb{A})$ at a smooth cubic C; and now the Hurwitz formula implies that through w pass exactly $d = 6$ lines $L_i \in \Lambda$, $i = 1, ..., 6$ which are tangent to C.

Lemma 2.10. *Let $L \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ be any simply 3-secant line to $\mathbb{D}_3(\mathbb{A})$, i.e. $L \in G(2, \mathcal{J}_3(\mathbb{A}))^o$.*

Then in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ there exists a unique simply 3-secant plane (or a reduction plane – see **1.5***) to the Severi variety $X_a = \mathbb{A}\mathbb{P}^2$ which passes through L. Denote this plane by*

$$\mathbb{P}^2 = \mathbb{P}^2_\varepsilon = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle,$$

$\varepsilon_1, \varepsilon_2, \varepsilon_3$ being the three intersection points of $\mathbb{P}^2$ with X_a. The intersection

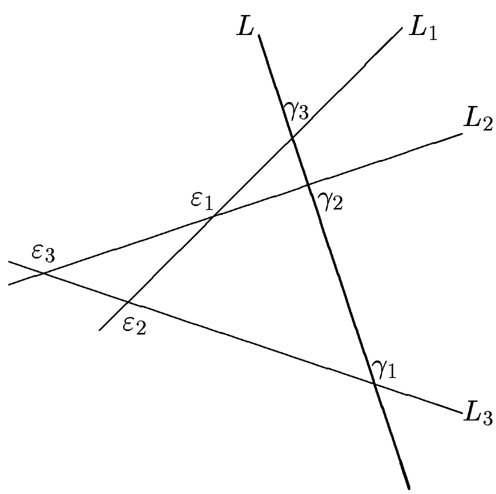
$$\Delta_\varepsilon = \mathbb{P}^2_\varepsilon \cap D_3(\mathbb{A})$$

is a triangle with sides $L_k = \langle \varepsilon_i, \varepsilon_j \rangle$, $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Let $\gamma_1, \gamma_2, \gamma_3$ be the three intersection points of L and $\mathbb{D}_3(\mathbb{A})$. By [31], for any γ_i, $i = 1, 2, 3$ there exists a unique subspace $\mathbb{P}^{a+1}_i \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ which passes through γ_i and intersects the Severi variety X_a along a smooth a-dimensional quadric $Q_i \subset \mathbb{P}^{a+1}_i$. Moreover any space $\mathbb{P}^{a+1}_i$ is swept out by all the lines through γ_i which are bisecant or tangent to X_a; and any two quadrics Q_i and Q_j intersect each other at a unique point ε_k, $\{i, j, k\} = \{1, 2, 3\}$.

We shall see that the plane $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ pases through L. Indeed, let e.g. $\varepsilon_3 \in X_a$ be the intersection point of the quadrics Q_1 and Q_2, let $\mathbb{P}^2_3$ be the plane spanned by L and ε_3, and let L_i be the line $\overline{\gamma_i \varepsilon_3}$, $i = 1, 2$.

Since the line L_i lies in $\mathbb{P}^{a+1}_1$ and passes through γ_i, it is bisecant to X_a: the intersection $L_i \cap X_a$ contains ε_3 and another point ε'_i.



Since $\mathbb{P}^2_3 \supset L$ and the general point of L has rank 3, the plane $\mathbb{P}^2_3$ can't lie entirely in the determinant cubic $\mathbb{D}_3(\mathbb{A})$. Therefore, it intersects $\mathbb{D}_3(\mathbb{A})$ along a 1-cycle Δ of degree 3. Being bisecant to X_a, the two lines L_1, L_2 are components of Δ. Moreover Δ passes through the rank 2 point $\gamma_3 \in \mathbb{P}^2_3$. Therefore, there exists another line $L_3 \subset \mathbb{P}^2_3$, passing through γ_3, such that the cubic cycle $\Delta = L_1 + L_2 + L_3$. This line is bisecant to X_a at its intersection points with L_1 and L_2, which must coincide with ε'_1 and ε'_2; and now the uniqueness of the triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ yields $\varepsilon'_i = \varepsilon_i$ and $\mathbb{P}^2_3 = \mathbb{P}^2_\varepsilon$.

In addition, if in the above construction one assumes that one of the lines, say L_1 degenerate to a tangent to X_a at ε_3, i.e. $\varepsilon'_2 = \varepsilon_3$, then the line L_2 will degenerate to a multiple (double or triple) component of the intersection cubic cycle Δ of $\mathbb{P}^2_3$, i.e. either $\Delta = L_1 + 2L_2$, or $L_1 = L_2$ and $\Delta = 3L_2$. But then L can't be simply 3-secant to $\mathbb{D}_3(\mathbb{A})$, since $L \subset \mathbb{P}^2_3$ and the intersection cycle $L \cap \mathbb{D}_3(\mathbb{A})$ must be contained (as a set) in the support of Δ. □

Corollary 2.11. *The group $SL_3(\mathbb{A})$ acts transitively on the set of reduction planes in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$.*

Proof. From the proof of the 3-secant lemma, we see that a reduction plane, i.e. a simply 3-secant plane to X_a, is uniquely defined by a simply 3-secant line to $\mathbb{D}_3(\mathbb{A})$. So we just need to prove that $SL_3(\mathbb{A})$ acts transitively on the set of these lines.

But it was observed in [28] that the action of $SL_3(\mathbb{A})$ on $\mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})$ is prehomogeneous, the dense orbit being given by the complement of the discriminant hypersurface, that is, precisely, the set of couples (x, y) in $\mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})$ which are independent, and such that the line $\overline{xy}$ is simply trisecant to $\mathbb{D}_3(\mathbb{A})$. Our claim obviously follows. □

Remark. Of course, the prehomogeneity of $\mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})$ is not an accident. Indeed, consider the exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. Their adjoint representations correspond to an extremal node of their Dynkin diagram. Delete its unique neighbour. Then the resulting (non connected) Dynkin diagram is that of $SL_3(\mathbb{A}) \times SL_2$, and if we darken the nodes which neighboured the one which has been erased, we get the weighted Dynkin diagram of its representation $\mathcal{J}_3(\mathbb{A}) \otimes \mathbb{C}^2 = \mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})$. By a Theorem of Vinberg, in such a situation, the action of $SL_3(\mathbb{A}) \times GL_2$ on $\mathcal{J}_3(\mathbb{A}) \oplus \mathcal{J}_3(\mathbb{A})$ is prehomogeneous (see [27]). In a sense, the previous Corollary is thus really a consequence of the existence of exceptional simple Lie algebras.

2.7. Lines in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$, reduction planes and triality varieties from the $\underline{0}$-column.

2.7.1. *Lines, reduction planes and triality subgroups.* In the hypotheses and notations of Lemmas 2.7 and 2.10, we shall see that the isotropic varieties X^a_w of all the points $w \in \mathbb{P}^2_\varepsilon - \Delta$ have a common subvariety – a *triality subvariety* Z_ε. We shall regard one of the rank 3 points $w \in L$ as fixed; this way identifying the unique 3-secant plane through $L \supset w$ to a non-singular reduction of w. Let $v \in L \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ be another point of L and let $\Delta = \mathbb{P}^2_\varepsilon \cap \mathbb{D}_3(\mathbb{A})$.

Lemma 2.12. *The isotropy subgroup $T^a_{w,v} = SO_3(\mathbb{A})_v \cap SO_3(\mathbb{A})_w$ of the pair (w, v) is, up to a finite group, a copy of the triality group $T(\mathbb{A})$.*

Recall from [20] that this triality group is defined as

$$T(\mathbb{A}) = \{g = (g_1, g_2, g_3) \in SO(\mathbb{A})^3,\ g_2(xy) = g_1(x)g_3(y)\ \forall x, y \in \mathbb{A}\}.$$

We have $T(\mathbb{A}) = 1, G_m \times G_m, SL_2 \times SL_2 \times SL_2, Spin_8$ respectively for $a = 1, 2, 4, 8$. Denote its Lie algebra by

$$\mathfrak{t}(\mathbb{A}) = \{u = (u_1, u_2, u_3) \in \mathfrak{so}(\mathbb{A})^3,\ u_1(xy) = u_2(x)y + xu_3(y)\ \forall x, y \in \mathbb{A}\}.$$

By construction, $\mathfrak{t}(\mathbb{A})$ has three natural actions on $\mathbb{A}$, which we denote $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$. By a special case of the triality construction of Freudenthal's magic square [18], we have a natural decomposition

$$\mathfrak{so}(\mathbb{A}) = Der\mathcal{J}_3(\mathbb{A}) = \mathfrak{t}(\mathbb{A}) \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3.$$

The Lie bracket can be explicitely described in this decomposition, but what we will need again and again in the sequel is the explicit action on $\mathcal{J}_3(\mathbb{A})$, which is given by the following formulas [16, 13]. Let $u = (u_1, u_2, u_3) \in \mathfrak{t}(\mathbb{A})$ and $a_i \in \mathbb{A}_i$. Then

$$(1) \qquad u\begin{pmatrix} r_1 & x_3 & x_2 \\ \overline{x_3} & r_2 & x_1 \\ \overline{x_2} & \overline{x_1} & r_3 \end{pmatrix} = \begin{pmatrix} 0 & u_3(x_3) & u_2(x_2) \\ \overline{u_3(x_3)} & 0 & u_1(x_1) \\ \overline{u_2(x_2)} & \overline{u_1(x_1)} & 0 \end{pmatrix},$$

$$(2) \qquad a_1\begin{pmatrix} r_1 & x_3 & x_2 \\ \overline{x_3} & r_2 & x_1 \\ \overline{x_2} & \overline{x_1} & r_3 \end{pmatrix} = \begin{pmatrix} 0 & -\bar{a}_1\bar{x}_2 & \bar{x}_3\bar{a}_1 \\ -x_2a_1 & -2q(a_1,x_1) & (r_2-r_3)a_1 \\ a_1x_3 & (r_2-r_3)\bar{a}_1 & 2q(a_1,x_1) \end{pmatrix},$$

$$(3) \qquad a_2\begin{pmatrix} r_1 & x_3 & x_2 \\ \overline{x_3} & r_2 & x_1 \\ \overline{x_2} & \overline{x_1} & r_3 \end{pmatrix} = \begin{pmatrix} 2q(a_2,x_2) & \bar{a}_2\bar{x}_1 & (r_3-r_1)a_2 \\ x_1a_2 & 0 & -\bar{x}_3\bar{a}_2 \\ (r_3-r_1)\bar{a}_2 & -a_2x_3 & -2q(a_2,x_2) \end{pmatrix},$$

$$(4) \qquad a_3\begin{pmatrix} r_1 & x_3 & x_2 \\ \overline{x_3} & r_2 & x_1 \\ \overline{x_2} & \overline{x_1} & r_3 \end{pmatrix} = \begin{pmatrix} 2q(a_3,x_3) & (r_2-r_1)a_3 & \bar{a}_3\bar{x}_1 \\ (r_2-r_1)\bar{a}_3 & -2q(a_3,x_3) & -\bar{x}_2\bar{a}_3 \\ x_1a_3 & -a_3x_2 & 0 \end{pmatrix}.$$

Recall that q is the natural scalar product on the complexified normed algebra $\mathbb{A}$. Using these formulas we can easily prove the Lemma.

Proof. Since the action of $SL_3(\mathbb{A})$ on the set of simply trisecant lines to $\mathbb{D}_3(\mathbb{A})$, we can suppose that L is the line of trace zero diagonal matrices, whose intersection with $\mathbb{D}_3(\mathbb{A})$ is the triple of points in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ defined by the matrices $diag(0,1,-1), diag(1,0,-1)$ and $diag(1,-1,0)$. We then read off the previous formulas that the subalgebra of $\mathfrak{so}(\mathbb{A})$ consisting of operators that kill every diagonal (traceless) matrices, is exactly $\mathfrak{t}(\mathbb{A})$. This implies the claim. □

Note that $T^a_{w,v} = T^a_{w',v'}$ for any other couple (w', v') of rank 3 points of $\mathbb{P}^2_\varepsilon$, such that the line $L' = \overline{w'v'}$ does not pass through a vertex ε_i of Δ. Indeed, the induced action of $T^a_{w,v} \subset SL_3(\mathbb{A})$ on $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ must fix together with w, v also all the lines (hence – all the points) in the unique reduction plane $\mathbb{P}^2_\varepsilon$ passing through the line $L = \overline{wv}$.

Therefore, for any reduction plane $\mathbb{P}^2_\varepsilon \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ the intersection

$$T^a_\varepsilon = \cap\{SO_3(\mathbb{A})_u : u \in \mathbb{P}^2_\varepsilon - \Delta\}$$

is, up to a finite group, a copy of the triality group of the $\underline{0}$-column of the Freudenthal square. Moreover,

$$T^a_\varepsilon = T^a_{w,v} = SO_3(\mathbb{A})_w \cap SO_3(\mathbb{A})_v$$

for any pair $w, v \in \mathbb{P}^2_\varepsilon$, $w \neq v$, such that the line $L = \overline{wv}$ does not pass through a vertex $\varepsilon_i \in \mathbb{AP}^2$ of the triangle $\Delta = \mathbb{P}^2_\varepsilon \cap \mathbb{D}_3(\mathbb{A})$.

2.7.2. *Lines, reduction planes and triality subspaces.* Let $w, v \in \mathbb{P}_\varepsilon - \Delta$ be as above, and let $\mathbb{P}(V^a_w)$ and $\mathbb{P}(V^a_v)$ be their isotropic subspaces of $\subset \mathbb{P}(V^a)$. Then the intersection subspace $\mathbb{P}(V^a_\varepsilon) = \mathbb{P}(V^a_w) \cap \mathbb{P}(V^a_v) = \cap\{\mathbb{P}(V^a_u) : u \in \mathbb{P}^2_\varepsilon - \Delta\}$ in $\mathbb{P}(V^a)$ is a projective representation space of their common triality subgroup $T^a_{w,v} = T^a_\varepsilon$.

Definition 2.13. (Triality subspaces) *For any reduction plane $\mathbb{P}^2_\varepsilon = \langle \varepsilon_1; \varepsilon_2, \varepsilon_3 \rangle \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ we call the projective subspace $\mathbb{P}(V^a_\varepsilon) \subset \mathbb{P}(V^a)$ the triality subspace of $\mathbb{P}^2_\varepsilon$.*

Proposition 2.14. *Let $a > 1$, with the same notations as above.*

(1) *$Z^a_{w,v} := X^a_w \cap X^a_v = Z(w) \cap Z(v)$ is a copy of the triality variety $Z^a = \vee^3\mathbb{P}^o, \times^3\mathbb{P}^1, D^{ad}_4$ from the $\underline{0}$-th column of the Freudenthal square.*

(2) *$Z^a_{w,v} = Z^a_\varepsilon := \cap\{X^a_u : u \in \mathbb{P}^2_\varepsilon - \Delta\} = \cap\{Z(u) : u \in \mathbb{P}^2_\varepsilon - \Delta\}$.*

Proof. The second assertion follows from the previous discussion. Let us prove the first assertion. The case $a = 2$ is easy, so we begin with $a = 4$: here w, v are generic symplectic forms on $\mathbb{C}^6$, and $Z^4_{w,v}$ is the intersection in $G(3,6)$ of the lagrangian grassmannians $LG_w(3,6)$ and $LG_v(3,6)$. It is a classical fact that w and v can be simultaneously diagonalized, which means that we can find three planes P_1, P_2, P_3 in general position in $\mathbb{C}^6$, which are orthogonal with respect to both w and v. It is then easy to see that a 3-plane which is isotropic with respect to w and v must be generated by three lines $l_i \subset P_i$. Hence the isomorphism $LG_w(3,6) \cap LG_v(3,6) \cong \times^3\mathbb{P}^1$. See [15] for more details.

Suppose now that $a = 8$, and let again $L = \overline{wv}$ be the line of traceless diagonal matrices in $\mathcal{J}_3(\mathbb{O})$. By the description of $\mathcal{E}^8$ given in **2.5.2**, we have $Z(w) \cap Z(v) = E^{ad}_6 \cap \mathbb{P}\mathfrak{so}_8 = D^{ad}_4$. □

2.8. **Triality varieties as zero-sets.** Let $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$, $a > 1$; and denote by $\mathcal{F}^a_w = \mathcal{E}^a|_{X^a_w}$ the restriction of the homogeneous vector bundle $\mathcal{E}^a \to X^a$, to the isotropic subvariety $X^a_w \subset X^a$. One can check case-by-case that $\mathcal{F}^a_w$ is an irreducible homogeneous vector bundle on X^a_w, and that its space of global sections is the polar hyperplane $\mathcal{J}_3(\mathbb{A})_w$. Since X^a_w is defined as the zero-locus of w, considered as a global section of $\mathcal{E}^a$ on X^a, this isomorphism comes from the natural maps

$$\mathcal{J}_3(\mathbb{A})_w \simeq \mathcal{J}_3(\mathbb{A})/\mathbf{C}w \simeq H^0(X^a, \mathcal{E}^a)/\mathbf{C}w \stackrel{res_{X^a_w}}{\longrightarrow} H^0(X^a_w, \mathcal{F}^a_w).$$

For any $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ the polar hyperplane $\mathbb{P}\mathcal{J}_3(\mathbb{A})_w \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ to w does not pass through w, and we can identify the base of the rational projection p_w of $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ from w with $\mathbb{P}\mathcal{J}_3(\mathbb{A})_w$. This identifies the lines $L \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ which pass through w and the projective sections $w_L \in \mathbb{P}(X^a_w, \mathcal{F}^a_w)$, i.e.

$$p_w : \mathbb{P}\mathcal{J}_3(\mathbb{A}) \to \mathbb{P}\mathcal{J}_3(\mathbb{A})_w = \mathbb{P}(X^a_w, \mathcal{F}^a_w), \quad L \mapsto w_L := p_w(L).$$

We shall denote by $Z(w_L) \subset X^a_w$ the zero-set of the projective section w_L of $\mathcal{F}^a_w$.

For the fixed $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$, let $U_w \subset G(2, \mathcal{J}_3(\mathbb{A}))$ be the subset of all simply 3-secant lines to $\mathbb{D}_3(\mathbb{A})$ which pass through w. In particular p_w embeds U_w in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_w$, and its complement

$$\Delta(\mathbb{A})_w = \mathbb{P}\mathcal{J}_3(\mathbb{A})_w - U_w \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$$

is a hypersurface of degree 6 in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_w$ – the *discriminant* hypersurface of w.

Let $L \in U_w$. By the preceding one can identify L with a projective section $w_L \in U_w \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_w = \mathbb{P}H^0(X^a_w, \mathcal{F}^a_w)$; and we shall identify the zero-set $Z(w_L) \subset X^a_w$ of w_L. First, by Lemma 2.10, through the line $L \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ passes a unique simply 3-secant plane $\mathbb{P}^2_\varepsilon = \mathbb{P}^2_{\varepsilon(L)}$ to X_a; let $\mathbb{P}^1_{\varepsilon(L)} \in Y^o_a$ be the proper p_w-image of $\mathbb{P}^2_{\varepsilon(L)}$ in $G(2, \mathcal{J}_3(\mathbb{A})_w)$. Second, by Proposition 2.9, for any $v \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ the zero-set $Z(v) \subset X^a$ coincides with the isotropic subvariety $X^a_v \subset X^a$. Third, by Proposition 2.14, for any $v \in L^o = L \cap \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$ the intersection $Z^a_L := X^a_w \cap X^a_v$ is a copy of the triality variety Z^a, embedded in the isotropy subvariety $X^a_w \subset X^a$; equivalently

$$Z^a_L = Z(w_L) = \cap\{Z(u) : u \in L^o\}$$

is a copy of the triality variety $Z^a \subset X^a_w$.

2.9. **The zero-set map.** We can identify the lines L_t in $\mathbb{P}^2_{\varepsilon(L)}$ passing through the point w, with the elements of the line $\mathbb{P}^1_{\varepsilon(L)} = p_w(\mathbb{P}^2_{\varepsilon(L)}) \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$. Among these lines only three – the lines $L_{\varepsilon_i} = \overline{w\varepsilon_i}, i = 1, 2, 3$ do not belong to U_w. Let $\overline{\varepsilon}_i = w_{L_{\varepsilon_i}}$ be their projections to $\mathbb{P}\mathcal{J}_3(\mathbb{A})_w$. Then

$$Z^a_L = \cap\{Z(w_{L_t}) : p_w(L_t) \in \mathbb{P}^1_{\varepsilon(L)} - \overline{\varepsilon}(L)\}$$

where $\overline{\varepsilon}(L) = \{\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3\}$.

Through any point $w_L \in \mathbb{P}H^0(X^a_w, \mathcal{F}^a_w)^o$ passes a *unique* 3-secant line $\mathbb{P}^1_\varepsilon = \mathbb{P}^1_{\varepsilon(L)}$ – the proper p_w-image of the unique 3-secant plane $\mathbb{P}^2_{\varepsilon(L)}$ which passes through the line L; and this plane is simply 3-secant to the Severi variety X^a. In other words, if $\Delta(\mathbb{A})_w \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$ is the discriminant sextic, then the open subset

$$\mathbb{P}H^0(X^a_w, \mathcal{F}^a_w)^o = \mathbb{P}\mathcal{J}_3(\mathbb{A})^o_w = \mathbb{P}\mathcal{J}_3(\mathbb{A})_w - \Delta(\mathbb{A})_w$$

is swept out <u>once</u> by the family of 3-secant lines $\mathbb{P}^1_\varepsilon$ to $\overline{X}^a_w$; and the unique line which passes through a point of $\mathbb{P}H^0(X^a_w, \mathcal{F}^a_w)^o$ belongs to the open subset $Y^o_a \subset Y_a$ of non-singular reductions of w. One can regard such a 3-secant line $\mathbb{P}^1_\varepsilon$ equivalently as a reduction ε of w. By the preceding, all the points on such 3-secant line $\mathbb{P}^1_\varepsilon$, except the 3 points of intersection of $\mathbb{P}^1_\varepsilon$ with $\overline{X}_a$, are projective sections of $\mathcal{F}^a_w$ with the same zero-set – a triality variety Z_ε, defined uniquely by the reduction ε. Moreover, two different simply 3-secant lines $\mathbb{P}^1_\varepsilon$ and $\mathbb{P}^1_\sigma$ (equivalently – two different non-singular reductions $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of w) have different triality varieties Z_ε and Z_σ. Indeed, two different reduction planes $\mathbb{P}^2_\varepsilon = \langle\varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$ and $\mathbb{P}^2_\sigma = \langle\sigma_1, \sigma_2, \sigma_3\rangle$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ define two different triality subgroups T^a_ε and T^a_σ of $SO_3(\mathbb{A})$, hence two different triality subspaces $\mathbb{P}(V^a_\varepsilon)$ and $\mathbb{P}(V^a_\sigma)$, and two different triality subvarieties $Z^a_\varepsilon \subset \mathbb{P}(V^a_\varepsilon)$ and $Z^a_\sigma \subset \mathbb{P}(V^a_\sigma)$.

We collect all these observations in the following

Proposition 2.15. *Let $a > 1$, and let $w \in \mathbb{P}\mathcal{J}_3(\mathbb{A})^o$. Let $p_w : \mathbb{P}\mathcal{J}_3(\mathbb{A}) \to \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$ be the rational projection from w to the polar hyperplane section to w, and let $\overline{X}_a \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$ be the isomorphic projection of the Severi variety*

$X_a \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ from w. Let $Y_a^o \subset Y^a$ be the open subset of non-singular reductions $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of w; and denote by

$$\mathbb{P}^2_\varepsilon = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \subset \mathbb{P}\mathcal{J}_3(\mathbb{A}) \quad \textit{and} \quad \mathbb{P}^1_\varepsilon = p_w(\mathbb{P}^2_\varepsilon) \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$$

correspondingly the plane and the line of the non-singular reduction $\varepsilon \in Y_a^o$.

Denote by $\mathcal{Z}(X_w^a)^o$ the (non closed) family of zero-sets $Z(v)$, for projective sections $v \in \mathbb{P}H^0(X_w^a, \mathcal{F}_w^a)^o$. Then any $Z(v) \in \mathcal{Z}(X_w^a)^o$ is a copy of the triality variety Z^a in X_w^a; and there exists a 1:1-correspondence

$$Y_a^o \cong \mathcal{Z}(X_w^a)^o, \quad \varepsilon \leftrightarrow Z_\varepsilon$$

described as follows:

1. *If $\varepsilon \in Y_a^o$, then $Z_\varepsilon = \cap\{Z(v) : v \in \mathbb{P}^1_\varepsilon - \overline{\varepsilon}\}$.*

2. *If $Z \in \mathcal{Z}(X_w^a)^o$, then the closure of*

$$\{v \in \mathbb{P}(H^0(X_w^a, \mathcal{F}_w^a)) : Z(v) = Z_\varepsilon\} \ \subset \mathbb{P}H^0(X_w^a, \mathcal{F}_w^a) = \mathbb{P}\mathcal{J}_3(\mathbb{A})_w$$

is a 3-secant line $\mathbb{P}^1_\varepsilon$ to $\overline{X}_w$, corresponding to a non-singular reduction $\varepsilon \in Y_a^o$ such that $Z = Z_\varepsilon$.

This suggests to compactify Y_a^0 by embedding it into the Hilbert scheme of X_w^a and taking the closure, but we will not do that.

We can also look at the sections $Z(v)$ of projective sections $v \in \mathbb{P}H^0(X_w^a, \mathcal{F}_w^a)$ that do not belong to the open subset $\mathbb{P}H^0(X_w^a, \mathcal{F}_w^a)$. An interesting "degeneration" of this kind occurs when v is on a line joining, in some reduction plane $\mathbb{P}^2_\varepsilon$, w to one of the three vertices ε_i of the simplex $\Delta = L_1 + L_2 + L_3$. We obtain the following picture:

a	2	4	8
Z_ε	$\vee^3\mathbb{P}^0$	$\times^3\mathbb{P}^1$	D_4^{ad}
$Z(v)$	$\mathbb{P}^1 \vee \mathbb{P}^0$	$Q^3 \times \mathbb{P}^1$	B_4^{ad}

Note that we have three vertices of Δ, hence three varieties $Z(v_1), Z(v_2), Z(v_3)$ containing Z_ε. For $a = 2$, we just have three points and we join two of them by a line. For $a = 4$ we have a product of three $\mathbb{P}^1$'s, and we embed the product of two of them into a three dimensional quadric. For $a = 8$, $\mathfrak{t}(\mathbb{O}) = \mathfrak{so}_8 \subset \mathfrak{t}(\mathbb{O}) \oplus \mathbb{O}_i \cong \mathfrak{so}_9 \subset \mathfrak{f}_4$ for each $i = 1, 2, 3$, so there are naturally three copies of $Spin_9$ in F_4 containing a given $Spin_8$, and there are three ways, inside F_4^{ad}, to embed the adjoint variety $D_4^{ad} = G_Q(2, 8)$ in a copy of $B_4^{ad} = G_Q(2, 9)$. Once again, triality leads the game.

3. Geometry of the varieties of reductions

In this section we make a detailed study of our varieties of reductions.

3.1. **Varieties of reductions are linear sections of Grassmannians.**

Our first result is that, as it is well-known for the Fano threefold Y_1, the varieties of reductions are linear sections of the ambient Grassmannians. But there is a first surprise:

Proposition 3.1. *The variety of reductions Y_a is a linear section of the Grassmannian $G(2, \mathcal{J}_3(\mathbb{A})_0)$, but non transverse for $a > 1$, and not even of the expected dimension.*

This will be proved in the next section.

The linear section is defined as follows. Recall that the automorphism group $\mathrm{Aut}\mathcal{J}_3(\mathbb{A}) = SO_3(\mathbb{A})_I$ preserves the quadratic form $Q(M) = \mathrm{trace}(M^2)$ on $\mathcal{J}_3(\mathbb{A})_0$, At the infinitesimal level, this implies that the action of $SO_3(\mathbb{A})_I$ on $\mathcal{J}_3(\mathbb{A})_0$ induces a map from the Lie algebra $\mathfrak{so}_3(\mathbb{A})$ of $SO_3(\mathbb{A})$ to the space of skew-symmetric endomorphisms $\wedge^2\mathcal{J}_3(\mathbb{A})_0$. Explicitely, choose any orthonormal basis X_i of $\mathcal{J}_3(\mathbb{A})_0$; then the map is

$$u \in \mathfrak{so}_3(\mathbb{A}) \mapsto \sum_i X_i \wedge uX_i.$$

In particular, by Schur's lemma the wedge power $\wedge^2\mathcal{J}_3(\mathbb{A})_0$ contains a copy of $\mathfrak{so}_3(\mathbb{A})$ as submodule. It turns out that there is a very simple decomposition

$$\wedge^2\mathcal{J}_3(\mathbb{A})_0 = \mathfrak{so}_3(\mathbb{A}) \oplus U_a,$$

where the module U_a is given as follows:

a	1	2	4	8
$\mathfrak{so}_3(\mathbb{A})$	$\mathfrak{sl}_2$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
U_a	$V_{6\omega_1}$	$V_{3\omega_1} \oplus V_{3\omega_2}$	$V_{\omega_1+\omega_3}$	V_{ω_3}

Here we denoted by V_ω the irreducible $\mathfrak{so}_3(\mathbb{A})$-module of highest weight ω, and we used the same indexing of the weights as [3].

We can define the projection map $\pi : \wedge^2\mathcal{J}_3(\mathbb{A})_0 \to \mathfrak{so}_3(\mathbb{A})$ by chosing two bases u_i, v_i of $\mathfrak{so}_3(\mathbb{A})$ which are dual one to each other with respect to the Killing form. Then we can let, for $X, Y \in \mathcal{J}_3(\mathbb{A})_0$,

$$\pi(X \wedge Y) = \sum_i Q(X, u_iY)v_i \in \mathfrak{so}_3(\mathbb{A}).$$

It is quite clear that, the quadratic form Q being $\mathfrak{so}_3(\mathbb{A})$-invariant, this map is well-defined, and equivariant. But by Schur's lemma there is only one such map, up to scalar, so π must be the projection, up to scalar. This gives in particular a simple characterization of U_a, since it is precisely the kernel of π: it is the subspace of $\wedge^2\mathcal{J}_3(\mathbb{A})_0$ generated by the skew tensors $X \wedge Y$ such that $Q(X, uY) = 0$ for all $u \in \mathfrak{so}_3(\mathbb{A})$.

Note also that U_a is always irreducible as a $\mathfrak{so}_3(\mathbb{A}) \times H_a$-module, where H_a is the finite group defined as follows (it is non trivial only for $a = 2$, in which case $H_a = \mathbb{Z}_2$) : let D_a be the Dynkin diagram of $\mathfrak{so}_3(\mathbb{A})$, let $d_a \subset D_a$ be the set of nodes supporting the highest weight of $\mathcal{J}_3(\mathbb{A})_0$; then H_a is the group of diagram automorphisms of D_a preserving d_a.

The fact that $Y_a \subset \mathbb{P}U_a$ can be seen as follows: if $\mathbb{P}^2_\varepsilon$ is a simple trisecant plane to X_a passing though I, and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ denote the three intersection points, its representative in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0 \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})$ is easily computed to be

$$\omega_\varepsilon = \mathrm{trace}(\varepsilon_1)\varepsilon_2 \wedge \varepsilon_3 + \mathrm{trace}(\varepsilon_2)\varepsilon_3 \wedge \varepsilon_1 + \mathrm{trace}(\varepsilon_3)\varepsilon_1 \wedge \varepsilon_2.$$

Suppose that $\mathbb{P}^2_\varepsilon$ is the plane of diagonal matrices. Then we immediately read off the formulas **2.7.1** (1–4) that $Q(\varepsilon_i, u\varepsilon_j) = 0$ for all i, j and all $u \in \mathfrak{so}_3(\mathbb{A})$. But the set of simple reduction planes is $SO_3(\mathbb{A})$-homogeneous, so this is true in general. Thus each $\varepsilon_i \wedge \varepsilon_j$ is contained in U_a, and a fortiori ω_ε also is.

3.2. **Orbit structure.** The stabilizer $SO_3(\mathbb{A}) = \mathrm{Aut}\mathcal{J}_3(\mathbb{A})$ of the identity element of $\mathcal{J}_3(\mathbb{A})$ acts on the variety of reductions. We prove that under this action, Y_a only has a finite number of orbits. More precisely:

Proposition 3.2. *The variety of reductions Y_a is irreducible of dimension $3a$. It is the union of four $SO_3(\mathbb{A})$-orbits of respective codimensions 0, 1, 2 and 4. (The codimension 4 orbit is empty for $a = 1$.)*

Proof. We prove both Propositions simultaneously: we let

$$\tilde{Y}_a := G(2, \mathcal{J}_3(\mathbb{A})_0) \cap \mathbb{P}U_a$$

and prove that is has four $SO_3(\mathbb{A})$-orbits of codimensions 0, 1, 2 and 4. This will imply that $\tilde{Y}_a$ is irreducible. In particular it is equal to the closure of its open orbit Y_a^0, the subset of reduction lines with three simple contacts with $\bar{X}_a$. Since Y_a is precisely defined as the closure of this set of lines, it is equal to $\tilde{Y}_a$ and Propositions 3.1 and 3.2 follow.

Lemma 3.3. *Let $X \in \mathbb{P}\mathcal{J}_3(\mathbb{A})_0 - \Delta(\mathbb{A})$. Then there exists $g \in SO(\mathbb{A})_w$ such that gX is a diagonal.*

Proof. We first recall that the Cayley-Hamilton theorem holds in $\mathcal{J}_3(\mathbb{A})$ [7]: any matrix $Y \in \mathcal{J}_3(\mathbb{A})$ satisfies the identity

$$Y^3 - \mathrm{trace}(Y)Y^2 + Q'(Y)Y - \det(Y)I = 0,$$

with $Q'(Y) = \frac{1}{2}(\mathrm{trace}(Y)^2 - \mathrm{trace}(Y^2))$. In particular, if Y belongs to $\mathcal{J}_3(\mathbb{A})_0$, then $Y^3 - \frac{1}{2}Q(Y)Y - \det(Y)I = 0$. Moreover, the discriminant hypersurface can be defined, as usual, by the condition that the characteristic polynomial has a multiple root.

Let $\alpha_1, \alpha_2, \alpha_3$ denote the roots of the characteristic polynomial of X. Since $X \notin \Delta(\mathbb{A})$, they are distinct. Let

$$\pi_1 = \frac{(X-\alpha_2 I)(X-\alpha_3 I)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}, \quad \pi_2 = \frac{(X-\alpha_1 I)(X-\alpha_3 I)}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}, \quad \pi_3 = \frac{(X-\alpha_1 I)(X-\alpha_2 I)}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}.$$

A little computation shows that trace $(\pi_i) = 1$ and $\pi_i^2 = \pi_i$. Moreover, $\pi_1 + \pi_2 + \pi_3 = I$. In particular, $\langle \pi_1, \pi_2, \pi_3 \rangle$ is a plane through I with three simple contacts on X_a.

But we know that $SO_3(\mathbb{A})$ acts transitively on this set of planes, so that there exists $g \in SO_3(\mathbb{A})$ such that $g\langle \pi_1, \pi_2, \pi_3 \rangle$ is the plane of diagonal matrices. Since $X = \alpha_1\pi_1 + \alpha_2\pi_2 + \alpha_3\pi_3$, the matrix gX is diagonal. ∎

Let $l \in \tilde{Y}_a$ be a line which is not contained in the discriminant hypersurface $\Delta(\mathbb{A})$. Choose a point $X \in l - \Delta(\mathbb{A})$. By the lemma, we can suppose that X is diagonal. Then its diagonal coefficients are different, and the formulas **2.7.1** (1–4) imply that

$$\mathfrak{so}_3(\mathbb{A})X = \left\{ \begin{pmatrix} 0 & a_3 & a_2 \\ \bar{a_3} & 0 & a_1 \\ \bar{a_2} & \bar{a_1} & 0 \end{pmatrix}, \quad a_1, a_2, a_3 \in \mathbb{A} \right\}.$$

Thus $\mathbb{P}(\mathfrak{so}_3(\mathbb{A})X)^{\perp}$ is the line of trace zero diagonal matrices (orthogonality is taken with respect to the invariant quadratic form). This line is the projection of the plane of diagonal matrices, which is simply trisecant to the Severi variety $\overline{X_a}$.

Therefore, the open subset of $\tilde{Y}_a$, of lines not contained in the discriminant hypersurface, is an $SO(\mathbb{A})_w$-orbit isomorphic to the open set of simply trisecant planes in Y_a.

Let now $l \in \tilde{Y}_a$ be a line contained in the discriminant hypersurface $\Delta(\mathbb{A})$. The plane generated by l and the identity I has a double contact at least with X_a, at some point $Z \in X_a^0$. Since $SO_3(\mathbb{A})$ acts transitively on X_a^0 we may suppose that

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_Z X_a = \left\{ \begin{pmatrix} r - iu_0 & u & v \\ \bar{u} & r + iu_0 & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix}, \quad r \in \mathbf{C}, u, v \in \mathbb{A} \right\}.$$

Here u_0 denotes the real part of u.

Let us choose a tangent line generated by a pair (u, v), and consider the plane P generated by this line and the identity. Observe that if this plane is in $\tilde{Y}_a$, then u must be real, i.e. equal to its real part u_0. Indeed, the projection of P to $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ is the line joining Z to the matrix

$$Y = \begin{pmatrix} \frac{r}{3} - iu_0 & u & v \\ \bar{u} & \frac{r}{3} + iu_0 & iv \\ \bar{v} & i\bar{v} & -\frac{2r}{3} \end{pmatrix}.$$

For $s = (s_1, s_2, s_3) \in \mathfrak{t}(\mathbb{A}) \subset \mathfrak{so}_3(\mathbb{A})$, we have $Q(sZ, Y) = iq(s_3(1), u)$. Since $s_3(1)$, the image of the unit element $1 \in \mathbb{A}$, by the skew-symmetric endomorphism $s_3 \in \mathfrak{so}(\mathbb{A})$, can be any imaginary vector in $\mathbb{A}$, this scalar product is identically zero if and only if u is real. Changing Y into $Y + u_0 Z$ we can then suppose that $u = 0$. We call the set of such tangent directions through Z the *restricted set of tangents.*

Now, a simple computation shows that $det(aI + bY + cZ) = c(c + rb)^2$. For $r \neq 0$, the intersection $P \cap \mathbb{D}_3(\mathbb{A})$ is the union of a tangent line to X_a and a double non tangent line, both through Y. The non tangent line cuts X_a again, outside X_a^0, at the unique point

$$X = \begin{pmatrix} q(v) & iq(v) & -rv \\ iq(v) & -q(v) & -irv \\ -r\bar{v} & -ir\bar{v} & r^2 \end{pmatrix}.$$

For $r = 0$, $P \cap Sec(X_a)$ is a triple line through Y tangent to X_a. If $q(v) \neq 0$, this line meets X_a only at Y, but if $q(v) = 0$ it is contained in X_a.

This gives three cases, and we must check that we obtain correspondingly three $SO(\mathbb{A})$-orbits in $\tilde{Y}_a$, and no more.

Lemma 3.4. *The isotropy group of Z in $SO(\mathbb{A})$ acts on the restricted set of tangent directions through Z with exactly three orbits, respectively of codimension* 0, 1 *and* 2.

Proof. The isotropy subalgebra of (the line directed by) Z in $\mathfrak{so}_3(\mathbb{A})$ is

$$Iso_Z(\mathfrak{so}_3(\mathbb{A})) = \Big\{ (s, a_1, a_2, a_3) \in \mathfrak{so}_3(\mathbb{A}) = \mathfrak{t}(\mathbb{A}) \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3, \\ a_1 = ia_2, is_3(1) = 2Im(a_3) \Big\}.$$

Its action on the restricted set of tangent directions is given by the formulas

$$(s,0,0,b)\begin{pmatrix} r & 0 & v \\ 0 & r & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix} = iRe(b)\begin{pmatrix} 0 & 0 & v \\ 0 & 0 & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & s_2(v)-\frac{1}{2}s_3(1)v \\ 0 & 0 & is_1(v)+\frac{i}{2}s_3(1)v \\ * & * & 0 \end{pmatrix},$$

$$(0,ia,a,0)\begin{pmatrix} r & 0 & v \\ 0 & r & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -ra \\ 0 & 0 & -ira \\ -r\bar{a} & -ir\bar{a} & 0 \end{pmatrix},$$

(with $is_3(1) = 2Im(b)$, and the $*$ being the conjugates of the entries in symmetric position). On the second formula, we can already see that the isotropy group of Z acts transitively on the set of restricted tangent directions for which $r \neq 0$.

When $r = 0$, we have to use the first formula, and study the rank of the map $\varphi_v : \mathfrak{t}(\mathbb{A}) \to \mathbb{A}$ sending s to $s_2(v) - \frac{1}{2}s_3(1)v$. We claim that this map is surjective when $q(v) \neq 0$: this will ensure that the isotropy group of Z acts transitively on the set of tangent directions for which $r = 0$ and $q(v) \neq 0$.

Recall from [20] that the triality algebra $\mathfrak{t}(\mathbb{A})$ is isomorphic to the direct sum of the derivation algebra $Der(\mathbb{A})$, with two copies of Im$\mathbb{A}$. Explicitely, the map sending $(D, u, v, w) \in Der(\mathbb{A}) \oplus (\mathrm{Im}\mathbb{A})^3$, with $u + v + w = 0$, to the triple $s = (D + L_v - R_w, D + L_w - R_u, D + L_u - R_v)$, is an isomorphism onto $\mathfrak{t}(\mathbb{A}) \subset \mathfrak{so}(\mathbb{A})^3$. (We denoted by L_z and R_z the operators of left and right multiplication by z in $\mathbb{A}$.) We have

$$s_2(v) - \frac{1}{2}s_3(1)v = Dv - \frac{1}{2}(tv + 2vt),$$

so that the corank of φ_v is equal to the corank of the endomorphism ψ_v of $\mathbb{A}$ defined by $\psi_v(t) = tv + 2vt$.

Suppose that $t \in Ker(\psi_v)$. Since $\mathbb{A}$ is always alternative, the subalgebra generated by t and v is associative and we deduce that $2vtv = -tv^2 = -4v^2t$. But since v is imaginary, $v^2 = -q(v)$, thus when $q(v) \neq 0$ we get $t = 0$, as claimed. It follows that ψ_v and φ_v are surjective.

Now suppose that $q(v) = 0$. Then the corresponding tangent direction is in fact the direction of a line which is contained in X_a^0. The family of such lines is empty for $a = 1$, and for $a = 2$ it is the union of a projective plane and its dual. For $a > 2$, we know from [19], Theorem 4.3, that the family of lines in X_a^0 through the point Z is irreducible, but splits into two orbits of the isotropy group, giving two types of lines which we called *general* and *special*, respectively. Already for dimensional reasons we can see that the restricted tangent directions generate special lines only, hence that the isotropy group acts transitively on the set of tangent directions for which $r = q(v) = 0$. □

We have thus obtained four orbits in $\tilde{Y}_a$, of codimension 0, 1, 2 and 4. It is clear from the proof that each orbit is in the closure of any other orbit of larger dimension. In particular, $\tilde{Y}_a$ is irreducible. This concludes the proof of Propositions 3.1 and 3.2. □

Note that the identity $Y_a \cong \tilde{Y}_a$ implies the following characterization of lines belonging to Y_a, which we will use over and over in the sequel.

Corollary 3.5. *A line $\overline{XY} \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ defines a point of the variety of reductions $Y_a \subset G(2, \mathcal{J}_3(\mathbb{A})_0)$, if and only if*

$$Q(X, uY) = 0 \quad \forall u \in \mathfrak{so}_3(\mathbb{A}).$$

For future use we retain the following description of the $SO(\mathbb{A})_w$-orbits in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$. We denote by $\mathrm{Tan}^0(X_a^0) \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ the union of the tangent lines to X_a^0 corresponding to the codimension one orbits of the isotropy groups of the points of X_a^0, see the previous Lemma.

Proposition 3.6. *The orbits of $SO(\mathbb{A})_w$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ which are not contained in the discriminant hypersurface are the hypersurfaces $D_{[s,t]}(\mathbb{A}) = \{X \in \mathbb{P}\mathcal{J}_3(\mathbb{A})_0,\ 6t\det(X)^2 = sQ(X)^3\}$, where $[s,t] \in \mathbb{P}^1 - \{[1,9]\}$.*

The orbits of $SO(\mathbb{A})_w$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ which are contained in the discriminant hypersurface $\Delta(\mathbb{A}) = D_{[1,9]}(\mathbb{A})$ are:

$$X_a^0, \quad \overline{X_a} - X_a^0, \quad \mathrm{Tan}^0(X_a^0) - X_a^0, \quad \Delta(\mathbb{A}) - \overline{X_a} \cup \mathrm{Tan}(X_a^0).$$

Proof. The first assertion follows from Lemma 3.3. To prove the second assertion, we first recall that the action of $SO_3(\mathbb{A})$ on X_a (or $\overline{X_a}$) has exactly two orbits: the hyperplane section X_a^0 and its complement, see Lemma 2.5. Let now $X \in \Delta(\mathbb{A})$ with $Q(X) \neq 0$. Then $Q(X) = 6t^2$ and $\det(X) = -2t^3$ for a unique scalar t, and $(X-tI)^2(X+2tI) = 0$. Let $Z = (X-tI)(X+2tI)$. If $Z = 0$, then $X - tI$ is in X_a, thus X belongs to $\overline{X_a}$. If $Z \neq 0$, $\mathrm{trace}(Z) = \mathrm{trace}(X^2) - 6t^2 = 0$ and $Z^2 = 0$, so that Z defines a point of X_a^0. Moreover, if $U = X + 2tI$, we have $(U - 3tI)Z = 0$, hence $2UZ = 6tZ = \mathrm{trace}(U)Z$, which means that U belongs to $T_Z X_a$. Since X_a^0 is $SO_3(\mathbb{A})$-homogeneous, we may suppose that

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} r - iu_0 & u & v \\ \bar{u} & r + iu_0 & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix},$$

for some $r \in \mathbf{C}, u, v \in \mathbb{A}$. Then we compute that the identity $Z = (X - tI)(X+2tI) = (U-3tI)U$ is equivalent to the relations $Im(u) = 0$, $q(v) = 0$, $r = 3t$, $3tu_0 = i$. Then we can write

$$X = \frac{i}{3t}Z + \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix} + t\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

where $q(v) = 0$. If $v = 0$, then again X belongs to $\overline{X_a}$. If $v \neq 0$, by the proof of Proposition 3.2 the isotropy group of Z acts transitively on the special lines of X_a^0 passing through Z. Looking more carefully at the explicit action, we can see that in fact, it acts transitively on the cone of X_a^0 generated by these lines (minus the vertex Z, of course). We conclude that $SO_3(\mathbb{A})$ acts transitively on $\Delta(\mathbb{A}) - Tan^0(X_a^0) \cup \overline{X_a}$.

Let us consider now some $X \in \Delta(\mathbb{A})$ such that $Q(X) = \det(X) = 0$. In particular, $X^3 = 0$, and X belongs to X_a^0 if $X^2 = 0$. Suppose this is not the case. Then $Z = X^2$ defines a point of X_a^0, and $U = X$ belongs to $T_Z X_a^0$. We can chose Z to be the same matrix as in the previous case, as well as U, but with $r = 0$ since $\mathrm{trace}(U) = \mathrm{trace}(X) = 0$. The equation $Z = U^2$ gives the relations $q(v) = 1$ and $Im(u)v = 0$, hence $Im(u) = 0$ since v is invertible. Again, we check that the isotropy group of Z acts transitively on the (pointed) cone generated by the codimension one orbit of restricted tangent directions through Z. We conclude that $SO_3(\mathbb{A})$ acts transitively on $\mathrm{Tan}^0(X_a^0) - X_a^0$, and the proof is complete. □

Explicit representatives of the four orbits in $\Delta(\mathbb{A})$ are, respectively:

$$\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & i & 1 \\ i & -1 & i \\ 1 & i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1+iI & 0 \\ 1-iI & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

3.3. **Geometric description of the orbits.** We denote by Y_a^i the orbit of codimension i in Y_a.

a. *The open orbit.*

Proposition 3.7. *The open $SO_3(\mathbb{A})$-orbit in Y_a is $Y_a^0 \simeq SO_3(\mathbb{A})/T_a$, with $Lie(T_a) = \mathfrak{t}(\mathbb{A})$. In particular, it is an affine variety.*

Proof. A point in the open orbit Y_a^0 is given by the line of traceless diagonal matrices in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$. One can read off the explicit action of $\mathfrak{so}_3(\mathbb{A})$ on $\mathcal{J}_3(\mathbb{A})_0$ that the stabilizer of this line is $\mathfrak{t}(\mathbb{A})$, which implies the first assertion. Since $\mathfrak{t}(\mathbb{A})$ is reductive, the second assertion follows from a theorem of Matsushima, following which the quotient of a reductive group by a reductive subgroup is affine [24]. □

The open orbit Y_a^0 consists in planes in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ having three simple contacts with X_a. Let $\langle\varepsilon_1, \varepsilon_2, \varepsilon_3\rangle$ be such a plane, and suppose that $\varepsilon_1+\varepsilon_2+\varepsilon_3 = I$. By Corollary 2.11, this triple is projectively equivalent to the triple of rank one diagonal matrices, and since $SO_3(\mathbb{A}) = \mathrm{Aut}\mathcal{J}_3(\mathbb{A})$ this implies that $\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = \varepsilon_2\varepsilon_3 = 0$. Multiplying the previous identity by ε_i, we thus get $\varepsilon_i^2 = \varepsilon_i$, hence $\mathrm{trace}(\varepsilon_i) = 1$ by Lemma 2.2. The triple $\varepsilon_1, \varepsilon_2, \varepsilon_3$ is therefore what algebraists call a *Pierce decomposition* of the Jordan algebra $\mathcal{J}_3(\mathbb{A})$ [16].

These observations lead to the slightly more precise statement:

Proposition 3.8. *The generic isotropy group T_a is the semi-direct product of the triality group $T(\mathbb{A})$ with the symmetric group $\mathfrak{S}_4$.*

Proof. The generic isotropy group T_a is the stabilizer of the line of traceless diagonal matrices in $\mathcal{J}_3(\mathbb{A})_0$, or equivalently to the plane of diagonal matrices in $\mathcal{J}_3(\mathbb{A})$, which is generated by the three diagonal idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3$. These three idempotents are permuted by the elements of T_a, giving a morphism $\nu : T_a \to \mathfrak{S}_3$. Note that $\mathfrak{S}_3$ is the quotient of $\mathfrak{S}_4$ by the normal subgroup generated by the permutations which are products of two disjoint transposition. This subgroup is a copy of $\mathbb{Z}_2 \times \mathbb{Z}_2$. We must therefore prove that the morphism ν is surjective, and that its kernel T_a^0 coincides with the semi-direct product of $T(\mathbb{A})$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$.

To prove the surjectivity of ν, we define two endomorphisms σ_1 and σ_2 of $\mathcal{J}_3(\mathbb{A})$ by the formulas

$$\sigma_1\begin{pmatrix} c_1 & x_3 & x_2 \\ \bar{x}_3 & c_2 & x_1 \\ \bar{x}_2 & \bar{x}_1 & c_3 \end{pmatrix} = \begin{pmatrix} c_2 & \bar{x}_3 & x_2 \\ x_3 & c_1 & x_1 \\ \bar{x}_2 & \bar{x}_1 & c_3 \end{pmatrix},$$

$$\sigma_2\begin{pmatrix} c_1 & x_3 & x_2 \\ \bar{x}_3 & c_2 & x_1 \\ \bar{x}_2 & \bar{x}_1 & c_3 \end{pmatrix} = \begin{pmatrix} c_1 & x_3 & x_2 \\ \bar{x}_3 & c_3 & \bar{x}_1 \\ \bar{x}_2 & x_1 & c_2 \end{pmatrix}.$$

It is easy to check that these endomorphisms are in fact automorphisms of the Jordan algebra $\mathcal{J}_3(\mathbb{A})$. Moreover, they belong to T_a, and their images by ν are the two simple generators of $\mathfrak{S}_3$, proving that ν is surjective.

Let now $t \in T_a^0$, so that t fixes each ε_i. Being an automorphism of $\mathcal{J}_3(\mathbb{A})$, t also preserves the subspace $\varepsilon_i\mathcal{J}_3(\mathbb{A})\varepsilon_j$, for $1 \leq i, j \leq 3$. But this is the space of matrices whose entries are zero except possibly that on the i-th line and j-th column, and symmetrically that on the j-th line and i-th column. Therefore, there exists scalars $\zeta_1, \zeta_2, \zeta_3$, and endomorphisms τ_1, τ_2, τ_3 of $\mathbb{A}$, such that

$$t\begin{pmatrix} c_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & c_2 & x_1 \\ x_2 & \bar{x}_1 & c_3 \end{pmatrix} = \begin{pmatrix} \zeta_1 c_1 & \tau_3(x_3) & \tau_2(x_2) \\ \overline{\tau_3(x_3)} & \zeta_2 c_2 & \tau_1(x_1) \\ \overline{\tau_2(x_2)} & \overline{\tau_1(x_1)} & \zeta_3 c_3 \end{pmatrix}.$$

But $t(I) = I$, hence $\zeta_1 = \zeta_2 = \zeta_3 = 1$. Moreover, a straightforward computation shows that this is an automorphism of $\mathcal{J}_3(\mathbb{A})$ if and only if

$$\tau_2(xy) = \tau_3(x)\tau_1(y) \qquad \forall x, y \in \mathbb{A}.$$

This is precisely the definition of $T(\mathbb{A})$, except that we don't ask the τ_i to belong to $SO(\mathbb{A})$. They will automatically belong to the orthogonal group $O(\mathbb{A})$, but the sign ambiguity explains the appearance of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ factor. This concludes the proof. (The heart of this argument can be found in [16], see also [28]). □

b. The orbit of codimension one.

The codimension one orbit Y_a^1 is made of planes in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ having a simple contact with X_a outside X_a^0, and a double contact on X_a^0. This implies the existence of two fibrations p and p_0, the first one over $X_a - X_a^0$, the second one over X_a^0.

Lemma 3.9. (Point-line polarity in $X_a = \mathbb{AP}^2$). *Let Z be a point of X_a. The intersection $X_a \cap (T_Z X_a)^\perp$ is an a-dimensional quadric Q_Z^a, an $\mathbb{A}$-line in X_a. This quadric contains Z if and only if Z belongs to X_a^0.*

A point Y belongs to Q_Z^a if and only if there exists a reduction plane $P \in Y_a$ passing through Y and Z. In particular,

$$Y \in Q_Z^a \iff Z \in Q_Y^a.$$

Proof. If $Z \notin X_a^0$, the line $\overline{ZI}$ meets the determinant hypersurface $\mathbb{D}_3(\mathbb{A})$ at a unique point $M \notin X_a$, and the set of secant (or tangents) lines to X_a passing through M cuts a smooth a-dimensional quadric $\tilde{Q}_Z^a$ on X_a (the entry-locus of M, see e.g. [31]). A reduction plane $P \in Y_a$ through Z is then generated, with Z an I, by a point Y of that quadric. A simple computation shows that actually, $\tilde{Q}^a_Z = Q^a_Z$, and our claim follows for all points outside X^0_a.

To conclude the proof, we check than also for $Z \in X^0_a$, the intersection $X_a \cap (T_Z X_a)^\perp$ is a smooth quadric: this is a straightforward computation. Then the last assertion of the Lemma follows by continuity. □

Remark. If $Z \notin X^0_a$, one can define the quadric Q^a_Z as the set of points $Y \in X_a$ such that $YZ = 0$. Nevertheless, for $Z \in X^0_a$ this condition defines a larger set than Q^a_Z.

This Lemma allows a simple description of the fibers of the two projections p and p_0. Indeed, one easily checks that if $Z \in X^0_a$ and $X \in X_a - X^0_a$,

$$p_0^{-1}(Z) = Q^a_Z - Q^a_Z \cap X^0_a, \qquad \text{and} \quad p^{-1}(X) = Q^a_X \cap X^0_a.$$

We'll see below that $Q^a_Z \cap X^0_a$ is a singular section of the quadric Q^a_Z, with a unique singularity at Z, so that $p_0^{-1}(Z) \simeq \mathbb{C}^a$.

An explicit representative of Y^1_a is the line generated by

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

c. The orbit of codimension two.

The elements of the codimension two orbit Y^2_a are the planes in $\mathbb{P}\mathcal{J}_3(\mathbb{A})$ with a triple contact with X_a on X^0_a. This defines a fibration p over X^0_a, and by the proof of Proposition 3.2, the fiber of p over Z is the set of tangent directions generated by matrices of the form

$$Y = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix} \qquad \text{if} \quad Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $q(v) \neq 0$. An easy computation shows that

$$Q^a_Z = \left\{ \begin{pmatrix} r_1 & ir_1 & x_2 \\ ir_1 & -r_1 & ix_2 \\ \bar{x_2} & i\bar{x_2} & r_3 \end{pmatrix}, \ r_1, r_3 \in \mathbf{C}, x_2 \in \mathbb{A}, \ r_1 r_3 = q(x_2) \right\}.$$

This is a smooth quadric which is tangent to X^0_a at Z, hence $Q^a_Z \cap X^0_a$ is a quadratic cone with vertex Z, which is its unique singular point. The fiber $p^{-1}(Z)$ can then be described as the set of lines through Z in the linear subspace of $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ spanned by $Q^a_Z \cap X^0_a$, which are not contained in that cone. This shows that $p^{-1}(Z)$ is the complement of a smooth quadric hypersurface in a $\mathbb{P}^{a-1}$.

An explicit representative of Y^2_a is the line generated by

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}.$$

d. The closed orbit.

Finally, the codimension four orbit Y^4_a is made of planes containing a line of X^0_a, and such planes are completely determined by their corresponding line, which we noticed to be special when $a > 2$. With the convention that any line on $\times^2\mathbb{P}^2$ is special, we get:

Proposition 3.10. *The closed orbit Y_a^4 in Y_a is isomorphic to the orbit of special lines in X_a^0.*

Specifically, we have

$$Y_1^4 = \emptyset, \quad Y_2^4 = \mathbb{P}^2 \sqcup \check{\mathbb{P}}^2, \quad Y_4^4 = F_\omega(1,3;6) = Sp_6/P_{1,3}, \quad Y_8^4 = F_4/P_3.$$

An explicit representative of Y_a^4 is the line generated by

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 1-iI \\ 0 & 0 & i+I \\ 1+iI & i-I & 0 \end{pmatrix}.$$

3.4. **Smoothness.** We have seen in Proposition 3.1 that the varieties of reductions, for $a > 1$, are non transverse linear sections of their ambient Grassmannians. The following Theorem is therefore rather surprising.

Theorem 3.11. *The varieties of reductions Y_a are smooth.*

Proof. This is already known for $a = 1$. For $a > 1$, we check that a point of the codimension 4 orbit is smooth, which is enough to prove the theorem. We have just seen that a point of Y_a^4 is the line generated by

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 1-iI \\ 0 & 0 & i+I \\ 1+iI & i-I & 0 \end{pmatrix}.$$

We choose a basis $e_1 = 1 + iI, e_2, \ldots, e_a$ of $\mathbb{A}$. Then we can complete these two matrices Z, Y into a basis of $\mathcal{J}_3(\mathbb{A})_0$,

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_j^+ = \begin{pmatrix} 0 & 0 & e_j \\ 0 & 0 & ie_j \\ \bar{e}_j & i\bar{e}_j & 0 \end{pmatrix}, \quad Y_j^- = \begin{pmatrix} 0 & 0 & e_j \\ 0 & 0 & -ie_j \\ \bar{e}_j & -i\bar{e}_j & 0 \end{pmatrix},$$

$$X_j = \begin{pmatrix} 0 & e_j & 0 \\ \bar{e}_j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This provides us with a system of local coordinates on $G(2, \mathcal{J}_3(\mathbb{A})_0)$ around the line $\overline{ZY}$: a line in a certain neighbourhood of $\overline{ZY}$ has a unique basis A, B of the form

$$\begin{aligned} A &= Z + \textstyle\sum_{j>1} a_j^+ Y_j^+ + \sum_{j\geq 1} a_j^- Y_j^- + \sum_{j\geq 1} r_j X_j + uW, \\ B &= Y + \textstyle\sum_{j>1} b_j^+ Y_j^+ + \sum_{j\geq 1} b_j^- Y_j^- + \sum_{j\geq 1} s_j X_j + vW. \end{aligned}$$

By Lemma 3.5, such a line belongs to Y_a if and only if $Q(A, uB) = 0$ for all $u \in \mathfrak{so}_3(\mathbb{A})$. If we write $A = Z + \delta A$ and $B = Y + \delta B$, we get the infinitesimal equations

$$Q(uZ, \delta B) = Q(uY, \delta A) \qquad \forall u \in \mathfrak{so}_3(\mathbb{A}).$$

Using the explicit action of $m_i \in \mathbb{A}_i \subset \mathfrak{so}_3(\mathbb{A})$, we obtain the following three sets of a equations,

$$\begin{array}{rcl}
2i\sum_{j\geq 1} q(m_1,e_j)b_j^- & = & \sum_{j\geq 1} q(e_1\bar{m}_1,e_j)r_j + 4iq(m_1,e_1)u, \\
2i\sum_{j\geq 1} q(m_2,e_j)b_j^- & = & -\sum_{j\geq 1} iq(m_2\bar{e}_1,e_j)r_j + \sum_{j>1} iq(m_2,e_j)a_j^+ \\
 & & +\sum_{j\geq 1} q(m_2,e_j)a_j^- + 2iq(m_2,e_1)u, \\
-2\sum_{j\geq 1} q(m_3,e_j)s_j - 2iq(m_3,1)v & = & \sum_{j>1} iq(m_3\bar{e}_1,e_j)a_j^+ \\
 & & +\sum_{j\geq 1} q(m_3\bar{e}_1,e_j)a_j^-.
\end{array}$$

The first set of equations gives the b_j^- in terms of u and the r_j, because the coefficient of b_j^- is $q(m_1,e_j)$, and m_1 can be chosen arbitrarily . Then the second set of equations gives the a_j^- in terms of u, the r_j and the a_j^+. Finally, the third set of equations gives the s_j in terms of u,v, the r_j and the a_j^+.

This proves that the Zariski tangent space of Y_a at $\overline{ZY}$ has codimension at least $3a$ in that of $G(2,\mathcal{J}_3(\mathbb{A})_0)$. But Y_a has dimension $3a$ and $G(2,\mathcal{J}_3(\mathbb{A})_0)$ has dimension $6a$, so $\overline{ZY}$ must be a smooth point of Y_a. □

3.5. **Linear spaces in the varieties of reductions.** In a Grassmannian $G(2,n)$ of projective lines, there are two types of linear spaces. Those *of the first kind* are made of the lines containing a fixed point and contained in a fixed subspace. Those *of the second kind* have dimension two only; they are made of the lines contained in a fixed plane.

By Lemma 3.5, the maximal linear spaces of the first kind that are contained in Y_a are defined as follows: take some point $X \in \mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ and consider the space of lines L such that

$$X \in L \subset \mathbb{P}(\mathfrak{so}_3(\mathbb{A})X)^\perp.$$

Proposition 3.12. *The space $(\mathfrak{so}_3(\mathbb{A})X)^\perp$ has dimension $a+2$ if X belongs to the projection $\overline{X_a}$ of the Severi variety X_a, and dimension 2 otherwise.*

Proof. The dimension of $\mathfrak{so}_3(\mathbb{A})X$ is the dimension of the $SO(\mathbb{A})_w$-orbit of X in $\mathcal{J}_3(\mathbb{A})_0$ (not in the projectivisation $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$!). We consider the different cases obtained in Proposition 3.6.

If $X \notin \Delta(\mathbb{A})$, its orbit is, by Lemma 3.3, the set of matrices with the same characteristic polynomial: its codimension is two, hence $(\mathfrak{so}_3(\mathbb{A})X)^\perp$ has dimension two.

If X belongs to (the cone over) the open orbit in $\Delta(\mathbb{A})$, again its orbit in $\mathcal{J}_3(\mathbb{A})_0$ depends on its characteristic polynomial, so it must be of codimension two and again $(\mathfrak{so}_3(\mathbb{A})X)^\perp$ has dimension two.

The (pointed) cone over $\mathrm{Tan}^0(X_a^0) - X_a^0$ either is an $SO_3(\mathbb{A})$-orbit, or the union of a one dimensional family of codimension one orbits. An explicit computation of the infinitesimal action shows that we are in fact in the first situation. Indeed, By Proposition 3.6 and its comment we can let

$$X = \begin{pmatrix} 1 & i & 1 \\ i & -1 & i \\ 1 & i & 0 \end{pmatrix},$$

and determine its centralizer using formulas (1–4) in **2.7.1**. If $(u,a_1,a_2,a_3) \in \mathfrak{so}_3(\mathbb{A})$ annihilates X, looking at the diagonal coefficients we first see that

a_1, a_2 and a_3 must be imaginary. The non diagonal coefficients then give the equations

$$\begin{aligned} u_1(1) &= -ia_1 - a_2 - ia_3, \\ u_2(1) &= -a_1 + a_2 + ia_3, \\ u_1(1) &= ia_1 + a_2 - 2ia_3. \end{aligned}$$

The matrix formed by the coefficients of a_1, a_2, a_3 is easiliy seen to be invertible. We conclude that a_1, a_2 and a_3 are uniquely determined by u, which can be arbitrary. Thus the stabilizer of X has codimension $3a$ in $SO(\mathbb{A})$, which implies that the orbit of X has dimension $3a$, which is also the dimension of the cone over $\mathrm{Tan}^0(X_a^0)$. This proves our claim.

Hence if X belongs $\mathrm{Tan}^0(X_a^0) - X_a^0$, since it is a codimension two orbit in $\mathcal{J}_3(\mathbb{A})_0$, $(\mathfrak{so}_3(\mathbb{A})X)^\perp$ has dimension two again.

If X belongs to the cone over $\overline{X_a} - X_a^0$, its $SO_3(\mathbb{A})$-orbit depends on its characteristic polynomial, so its orbit has dimension $2a$. Finally, the (pointed) cone over X_a^0 is a full $SO_3(\mathbb{A})$-orbit of dimension $2a$. Thus if X belongs to the cone over $\overline{X_a}$, the dimension of $(\mathfrak{so}_3(\mathbb{A})X)^\perp$ is $a+2$, independently of the fact that X belongs to X_a^0 or not. □

Corollary 3.13. *The variety $Y_a \subset G(2, \mathcal{J}_3(\mathbb{A})_0)$ does not contain any plane of the second kind.*

Proof. A plane of the second kind in $G(2, \mathcal{J}_3(\mathbb{A})_0)$ is a space of lines contained in the projectivization of some three-dimensional subspace K of $\mathcal{J}_3(\mathbb{A})_0$. Let such a plane be contained in Y_a, and consider a point of that plane, which represents a line $\overline{xy}$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$. Then we must have $K \subset (\mathfrak{so}_3(\mathbb{A})x)^\perp \cap (\mathfrak{so}_3(\mathbb{A})y)^\perp$. In particular the line $\overline{xy} \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ must be contained in $\overline{X_a}$: otherwise, if $p = \alpha x + \beta y \notin \overline{X_a}$, then $(\mathfrak{so}_3(\mathbb{A})x)^\perp \cap (\mathfrak{so}_3(\mathbb{A})y)^\perp \subset (\mathfrak{so}_3(\mathbb{A})p)^\perp$, which is two-dimensional.

Suppose that $x \in \overline{X_a} - X_a^0$. Since this space is homogeneous, we can let

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \text{thus} \quad (\mathfrak{so}_3(\mathbb{A})x)^\perp = \left\{ \begin{pmatrix} r_1 & x_3 & 0 \\ \bar{x_3} & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, r_1+r_2+r_3 = 0 \right\}.$$

A straightforward computation shows that $\mathbb{P}(\mathfrak{so}_3(\mathbb{A})x)^\perp \cap \overline{X_a}$ is

$$\left\{ \begin{pmatrix} r_1 & x_3 & 0 \\ \bar{x_3} & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \; r_1 + r_2 + r_3 = 0, q(x_3) = (r_1 - r_3)(r_2 - r_3) \right\} \cup \{x\}.$$

In particular, x is an isolated point of that intersection, which can therefore contain no line through x.

Suppose now that $x \in X_a^0$, and we can even suppose that the whole line $\overline{xy}$ is contained in X_a^0. By the proof of Proposition 3.2, we can suppose that

$$x = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix},$$

where $q(v) = 0$. Another computation, which we leave to the reader, shows that again $(\mathfrak{so}_3(\mathbb{A})x)^\perp \cap (\mathfrak{so}_3(\mathbb{A})y)^\perp$ is only two-dimensional. □

Corollary 3.14. *The maximal linear spaces in Y_a are $\mathbb{P}^a$'s parametrized by X_a.*

Recall (Lemma 2.5) that $SO_3(\mathbb{A})$ has exactly two orbits inside $X_a \simeq \overline{X_a}$: the closed orbit X_a^0, which is the hyperplane section of X_a by $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$, and its complement. Correspondingly, there are two types of $\mathbb{P}^a$'s inside Y_a : *special* ones, for $x \in X_a^0$, and *general* ones, for $x \notin X_a^0$.

Proposition 3.15. *The numbers of general and special $\mathbb{P}^a$'s through a point of the codimension i orbit Y_a^i of Y_a is given as follows:*

	general	*special*
Y_a^0	3	0
Y_a^1	1	1
Y_a^2	0	1
Y_a^4	0	∞^1

Note that the fact that there are exactly three $\mathbb{P}^a$'s through a point of the open orbit is a genuine geometric manifestation of triality ! Indeed, we know that the tangent space to a point of the open orbit is equal to $\mathfrak{so}_3(\mathbb{A})/\mathfrak{t}(\mathbb{A}) = \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3$ as a module over the stabilizer Lie algebra $\mathfrak{t}(\mathbb{A})$. The three copies of $\mathbb{A}$ correspond to the directions of the three $\mathbb{P}^a$'s. Moreover, by Proposition 3.8 the isotropy group of a generic point contains a copy of $\mathfrak{S}_3$, which permutes these three spaces.

Proof. A point of Y_a^0 is given by the line of diagonal matrices in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$. If this point is contained in a maximal linear subspace $\mathbb{P}_x^a$ of Y_a, then x is diagonal and belongs to $\overline{X_a}$. There are exactly three such matrices (up to scalar),

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}, \qquad \begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix}-2&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

the projections of the three diagonal matrices of rank one. Thus a point of Y_a^0 belongs to exactly three lines, and they are all general.

A point of Y_a^1 is given by the line in $\mathcal{J}_3(\mathbb{A})_0$ generated by

$$Z = \begin{pmatrix}1&i&0\\i&-1&0\\0&0&0\end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

This line meets $\overline{X_a}$ only at Z and Y. Since Y belongs to X_a^0 and Z does not, this implies that a point of Y_a^1 belongs exactly to one special $\mathbb{P}^a$ and one general $\mathbb{P}^a$ of Y_a.

A point of Y_a^2 is given by the line generated by

$$Z = \begin{pmatrix}1&i&0\\i&-1&0\\0&0&0\end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix}0&0&1\\0&0&i\\1&i&0\end{pmatrix}.$$

This is a tangent line to X_a^0 at Z, and it meets $\overline{X_a}$ only at this point. Thus a point in Y_a^2 belongs to a unique maximal linear subspace of Y_a, which is special.

Finally, a point of Y_a^4 is given by a special line in X_a^0, and each point of this line defines a special maximal linear subspace of Y_a through that point of Y_a^4. □

3.6. **Varieties of reductions are rational Fano manifolds.** Let Z_a denote the space of incident points and lines $p \subset l$, with $p \in \mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ and $l \in Y_a$.

Proposition 3.16. *There is a commutative diagram*

$$\begin{array}{ccc} & Z_a = \mathbb{P}_{Y_a}(S) & \\ \swarrow^{\sigma} & & \searrow^{p} \\ X_a^0 \subset \overline{X_a} \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_0 & \overset{\phi}{\dashrightarrow} & Y_a \subset \mathbb{P}U_a, \end{array}$$

where p is the $\mathbb{P}^1$-bundle defined by the restriction to Y_a of the tautological rank two bundle S over $G(2, \mathcal{J}_3(\mathbb{A})_0)$, and σ is the blow-up of $\overline{X_a}$.

Proof. It follows from Proposition 3.12 that the projection σ to $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ is an isomorphism over the complement of $\overline{X_a}$. Moreover, the fiber of a point of $\overline{X_a}$ is a $\mathbb{P}^a$, which is mapped isomorphically by p to a maximal linear subspace of Y_a. This implies in particular that $\sigma^{-1}(\overline{X_a})$ is a smooth irreducible divisor E in Z_a, which is itself smooth since Y_a is smooth. By [8], Theorem 1, this is enough to ensure that σ is the blow-up of $\overline{X_a}$. □

Corollary 3.17. *The variety of reductions Y_a is a rational Fano manifold of index $a+1$, with Picard group $Pic(Y_a) = \mathbb{Z}\mathcal{O}(1)$.*

Proof. The claim on the Picard group is clear. To compute the index of Y_a, let again E denote the exceptional divisor of σ, and H the pull-back of the hyperplane class. Since there are three $\mathbb{P}^a$'s through the general point of Y_a, we have $E.f = 3$ if f denotes the class of a fiber of p. Also $H.f = 1$, hence $p^*\mathcal{O}(1) = 3H - E$. On the other hand it is easy to see that $\mathcal{O}_S(1) \otimes p^*\mathcal{O}(1) = H$, hence $\mathcal{O}_S(1) = E - 2H$. Now one can compute the canonical divisor of Z_a in two ways:

$$\begin{aligned} K_{Z_a} &= \sigma^* K_{\mathbb{P}\mathcal{J}_3(\mathbb{A})_0} + aE = -(3a+2)H + aE \\ &= p^*(K_{Y_a} \otimes \det S) \otimes \mathcal{O}_S(-2) = p^*K_{Y_a} + H - E, \end{aligned}$$

so that $p^*K_{Y_a} = -(a+1)p^*\mathcal{O}(1)$ and $K_{Y_a} = \mathcal{O}(-a-1)$. Since the Picard group of Y_a is generated by $\mathcal{O}(1)$, this implies that Y_a is a Fano manifold of index $a+1$.

The fact that it is rational follows from the diagram above: if L is a hyperplane in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$, it is birational to its strict transform by p, which is itself birational to Y_a via σ, since a general line of Y_a meets L. □

Remark. The diagram of Proposition 3.16 can be interpreted in terms of the study we made in section 2. Indeed, we can complete it as follows:

$$\begin{array}{ccccc} & & Z_a = \mathbb{P}_{Y_a}(S) & & \\ & \swarrow^{\sigma} & & \searrow^{p} & \\ X_a^0 \subset \overline{X_a} \subset \mathbb{P}\mathcal{J}_3(\mathbb{A})_0 & & \overset{\phi}{\dashrightarrow} & & Y_a \subset \mathbb{P}U_a, \\ \cup & & & & \cup \\ \mathbb{P}H^0(X_I^a, \mathcal{F}_I^a)^0 & & \xrightarrow{\textit{zero-set map}} & & Y_a^0 \end{array}$$

Remember from **2.8** and **2.9** that we defined on X_I^a a homogeneous vector bundle $\mathcal{F}_I^a$, whose space of global sections was isomorphic to $\mathcal{J}_3(\mathbb{A})_I = \mathcal{J}_3(\mathbb{A})_0$ (we take $w = I$ here, which is harmless). On the complement $\mathbb{P}H^0(X_I^a, \mathcal{F}_I^a)^0$ of the discriminant hypersurface, the zero-locus of a projective section was a triality subvariety Z_ε^a, and we proved (see Proposition 2.15) that the family of these triality varieties was parametrized by the open orbit Y_a^0 of the variety of reductions. The map ϕ is nothing but the zero-set map considered as a rational map.

We now turn to a different direction. We use Proposition 3.16 to compute the Betti numbers of the Y_a's.

Corollary 3.18. *Y_a has pure cohomology (i.e. its Hodge numbers $h^{p,q}(Y_a) = 0$ for $p \neq q$), and for $a \geq 2$ its Betti numbers can be deduced from those of X_a by the formula*

$$b_{2p}(Y_a) = \frac{1 + (-1)^p}{2} + \sum_{0 \leq 2j < a} b_{2p-4j-2}(X_a).$$

The topological Euler charateristic of Y_a is $e(Y_a) = 3\frac{a(a+2)}{2} + 1$, again for $a \geq 2$.

Proof. A simple computation, using the formulas giving the Betti numbers of a blow-up that can be found in [12], page 605. □

The Betti numbers of X_a present a nice regular pattern at least for $a \geq 2$:

$$b_{2p}(X_a) = \begin{cases} 1 & \text{for } 0 \leq p < \frac{a}{2} \text{ or } \frac{3a}{2} < p \leq 2a, \\ 2 & \text{for } \frac{a}{2} \leq p < a \text{ or } a < p \leq \frac{3a}{2}, \\ 3 & \text{for } p = a. \end{cases} .$$

In particular $e(X_a) = 3a + 3$. From this fact and the recursive formula of the Corollary we can easily deduce the explicit Betti numbers of our varieties of reductions:

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$b_{2p}(Y_1)$	1	1	1	1																					
$b_{2p}(Y_2)$	1	1	3	3	3	1	1																		
$b_{2p}(Y_4)$	1	1	2	3	4	5	5	5	4	3	2	1	1												
$b_{2p}(Y_8)$	1	1	2	2	3	4	5	6	7	8	8	9	9	9	8	8	7	6	5	4	3	2	2	1	1

Correspondingly, the Euler characteristics are

$$e(Y_1) = 4, \quad e(Y_2) = 13, \quad e(Y_4) = 37, \quad e(Y_8) = 121.$$

Now we identify the rational map ϕ. Since $p^*\mathcal{O}(1) = 3H - E$, it must be defined by a system of cubics on $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$.

Proposition 3.19. *The space of cubics on $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ vanishing on $\overline{X_a}$ is isomorphic to U_a.*

Proof. A case-by-case verification with [23] shows that the space of cubics on $\mathcal{J}_3(\mathbb{A})_0$ is

$$S^3\mathcal{J}_3(\mathbb{A})_0^* = S^3\mathcal{J}_3(\mathbb{A})_0 = S^{(3)}\mathcal{J}_3(\mathbb{A})_0 \oplus S^2\mathcal{J}_3(\mathbb{A})_0 \oplus U_a.$$

The embedding of $S^2\mathcal{J}_3(\mathbb{A})_0$ inside $S^3\mathcal{J}_3(\mathbb{A})_0$ is given as follows: to $A, B \in \mathcal{J}_3(\mathbb{A})_0$, we associate the cubic form $p_{A,B}(X) = \mathrm{trace}(X(AX)(BX))$. One can check that these cubics vanish on X_a^0, as expected, but not identically on $\overline{X_a}$.

The embedding of U_a is deduced from the map $\wedge^2\mathcal{J}_3(\mathbb{A})_0 \to S^3\mathcal{J}_3(\mathbb{A})_0$ defined as follows: to a skew-symmetric form θ, we associate the cubic form

$$p_\theta(X) = \theta(X, X^2 - \frac{1}{3}\mathrm{trace}(X^2)I).$$

Such cubics vanish on $\overline{X_a}$. Indeed, let X be the projection of some $Z \in X_a$, that is $X = Z - \frac{z}{3}I$ and $Z^2 = zZ$, where $z = \mathrm{trace}\ (Z)$. Then $X^2 - \frac{1}{3}\mathrm{trace}\ (X^2)I = \frac{z}{3}X$, so clearly $p_\theta(X) = 0$.

To conclude that the base locus of $U_a \subset S^3\mathcal{J}_3(\mathbb{A})_0 \simeq S^3\mathcal{J}_3(\mathbb{A})_0^*$ is exactly $\overline{X_a}$, we first observe that these cubics cannot vanish identically on the discriminant hypersurface, which is irreducible of degree 6. By Proposition 3.6, what remains to check is that they don't vanish identically on $\mathrm{Tan}^0(X_a^0)$. Remember from the proof of Proposition 3.2 that the tangent space of X_a^0 at

$$Z = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{is}$$

$$T_Z X_a = \left\{ X = \begin{pmatrix} r - iu_0 & u & v \\ \bar{u} & r + iu_0 & iv \\ \bar{v} & i\bar{v} & 0 \end{pmatrix}, \quad r \in \mathbf{C}, u, v \in \mathbb{A} \right\}.$$

Now consider the line of diagonal matrices in $\mathcal{J}_3(\mathbb{A})_0$, seen as a point $\theta \in U_a$. A straightforward computation shows that $p_\theta(X) = -\frac{2}{3}iu_0 q(Im(u)) \neq 0$. □

Corollary 3.20. *The rational map ϕ is the map defined by the linear system of cubics vanishing on the projected Severi variety $\overline{X_a}$.*

Finally, we use Proposition 3.16 to compute the degrees of the varieties of reductions. We also provide the degrees of the Grassmannians $G(2, 3a+2)$, which are well-known to be the Catalan numbers $\frac{1}{3a+1}\binom{6a}{3a}$, to show that although the degrees of the Y_a can be quite big, they are relatively small compared to those of their ambient Grassmannians.

Theorem 3.21. *The degrees of the varieties Y_a, and of the Grassmannians $G(2, \mathcal{J}_3(\mathbb{A})_0)$ are:*

$\deg Y_1$	$=$	5	$\deg G(2,5)$	$=$	5
$\deg Y_2$	$=$	57	$\deg G(2,8)$	$=$	132
$\deg Y_4$	$=$	$12\,273$	$\deg G(2,14)$	$=$	$208\,012$
$\deg Y_8$	$=$	$1\,047\,361\,761$	$\deg G(2,26)$	$=$	$1\,289\,904\,147\,324$

Proof. Using Proposition 3.16, the degree of Y_a can be computed once the normal bundle of $\overline{X_a}$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$, and the Chow ring of X_a are understood. The case of $a = 2$ is explained in the next section.

The most complicated case is of course that of Y_8. We give a detailed description of the Chow ring of the Cayley plane $X_8 = \mathbb{OP}^2$ in the Appendix, and show how the degree of Y_8 can be computed. □

3.7. **Varieties of reductions are compactifications of affine spaces.** The fact that Y_1 is a compactification of $\mathbb{C}^3$ is due to Furushima [11], who gave several geometric proofs of this property. We show that the varieties of reductions are always compactifications of affine spaces (and indeed *minimal compactifications*, since the Picard group is cyclic). Actually, this will directly follow from the fact that there is only a finite number of orbits.

Theorem 3.22. *The variety of reductions Y_a is a compactification of $\mathbb{C}^{3a}$.*

Proof. Since Y_a is a smooth projective variety, it is enough to prove that $SO_3(\mathbb{A})$ contains a one-dimensional torus T acting on Y_a with a finite number of fixed points. By the work of Byalinicki-Birula ([4], Theorem 4.4), this will ensure that Y_a contains a dense affine cell. More precisely, one can attach to each fixed point $z \in Y_a$ the subset of Y_a defined as the union of the points that are attracted by z through the action of $t \in T$, when t tends to zero, and this provides a cell decomposition of the variety.

For $a > 1$, we have seen in Proposition 3.8 that $SO_3(\mathbb{A})$ contains the triality group $T(\mathbb{A})$, a reductive subgroup of maximal rank. We choose a maximal torus H of $SO(\mathbb{A})$ contained in $T(\mathbb{A})$.

As a $\mathfrak{t}(\mathbb{A})$-module, $\mathcal{J}_3(\mathbb{A})_0 = \mathbb{C}^2 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2 \oplus \mathbb{A}_3$. In particular the set of weights of $\mathcal{J}_3(\mathbb{A})_0$ is very easy to describe: there are $3a$ non-zero weights of multiplicity one (which are all conjugate under the action of the Weyl group), and the weight zero, whose multiplicity equals two. Let us denote by $L_1, \dots, L_{3a}$ the one-dimensional weight spaces, and by P the plane of weight zero.

Let T be a generic one-dimensional subtorus of H. Then the set $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0^T$ of fixed points of T in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ is the union of $3a$ points $e_1, \dots, e_{3a}$, and a projective line d. For the induced action on the Grassmannian $G(2, \mathcal{J}_3(\mathbb{A})_0)$, the set of fixed points $G(2, \mathcal{J}_3(\mathbb{A})_0)^T$ is then the union of $\binom{3a}{2}$ points, the lines $\overline{e_i e_j}$ in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$, another point, the line d in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$, and $3a$ projective lines d_i, given by the set of lines in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$ joining e_i to some point of d.

We need to check that none of these lines is contained in Y_a. Suppose that $d_i \subset Y_a$. This would mean that $L \subset (\mathfrak{so}_3(\mathbb{A}) l_i)^\perp$. But l_i is generated by a weight vector X_i contained in some $\mathbb{A}_j \subset \mathcal{J}_3(\mathbb{A})_0$: for example, if $j=3$,

$$X_i = \begin{pmatrix} 0 & z & 0 \\ \bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{so}$$

$$\mathfrak{so}_3(\mathbb{A})X_i = \left\{ \begin{pmatrix} 2q(z,a_3) & g_3(z) & -za_1 \\ \overline{g_3(z)} & -2q(z,a_3) & -\bar{z}a_2 \\ \bar{a_1}\bar{z} & -\bar{a_2}z & 0 \end{pmatrix}, \; g \in \mathfrak{t}(\mathbb{A}), a_1, a_2, a_3 \in \mathbb{A} \right\}.$$

Since the diagonal coefficients of these matrices are not identically zero, $\mathfrak{so}_3(\mathbb{A})X_i$ is not orthogonal to P.

Therefore the fixed lines d_i are not contained in Y_a, which means that the action of T on Y_a has a finite number of fixed points. □

We can analyze a little more carefully the set of fixed points of T contained in Y_a. First note that the line d is certainly one of them. If X_i is as in the proof above, then $\mathfrak{so}_3(\mathbb{A})X_i$ is orthogonal to exactly one point of d (more precisely, one of the three points of $d \cap \overline{X_a}$). This proves that d_i cuts Y_a at exactly one point, giving $3a$ fixed point of T in Y_a.

Finally, we have to decide which of the lines $\overline{e_ie_j}$ are contained in Y_a. Suppose that e_i is generated by the vector X_i,above, which we denote by $A_3(z)$. If e_j also belong to $\mathbb{A}_3$ and is represented by $X_j = A_3(z')$, then the line $\overline{e_ie_j}$ represents a point of Y_a if $q(z', g_3(z)) = 0$ for all $g \in \mathfrak{t}(\mathbb{A})$. But $g_3(z)$ can be any vector in $\mathbb{A}$ orthogonal to z, so this would imply that z and z' are parallel, hence $e_i = e_j$. Now suppose that e_j does not belong to $\mathbb{A}_3$ like e_i, but for example to $\mathbb{A}_2$. Then the line $\overline{e_ie_j}$ represents a point of Y_a if $q(z', za_1) = 0$ for all $a_1 \in \mathbb{A}$. But z is isotropic, so this means that $z' \in L(z)$. Since $L(z)$ is, like z, preserved by the torus action, it has a basis of eigenvectors of the torus, and we have $\lambda = \dim L(z)$ possibilities for the choice of e_j in $\mathbb{A}_2$, and also in $\mathbb{A}_3$. This gives $3a\lambda$ new fixed points of T in Y_a. Since $\lambda = 0$ when $a = 1$ and $\lambda = a/2$ when $a \geq 2$, we get a total of 4 fixed points when $a = 1$, and $3a^2/2 + 3a + 1$ fixed points when $a \geq 2$. Note that we know from [4] that this number of fixed points is just the topological Euler characteristic of Y_a, which we have thus recovered.

A more interesting consequence is the fact that $\mathbb{C}^{3a}$ is the complement in Y_a of a hyperplane section.

Proposition 3.23. *For $a > 1$, let x be a point of the closed orbit Y_a^4 of Y_a, and let H_x denote the polar hyperplane. Then $Y_a - Y_a \cap H_x \simeq \mathbb{C}^{3a}$.*

Proof. A case by case examination with the help of [23] shows that the highest weight of U_a has exactly $3a^2/2$ conjugate under the Weyl group action: they are the weights $\mu_i + \mu_j$ of the lines $\overline{e_ie_j}$ we have just described. In particular, the highest weight of U_a is of this type, and we can find a one-parameter subgroup whose attractive point in $\mathbb{P}U_a$ is the line of highest weight. This point x, which belongs to the closed orbit in Y_a (and can be chosen, by homogeneity, to be any point in Y_a^4), attracts the complement of a hyperplane of $\mathbb{P}U_a$ generated by the remaining weight vectors. This is precisely the polar hyperplane H_x of x. If we restrict the action to U_a, the point x attracts the complement to the hyperplane section $Y_a \cap H_x$, and by Bialiynicki-Birula's theorem this is a copy of $\mathbb{C}^{3a}$. □

Note that a point x of Y_a^4 is a special line l_x on X_a^0. The polar subspace L_x has codimension 2 in $\mathbb{P}\mathcal{J}_3(\mathbb{A})_0$, and we get:

Proposition 3.24. *The singular section $Y_a \cap H_x$ is the Schubert variety of lines that meet L_x, and belong to Y_a.*

Remark. When $a = 1$ the analysis is of course different, since the codimension 4 orbit is empty. Nevertheless, one can check that Byalinicki-Birula's method still applies, giving a family of singular hyperplane sections of Y_1 parametrized by the closed orbit (a sextic curve – or equivalently by the special lines on Y_1), whose complements are isomorphic to $\mathbb{C}^3$. Nevertheless, Furushima proved that there exists another family of hyperplane sections of Y_1 (parametrized by the general lines), whose complements are affine cells. We do not know whether a similar phenomenon holds for the other varieties of reductions Y_a, $a > 1$.

4. The case of $\mathbb{P}^2 \times \mathbb{P}^2$

The case $a = 2$, $\mathbb{A} = \mathbb{C}$ deserves special attention because the variety of reductions Y_2 is a smooth compactification of the space of independent triples in $\mathbb{P}^2$. But we already know such a compactification : the Hilbert scheme $\mathrm{Hilb}^3\mathbb{P}^2$.

4.1. Y_2 and the Hilbert scheme.

The second Severi variety $X_2 = \mathbb{P}^2 \times \mathbb{P}^2$ is embedded into $\mathbb{P}\mathcal{J}_3(\mathbb{C}) = \mathbb{P}M_3(\mathbb{C})$, and is homogeneous under the action of $SL_3(\mathbb{C}) = SL_3 \times SL_3$. Its hyperplane section $X_2^0 = \mathbb{P}T_{\mathbb{P}^2} = \mathbb{F}(1, 2; 3) \subset \mathbb{P}\mathcal{J}_3(\mathbb{C})_0 = \mathbb{P}\mathfrak{sl}_3$, the variety of complete flags in $\mathbb{C}^3$, is homogeneous under $SO_3(\mathbb{C}) = SL_3$. We thus have a coincidence between $\mathcal{J}_3(\mathbb{C})_0$ and $\mathfrak{so}_3(\mathbb{C})$, and the map $\wedge^2\mathcal{J}_3(\mathbb{C})_0 = \wedge^2\mathfrak{sl}_3 \longrightarrow \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_3$ is just the Lie bracket. Therefore:

Proposition 4.1. *The second variety of reductions $Y_2 \subset G(2, \mathfrak{sl}_3)$, is the variety of abelian planes in $\mathfrak{sl}_3$.*

The open SL_3-orbit of Y_2 is the set of planes of matrices which are diagonal in a given basis. This basis is unique up to multiplication by scalars, so that a point in Y_2^0 is actually just an unordered triple of independent points in $\mathbb{P}^2$. In particular, Y_2 is birational to $\mathrm{Hilb}^3\mathbb{P}^2$, the punctual Hilbert scheme of length three subschemes of $\mathbb{P}^2$. But we can be much more precise:

Theorem 4.2. *The Hilbert scheme* $\mathrm{Hilb}^3\mathbb{P}^2$ *has two extremal contractions. One is the Hilbert-Chow morphism onto $Sym^3\mathbb{P}^2$. The variety Y_2 is isomorphic to the image of the other one.*

Proof. The extremal contraction of $\mathrm{Hilb}^3\mathbb{P}^2$ which is not the Hilbert-Chow morphism was constructed in [22] as follows: this is the morphism

$$\varphi_1 : \mathrm{Hilb}^3\mathbb{P}^2 \longrightarrow G(3, S^2\mathbb{C}^3)$$

mapping a length three subscheme Z of $\mathbb{P}^2$ to the dimension 3 system of conics that contains it.

Lemma 4.3. $U_2 \simeq \wedge^3(S^2\mathbb{C}^3)$.

Proof. The space $\wedge^3(S^2\mathbb{C}^3)$ is generated by decomposable tensors of the form $e^2\wedge f^2\wedge g^2$. If we identify the dual of $\mathbb{C}^3$ with its second wedge power, we can associate to such a tensor the 3-dimensional subspace of $\mathfrak{gl}_3$ generated by $e\wedge f\otimes g$, $f\wedge g\otimes e$ and $g\wedge e\otimes f$. These three morphism commute and their sum is $e\wedge f\wedge g$ times the identity, hence their projection to $\mathfrak{sl}_3$ from the identity defines a point in U_2. The morphism so defined is clearly $\mathfrak{sl}_3$ and H_2 equivariant (see **3.1** for the definition of H_2), hence an isomorphism. □

This implies that we can identify Y_2 with the subvariety of $G(3,S^2\mathbb{C}^3)$ defined as the closure of the planes $e^2\wedge f^2\wedge g^2$. This is not exactly the image of φ_1, which is the closure of the planes $ef\wedge fg\wedge ge$. But the theorem follows from the following lemma.

Lemma 4.4. *The endomorphism of $\wedge^3(S^2\mathbb{C}^3)$ mapping $e^2\wedge f^2\wedge g^2$ to $ef\wedge fg\wedge ge$, is an isomorphism.*

Proof of the lemma. Let μ denote this endomorphism. We prove that 2μ is an involution. We have $64\mu(ef\wedge fg\wedge ge)$ is equal to

$$\begin{aligned}
\mu\Big([(e+f)^2-(e-f)^2]\wedge[(f+g)^2-(f-g)^2]\wedge[(g+e)^2-(g-e)^2]\Big)&\\
= \textstyle\sum_{\varepsilon,\varepsilon',\varepsilon''=\pm1}(f+\varepsilon e)(g+\varepsilon' f)\wedge(g+\varepsilon' f)(e+\varepsilon'' g)\wedge(e+\varepsilon'' g)(f+\varepsilon e)&\\
= \textstyle\sum_{\varepsilon,\varepsilon',\varepsilon''=\pm1}(fg\wedge ge\wedge ef+ef\wedge fg\wedge ge)&\\
= 16ef\wedge fg\wedge ge.&
\end{aligned}$$

Indeed, in the previous sum we need only keep terms which have even degree in each of the $\varepsilon,\varepsilon',\varepsilon''$ (the other ones clearly cancel), and there are only two of them. □

Note that we have obtained, by the way, another interpretation of Y_2.

Proposition 4.5. *The second variety of reductions Y_2 is isomorphic to the variety of trisecant planes to the Veronese surface $v_2(\mathbb{P}^2)\subset\mathbb{P}^5$.*

It was proved in [22] that φ_1 contracts those subschemes of $\mathbb{P}^2$ contained in a line, to a $\mathbb{P}^2$ parametrizing precisely these lines. At the level of SL_3-orbits, the Hilbert scheme contains seven orbits, and 3 of them are contracted onto one the two $\mathfrak{sl}_3$-invariant $\mathbb{P}^2$'s inside Y_2. Actually, φ_1 is just the blow-up of this $\mathbb{P}^2$. The fact that the image of φ_1 is smooth, which is the most surprising point here, is not mentionned in [22].

Proposition 4.6. *The degree of Y_2 is* 57.

First proof. From the preceding theorem we get that

$$\deg Y_2=(\varphi_1^*\mathcal{O}(1))^6,$$

and we are reduced to a computation on $\mathrm{Hilb}^3\mathbb{P}^2$, whose Chow ring has been described in detail in [9]. With their notations, it is easy to see that $\varphi_1^*\mathcal{O}(1)=A+H$. Since they computed that $H^6=15$, $H^5A=15$, $H^4A^2=3$, $H^3A^3=-12$, $H^2A^4=12$, $HA^5=-3$ and $A^6=-15$, we get

$$\varphi_1^*\mathcal{O}(1)^6=H^6+6H^5A+15H^4A^2+20H^3A^3+15H^2A^4+6HA^5+A^6=57.$$

Second proof. We can use Theorem 3.16 and the structure of Z_2 to compute the degree of Y_2 as follows. First notice that $H.f=1$ implies that

$$\deg Y_2=H(3H-E)^6$$

(recall that H is the pull-back of the hyperplane class by σ, and E the exceptional divisor). The intersection numbers on Z_2 can be computed explicitely, using the fact that σ is a blow-up with smooth center $\overline{X_2}$, once we know the Chern classes of its normal bundle. First note that the normal bundle of $X_2 = \mathbb{P}^2 \times \mathbb{P}^2$ is $Q(1) \otimes Q'(1)$, where Q, Q' denote the rank two tautological quotient bundles on the two copies of $\mathbb{P}^2$. Since $\overline{X_2}$ is an isomorphic linear projection of X_2, we have an exact sequence $0 \to \mathcal{O}(1,1) \to N_{X_2} \to N_{\overline{X_2}} \to 0$, from which we deduce that the Chern classes of $N_{\overline{X_2}}$ are

$$\begin{aligned} c_1(N_{\overline{X_2}}) &= 5h + 5h', \\ c_2(N_{\overline{X_2}}) &= 10h^2 + 17hh' + 10h'^2, \\ c_3(N_{\overline{X_2}}) &= 18h^2h' + 18hh'^2, \end{aligned}$$

if h and h' denote the two hyperplane classes on our two copies of $\mathbb{P}^2$. Therefore, the Chow ring of the exceptional divisor E is the quotient of $\mathbb{Z}[h, h', e]$ by the relations $h^3 = h'^3 = 0$ and $e^3 - 5(h+h')e^2 + (10h^2 + 17hh' + 10h'^2)e - 18(h^2h' + hh'^2)$ (see e.g. [12]). This being given, we compute (note that $h^2h'^2e^2 = 1$, since $h^2h'^2$ is the class of a fiber of $E \to \mathbb{P}^2 \times \mathbb{P}^2$, over which e restricts to the hyperplane class on $\mathbb{P}^2$),

$$\begin{aligned} &H^7 = 1, \quad H^6E = 0, \quad H^5E^2 = 0, \quad H^4E^3 = \deg \overline{X_2} = 6, \\ &H^3E^4 = (h + h')^3e^3 = 5(h + h')^4e^2 = 30, \\ &H^2E^5 = (h + h')^2e^4 = 5(h + h')^3e^3 - (h + h')^2(10h^2 + 17hh' + 10h'^2)e^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 150 - 54 = 96, \\ &HE^6 = (h + h')e^5 = 5(h + h')^2e^4 - (h + h')(10h^2 + 17hh' + 10h'^2)e^3 \\ &\qquad\qquad +18(h + h')(h^2h' + hh'^2)e^2 = 480 - 270 + 36 = 246, \end{aligned}$$

hence $\deg Y_2 = 3^6 - 3^3\binom{6}{3}6 + 3^2\binom{6}{2}30 - 3\binom{6}{1}96 + 246 = 57.$ □

Corollary 4.7. *As a subvariety of $G(3, S^2\mathbb{C}^3)$, the homology class of Y_2 is Poincaré dual to $\sigma_3 + 2\sigma_{21} + 4\sigma_{111}$.*

Proof. The Schubert class σ_{111} is dual to the space of 3-planes in $S^2\mathbb{C}^3$ contained in a fixed 4-plane. This defines a $\mathbb{P}^3$ cutting the Veronese surface in $\mathbb{P}^5 = \mathbb{P}(S^2\mathbb{C}^3)$ in four points. There are four ways to choose three of these four points, so that σ_{111} cuts Y_2 at four points.

Similarly, the Schubert class σ_3 is dual to the space of 3-planes in $S^2\mathbb{C}^3$ containing a fixed 2-plane. For such a generic plane, there is a unique basis of $\mathbb{C}^3$ diagonalizing each of the quadratic forms it parametrizes, so that σ_2 cuts Y_2 at a single point.

To compute the last coefficient, we just note that a variety whose homology class is Poincaré dual to $x\sigma_3 + y\sigma_{21} + z\sigma_{111}$ has degree $d = 5x + 16y + 5z$. For $d = 57$, $x = 1$ and $z = 4$ imply $y = 2$. □

4.2. **A Calabi-Yau linear section.** Since Y_2 is a Fano variety of dimension 6 and index 3, we can take smooth linear sections of dimension 3 to get a family of Calabi-Yau manifolds C. It was communicated to us by K. Ranestad that these Calabi-Yau's had first been considered in [30], who computed the corresponding Gromov-Witten invariants. The following result seems to be new:

Proposition 4.8. *The Betti numbers of C are $b_0 = b_2 = b_4 = b_6 = 1$, $b_1 = b_5 = 0$, $b_3 = 2140$.*

Proof. We know the Betti numbers of Y_2, and then all the Betti numbers of C except b_3 are given by Lefschetz theorem. To compute b_3, we need to compute the Euler number

$$e(C) = \int_C c_3(C) = \int_{Y_2} \frac{h^3}{(1+h)^3} c(Y_2),$$

where h denotes the hyperplane class. To do this, we can pull everything back to $\mathrm{Hilb}^3\mathbb{P}^2$. Note that since the the Hilbert scheme is the blow-up of Y_2 along a $\mathbb{P}^2$ which will not be cut by a generic linear section of codimension 3, we have

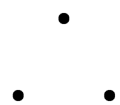
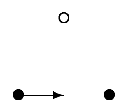
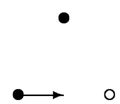
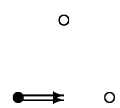
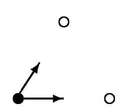
$$e(C) = e(\varphi_1^{-1}(C)) = \int_{\mathrm{Hilb}^3\mathbb{P}^2} \frac{l^3}{(1+l)^3} c(\mathrm{Hilb}^3\mathbb{P}^2),$$

where $l = \varphi_1^{-1}h$. To compute this, we use Bott's fixed-point formula as in [10], taking profit of the natural action on $\mathbb{P}^2$, hence on $\mathrm{Hilb}^3\mathbb{P}^2$, of the diagonal torus $D = \{diag(x_0, x_1, x_2)\}$ of GL_3.

The fixed points in $\mathrm{Hilb}^3\mathbb{P}^2$ are unions of monomial ideals supported on the three fixed points in $\mathbb{P}^2$. There are 22 of them, divided into five classes of cardinality 1, 6, 6, 6 and 3 respectively. These classes are described below with the induced action of the torus on the tangent space of the Hilbert scheme at the fixed points.

The drawings on the left column of the table below represent the different types of length three subschemes of $\mathbb{P}^2$ which are fixed points of the torus action: 1) the union of the three fixed points, 2) a length two subscheme supported on a fixed point, given by a tangent line pointing to another fixed point, plus that point, 3) a length two subscheme supported on a fixed point, given by a tangent line pointing to another fixed point, plus the other fixed point, 4) a length three curvilinear subscheme supported on a fixed point, with a prefered direction pointing to another fixed point, 5) a fattened fixed point, i.e. the square of the maximal ideal of a fixed point.

For each fixed point Z, the torus D acts on the tangent space $T_Z\mathrm{Hilb}^3\mathbb{P}^2$. The character $\mathrm{ch}_D(T_Z\mathrm{Hilb}^3\mathbb{P}^2)$ of this module has been computed with the help of formula (4.7) in [10].

Z	$H^0(\mathcal{O}_Z)$	$\mathrm{ch}_D(T_Z\mathrm{Hilb}^3\mathbb{P}^2)$
	x_0^3, x_1^3, x_2^3	$\frac{x_0}{x_1} + \frac{x_0}{x_2} + \frac{x_1}{x_0} + \frac{x_1}{x_2} + \frac{x_2}{x_0} + \frac{x_2}{x_1}$
	$x_0^3, x_2^3, x_2^2x_0$	$2\frac{x_0}{x_1} + \frac{x_0}{x_2} + \frac{x_2}{x_0} + \frac{x_2}{x_1} + (\frac{x_2}{x_0})^2$
	$x_0^3, x_2^3, x_2^2x_1$	$\frac{x_0}{x_1} + \frac{x_1}{x_0} + \frac{x_0}{x_2} + \frac{x_2}{x_0} + \frac{x_2}{x_1} + (\frac{x_2}{x_1})^2$
	$x_0^3, x_0^2x_1, x_0x_1^2$	$\frac{x_0}{x_1} + \frac{x_1}{x_2} + \frac{x_0}{x_2} + \frac{x_1^2}{x_0x_2} + (\frac{x_0}{x_1})^2 + (\frac{x_0}{x_1})^3$
	$x_0^3, x_0^2x_1, x_0^2x_2$	$2\frac{x_0}{x_1} + \frac{x_0x_1}{x_2^2} + 2\frac{x_0}{x_2} + \frac{x_0x_2}{x_1^2}$

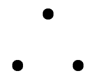
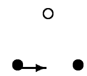
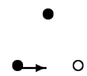
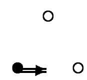
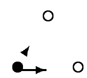
We now choose a one dimensional subtorus

$$T = \left\{ \begin{pmatrix} t^{w_0} & 0 & 0 \\ 0 & t^{w_1} & 0 \\ 0 & 0 & t^{w_2} \end{pmatrix}, t \in \mathbf{C}^* \right\} \subset D = \left\{ \begin{pmatrix} x_0 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_2 \end{pmatrix}, x_0, x_1, x_2 \in \mathbf{C}^* \right\},$$

with the same fixed points. This condition is achieved when the tangent spaces have no zero weight space: the formulas above show that the weights w_0, w_1, w_2 must be such that $w_i \neq w_j$ and $2w_i \neq w_j + w_k$ for all permutations i, j, k of $0, 1, 2$. We let $w_0 = 0, w_1 = 1, w_2 = 3$. The table below gives the character $\mathrm{ch}_T(T_Z\mathrm{Hilb}^3\mathbb{P}^2)$, that is, the set of integers $m_1, \dots, m_6$ such that

$$\mathrm{ch}_T(T_Z\mathrm{Hilb}^3\mathbb{P}^2) = t^{m_1} + t^{m_2} + t^{m_3} + t^{m_4} + t^{m_5} + t^{m_6}.$$

To complete the computation, we will apply Bott's fixed point formula in the form given by Theorem 2.2 in [10]. The intersection numbers we need to compute are given by this formula as sums of rational numbers which are contributions of the 22 fixed points. Part of these contributions come from the Chern classes of the tangent bundle, and are easily deduced from its character at each fixed point. The contribution of the class l can be computed as follows: recall that $l = \varphi_1^{-1}h = 2H + E$, where H denotes the pull-back of the hyperplane class by the Hilbert-Chow morphism, and E is the exceptional divisor of that morphism. If $\mathcal{E}_n$ is the vector bundle whose fiber at a scheme Z is given by $H^0(\mathcal{O}_Z(n))$, we have $\det \mathcal{E}_n = nH + E$, so we just need to compute $\det \mathcal{E}_0$ and $\det \mathcal{E}_1$, which is straightforward. We get:

Z	$\det \mathcal{E}_0$	$\det \mathcal{E}_1$	l
	1	$x_0x_1x_2$	$x_0^2x_1^2x_2^2$
	$\frac{x_0}{x_2}$	$x_0^2x_2$	$x_0^3x_2^3$
	$\frac{x_2}{x_1}$	$x_0x_1x_2$	$x_0^2x_1^3x_2$
	$\frac{x_1^3}{x_0^3}$	x_1^3	$x_0^3x_1^3$
	$\frac{x_1x_2}{x_0^2}$	$x_0x_1x_2$	$x_0^4x_1x_2.$

Finally, we list in the table below, for our choice of T, and for each fixed point Z, the integers $m_1, \ldots, m_6$ giving the character of the tangent space, the integers c_i corresponding to the Chern classes $c_i(\mathrm{Hilb}^3\mathbb{P}^2)$ for $i = 1, 2, 3, 6$ (these are just the i-th elementary symmetric functions of $m_1, \ldots, m_6$), as well as the weights λ of l.

From these data we can compute that

$$\int_{\mathrm{Hilb}^3\mathbb{P}^2} c_3(\mathrm{Hilb}^3\mathbb{P}^2)l^3 = \sum_Z \frac{c_3}{c_6}\lambda^3 = 243,$$

$$\int_{\mathrm{Hilb}^3\mathbb{P}^2} c_2(\mathrm{Hilb}^3\mathbb{P}^2)l^4 = \sum_Z \frac{c_2}{c_6}\lambda^4 = 261,$$

$$\int_{\mathrm{Hilb}^3\mathbb{P}^2} c_1(\mathrm{Hilb}^3\mathbb{P}^2)l^5 = \sum_Z \frac{c_1}{c_6}\lambda^5 = -171,$$

hence $e(C) = 243 - 3 \times 261 - 6 \times 171 - 10 \times 57 = -2136$. Since $e(C) = 4 - b_3$, this implies our claim. □

Since $b_3(C) = 2 + 2h^{2,1}(C)$, we get $h^{2,1}(C) = 1069 = \dim H^1(C, TC)$. This is the dimension of the space of deformations of C, and is much larger than the number of available parameters for the codimension three linear section which defines C. Therefore:

Corollary 4.9. *A general deformation of C is not a linear section of Y_2.*

$T_Z \mathrm{Hilb}^3\mathbb{P}^2$	c_1	c_2	c_3	c_6	λ
$1,-1,2,-2,3,-3$	0	−14	0	−36	8
$-1,-1,2,3,-3,6$	6	−12	−70	−108	9
$1,1,2,-2,3,4$	9	23	−5	−48	12
$1,-1,2,-2,-3,-3$	−6	4	30	36	3
$1,-1,-2,-2,-2,-3$	−9	29	−35	−24	3
$1,2,-2,3,3,-4$	3	−17	−63	144	12
$-1,2,2,3,-3,-6$	−3	−27	23	−216	9
$1,-1,2,3,-3,4$	6	−2	−60	72	10
$1,-1,-2,3,-3,-4$	−6	−2	60	72	6
$1,-1,2,-2,3,6$	9	13	−45	72	11
$1,-1,2,-2,-3,-6$	−9	13	45	72	5
$1,2,2,-2,3,-3$	3	−11	−39	72	9
$-1,2,-2,-2,3,-3$	−3	−11	39	72	7
$-1,-1,-2,-2,-3,-3$	−12	58	−144	36	3
$1,2,-2,3,-3,-4$	−3	−17	39	−144	3
$-1,2,-3,5,-6,-9$	−12	−6	368	1620	9
$-1,-2,3,-4,5,-6$	−3	−41	87	−720	12
$-1,2,3,-4,6,9$	15	39	−235	1296	9
$1,-1,2,3,4,6$	15	79	165	−144	12
$1,-1,-1,-3,-3,-5$	−12	49	−72	−45	4
$1,1,-2,-2,4,-5$	−3	−21	47	−80	7
$1,2,2,3,3,4$	15	91	285	144	13

Appendix. The Chow ring of the Cayley plane

A.1. Introduction

In this appendix we give a detailed description of the Chow ring of the complex Cayley plane[2] $X_8 = \mathbb{OP}^2$, the fourth Severi variety. This is a smooth complex projective variety of dimension 16, homogeneous under the action of the adjoint group of type E_6. It can be described as the closed orbit in the projectivization $\mathbb{P}^{26}$ of the minimal representation of E_6.

The Chow ring of a projective homogeneous variety G/P has been described classically in two different ways.

First, it can be described has a quotient of a ring of invariants. Namely, we have to consider the action of the Weyl group of P on the character ring, take the invariant subring, and mod out by the homogeneous ideal generated by the invariants (of positive degree) of the full Weyl group of G. This is the *Borel presentation*.

Second, the Chow ring has a basis given by the *Schubert classes*, the classes of the closures of the B-orbits for some Borel subgroup B of E_6. These varieties are the Schubert varieties. Their intersection products can in principle be computed by using Demazure operators [2]. This is the *Schubert presentation*.

We give a detailed description of the Schubert presentation of the Chow ring $A^*(\mathbb{OP}^2)$ of the Cayley plane. We describe explicitely the most interesting Schubert cycles, after having explained how to understand geometrically a Borel subgroup of E_6. Then we compute the intersection numbers. In the final section, we turn to the Borel presentation and determine the classes of some invariants of the partial Weyl group in terms of Schubert classes, from which we deduce the Chern classes of the normal bundle of $X_8 = \mathbb{OP}^2$ in $\mathbb{P}^{26}$. This allows us to compute the degree of the variety of reductions $Y_8 \subset \mathbb{P}^{272}$.

A.2. The Cayley plane

Let $\mathbf{O}$ denote the normed algebra of (real) octonions, and let $\mathbb{O}$ be its complexification. The space

$$\mathcal{J}_3(\mathbb{O}) = \left\{ \begin{pmatrix} c_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & c_2 & x_1 \\ x_2 & \bar{x}_1 & c_3 \end{pmatrix} : c_i \in \mathbf{C}, x_i \in \mathbb{O} \right\} \cong \mathbf{C}^{27}$$

of $\mathbb{O}$-Hermitian matrices of order 3, is the exceptional simple complex Jordan algebra, for the Jordan multiplication $A \circ B = \frac{1}{2}(AB + BA)$.

The subgroup $SL_3(\mathbb{O})$ of $GL(\mathcal{J}_3(\mathbb{O}))$ consisting in automorphisms preserving the determinant is the adjoint group of type E_6. The Jordan algebra $\mathcal{J}_3(\mathbb{O})$ and its dual are the minimal representations of this group.

The action of E_6 on the projectivization $\mathbb{P}\mathcal{J}_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface, the regular part of this hypersurface, and its singular part which is the closed E_6-orbit. These three orbits are the sets of matrices of rank three, two, and one respectively.

[2]Not to be confused with the real Cayley plane $F_4/Spin_9$, the real part of $\mathbb{OP}^2$, which admits a cell decomposition $\mathbb{R}^0 \cup \mathbb{R}^8 \cup \mathbb{R}^{16}$ and is topologically much simpler.

The closed orbit, i.e. the (projectivization of) the set of rank one matrices, is the *Cayley plane*. It can be defined by the quadratic equation

$$X^2 = \mathrm{trace}(X)X, \qquad X \in \mathcal{J}_3(\mathbb{O}),$$

or as the closure of the affine cell

$$\mathbb{OP}^2_1 = \left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & x\bar{x} & y\bar{x} \\ \bar{y} & x\bar{y} & y\bar{y} \end{pmatrix}, \quad x, y \in \mathbb{O} \right\} \cong \mathbf{C}^{16}.$$

It is also the closure of the two similar cells

$$\mathbb{OP}^2_2 = \left\{ \begin{pmatrix} \bar{u}u & u & vu \\ \bar{u} & 1 & v \\ \bar{u}\bar{v} & \bar{v} & v\bar{v} \end{pmatrix}, \quad u, v \in \mathbb{O} \right\} \cong \mathbf{C}^{16},$$

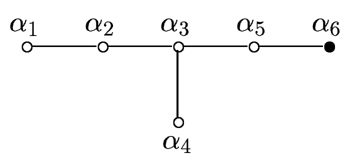
$$\mathbb{OP}^2_3 = \left\{ \begin{pmatrix} \bar{t}t & \bar{s}t & t \\ \bar{t}s & \bar{s}s & s \\ \bar{t} & \bar{s} & 1 \end{pmatrix}, \quad s, t \in \mathbb{O} \right\} \cong \mathbf{C}^{16}.$$

Unlike the ordinary projective plane, these three affine cells do not cover $\mathbb{OP}^2$. The complement of their union is

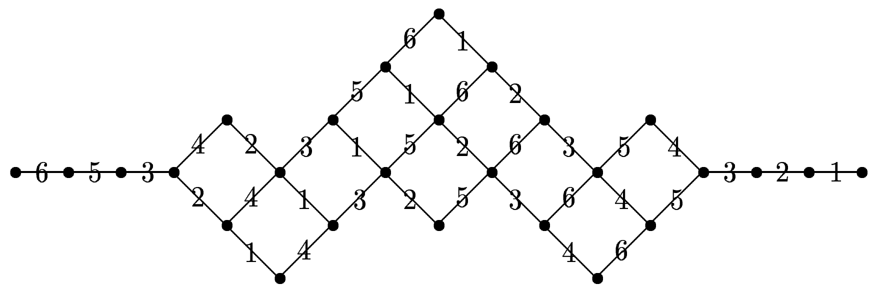
$$\mathbb{OP}^2_\infty = \left\{ \begin{pmatrix} 0 & x_3 & x_2 \\ \bar{x_3} & 0 & x_1 \\ \bar{x_2} & \bar{x_1} & 0 \end{pmatrix}, \quad \begin{matrix} q(x_1) = q(x_2) = q(x_3) = 0, \\ x_2\bar{x_3} = x_1x_3 = \bar{x_1}x_2 = 0 \end{matrix} \right\},$$

a singular codimension three linear section. Here, $q(x) = x\bar{x}$ denotes the non degenerate quadratic form on $\mathbb{O}$ obtained by complexification of the norm of $\mathbf{O}$.

Since the Cayley plane is a closed orbit of E_6, it can also be identified with the quotient of E_6 by a parabolic subgroup, namely the maximal parabolic subgroup defined by the simple root α_6 in the notation below. The semi-simple part of this maximal parabolic is isomorphic to Spin_{10}.



The E_6-module $\mathcal{J}_3(\mathbb{O})$ is *minuscule*, meaning that its weights with respect to any maximal torus of E_6, are all conjugate under the Weyl group action. We can easily list these weights as follows. Once we have fixed a set of simple roots of the Lie algebra, we can define the height of any weight ω as the sum of its coeficients when we express ω on the basis of simple roots. Alternatively, this is just the scalar product (ρ, ω), if ρ denotes, as usual, the sum of the fundamental weights, and the scalar product is dual to the Killing form. The highest weight ω_6 of $\mathcal{J}_3(\mathbb{O})$ is the unique weight with maximal height. We can obtain the other weights using the following process: if we have some weight ω of $\mathcal{J}_3(\mathbb{O})$, we express it in the basis of fundamental weights. For each fundamental weight ω_i on which the coefficient of ω is positive, we apply the corresponding simple reflection s_i. The result is a weight of $\mathcal{J}_3(\mathbb{O})$ of height smaller than that of ω, and we obtain all the weights in this way. The following diagram is the result of this process. We do not write down the weights explicitely, but we keep track of the action of the simple reflections: if we apply s_i to go from a weight to another one, we draw an edge between them, labeled with an i.



A.3. The Hasse diagram of Schubert cycles

Schubert cycles in $\mathbb{OP}^2$ are indexed by a subset W^0 of the Weyl group W of E_6, the elements of which are minimal length representatives of the W_0-cosets in W. Here W_0 denotes the Weyl group of the maximal parabolic $P_6 \in E_6$: it is the subgroup of W generated by the simple reflections $s_1, \ldots, s_5$, thus isomorphic to the Weyl group of Spin_{10}.

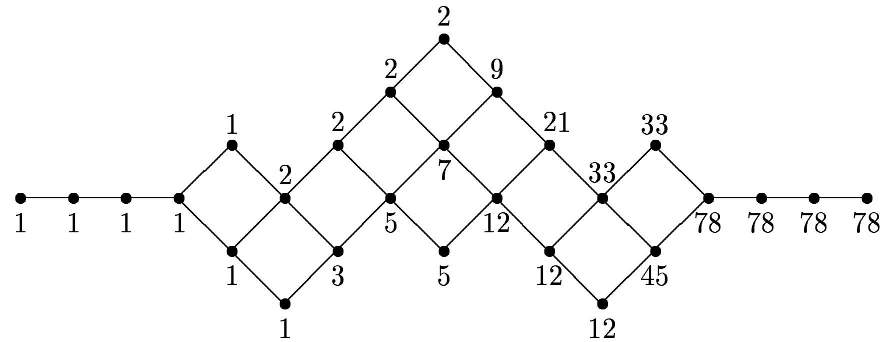
But W_0 is also the stabilizer in W of the weight ω_6. Therefore, the weights of $\mathcal{J}_3(\mathbb{O})$ are in natural correspondance with the elements of W^0, and we can obtain very explicitely, from the picture above, the elements of W^0. Indeed, choose any vertex of the diagram, and any chain of minimal length joining this vertex to the leftmost one. Let $i_1, \ldots, i_k$ be the consecutive labels on the edges of this chain; then $s_{i_1} \cdots s_{i_k}$ is a minimal decomposition of the corresponding elements of W^0, and every such decomposition is obtained in this way.

For any $w \in W^0$, denote by σ_w the corresponding Schubert cycle of $\mathbb{OP}^2$. This cycle σ_w belongs to $A^{l(w)}(\mathbb{OP}^2)$, where $l(w)$ denotes the length of w. We have just seen that this length is equal to the distance of the point corresponding to w in the picture above, to the leftmost vertex. In particular, the dimension of $A^k(\mathbb{OP}^2)$ is equal to one for $0 \leq k \leq 3$, to two for $4 \leq k \leq 7$, to three for $k = 8$ (and by duality, this dimension is of course unaltered when k is changed into $16 - k$).

The degree of each Schubert class can be deduced from the Pieri formula, which is particularly simple in the minuscule case. Indeed, we have ([14], Corollary 3.3), if H denotes the hyperplane class:

$$\sigma_w . H^k = \sum_{l(v)=l(w)+k} \kappa(w, v) \sigma_v,$$

where $\kappa(w, v)$ denotes the number of path from w to v in the diagram above; that is, the number of chains $w = u_0 \to u_1 \to \cdots \to u_k = v$ in W^0 such that $l(u_i) = l(w) + i$ and $u_{i+1} u_i^{-1}$ is a simple reflection. In particular, the degree of σ_w is just $\kappa(w, w^0)$, where w^0 denotes the longest element of W^0, which corresponds to the leftmost vertex of the diagram. We include these degrees in the following picture, the Hasse diagram of $\mathbb{OP}^2$. Note that they can very quickly be computed inductively, beginning from the left: the degree of each cycle is the sum of the degrees of the cycles connected to it in one dimension less.



Degrees of the Schubert cycles

We can already read several interesting informations on this diagram.

(1) The degree of $\mathbb{OP}^2 \subset \mathbb{P}^{26}$ is 78. This is precisely the dimension of E_6. Is there a natural explanation of this coincidence ?

(2) One of the three Schubert varieties of dimension 8 is a quadric. This must be an $\mathbb{O}$-line in $\mathbb{OP}^2$, i.e. a copy of $\mathbb{OP}^1 \simeq \mathbb{Q}^8$. Indeed, E_6 acts transitively on the family of these lines, which is actually parametrized by $\mathbb{OP}^2$ itself. In particular, a Borel subgroup has a fixed point in this family, which must be a Schubert variety.

(3) The Cayley plane contains two families of Schubert cycles which are maximal linear subspaces: a family of $\mathbb{P}^4$'s, which are maximal linear subspaces in some $\mathbb{O}$-line, and a family of $\mathbb{P}^5$'s which are not contained in any $\mathbb{O}$-line. We thus recover the results of [19], from which we also know that these two families of linear spaces in $\mathbb{OP}^2$ are homogeneous. Explicitely, we can describe both types in the following way.

Let $z \in \mathbb{O}$ be a non zero octonion such that $q(z) = 0$. Denote by $R(z)$ and $L(z)$ the spaces of elements of $\mathbb{O}$ defined as the images of the right and left multiplication by z, respectively. Similarly, if $l \subset \mathbb{O}$ is an isotropic line, denote by $R(l)$ and $L(l)$ the spaces $R(z)$ and $L(z)$, if z is a generator of l. When l varies, $R(l)$ and $L(l)$ describe the two families of maximal isotropic subspaces of $\mathbb{O}$ (this is a geometric version of triality, see e.g. [7]). Consider the sets

$$\left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & 0 & 0 \\ \bar{y} & 0 & 0 \end{pmatrix}, \; y \in l, x \in L(l) \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; x \in R(l) \right\}.$$

Their closures in $\mathbb{P}\mathcal{J}_3(\mathbb{O})$ are maximal linear subspaces of $\mathbb{OP}^2$ of respective dimensions 5 and 4.

A.4. What is a Borel subgroup of E_6 ?

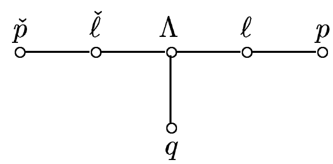
The Schubert varieties in $\mathbb{OP}^2$, by definition, are the closures of the B-orbits, where B denotes a Borel subgroup of E_6. To identify the Schubert varieties geometrically, we need to understand these Borel subgroups better.

The Cayley plane $\mathbb{OP}^2 = E_6/P_6 \subset \mathbb{P}\mathcal{J}_3(\mathbb{O})$ is one of the E_6-grassmannians, if we mean by this a quotient of E_6 by a maximal parabolic subgroup. It is isomorphic to the dual plane $\check{\mathbb{OP}}^2 = E_6/P_1 \subset \check{\mathbb{P}}\mathcal{J}_3(\mathbb{O})$, the closed orbit of the projectivized dual representation. By [19], we can identify E_6/P_5 and E_6/P_3 to the varieties $G(\mathbb{P}^1, \mathbb{OP}^2)$ and $G(\mathbb{P}^2, \mathbb{OP}^2)$ of projective lines and planes contained in $\mathbb{OP}^2$. Similarly, E_6/P_2 and E_6/P_3 can be interpreted as the varieties of projective lines and planes contained in $\check{\mathbb{OP}}^2$.

The remaining E_6-grassmannian E_6/P_4 is the adjoint variety E_6^{ad}, the closed orbit in the projectivization $\mathbb{P}\mathfrak{e}_6$ of the adjoint representation. By [19] again, E_6/P_3 can be identified to the variety $G(\mathbb{P}^1, E_6^{ad})$ of projective lines contained in $E_6^{ad} \subset \mathbb{P}\mathfrak{e}_6$.

Now, a Borel subgroup B in E_6 is the intersection of the maximal parabolic subgroups that it contains, and there is one such group for each simple root. Each of these maximal parabolics can be seen as a point on an E_6-grassmannian, and the fact that these parabolic subgroups have a Borel subgroup in common, means that these points are *incident* in the sense of Tits geometries [29].

Concretely, a point of E_6/P_3 defines a projective plane Π in $\mathbb{OP}^2$, a dual plane $\check{\Pi}$ in $\check{\mathbb{OP}}^2$, and a line Λ in E_6^{ad}. Choose a point p and a line ℓ in $\mathbb{OP}^2$ such that $p \in \ell \subset \Pi$, choose a point $\check{p}$ and a line $\check{\ell}$ in $\check{\mathbb{OP}}^2$ such that $\check{p} \in \check{\ell} \subset \check{\Pi}$, and finally a point $q \in \Lambda$.



We call this data a complete E_6-flag. By [29], there is a bijective correspondance between the set of Borel subgroups of E_6 and the set of complete E_6-flags: this is a direct generalization of the usual fact that a Borel subgroup of SL_n is the stabilizer of a unique flag of vector subspaces of $\mathbf{C}^n$.

We will not need this, but to complete the picture let us mention that the correspondance between Π, $\check{\Pi}$ and Λ can be described as follows:

$$\Pi = \bigcap_{z \in \check{\Pi}} (T_z \check{\mathbb{OP}}^2)^{\perp} = \bigcap_{y \in \Lambda} y\mathcal{J}_3(\mathbb{O}).$$

This description of Borel subgroups will be useful to construct Schubert varieties in $\mathbb{OP}^2$. Indeed, any subvariety of the Cayley plane that can defined in terms of a complete (or incomplete) E_6-flag, must be a finite union of Schubert varieties.

Let us apply this principle in small codimension. The data $\check{p}, \check{\ell}, \Lambda$ from our E_6-flag are respectively a point, a line and a plane in $\check{\mathbb{O}\mathbb{P}}^2$. They define special linear sections of $\mathbb{O}\mathbb{P}^2$, of respective codimensions 1, 2 and 3. We read on the Hasse diagram that these sections are irreducible Schubert varieties.

Something more interesting happens in codimension four, since we can read on the Hasse diagram that a well-chosen codimension four linear section of $\mathbb{O}\mathbb{P}^2$ should split into the union of two Schubert varieties, of degrees 33 and 45. The most degenerate codimension four sections must correspond to very special $\mathbb{P}^3$'s in $\check{\mathbb{O}\mathbb{P}}^2$. We know from [19] that $\check{\mathbb{O}\mathbb{P}}^2$ contains a whole family of $\mathbb{P}^3$'s, in fact a homogeneous family parametrized by $E_6/P_{2,4}$. In terms of our E_6-flag, that means that a unique member of this family is defined by the pair (q, ℓ).

We can describe explicitely a $\mathbb{P}^3$ in $\mathbb{O}\mathbb{P}^2$ in the following way. Choose a non-zero vector $z \in \mathbb{O}$, of zero norm. Then the closure of the set

$$\left\{ \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x \in L(z) \right\},$$

is a three dimensional projective space $\mathbb{P}^3_z$ in $\mathbb{O}\mathbb{P}^2$. Let us take the orthogonal of this space with respect to the quadratic form $Q(X) = \mathrm{trace}(X^2)$, and cut it with $\mathbb{O}\mathbb{P}^2$. We obtain two codimension 4 subvarieties Z_1 and Z_2, respectively the closures of the following affine cells Z_1^0 and Z_2^0:

$$Z_1^0 = \left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & 0 & \bar{x}y \\ \bar{y} & \bar{y}x & \bar{y}y \end{pmatrix}, \quad x \in L(z), y \in \mathbb{O} \right\}, \tag{5}$$

$$Z_2^0 = \left\{ \begin{pmatrix} 0 & u & uv \\ \bar{u} & 1 & v \\ \bar{v}\bar{u} & \bar{v} & \bar{v}v \end{pmatrix}, \quad u \in L(z), v \in \mathbb{O} \right\}. \tag{6}$$

The sum of the degrees of these two varieties is equal to 78. The corresponding cycles are linear combinations of Schubert cycles with non negative coefficients. But in codimension 4 we have only two such cycles, σ_4' and σ_4'', of respective degrees 33 and 45. The only possibility is that the cycles $[Z_1]$ and $[Z_2]$ coincide, up to the order, with σ_4' and σ_4''.

To decide which is which, let us cut Z_1 with $H_1 = \{c_1 = 0\}$.

Lemma 4.10. *The hyperplane section $Y_1 = Z_1 \cap H_1$ has two components $Y_{1,1}$ and $Y_{1,2}$. One of these two components, say $Y_{1,1}$, is the closure of*

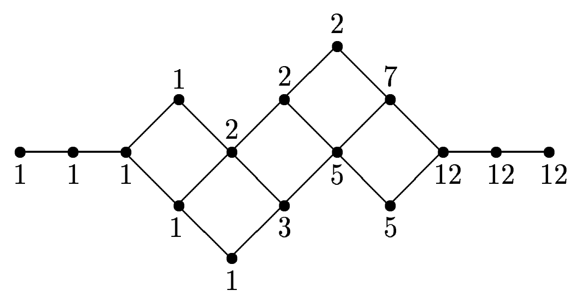
$$Y_{1,1}^0 = \left\{ \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ \bar{t} & \bar{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0, \ \bar{s}t = 0 \right\}.$$

It is a cone over the spinor variety $\mathbb{S}_{10} \subset \mathbb{P}^{15}$.

Recall that the spinor variety $\mathbb{S}_{10}$ is one of the two families of maximal isotropic subspaces of a smooth eight-dimensional quadric. Its appearance is not surprising, since we have seen on the weighted Dynkin diagram of $\mathbb{O}\mathbb{P}^2 = E_6/P_6$ that the semi-simple part of P_6 is a copy of Spin_{10}. At a given point of $p \in \mathbb{O}\mathbb{P}^2$, the stabilizer P_6 and its subgroup Spin_{10} act on the tangent space, which is isomorphic as a Spin_{10}-module to a half-spin representation, say Δ_+. From [19], we know that the family of lines through p, that are contained in $\mathbb{OP}^2$, is isomorphic to the spinor variety $\mathbb{S}_{10}$, since it is the closed Spin_{10}-orbit in $\mathbb{P}\Delta_+$.

In particular, to each point p of $\mathbb{OP}^2$ we can associate a subvariety, the union of lines through that point, which is a cone $\mathcal{C}(\mathbb{S}_{10})$ over the spinor variety. This is precisely what is $Y_{1,1}$. Note that we get a Schubert variety in the Cayley plane. Moreover, since we can choose a Borel subgroup of Spin_{10} inside a Borel subgroup of E_6 contained in P_6, we obtain a whole series of Schubert varieties which are isomorphic to cones over the Schubert subvarieties of $\mathbb{S}_{10}$. These Schubert varieties can be described in terms of incidence relations with an isotropic reference flag which in principle can be deduced from our reference E_6-flag.

The Hasse diagram of Schubert varieties in $\mathbb{S}_{10}$ is the following:



The identification of the cone of lines in $\mathbb{OP}^2$ through some given point, with the spinor variety $\mathbb{S}_{10}$, is not so obvious. Consider the map

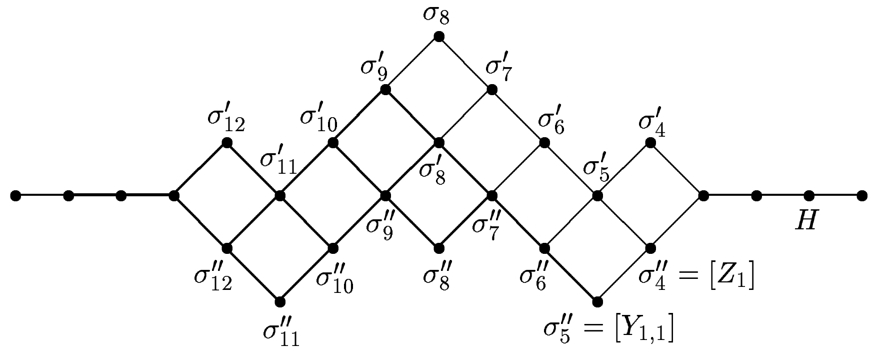
$$\nu_2 : \mathbb{O} \oplus \mathbb{O} \to \mathcal{J}_2(\mathbb{O}), \qquad \nu_2(x,y) = \begin{pmatrix} x\bar{x} & x\bar{y} \\ y\bar{x} & y\bar{y} \end{pmatrix}.$$

We want to identify $\mathbb{P}\nu_2^{-1}(0)$ with $\mathbb{S}_{10}$. The following result is due to P.E. Chaput:

Proposition 4.11. *Let $(x,y) \in \nu_2^{-1}(0)$. The image of the tangent map to ν_2 at (x,y) is a 5-dimensional subspace of $\mathcal{J}_2(\mathbb{O})$, which is isotropic with respect to the determinantal quadratic form on $\mathcal{J}_2(\mathbb{O})$. Moreovoer, this induces an isomorphism between $\mathbb{P}\nu_2^{-1}(0)$ and the spinor variety $\mathbb{S}_{10}$.*

In fact, we can obtain this way the two families of maximal isotropic subspaces in $\mathcal{J}_2(\mathbb{O})$, just by switching the two diagonal coefficients in the definition of ν_2. The spin group Spin_{10} can also be described very nicely.

But let's come back to the Schubert varieties in $\mathbb{S}_{10}$. Taking cones over them, we get Schubert subvarieties which define a subdiagram of the Hasse diagram of $\mathbb{OP}^2$. We drew this subdiagram in thicklines on the picture below.



In principle, we are able to describe any of these Schubert varieties geometrically in terms of our reference E_6-flag.

Proof of Lemma 4.1. First note that Y_1 does not meet the two affine cells $\mathbb{OP}^2_1$ and $\mathbb{OP}^2_2$ (see section 2). Moreover, it is easy to check that $Y_1 \cap \mathbb{OP}^2_\infty$ has dimension at most ten, hence strictly smaller dimension than Y_1. Therefore, Y_1 is the closure of its intersection with $\mathbb{OP}^2_3$, namely

$$Y_1 \cap \mathbb{OP}^2_3 = \left\{ \begin{pmatrix} 0 & \bar{s}t & t \\ \bar{t}s & 0 & s \\ \bar{t} & \bar{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0, \ \bar{s}t \in L(z) \right\}.$$

For a given non zero s, the product $\bar{s}t$ must belong to $L(z) \cap L(\bar{s})$, the intersection of two maximal isotropic spaces of the same family. In particular, this intersection has even dimension.

Generically, the intersection $L(z) \cap L(\bar{s}) = 0$, and we obtain

$$Y^0_{1,1} = \left\{ \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ \bar{t} & \bar{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0, \ \bar{s}t = 0 \right\} \subset Y_1.$$

We have seven parameters for s, and for each $s \neq 0$, t must belong to $L(s)$, which gives four parameters. In particular, $Y^0_{1,1}$ is irreducible of dimension 11, and its closure is an irreducible component of Y_1.

The intersection $L(z) \cap L(\bar{s})$ has dimension two exactly when the line joining z to $\bar{s}$ is isotropic, which means that $\bar{s}$ belongs to the intersection of the quadric $q = 0$ with its tangent hyperplane at z. This gives six parameters for s, and for each s, five parameters for t, which must be contained in the intersection of the quadric with a six-dimensional linear space. Therefore, the closure of

$$Y^0_{1,2} = \left\{ \begin{pmatrix} 0 & \bar{s}t & t \\ \bar{t}s & 0 & s \\ \bar{t} & \bar{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0, \ \dim L(z) \cap L(\bar{s}) = 2 \right\},$$

is another component of Y_1.

The remaining possibility is that s be a multiple of z, but the corresponding subset of has dimension smaller than eleven. Hence $Y_1 = Y_{1,1} \cup Y_{1,2}$. □

We conclude that Z_1 has degree 45, while Z_2 has degree 33. Indeed, if Z_1 had degree 33, we would read on the Hasse diagram that its proper hyperplane sections are always irreducible, and we have just verified that this is not the case.

Note that Z_1 and Z_2 look very similar at first sight. Nevertheless, a computation similar to the one we have just done shows that if we cut Z_2 by the hyperplane $H_2 = \{c_2 = 0\}$, we get an irreducible variety, the difference with Z_1 coming from the fact that we now have to deal with maximal isotropic subspaces which are not on the same family. The difference between Z_1 and Z_2 is therefore just a question of spin...

A.5. Intersection numbers

We now determine the multiplicative structure of the Chow ring $A^*(\mathbb{OP}^2)$. A priori, we have several interesting informations on that ring structure. We have already seen in section 2 that the Pieri formula determines combinatorially the product with the hyperplane class. Another important property is that Poincaré duality has a very simple form in terms of Schubert cycles: the basis $(\sigma_w)_{w\in W^0}$ is, up to order, self-dual; more precisely its dual basis is $(\sigma_{w^*})_{w\in W^0}$, where the involution $w \mapsto w^*$ is very simple to define on the Hasse diagram: it is just the symmetry with respect to the vertical line passing through the cycles of middle dimension. Finally, we know from Poincaré duality and general transversality arguments that any *effective* cycle must be a linear combination of Schubert cycles with non negative coefficients.

This is the information we have on any rational homogeneous space. For what concerns the Cayley plane, we begin with an obvious observation:

Proposition 4.12. *The Chow ring $A^*(\mathbb{OP}^2)$ is generated by the hyperplane class H, the class σ_4', and the class σ_8 of an $\mathbb{O}$-line.*

More precisely, one can directly read on the Hasse diagram and from the Pieri formula that as a vector space, the Chow ring is generated by classes of type H^i, $\sigma_4' H^j$ and $\sigma_8 H^k$. For example, we have the relations

$$H^4 = \sigma_4' + \sigma_4'', \tag{7}$$
$$\sigma_4' H^4 = \sigma_8 + 3\sigma_8' + 2\sigma_8'', \tag{8}$$
$$\sigma_4'' H^4 = \sigma_8 + 4\sigma_8' + 3\sigma_8'', \tag{9}$$
$$\sigma_8 H^4 = \sigma_{12}' + \sigma_{12}'', \tag{10}$$
$$\sigma_8' H^4 = 3\sigma_{12}' + 4\sigma_{12}'', \tag{11}$$
$$\sigma_8'' H^4 = 2\sigma_{12}' + 3\sigma_{12}''. \tag{12}$$

As a consequence, the multiplicative structure of the Chow ring will be completely determined once we'll have computed the intersection products $(\sigma^8)^2, \sigma_4'\sigma_8$ and $(\sigma_4')^2$. (Note that the Hasse diagram and the Pieri formula can be used to derive relations in dimension 9 and 13, but these relations are not sufficient to determine the whole ring structure.)

Proposition 4.13. *We have the following relations in the Chow ring:*

$$\sigma_8^2 = 1, \tag{13}$$
$$\sigma_4'\sigma_8 = \sigma_{12}', \tag{14}$$
$$\sigma_4''\sigma_8 = \sigma_{12}''. \tag{15}$$

Proof. Recall that σ_8 is the class of an $\mathbb{O}$-line in $\mathbb{OP}^2$, and that we know that the geometry of these lines is similar to the usual line geometry in

$\mathbb{P}^2$: namely, two generic lines meet transversely in one point. This implies immediately that $\sigma_8^2 = 1$.

To compute $\sigma_4'\sigma_8$ and $\sigma_4''\sigma_8$, we cut the Schubert varieties Z_1 and Z_2 introduced in section 4, whose class we know to be σ_4' and σ_4'', with the $\mathbb{O}$-line L defined in $\mathbb{OP}^2$ by the conditions $x_1 = x_2 = r_3 = 0$. We get transverse intersections

$$Z_1 \cap L = \left\{ \begin{pmatrix} r & y & 0 \\ \bar{y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \in L(z) \right\},$$

$$Z_2 \cap L = \left\{ \begin{pmatrix} 0 & y & 0 \\ \bar{y} & r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \in L(z) \right\}.$$

These are two four dimensional projective spaces $\mathbb{P}_1^4$ and $\mathbb{P}_2^4$ inside $\mathbb{OP}^2$, which look very similar. But there is actually a big difference: $\mathbb{P}_1^4$ is extendable, but $\mathbb{P}_2^4$ is not ! Indeed, a $\mathbb{P}^5$ in $\mathbb{OP}^2$ containing $\mathbb{P}_1^4$ or $\mathbb{P}_2^4$ must be of the form, respectively:

$$\left\{ \begin{pmatrix} r & y & s \\ \bar{y} & 0 & 0 \\ \bar{s} & 0 & 0 \end{pmatrix}, \ y \in L(z) \right\}, \quad \text{and} \quad \left\{ \begin{pmatrix} 0 & y & 0 \\ \bar{y} & r & s \\ 0 & \bar{s} & 0 \end{pmatrix}, \ y \in L(z) \right\},$$

where s describes some line in $\mathbb{O}$. In the second case, the equation $sy = 0$ must be verified identically, and we can take s on the line $\mathbf{C}\bar{z}$: thus $\mathbb{P}_2^4$ is extendable. But in the first case, we need the identity $y\bar{s} = 0$ for all $y \in L(z)$, which would imply that $L(z) \subset R(s)$: this is impossible, and $\mathbb{P}_1^4$ is not extendable. The proposition follows – see the third observation at the end of section 3. □

We now have enough information to complete the multiplication table. First, we know by Poincaré duality that

$$(\sigma_8)^2 = (\sigma_8')^2 = (\sigma_8'')^2 = 1, \tag{16}$$

$$\sigma_8\sigma_8' = \sigma_8'\sigma_8'' = \sigma_8\sigma_8'' = 0, \tag{17}$$

$$\sigma_4'\sigma_{12}' = \sigma_4''\sigma_{12}'' = 1, \tag{18}$$

$$\sigma_4'\sigma_{12}'' = \sigma_4''\sigma_{12}' = 0. \tag{19}$$

Suppose that we have

$$\begin{aligned} (\sigma_4')^2 &= \mu_0\sigma_8 + \mu_1\sigma_8' + \mu_2\sigma_8'', \\ (\sigma_4'')^2 &= \nu_0\sigma_8 + \nu_1\sigma_8' + \nu_2\sigma_8'', \\ \sigma_4'\sigma_4'' &= \gamma_0\sigma_8 + \gamma_1\sigma_8' + \gamma_2\sigma_8'', \end{aligned}$$

for some coefficients to be determined. Cutting with σ_8, we get $\mu_0 = \nu_0 = 1$. The equations (3), (4), (5) give the relations

$$\begin{aligned} &\mu_0 + \gamma_0 = 1, \quad \mu_1 + \gamma_1 = 3, \quad \mu_2 + \gamma_2 = 2, \\ &\nu_0 + \gamma_0 = 1, \quad \nu_1 + \gamma_1 = 4, \quad \nu_2 + \gamma_2 = 3. \end{aligned}$$

In particular, $\gamma_0 = 0$. Now, we compute $(\sigma_4')^2(\sigma_4'')^2$ in two ways to obtain the relation

$$\gamma_1^2 + \gamma_2^2 = \mu_0\nu_0 + \mu_1\nu_1 + \mu_2\nu_2.$$

Eliminating the μ_i's and ν_i's, we get that $7\gamma_1 + 5\gamma_2 = 19$. But γ_1 and γ_2 are non negative integers, so the only possibility is that $\gamma_1 = 2$, $\gamma_2 = 1$. Thus:

$$(\sigma_4')^2 = \sigma_8 + \sigma_8' + \sigma_8'', \tag{20}$$
$$(\sigma_4'')^2 = \sigma_8 + 2\sigma_8' + 2\sigma_8'', \tag{21}$$
$$\sigma_4'\sigma_4'' = 2\sigma_8' + \sigma_8''. \tag{22}$$

And this easily implies that

$$\sigma_4'\sigma_8' = \sigma_{12}' + 2\sigma_{12}'', \tag{23}$$
$$\sigma_4'\sigma_8'' = \sigma_{12}' + \sigma_{12}'', \tag{24}$$
$$\sigma_4''\sigma_8' = 2\sigma_{12}' + 2\sigma_{12}'', \tag{25}$$
$$\sigma_4''\sigma_8'' = \sigma_{12}' + 2\sigma_{12}''. \tag{26}$$

A.6. The Borel presentation

We now turn to the Borel presentation of the Chow ring of $\mathbb{OP}^2$. This is the ring isomorphism

$$A^*(\mathbb{OP}^2)_{\mathbb{Q}} \simeq \mathbb{Q}[\mathcal{P}]^{W_0}/\mathbb{Q}[\mathcal{P}]^W_+,$$

where $\mathbb{Q}[\mathcal{P}]^{W_0}$ denotes the ring of W_0-invariants polynomials on the weight lattice, and $\mathbb{Q}[\mathcal{P}]^W_+$ is the ideal of $\mathbb{Q}[\mathcal{P}]^{W_0}$ generated by W-invariants without constant term (see [14], ??).

The ring $\mathbb{Q}[\mathcal{P}]^{W_0}$ is easily determined: it is generated by ω_6, and the subring of W_0 invariants in the weight lattice of Spin_{10}. It is therefore the polynomial ring in the elementary symmetric functions $e_{2i} = c_i(\varepsilon_1, \dots, \varepsilon_5)$, $1 \le i \le 4$, and in $e_5 = \varepsilon_1 \cdots \varepsilon_5$.

The invariants of W, the full Weyl group of E_6, are more difficult to determine, although we know their fundamental degrees. But since we know how to compute the intersection products of any two Schubert cycles, we just need to express the W_0-invariants in terms of the Schubert classes. This can be achieved, following [2], by applying suitable difference operators to these invariants.

Since we give a prominent role to the subsystem of E_6 of type D_5, it is natural to choose for the first five simple roots the usual simple roots of D_5, that is, in a euclidian 6-dimensional space with orthonormal basis $\varepsilon_1, \dots, \varepsilon_6$,

$$\begin{aligned}
\alpha_1 &= \varepsilon_1 - \varepsilon_2, \\
\alpha_2 &= \varepsilon_2 - \varepsilon_3, \\
\alpha_3 &= \varepsilon_3 - \varepsilon_4, \\
\alpha_4 &= \varepsilon_4 - \varepsilon_5, \\
\alpha_5 &= \varepsilon_4 + \varepsilon_5, \\
\alpha_6 &= -\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \tfrac{\sqrt{3}}{2}\varepsilon_6.
\end{aligned}$$

The fundamental weights are given by the dual basis:

$$\begin{aligned}
\omega_1 &= \varepsilon_1 + \frac{1}{\sqrt{3}}\varepsilon_6, \\
\omega_2 &= \varepsilon_1 + \varepsilon_2 + \frac{2}{\sqrt{3}}\varepsilon_6, \\
\omega_3 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \frac{3}{\sqrt{3}}\varepsilon_6, \\
\omega_4 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5) + \frac{\sqrt{3}}{2}\varepsilon_6, \\
\omega_5 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \frac{5}{\sqrt{3}}\varepsilon_6, \\
\omega_6 &= -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \frac{\sqrt{3}}{2}\varepsilon_6.
\end{aligned}$$

The action of the fundamental reflexions on the weight lattice is specially simple in the basis $\varepsilon_1, \ldots, \varepsilon_5, \alpha_6$. Indeed, s_1, s_2, s_3 and s_4 are just the transpositions $(12), (23), (34), (45)$. The reflection s_5 affects $\varepsilon_4, \varepsilon_5$ and α_6, which are changed into $-\varepsilon_5, -\varepsilon_4$ and $\alpha_6 + \varepsilon_4 + \varepsilon_5$. Finally, s_6 changes each ε_i into $\varepsilon_i + \alpha_6/2$, and of course α_6 into $-\alpha_6$.

It is then reasonably simple to compute the corresponding divided differences with MAPLE. We obtain:

Proposition 4.14. *The fundamental W_0-invariants are given, in the Chow ring of the Cayley plane, in terms of Schubert cycles by;*

$$e_2 = -\frac{3}{4}H^2, \tag{27}$$

$$e_4 = -\frac{27}{8}\sigma_4' + \frac{21}{8}\sigma_4'', \tag{28}$$

$$e_5 = \frac{3}{16}\sigma_5' - \frac{21}{32}\sigma_5'', \tag{29}$$

$$e_6 = -\frac{27}{16}\sigma_6' + \frac{87}{32}\sigma_6'', \tag{30}$$

$$e_8 = \frac{21}{128}\sigma_8 + \frac{291}{256}\sigma_8' - \frac{519}{256}\sigma_8''. \tag{31}$$

This allows to compute any product in the Borel presentation of the Chow ring of $\mathbb{OP}^2$.

A.7. Chern classes of the normal bundle

Let $\mathcal{N}$ denote the normal bundle to the Cayley plane $\mathbb{OP}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{O})$. We want to compute its Chern classes.

First note that the restriction of $\mathcal{J}_3(\mathbb{O})$ to the Levi part $L \simeq \mathrm{Spin}_{10} \times \mathbf{C}^*$ of the parabolic subgroup P_6 of E_6, is

$$\mathcal{J}_3(\mathbb{O})_{|L} \simeq W_{\omega_6} \oplus W_{\omega_5 - \omega_6} \oplus W_{\omega_1 - \omega_6}.$$

Indeed, there is certainly the line generated by the highest weight vector, which gives a stable line on which L acts through the character ω_6. After ω_6, there is a in $\mathcal{J}_3(\mathbb{O})$ a unique highest weight, $\omega_5 - \omega_6$, which generates a 16-dimensional half-spin module. Finally, the lowest weight of $\mathcal{J}_3(\mathbb{O})$ is $-\omega_1$, whose highest W_0-conjugate is $\omega_1 - \omega_6$ and generates a copy of the natural 10-dimensional representation of Spin_{10}. Since these three modules give $1+16+10=27$ dimensions, we have the full decomposition.

Geometrically, this decomposition of $\mathcal{J}_3(\mathbb{O})$ must be interpreted as follows. We have chosen a point p of $\mathbb{OP}^2$, corresponding to the line $\hat{p} = W_{\omega_6}$. The tangent space to $\mathbb{OP}^2$ at that point is given by the factor $W_{\omega_5-\omega_6}$. (More precisely, only the affine tangent space $\hat{T}$ is a well-defined P_6-submodule of $\mathcal{J}_3(\mathbb{O})$, and it coincides with $W_{\omega_6} \oplus W_{\omega_5-\omega_6}$.) The remaining term $W_{\omega_1-\omega_6}$ corresponds to the normal bundle. To be precise, if $\mathcal{N}_p$ denotes the normal space to $\mathbb{OP}^2$ at p, there is a canonical identification

$$\mathcal{N}_p \simeq Hom(\hat{p}, \mathcal{J}_3(\mathbb{O})/\hat{T}) = Hom(W_{\omega_6}, W_{\omega_1-\omega_6}).$$

In other words, the normal bundle $\mathcal{N}$ to $\mathbb{OP}^2$ is the homogeneous bundle $\mathcal{E}_{\omega_1-2\omega_6}$ defined by the irreducible P_6-module $W_{\omega_1-2\omega_6}$.

Since $\omega_1 = \varepsilon_1 + \frac{1}{2}\omega_6$, the weights of the normal bundle are the $\pm\varepsilon_i - \frac{3}{2}\omega_6$, and its Chern class is

$$\begin{aligned} c(\mathcal{N}) = & \prod_{i=1}^5 (1+\varepsilon_i - \tfrac{3}{2}\omega_6)(1-\varepsilon_i-\tfrac{3}{2}\omega_6) \\ = & \prod_{i=1}^5 \left((1+\tfrac{3}{2}H)^2 - \varepsilon_i^2\right) \\ = & \sum_{i=0}^5 (-1)^i (1+\tfrac{3}{2}H)^{10-2i} e_{2i}, \end{aligned}$$

where $e_{10} = e_5^2$. We know how to express this in terms of Schubert classes, and the result is as follows.

Proposition 4.15. *In terms of Schubert cycles, the Chern classes of the normal bundle to $\mathbb{OP}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{O})$ are:*

$$\begin{aligned} c_1(\mathcal{N}) &= 15H \\ c_2(\mathcal{N}) &= 102H^2 \\ c_3(\mathcal{N}) &= 414H^3 \\ c_4(\mathcal{N}) &= 1107\sigma_4' + 1113\sigma_4'' \\ c_5(\mathcal{N}) &= 2025\sigma_4'H + 2079\sigma_4''H \\ c_6(\mathcal{N}) &= 5292\sigma_6' + 8034\sigma_6'' \\ c_7(\mathcal{N}) &= 4698\sigma_6'H + 7218\sigma_6''H \\ c_8(\mathcal{N}) &= 2751\sigma_8 + 9786\sigma_8' + 7032\sigma_8'' \\ c_9(\mathcal{N}) &= 963\sigma_8 H + 3438\sigma_8' H + 2466\sigma_8'' H \\ c_{10}(\mathcal{N}) &= 153\sigma_8 H^2 + 549\sigma_8' H^2 + 387\sigma_8'' H^2 \end{aligned}$$

Note that as expected, we get integer coefficients, while the fundamental W^0-invariants are only rational combinations of the Schubert cycles. This is a strong indication that our computations are correct.

A.8. The final computation

We want to compute the degree of the variety of reductions Y_8. Recall that this variety Y_8 is a smooth projective variety of dimension 24, embedded in $\mathbb{P}^{272}$. A $\mathbb{P}^1$-bundle Z_8 over Y_8 can be identified with the blow-up of the projected Cayley plane $\overline{X_8}$ in $\mathbb{P}\mathcal{J}_3(\mathbb{O})_0$, the projective space of trace zero Hermitian matrices of order three, with coefficients in the Cayley octonions.

Let H denote the pull-back to Z_8 of the hyperplane class of $\mathbb{P}\mathcal{J}_3(\mathbb{O})_0$, and E the exceptional divisor of the blow-up. We want to compute

$$\deg Y_8 = H(3H-E)^{24}.$$

We use the fact that the Chow ring of the exceptional divisor $E \subset Z_8$, since it is the projectivization of the normal is the quotient of the ring $A^*(\mathbb{OP}^2)[e]$ by the relation given by the Chern classes of the normal bundle $\bar{\mathcal{N}}$ of $\overline{X_8}$, namely

$$e^9 + \sum_{i=1}^{9}(-1)^i c_i(\bar{\mathcal{N}})e^{9-i} = 0.$$

The normal bundle $\bar{\mathcal{N}}$ of $\overline{X_8}$ is related to the normal bundle $\mathcal{N}$ of $X_8 = \mathbb{OP}^2$ by an exact sequence $0 \to \mathcal{O}(1) \to \mathcal{N} \to \bar{\mathcal{N}} \to 0$, from which we can compute the Chern classes of $\bar{\mathcal{N}}$:

$$\begin{aligned}
c_1(\bar{\mathcal{N}}) &= 14H \\
c_2(\bar{\mathcal{N}}) &= 88H^2 \\
c_3(\bar{\mathcal{N}}) &= 326H^3 \\
c_4(\bar{\mathcal{N}}) &= 781\sigma_4' + 787\sigma_4'' \\
c_5(\bar{\mathcal{N}}) &= 1244\sigma_4'H + 1292\sigma_4''H = 2536\sigma_5' + 1292\sigma_5'' \\
c_6(\bar{\mathcal{N}}) &= 2756\sigma_6' + 4206\sigma_6'' \\
c_7(\bar{\mathcal{N}}) &= 1942\sigma_6'H + 3012\sigma_6''H = 1942\sigma_7' + 4954\sigma_7'' \\
c_8(\bar{\mathcal{N}}) &= 809\sigma_8 + 2890\sigma_8' + 2078\sigma_8'' \\
c_9(\bar{\mathcal{N}}) &= 154\sigma_8H + 548\sigma_8'H + 388\sigma_8''H = 702\sigma_9' + 936\sigma_9'' \\
c_{10}(\bar{\mathcal{N}}) &= -\sigma_8H^2 + \sigma_8'H^2 - \sigma_8''H^2 = 0\,!
\end{aligned}$$

The fact that we get $c_{10}(\bar{\mathcal{N}}) = 0$, which must hold since $\bar{\mathcal{N}}$ has rank 9, is again a strong indication that we did no mistake.

To complete our computation, we must compute the intersection products $H^{25-i}E^i$ in the Chow ring of Z_8. For $i > 0$, this can be computed on the exceptional divisor; since the restriction of the class E to the exceptional divisor is just the relative hyperplane section, that is, the class e, we have $H^{25-i}E^i = H^{25-i}e^{i-1}$, the later product being computed in $A^*(E)$. We still denoted by H the pull-back of the hyperplane section from $\mathbb{OP}^2$.

Lemma 4.16. *Let $\sigma \in A^{16-k}(\mathbb{OP}^2)$. Then $\sigma e^{8+k} = \sigma s_k(\bar{\mathcal{N}})$, where $s_k(\bar{\mathcal{N}})$ denotes the k-th Segre class of the normal bundle $\bar{\mathcal{N}}$. The former product is computed in $A^*(E)$, and the the later in $A^*(\mathbb{OP}^2)$.*

Proof. Induction, using the relation $e^9 + \sum_{i=1}^{9}(-1)^i c_i(\bar{\mathcal{N}})e^{9-i} = 0$, and the fact that the Segre classes are related to the Chern classes by the formally similar relation $s_k(\bar{\mathcal{N}}) + \sum_{i=1}^{9}(-1)^i c_i(\bar{\mathcal{N}})s_{k-i}(\bar{\mathcal{N}}) = 0$. □

We use the later relation to determine the Segre classes inductively. We obtain

$$\begin{aligned}
s_1(\bar{\mathcal{N}}) &= 14H, \\
s_2(\bar{\mathcal{N}}) &= 108H^2, \\
s_3(\bar{\mathcal{N}}) &= 606H^3, \\
s_4(\bar{\mathcal{N}}) &= 2763\sigma_4' + 2757\sigma_4'', \\
s_5(\bar{\mathcal{N}}) &= 21624\sigma_5' + 10752\sigma_5'' \\
s_6(\bar{\mathcal{N}}) &= 75492\sigma_6' + 112602\sigma_6'', \\
s_7(\bar{\mathcal{N}}) &= 240534\sigma_7' + 596598\sigma_7'', \\
s_8(\bar{\mathcal{N}}) &= 711489\sigma_8 + 2462397\sigma_8' + 1750947\sigma_8'', \\
s_9(\bar{\mathcal{N}}) &= 8768196\sigma_9' + 11600304\sigma_9'', \\
s_{10}(\bar{\mathcal{N}}) &= 53127900\sigma_{10}' + 30193704\sigma_{10}'', \\
s_{11}(\bar{\mathcal{N}}) &= 206857602\sigma_{11}' + 74823228\sigma_{11}'', \\
s_{12}(\bar{\mathcal{N}}) &= 491985531\sigma_{12}' + 669523221\sigma_{12}'', \\
s_{13}(\bar{\mathcal{N}}) &= 2657712312\sigma_{13}, \\
s_{14}(\bar{\mathcal{N}}) &= 5875513812\sigma_{14} \\
s_{15}(\bar{\mathcal{N}}) &= 12591161406\sigma_{15}.
\end{aligned}$$

This immediately gives the degree of Y_8,

$$\deg Y_8 = 3^{24} + \sum_{k=9}^{24} (-1)^k \binom{24}{k} 3^{24-k} H^{25-k} s_{k-9}(\bar{\mathcal{N}}).$$

Theorem 4.17. *The degree of the variety of reductions Y_8 is*

$$\deg Y_8 = 1\,047\,361\,761.$$

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