REMARKS ON THE SPECTRUM OF THE NEUMANN PROBLEM WITH MAGNETIC FIELD IN THE HALF SPACE

Abderemane MORAME and Françoise TRUC

Prépublication de l'Institut Fourier n° 607 (2003)
http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract

We consider a Schrödinger operator with a constant magnetic field in a half 3-dimensional space, with Neumann type boundary conditions. It is known from the works by Lu-Pan and Helffer-Morame that the lower bound of its spectrum is less than the intensity b of the magnetic field, provided that the magnetic field is not normal to the boundary.

We prove that the spectrum under b is a finite set of eigenvalues (each of infinite multiplicity).

In the case when the angle between the magnetic field and the boundary is small, we give a sharp asymptotic expansion of the number of these eigenvalues.

1 Introduction

Let us consider, for (t, x, s) in the half space $E = I\!R_+ \times I\!R^2$, the Neumann realization of the operator with magnetic field

$$H = (D_t - A_1)^2 + (D_x - A_2)^2 + (D_y - A_3)^2$$

where $D_s = -i(\frac{\partial}{\partial s})$.

We will assume that the magnetic field $B = dA$, seen as a 3-dimensional vector field, is not tangent to the boundary ∂E, and denote by θ the angle between B and the plane $t = 0$ and by b the norm of B.

This implies that a suitable choice for the gauge A is the 1-form

$$A = b(x \sin\theta - t\cos\theta)dy$$

Keywords: Neumann problem, magnetic field, spectral theory, eigenvalues.
Mathematics Subject Classification: 35P20.

(so that $A_1 = A_2 = 0$), since the condition $B = dA$ leads to the 2-form

$$B = b\sin\theta\ dx \wedge dy - b\cos\theta\ dt \wedge dy, \quad (\theta\ \in [0, \frac{\pi}{2}]) .$$

Now the operator H can be written as

$$H_\theta = D_t^2 + D_x^2 + (D_y - b(x\sin\theta - t\cos\theta))^2 \tag{1.1}$$

When $\theta = 0$, it is easy to see that the spectrum of the Neumann operator H_0 is absolutely continuous. More precisely one has :

$$\sigma(H_0) = \sigma_{ac}(H_0) = [b\mu_0, +\infty[\ ; \tag{1.2}$$

$$\mu_0 = \inf_{\xi\in\mathbb{R}} \mu(\xi)\ , \tag{1.3}$$

where $\mu(\xi)$ denotes the first eigenvalue of the Neumann operator
$Q_\xi = D_t^2 + (t-\xi)^2$ on $L^2(\mathbb{R}_+)$:

$$\mu(\xi) = \inf\ \sigma(Q_\xi) = \inf_{\|f\|_{L^2(\mathbb{R}_+)}=1} \int_{\mathbb{R}_+} \left[|D_t f|^2 + (t-\xi)^2|f|^2\right] dt\ . \tag{1.4}$$

It is also easy to see that, if $\theta = \frac{\pi}{2}$, the spectrum of $H_{\frac{\pi}{2}}$ is absolutely continuous:

$$\sigma(H_{\frac{\pi}{2}}) = \sigma_{ac} = [b, +\infty[\ . \tag{1.5}$$

When $\theta\ \in\ ]0, \frac{\pi}{2}[$, the spectrum of H_θ is no longer absolutely continuous as proved by K. Lu and X-B. Pan [LuPa], (see also [HeMo2]).
We are precisely interested in that case:

$$0 < \theta < \frac{\pi}{2}\ . \tag{1.6}$$

First, we observe that

$$\sigma(H_\theta) = \bigcup_{\tau\in\mathbb{R}} \sigma(H_{\theta,\tau})\ , \tag{1.7}$$

where $H_{\theta,\tau}$ denotes the Neumann realization in the half plane
$F = \mathbb{R}_+ \times \mathbb{R}$ of the operator

$$H_{\theta,\tau} = D_t^2 + D_x^2 + (\tau - b(x\sin\theta - t\cos\theta))^2\ . \tag{1.8}$$

Furthermore using for any τ the change of coordinates $x \to x - \frac{\tau}{b\sin\theta}$,
we see that $\sigma(H_{\theta,\tau}) = \sigma(H_{\theta,0})$,
and then the spectrum of H_θ is essential and given by :

$$\sigma(H_\theta) = \sigma_{ess}(H_\theta) = \sigma(H_{\theta,0}) = b \times \sigma(P_\theta)\ , \tag{1.9}$$

if $P_\theta = D_t^2 + D_x^2 + (t\cos\theta - x\sin\theta)^2$ is the Neumann operator on the half plane $F = \mathbb{R}_+ \times \mathbb{R}$.

In [LuPa], (see also [HeMo2]), it was proved that

$$\inf\ \sigma(P_\theta) = \nu(\theta) < 1 = \inf\ \sigma_{ess}(P_\theta)\ ; \tag{1.10}$$

so there exists a countable set of eigenvalues of P_θ, $(\nu_j(\theta))_{j\in I}$, $(I \subset I\!N)$, in $[\nu(\theta), 1[$. Each eigenvalue is of finite multiplicity, so we will assume that each one is repeated according to its multiplicity. The associated orthonormalized sequence of eigenfunctions will be denoted by $(\psi_{\theta,j})_{j\in I}$:

$$\begin{aligned} \nu(\theta) &= \nu_1(\theta) \le \nu_2(\theta) \le \ldots \nu_j(\theta) \le \nu_{j+1}(\theta) \le \ldots < 1, \\ P_\theta\psi_{\theta,j} &= \nu_j(\theta)\psi_{\theta,j}, \\ \langle\psi_{\theta,j}|\psi_{\theta,k}\rangle &= \delta_{jk}, \\ E_{]-\infty,1[}(P_\theta)f &= \sum_j \langle\psi_{\theta,j}|f\rangle\psi_{\theta,j}\ ; \end{aligned} \tag{1.11}$$

($\langle g\ |f\rangle = \int_F \overline{g} f dt dx$ and $E_J(P_\theta)$ denotes the spectral projection of P_θ on J). So

$$\sigma(H_\theta)\cap\ ]-\infty, b[= \{b\nu_1(\theta),\ b\nu_2(\theta), \ldots,\ b\nu_j(\theta),\ b\nu_{j+1}(\theta), \ldots\}\ , \tag{1.12}$$

(each $b\nu_j(\theta)$ is an eigenvalue of infinite multiplicity of H_θ).

For any $d \le 1$ let us denote by $N(d,\ P_\theta)$ the number of eigenvalues of P_θ in $]-\infty, d[$:

$$N(d,\ P_\theta) = Tr(E_{]-\infty,d[}(P_\theta) = \sharp\{j;\ \nu_j(\theta) < d\}\ . \tag{1.13}$$

The aim of this work is first to prove that for any $\theta \in]0, \frac{\pi}{2}[$, the number of eigenvalues of P_θ in $]-\infty,\ 1[$ is finite. This is the purpose of section 2. Another interesting question is to get the asymptotic behaviour of $N(d,\ P_\theta)$ as θ goes to zero, when $d < 1$. This is done in section 4. Section 3 is devoted to a survey of preliminary results about the function $\mu(\xi)$ defined in (1.4) , which are required in the computation of the asymptotics in section 4.

2 Finiteness of the discrete spectrum

The purpose of this section is to prove the following Theorem.

Theorem 2.1 *There exists a constant $C \ge 1$ such that, for any $\theta \in]0, \frac{\pi}{2}[$,*

$$N(1,\ P_\theta) \le \frac{C}{\sin\theta}\ . \tag{2.1}$$

Proof

Convention 2.2 *$\theta \in]0, \frac{\pi}{2}[$ is fixed.*

Convention 2.3 *From now on, any constant depending only on* θ *will be denoted invariably* C_θ *.*

If the constant does not depend on θ, *it will be denoted invariably* C.

Let us denote by q_θ the quadratic form associated to P_θ :

$$q_\theta(u) = \int_F [[D_t u|^2 + |D_x u|^2 + (t\cos\theta - x\sin\theta)^2 |u|^2] dtdx \ , \tag{2.2}$$

$\forall u \in H^1(F) \bigcap L^2(F;\ (t\cos\theta - x\sin\theta)^2 dtdx)$; $(F = I\!R_{+,t} \times I\!R_x\ )$.

There exists a partition of unity $(\chi_0(t),\ \chi_1(t))$ satisfying :

$$\begin{aligned} &\chi_0(t) = 1 \text{ if } t < 1, \\ &\chi_0(t) = 0 \text{ if } t > 2, \\ &\chi_0^2(t) + \chi_1^2(t) = 1\ . \end{aligned} \tag{2.3}$$

Let $R > 1$ be fixed. We consider the following covering of F :

$$\begin{aligned} O_{0,R} &= \{(t,x) \in I\!R_+ \times I\!R,\ 0 < t < 2R\} \\ O_{1,R} &= \{(t,x) \in I\!R_+ \times I\!R,\ R < t\} \end{aligned} \tag{2.4}$$

We define the partition of unity $(\chi_{0,R}(t),\ \chi_{1,R}(t))$ by :

$$\chi_{j,R}(t) = \chi_j(\frac{t}{R})\ . \tag{2.5}$$

Let us recall that

$$q_\theta(u) = \sum_j q_\theta(\chi_{j,R} u) - \sum_j \|\chi'_{j,R} u\|^2\ . \tag{2.6}$$

We define the following quadratic forms:

$$q_{\theta,0}(u) = \int_{O_{0,R}} [D_t u|^2 + |D_x u|^2 + (\ (t\cos\theta - x\sin\theta)^2 - V_R(t)\ )|u|^2] dtdx\ , \tag{2.7}$$

$\forall u \in H^1(O_{0,R}) \bigcap L^2(O_{0,R}; x^2 dtdx)$, $u/\{t = 2R\} = 0$, with $V_R(t) = \sum_j |\chi'_{j,R}(t)|^2$, and

$$q_{\theta,1}(u) = \int_{O_{1,R}} [D_t u|^2 + |D_x u|^2 + (\ (t\cos\theta - x\sin\theta)^2 - V_R(t)\ )|u|^2] dtdx\ , \tag{2.8}$$

$\forall\ u\ \in\ H^1(O_{1,R}) \bigcap L^2(O_{1,R}; (t\cos\theta - x\sin\theta)^2 dtdx)$, $u/\{t = R\} = 0$.

By min-max principle, we have

$$N(1,\ q_\theta) \leq N(1,\ q_{\theta,0}) + N(1,\ q_{\theta,1})\ . \tag{2.9}$$

This estimate remains if we change $O_{1,R}$ into $I\!R^2$ in the definition of $q_{\theta,1}$:

$$q_{\theta,1}(u) = \int_{I\!R^2} [D_t u|^2 + |D_x u|^2 + (\ (t\cos\theta - x\sin\theta)^2 - V_R(t)\ )|u|^2] dtdx\ , \tag{2.10}$$

$\forall\, u \;\in\; H^1(I\!\!R^2)\bigcap L^2(I\!\!R^2;(t\cos\theta - x\sin\theta)^2 dtdx)$.

As the operator $P_{\theta,0}$, associated to $q_{\theta,0}$ has compact resolvent, and

$$q_{\theta,0}(u) \geq \int_{O_{0,R}} \Big[D_t u|^2 + |D_x u|^2 + \Big(\frac{1}{2}x^2\sin^2\theta - 4R^2\cos^2\theta - \frac{C}{R^2}\Big)|u|^2\Big] dtdx\ ,$$

we get easily

$$N(1,\ q_{\theta,0}) \leq \frac{CR}{\sin\theta}\big[1 + R^2\cos^2\theta\big]^{3/2}\ . \tag{2.11}$$

Using the orthonormal change of coordinates :

$(t,x) \to (s,y)$ with $s = t\cos\theta - x\sin\theta$, and $y = t\sin\theta + x\cos\theta$,
we can take for $q_{\theta,1}$ the following expression :

$$q_{\theta,1}(u) = \int_{I\!\!R^2} [D_s u|^2 + |D_y u|^2 + (s^2 - V_R(s,y)\ )|u|^2] dsdy\ , \tag{2.12}$$

$\forall\, u \;\in\; H^1(I\!\!R^2)\bigcap L^2(I\!\!R^2;\ s^2 dsdy)$, with

$$V_R(s,y) = \frac{1}{R^2}\sum_j \Big|\chi_j'\Big(\frac{s\cos\theta + y\sin\theta}{R}\Big)\Big|^2\ . \tag{2.13}$$

Let us consider the orthogonal projections

$$\begin{aligned} \Pi_1(u)(s,y) &= e^{-s^2/2}\int_{I\!\!R} u(\tau,y)e^{-\tau^2/2}\frac{d\tau}{\sqrt{\pi}} \\ \Lambda_1 u &= u - \Pi_1 u\ , \end{aligned} \tag{2.14}$$

so that, for any $u \in L^2(I\!\!R^2)$, we get : $\|u\|^2 = \|\Pi_1 u\|^2 + \|\Lambda_1 u\|^2$.

Writing : $\Pi_1 u(s,y) = \frac{e^{-s^2/2}}{\pi^{1/4}}\psi(y)$ and

$$W_R(y) = \frac{1}{\sqrt{\pi}}\int_{I\!\!R} e^{-s^2} V_R(s,y) ds\ ,$$

we obtain that :

$$q_{\theta,1}(\Pi_1 u) = \int_{I\!\!R} [|D_y\psi|^2 + (1 - W_R(y))|\psi|^2] dy\ . \tag{2.15}$$

We have also

$$q_{\theta,1}(\Lambda_1 u) \geq \int_{I\!\!R^2} \Big[|D_y\Lambda_1 u|^2 + \Big(3 - \frac{C}{R^2}\Big)|\Lambda_1 u|^2\Big] dsdy\ . \tag{2.16}$$

But

$$q_{\theta,1}(u) = q_{\theta,1}(\Pi_1 u) + q_{\theta,1}(\Lambda_1 u) - 2\mathcal{R}e\int_{I\!\!R^2} V_R(s,y)\Pi_1 u.\overline{\Lambda_1 u} dsdy\ ; \tag{2.17}$$

so, for any $\epsilon \in]0,1]$,

$$q_{\theta,1}(u) \geq q_{\theta,1}(\Pi_1 u) - \frac{1}{\epsilon} \int_{I\!R^2} V_R^2(s,y)|\Pi_1 u|^2 ds dy + q_{\theta,1}(\Lambda_1 u) - \epsilon \|\Lambda_1 u\|^2 \ . \tag{2.18}$$

Thanks to (2.16), we can take $\epsilon = 1$ and R large enough such that

$$q_{\theta,1}(\Lambda_1 u) - \epsilon \|\Lambda_1 u\|^2 > \|\Lambda_1 u\|^2 \ , \tag{2.19}$$

for example R satisfying : $3 - \frac{C}{R^2} - \epsilon > 1$.
Then, by (2.15) (2.17) and (2.19), we get that

$$N(1,\ q_{\theta,1}) \leq N(0,\ q_{\theta,1,0}) \ , \tag{2.20}$$

if

$$q_{\theta,1,0}(\psi) = \int_{I\!R} \left[|D_y \psi|^2 - W_{R,1}(y)|\psi|^2\right] dy \ , \tag{2.21}$$

$\forall\, \psi \in H^1(I\!R)$, with

$$W_{R,1}(y) = \frac{1}{\sqrt{\pi}} \int_{I\!R} e^{-s^2} \left[V_R(s,y) + V_R^2(s,y)\right] ds \ .$$

From (2.3) and the formula (2.13), the following bound holds :

$$0 \leq W_{R,1}(y) \leq \frac{C}{R^2} \int_{\frac{R - y\sin\theta}{\cos\theta}}^{\frac{2R - y\sin\theta}{\cos\theta}} e^{-s^2} ds \ ,$$

(we used the fact that, for any fixed y, $V_R(s,y) = 0$ for s outside the interval defined by $R < s\cos\theta + y\sin\theta < 2R$), so

$$\begin{aligned} 0 &\leq W_{R,1}(y) \\ &\leq \frac{C}{R^2} \chi_{\left[\frac{R}{2\sin\theta},\ \frac{3R}{\sin\theta}\right]}(y) + \frac{C}{R\cos\theta}\left[e^{-\frac{(y\sin\theta - R)^2}{\cos^2\theta}} + e^{-\frac{(y\sin\theta - 2R)^2}{\cos^2\theta}}\right]. \end{aligned} \tag{2.22}$$

As the operators on $L^2(I\!R)$, $\frac{1}{3}D_y^2 - \frac{C}{R\cos\theta} e^{-\frac{(y\sin\theta - R)^2}{\cos^2\theta}}$, $\frac{1}{3}D_y^2 - \frac{C}{R\cos\theta} e^{-\frac{(y\sin\theta - 2R)^2}{\cos^2\theta}}$ and $\frac{\sin^2\theta}{3\cos^4\theta} D_y^2 - \frac{C}{R\cos\theta} e^{-y^2}$ have the same spectrum, we get from (2.20)–(2.22) that

$$N(1,\ q_{\theta,1}) \leq 2N(0,\ q_{\theta,1,1}) \ + \ N(0,\ q_{\theta,1,2}) \ , \tag{2.23}$$

if

$$q_{\theta,1,1}(\psi) = \int_{I\!R} \left[\frac{\sin^2\theta}{3\cos^4\theta}|D_y\psi|^2 - \frac{C}{R\cos\theta} e^{-y^2}|\psi|^2\right] dy \ , \tag{2.24}$$

and

$$q_{\theta,1,2}(\psi) = \int_{I\!R} \left[\frac{1}{3}|D_y\psi|^2 - \frac{C}{R^2}\chi_{\left[\frac{R}{2\sin\theta},\ \frac{3R}{\sin\theta}\right]}(y)|\psi|^2\right] dy \ , \tag{2.25}$$

$\forall\, \psi \in H^1(I\!R)$, for some $R > 1$ independent of $\theta \in]0, \frac{\pi}{2}[$.

It is well known that $N(0,\ q_{\theta,1,1}) \leq C\frac{\cos^{3/2}\theta}{\sin\theta}$
and $N(0,\ q_{\theta,1,2}) \leq \frac{C}{\sin\theta}$, so

$$N(1,\ q_{\theta,1}) \leq \frac{C}{\sin\theta} \ . \tag{2.26}$$

We conclude from (2.9), (2.11) and (2.26) that the estimate (2.1) is valid.

3 Some properties of $\mu(\xi)$

The properties of the first eigenvalue, $\mu(\xi)$, of the Neumann operator on $L^2(I\!R_+)$, $Q_\xi = D_t^2 + (t-\xi)^2$,
can be found in [DaHe], or in [BeSt], or in [HeMo1].
The main one is

$$\begin{array}{l} \mu \in C^\infty(I\!R), \\ \mu'(\xi) \neq 0 \quad \text{if } \xi \neq \xi_0, \\ \mu(\xi) > 1, \quad \text{if} \quad \xi < 0, \\ \mu(\xi) < 1, \quad \text{if} \quad \xi > 0, \\ \lim_{\xi \to -\infty} \mu(\xi) = +\infty, \quad \lim_{\xi \to +\infty} \mu(\xi) = 1 \ ; \end{array} \tag{3.1}$$

($\xi_0 > 0$ is such that $\xi_0^2 = \mu(\xi_0)$).
Let φ_ξ be a normalized eigenfunction associated to $\mu(\xi)$:

$$\|\varphi_\xi\|_{L^2(I\!R_+)} = 1, \quad \varphi'_\xi(0) = 0, \quad Q_\xi \varphi_\xi = \mu(\xi)\varphi_\xi \ , \tag{3.2}$$

then

$$\mu'(\xi) = -(\mu(\xi) - \xi^2)\varphi_\xi^2(0) \ . \tag{3.3}$$

It is easy to see that φ_ξ is exponentially decreasing. More precisely we have

Lemma 3.1 *For any* $\xi > 1$ *and* $\eta \in \,]0,1[$,

$$\int_{I\!R_+} \left[\eta(t-\xi)^2 - \mu(\xi)\right]_+ e^{(1-\eta)^{1/2}(t-\xi)^2} |\varphi_\xi(t)|^2 dt \leq \mu(\xi) e^{\mu(\xi)/\eta} \ ; \tag{3.4}$$

consequently, there exists $C_0 > 1$ *such that*

$$1 - C_0 e^{-|\xi|^2/C_0} \leq \mu(\xi) \ . \tag{3.5}$$

More over, if $\mu_2(\xi)$ *is the second eigenvalue of* Q_ξ , *then we have also*

$$3 - C_0 e^{-|\xi|^2/C_0} \leq \mu_2(\xi) \tag{3.6}$$

and then

$$\|\frac{\partial}{\partial \xi}\varphi_\xi(\ .\ )\|_{L^2(I\!R_+)} \leq C_0, \quad \forall\, \xi > 1 \ . \tag{3.7}$$

Proof We proceed as in [He1] to get (3.4). For any Lipschitz and real function Φ, with compact support,

$$\|D_t(e^\Phi \varphi_\xi)\|^2_{L^2(I\!R_+)} = \langle \left[\mu(\xi) - (t-\xi)^2 + (\Phi')^2\right] e^\Phi \varphi_\xi | e^\Phi \varphi_\xi \rangle$$

so

$$\langle \left[(t-\xi)^2 - \mu(\xi) - (\Phi')^2\right]_+ e^\Phi \varphi_\xi | e^\Phi \varphi_\xi \rangle \ \leq \ \langle \left[\mu(\xi) - (t-\xi)^2 + (\Phi')^2\right]_+ e^\Phi \varphi_\xi | e^\Phi \varphi_\xi \rangle \ .$$

This estimate is still valid for Φ with non compact support, provided that the right hand side of the inequality is finite; so we can take

$\Phi(t)=(1-\eta)^{1/2}(t-\xi)^2/2$ to get (3.4).
Now, let χ be a smooth cut-off function on $I\!R$:

$$\begin{array}{l} \chi \in C^\infty(I\!R), \\ \chi(t)=1, \quad \text{if} \quad -1<t<1, \\ \chi(t)=0, \quad \text{if} \quad |t|>2, \\ 0\le\chi\le 1\,. \end{array} \tag{3.8}$$

If $\xi>1$, we define the function $\widetilde{\varphi}_{1,\xi}(t)=\chi(4\frac{t-\xi}{\xi})\varphi_\xi(t)$.
So

$$\|D_t\widetilde{\varphi}_{1,\xi}\|^2+\|(t-\xi)\widetilde{\varphi}_{1,\xi}\|^2=\mu(\xi))\|\widetilde{\varphi}_{1,\xi}\|^2+\frac{16}{\xi^2}\|\chi'(4\frac{t-\xi}{\xi})\widetilde{\varphi}_\xi\|^2\,.$$

As $\widetilde{\varphi}_{1,\xi}$ is of compact support and the first eigenvalue of $D_t^2+(t-\xi)^2$ on $L^2(I\!R)$ is 1 , then

$$\|\widetilde{\varphi}_{1,\xi}\|^2\le\mu(\xi))\|\widetilde{\varphi}_{1,\xi}\|^2+\frac{16}{\xi^2}\|\chi'(4\frac{t-\xi}{\xi})\varphi_\xi\|^2\,;$$

then we use the estimate (3.4) to see that, for some constant $C>1$, $1\le\mu(\xi)+Ce^{-\xi^2/C}$: the estimate (3.5) follows.

If $\mu_2(\xi)$ is the second eigenvalue of Q_ξ and $\varphi_{2,\xi}$ the associated normalized eigenfunction, then we have in the same way, for any real function Φ,

$$\begin{aligned} &\langle\left[(t-\xi)^2-\mu_2(\xi)-(\Phi')^2\right]_+ e^\Phi\varphi_{2,\xi}|e^\Phi\varphi_{2,\xi}\rangle \\ &\qquad\qquad\le\langle\left[\mu_2(\xi)-(t-\xi)^2+(\Phi')^2\right]_+ e^\Phi\varphi_{2,\xi}|e^\Phi\varphi_{2,\xi}\rangle; \end{aligned}$$

so $\varphi_{2,\xi}$ is exponentially decreasing as φ_ξ, and if $\widetilde{\varphi}_{2,\xi}(t)=\chi(4\frac{t-\xi}{\xi})\varphi_{2,\xi}(t)$, then

$$\|D_t\widetilde{\varphi}_{2,\xi}\|^2+\|(t-\xi)\widetilde{\varphi}_{2,\xi}\|^2\le\mu_2(\xi)+Ce^{-\xi^2/C}$$

and

$$\begin{array}{l} |\;\|\widetilde{\varphi}_{2,\xi}\|-1|\le Ce^{-\xi^2/C}, \\ |\langle\widetilde{\varphi}_{1,\xi}|\widetilde{\varphi}_{2,\xi}\rangle|\le Ce^{-\xi^2/C}\,. \end{array} \tag{3.9}$$

Those estimates, (3.5) and the min-max principle show that $\mu_2(\xi)+Ce^{-\xi^2/C}$ is greater then the second eigenvalue of $D_t^2+(t-\xi)^2$ on $L^2(I\!R)$,
so $3\le\mu_2(\xi)+Ce^{-\xi^2/C}$: this ends the proof of the Lemma.

We recall that some elementary technique of perturbation shows that

$$\frac{\partial}{\partial\xi}\varphi_\xi(t)=2\left[Q_\xi-\mu(\xi)\right]^{-1}\psi_\xi\,, \tag{3.10}$$

with

$$\psi_\xi(t)=(t-\xi)\varphi_\xi(t)-\langle(t-\xi)\varphi_\xi|\varphi_\xi\rangle\varphi_\xi(t)\,.$$

As $\|D_t^2\varphi_\xi\|+\|(t-\xi)\varphi_\xi\|=\mu(\xi)\|\varphi_\xi\|=\mu(\xi)$, we get easily (3.7) from (3.10).

Remark 3.2 *As ξ goes to $-\infty$, the following asymptotic expansions hold*

$$\mu(\xi) = \xi^2 \;+\; (-2\xi)^{2/3}\rho_1 \;+\; \mathbf{O}(|\xi|^{-2/3}) \tag{3.11}$$

and

$$\mu_2(\xi) = \xi^2 \;+\; (-2\xi)^{2/3}\rho_2 \;+\; \mathbf{O}(|\xi|^{-2/3}) \tag{3.12}$$

if (ρ_j) is the increasing sequence of eigenvalues of the Neumann operator on $L^2(I\!R_+)$ associated to $D_t^2 + t$.

As a matter of fact, if $\xi < -1$, we have easily $\sigma(Q_\xi) = \xi^2\big[1 \;+\; \sigma(Q_{\xi,2})\big]$, if $Q_{\xi,2}$ is the Neumann operator on $L^2(I\!R_+)$ associated to $h^2D_t^2 + 2t + t^2$ with $h = \xi^{-2}$.

Using semi-classical method as in [Si1] or in [HeSj1] (or in [He1]), we get easily that the N-first eigenvalues of $Q_{\xi,2}$ are equal to $(2h)^{2/3}\rho_1,\ \ldots,\ (2h)^{2/3}\rho_N$ modulo a $\mathbf{O}(h^{4/3})$.

4 The case of small θ

We are still investigating the spectrum of the operator P_θ, defined in the introduction as following :

$$P_\theta = D_t^2 + D_x^2 + (t\cos\theta - x\sin\theta)^2 \ .$$

Performing the scaling $(t,x) \to (t\sqrt{\cos\theta}, -\frac{x sin\theta}{\sqrt{\cos\theta}})$, we observe that this operator has the same spectrum as :

$$P_\theta = \cos\theta\big[D_t^2 + (t-x)^2\big] + \frac{\sin^2\theta}{\cos\theta}D_x^2$$

(we keep on the same notation for simplification).

It has been proved [HeMo2] that for small values of $\theta > 0$ the following asymptotics hold :

$$\inf\ \sigma(P_\theta) \sim \mu_0 + \sum_{j\geq 1} c_j\theta^j \ .$$

Therefore let us consider a set

$$I_d = \ \big] -\infty,\ d\big[\ \text{ with }\ d \in \big]\,\mu_0, 1\,\big[\ .$$

The goal of this section is to get information about

$$N(d, P_\theta) = \sharp\sigma(P_\theta)\cap\big] -\infty,\ d\big[\,, \tag{4.1}$$

which denotes the number of eigenvalues of P_θ included in the set I_d.

Here may be one can apply the technique of Balazard-Konlein [Bal] to get the asymptotics of $N(d,P_\theta)$, but the result will be rough, compared to our result in Theorem 4.2: our remainder is an $\mathbf{O}(1)$ and the result of [Bal] would

give $\mathbf{O}(\sin^{-\rho}\theta)$ with $\rho > 1/2$.
More over the assumptions in [Bal] are not satisfied in our case.

For a fixed $a > 1$ let us consider the following sets

$$J_0 = \left[-a, +\infty\right[, \qquad J_1 = \left]-\infty, -\frac{a}{2}\right] ,$$

and a partition of unity :

$$\chi_0^2(x) + \chi_1^2(x) = 1, \quad \text{support } (\chi_j) \subset J_j, \; \sum_j |\chi_j'(x)|^2 < C \; .$$

For $j = 0, 1$ let us denote by Ω_j the domains $I\!\!R_+ \times J_j$:

$\Omega_0 = I\!\!R_+ \times \left]-a, +\infty\right[$ and $\Omega_1 = I\!\!R_+ \times \left]-\infty, -\frac{a}{2}\right[$.

We take now the realization of the operators P_θ^j on each domain Ω_j, associated to the quadratic form q_{Ω_j}, with Neumann conditions on $\Gamma_N = \{0\} \times J_j$ and Dirichlet conditions on $\Gamma_D = I\!\!R_+ \times \partial J_j$.

The quadratic forms are defined as follows :

$$q_{\Omega_j}(u) = \int_{\Omega_j} \{\cos\theta\left[|D_t u|^2 + (t-x)^2|u|^2\right] + \frac{\sin^2\theta}{\cos\theta}|D_x u|^2 - \frac{\sin^2\theta}{\cos\theta}\sum_{j=0}^{1}|\chi_j'(x)|^2|u|^2\} dt dx$$

Let us first explain why q_{Ω_1} will not give any contribution to the term $N(d, P_\theta)$.

According to section 3 we know that $\mu(x)$, the first eigenvalue of $D_t^2 + (t-x)^2$ is decreasing on J_1 , so we have :

$$q_{\Omega_1}(u) \geq \left[\mu(-\frac{a}{2})\cos\theta - C\,\frac{\sin^2\theta}{\cos\theta}\right]\|u\|^2_{L^2(\Omega_1)} \; .$$

But $\mu(-\frac{a}{2}) > 1$, so for small θ the preceding minoring ensues

$$q_{\Omega_1}(u) \;\geq\; \|u\|^2_{L^2(\Omega_1)} \; , \quad \text{if} \quad 0 < \theta < \theta_0 \; , \tag{4.2}$$

for some $\theta_0 \in \left]0, \frac{\pi}{4}\right[$.

In order to study the form q_{Ω_0}, it is convenient to use the normalized eigenfunction φ_x, associated to $\mu(x)$, in the following way.

Let us denote by $\Pi_0(u)$ the orthogonal projection on the set

$$F_0 = \{\varphi_x(t)\psi(x); \quad \psi \in L^2(J_0)\} \; , \tag{4.3}$$

defined by

$$\Pi_0(u) = \varphi_x(t)\left(\int_{I\!\!R_+} u(s, x)\varphi_x(s) ds\right) \; , \tag{4.4}$$

and by $F_1 = (F_0)^\perp$ the orthogonal set of F_0 . The corresponding orthogonal projection is

$$\Pi_1 = I - \Pi_0 \; .$$

A direct computation gives :

$$\partial_x(\Pi_0 u) = \Pi_0(\partial_x u) + R(u) \ ,$$

where R is defined by

$$R(u) = \varphi_x(t)\Big(\int_{\mathbb{R}_+} u(s,x)\partial_x\varphi_x(s)ds\Big) + \partial_x\varphi_x(t)\Big(\int_{\mathbb{R}_+} u(s,x)\varphi_x(s)ds\Big)\ . \quad (4.5)$$

The additional fact that

$$\partial_x(\Pi_1 u) = \Pi_1(\partial_x u) - R(u)$$

yields the following bounds :

$$\begin{aligned}(1-\epsilon)\big[\|\partial_x(\Pi_0 u)\|^2_{L^2(\Omega_0)} + \|\partial_x(\Pi_1 u)\|^2_{L^2(\Omega_0)}\big] + 2(1-\frac{1}{\epsilon})\|R(u)\|^2_{L^2(\Omega_0)} \\ \leq \|\partial_x u\|^2_{L^2(\Omega_0)} \\ \leq (1+\epsilon)\big[\|\partial_x(\Pi_0 u)\|^2_{L^2(\Omega_0)} + \|\partial_x(\Pi_1 u)\|^2_{L^2(\Omega_0)}\big] + 2(1+\frac{1}{\epsilon})\|R(u)\|^2_{L^2(\Omega_0)}\ .\end{aligned}$$

Using the result (3.10) in the Lemma 3.1, we get the following bound :

Lemma 4.1

$$\exists\ C_0 > 0,\ s.t.\ \ \forall\ u\ \in\ L^2(\Omega_0),\ \ \|R(u)\|_{L^2(\Omega_0)} \leq C_0\|u\|_{L^2(\Omega_0)}\ . \quad (4.6)$$

Let us sketch the proof of the lemma 4.1. Taking norms in (4.5) we have :

$$\begin{aligned}\|R(u)\|^2_{L^2(\Omega_0)} &\leq 2\int_{\Omega_0}|u(s,x)(\partial_x\varphi_x(s))|^2 dsdx + 2\sup_{x\in J_0}\|\partial_x\varphi_x(t)\|^2_{L^2(\mathbb{R}_+)}\|u\|^2_{L^2(\Omega_0)} \\ &\leq 4\sup_{x\in J_0}\|\partial_x\varphi_x(t)\|^2_{L^2(\mathbb{R}_+)}\|u\|^2_{L^2(\Omega_0)}\ .\end{aligned}$$

The lemma will then be proved if we show that $\sup_{x\in J_0}\|\partial_x\varphi_x(t)\|^2_{L^2(\mathbb{R}_+)}\|$ is finite.

Going back to the relation (3.10)

$$\frac{\partial}{\partial x}\varphi_x(t) = 2\big[Q_x - \mu(x)\big]^{-1}\psi_x\ , \quad (4.7)$$

with

$$\psi_x(t) = (t-x)\varphi_x(t) - \langle (t-x)\varphi_x|\varphi_x\rangle\varphi_x(t)\ ,$$

and using :

$$\|D_t^2\varphi_x\|^2_{L^2(\mathbb{R}^2_+)} + \|(t-x)\varphi_x\|^2_{L^2(\mathbb{R}^2_+)} = \mu(x)\|\varphi_x\|^2_{L^2(\mathbb{R}^2_+)} = \mu(x)\ \ ,$$

we get that :

$$\|(t-x)\varphi_x\|_{L^2(\mathbb{R}^2_+)} \leq \sqrt{\mu(x)}$$

and then

$$|\langle (t-x)\varphi_x|\varphi_x\rangle_{L^2(\mathbb{R}^2_+)}| \leq \sqrt{\mu(x)}\ ,$$

so

$$||\psi_x||_{L^2(\mathbb{R}^2_+)} \leq 2\sqrt{\mu(x)}\ \ .$$

Since ψ_x lives on the orthogonal space of φ_x , let us consider the norm N_x of the restriction of $\left[Q_x - \mu(x)\right]^{-1}$ to this orthogonal space. It is given by :

$$N_x = \frac{1}{\mu_2(x) - \mu_1(x)} = \frac{1}{\mu_2(x) - \mu(x)}\ \ ,$$

where $(\mu_j(x))_j$ is the increasing sequence of the eigenvalues of Q_x .

According to (3.1) and (3.6) , $\mu(x)$ and N_x are uniformly bounded on J_0 , so there exists $c_0 > 0$ such that

$$\sup_{x\in J_0} ||\partial_x \varphi_x(t)||_{L^2(\mathbb{R}_+)} \leq 2 \sup_{x\in J_0} \frac{\sqrt{\mu(x)}}{\mu_2(x) - \mu(x)} \leq C_0\ ,$$

so the Lemma 4.1 follows.

From Lemma 4.1, we see that we can find a constant $C_1 > 0$, such that, for any $\epsilon \in]0,1[$,

$$(1-\epsilon)\left[||\partial_x(\Pi_0 u)||^2_{L^2(\Omega_0)} + ||\partial_x(\Pi_1 u)||^2_{L^2(\Omega_0)}\right] - \frac{C_1}{\epsilon}||u||^2_{L^2(\Omega_0)}$$

$$\leq ||\partial_x u||^2_{L^2(\Omega_0)}$$

$$\leq (1+\epsilon)\left[||\partial_x(\Pi_0 u)||^2_{L^2(\Omega_0)} + ||\partial_x(\Pi_1 u)||^2_{L^2(\Omega_0)}\right] + \frac{C_1}{\epsilon}||u||^2_{L^2(\Omega_0)}$$

From that we obtain the corresponding bounds on the quadratic form q_{Ω_0} :

$$q^{\epsilon,-}_{\Omega_0}(\Pi_0 u) + q^{\epsilon,-}_{\Omega_0}(\Pi_1 u) \leq q_{\Omega_0}(u) \leq q^{\epsilon,+}_{\Omega_0}(\Pi_0 u) + q^{\epsilon,+}_{\Omega_0}(\Pi_1 u) \tag{4.8}$$

where we used the natural notations :

$$q^{\epsilon,-}_{\Omega_0}(u) = \int_{\Omega_0} \left\{\cos\theta\left[|D_t u|^2 + (t-x)^2|u|^2\right] + (1-\epsilon)\frac{\sin^2\theta}{\cos\theta}|D_x u|^2\right\}dtdx$$

$$- \frac{\sin^2\theta}{\cos\theta}\left[C + \frac{C_1}{\epsilon}\right]||u||^2_{L^2(\Omega_0)}$$

and

$$q^{\epsilon,+}_{\Omega_0}(u) = \int_{\Omega_0} \left\{\cos\theta\left[|D_t u|^2 + (t-x)^2|u|^2\right] + (1+\epsilon)\frac{\sin^2\theta}{\cos\theta}|D_x u|^2\right\}dtdx$$

$$+ \frac{\sin^2\theta}{\cos\theta}\frac{C_1}{\epsilon}||u||^2_{L^2(\Omega_0)}$$

Writing

$$h = \frac{\sin\theta}{\sqrt{\cos\theta}}\ , \tag{4.9}$$

taking into account (4.3), we define

$$W(x) = \int_{\mathbb{R}_+} |\frac{\partial}{\partial x}\varphi_x(t)|^2 dt,$$

and we get, using (4.4) that :

$$\begin{aligned} q_{\Omega_0}^{\epsilon,-}(\Pi_0 u) &= q^{\epsilon,-}(\psi) \\ &= \int_{J_0} \{ \big[\mu(x)\ \cos\theta + (1-\epsilon)h^2 W(x)\big] |\psi(x)|^2 \\ &\quad + (1-\epsilon)h^2 |D_x\psi(x)|^2 - h^2\Big[C + \frac{C_1}{\epsilon}\Big]|\psi(x)|^2\} dx. \end{aligned}$$

In the same way we have :

$$\begin{aligned} q_{\Omega_0}^{\epsilon,+}(\Pi_0 u) &= q^{\epsilon,+}(\psi) \\ &= \int_{J_0} \{ \big[\mu(x)\ \cos\theta + (1+\epsilon)h^2 W(x)\big] |\psi(x)|^2 \\ &\quad + (1+\epsilon)h^2 |D_x\psi(x)|^2 + h^2\frac{C_1}{\epsilon}|\psi(x)|^2\} dx. \end{aligned}$$

Now we have to deal with the terms involving the second projection $\Pi_1 u$. But the definition of $\Pi_1 u$, the min-max principle and the estimate $1-\cos\theta \leq h^2 C$ give the following lower bound :

$$q_{\Omega_0}^{\epsilon,\pm}(\Pi_1 u) \geq \big[\inf_{x\in J_0} \mu_2(x) - h^2\Big(C + \frac{C_1}{\epsilon}\Big)\|\Pi_1 u\|_{L^2(\Omega_0)}$$

where $\mu_2(x)$ denotes the second eigenvalue defined in the Lemma 3.1. This eigenvalue has to be greater than the first eigenvalue of the corresponding Dirichlet problem, so

$$\mu_2(x) > 1\ .$$

If we take for example $\epsilon = h$, we get that

$$N(d, q_{\Omega_0}^{\epsilon,\pm}) = N(d, q^{\epsilon,\pm})\ .$$

Let us take an extension $\widetilde{\mu}(x)$ of $\mu(x)$ outside of

$$J_{0,d} = \{x \in J_0,\ \mu(x) < d + (1-d)/2\}\ , \tag{4.10}$$

such that

$$\begin{array}{l} \widetilde{\mu}(x) = \mu(x),\ \text{if}\ x\ \in\ J_{0,d} \\ \widetilde{\mu}(x) \geq (1+d)/2\ ,\ \forall\, x \notin\ J_{0,d} \\ \widetilde{\mu}(x) = 1\ \text{if}\ |x| > C_d \end{array} \tag{4.11}$$

for some constant $C_d > 0$. Let us define

$$q_0^{\epsilon,\pm}(\psi) = \int_{\mathbb{R}} \left[\widetilde{\mu}(x)\ |\psi(x)|^2 + (1 \pm \epsilon)h^2|D_x\psi(x)|^2\right]dx\ .$$

The exponential localization of eigenfunctions of $q^{\epsilon,\pm}$ associated to eigenvalues in $]-\infty, d[$, (see [He1]), and the polynomial bound of $N(d, q^{\epsilon,\pm})$ and $N(d + hC, q_0^{\epsilon,\pm})$ obtained by Theorem 2.1, show as in [He1] that

$$N(d - hC, q_0^{\epsilon,\pm}) \leq N(d, q^{\epsilon,\pm}) \leq N(d + hC, q_0^{\epsilon,\pm})\ , \tag{4.12}$$

(we have used that $W(x)$ is bounded in J_0, thanks to (3.7)).

Applying a classical estimate of $N(d, q_0^{\epsilon,\pm})$, (see for example D. Robert's book [Ro]), (Theorem V-11, page 263), we have that, for any $\lambda < (1 + d)/2$, there exists $C_\lambda > 0$ such that, for any $h \in]0, 1[$ and any $\epsilon \in]0, 1/2[$,

$$\left| N(\lambda, q_0^{\epsilon,\pm})\ -\ \frac{1}{2\pi h\sqrt{1 \pm \epsilon}} \int_{\mathbb{R}} \left[\lambda - \mu(x)\right]_+^{1/2} dx \right| \leq C_\lambda\ . \tag{4.13}$$

Taking $\epsilon = h$, we get from (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) with $\lambda = d \pm hC$, that there exists $C_d > 0$, depending only on d, such

$$\left| N(d, q_{\Omega_0})\ -\ \frac{1}{2\pi \sin\theta} \int_{\mathbb{R}} \left[d - \mu(x)\right]_+^{1/2} dx \right| \leq C_d\ . \tag{4.14}$$

We get easily from the above discussion the following theorem :

Theorem 4.2 *For any $d \in]\mu_0,\ 1[$,
there exists $C_d > 0$ such that*

$$\left| N(d, P_\theta)\ -\ \frac{1}{2\pi \sin\theta} \int_{\mathbb{R}} \left[d - \mu(x)\right]_+^{1/2} dx \right| \leq C_d\ . \tag{4.15}$$

Remark 4.3 *The condition $\theta < \theta_0$ (4.2) can be removed since $N(d, P_\theta)$ is finite for fixed θ according to Theorem 2.1.*

Acknowledgements. The authors would like to thank Bernard Helffer for useful discussions.

References

[Bal] A. Balazard-Konlein : Asymptotique semi-classique du spectre pour des opérateurs à symbole opératoriel.
C.R. Acad. Sci. Paris, I, 301, (1985), pp. 903–906.

[BaPhTa] P. Bauman, D. Phillips, Q. Tang : Stable nucleation for the Ginzburg-Landau system with an applied magnetic field. IMA Preprint Series 1416 (1996). Arch. Rational Mech. Anal. 142 (1998), p. 1–43.

[BeSt] A. Bernoff and P. Sternberg : Onset of superconductivity in decreasing fields for general domains. J. Math. Phys. 39 (1998), p. 1272–1284.

[Bo] C. Bolley : Modélisation du champ de retard à la condensation d'un supraconducteur par un problème de bifurcation. M^2AN, Vol. 26, n° 2, (1992), p. 235–287.

[BoHe] C. Bolley and B. Helffer :

[CDS] J.M. Combes, P. Duclos and R. Seiler : The Born-Oppenheimer approximation in " Rigorous Atomics and Molecular Physics" (G. Velo and A. Wightman Editors) pp. 185–212, Plenum, New York 1981.

[CFKS] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon : Schrödinger Operators. Springer-Verlag, Berlin 1987.

[DaHe] M. Dauge, B. Helffer : Eigenvalues variation I, Neumann problem for Sturm-Liouville operators. Journal of Differential Equations, Vol. 104, n° 2, (1993), p. 243–262.

[He1] B. Helffer : Introduction to the semiclassical analysis for the Schrödinger operator and applications. *Springer lecture Notes in Math.*, n° 1336 (1988).

[He3] B. Helffer : Bouteilles magnétiques et supraconductivité (d'après Helffer-Morame, Lu-Pan et Helffer-Pan). Séminaire EDP de l'école Polytechnique 2000–2001.

[HeMo1] B. Helffer and A. Morame : Magnetic bottles in connection with superconductivity. J. of Functional Anal., 185 (2001), pp. 604–680.

[HeMo2] B. Helffer and A. Morame : Magnetic bottles for the Neumann problem: the case of dimension 3. Proc. Indian Acad. Sci. (Math. Sci.), Vol. 112(1) (2002), pp. 71–84.

[HeMo3] B. Helffer and A. Morame : Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case) Preprint March 2001: mp_ arc 02-145.

[HeSj1] B. Helffer and J. Sjöstrand : Multiple wells in the semi-classical limit I. Comm. in PDE, 9,(4), (1984), pp. 337–408.

[LuPa1] K. Lu, X-B. Pan : Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity. Physica D 127 (1999), p. 73–104.

[LuPa2] K. Lu, X-B. Pan : Eigenvalue problems of Ginzburg-Landau operator in bounded domains. Journal of Math. Physics, Vol. 40, n° 6, (1999), p. 2647–2670.

[LuPa4] K. Lu, X-B. Pan : Ginzburg-Landau system and surface nucleation of superconductivity. Proceeding of the IMS Workshop on Reaction-Diffusion systems. Chinese University of Hong-Kong, December 6–11 (1999).

[LuPa] K. Lu, X-B. Pan : Surface nucleation of superconductivity in 3-dimension. J. of Differential Equations 168(2) (2000), pp. 386–452.

[HePa3] B. Helffer, X-B. Pan : Upper critical field and location of surface nucleation of superconductivity. Preprint February 2001.

[MoTr] A. Morame and F. Truc : Semiclassical Eigenvalue Asymptotics for a Schrödinger Operator with Degenerate Potential. Asymptotic Anal., 22(1) (2000), pp. 39–49.

[Pan] X-B. Pan : Upper critical for superconductors with edges and corners. Preprint 2000. Submitted.

[PiFeSt] M. del Pino, P.L. Felmer and P. Sternberg : Boundary concentration for eigenvalue problems related to the onset of superconductivity. Comm. Math. Phys., 210 (2000), pp. 413–446.

[ReSi] M. Reed and B. Simon : Methods of modern Mathematical Physics, IV: Analysis of operators. Academic Press, New York, 1978.

[Ro] D. Robert : Autour de l'approximation semi-classique. Progress in Math., Birkhäuser, Berlin 1986.

[Si1] B. Simon : Semi-classical analysis of low lying eigenvalues I. Ann. Inst. H. Poincaré, 38 (1983), pp. 295–307.

Abderemane MORAME
Université de Nantes
Faculté des Sciences
Département de Mathématiques
UMR 6629 du CNRS, B.P. 99208
44322 Nantes Cedex 3, (FRANCE)

email: morame@math.univ-nantes.fr

and

Françoise TRUC
INSTITUT FOURIER
Laboratoire de Mathématiques
UMR5582 (UJF-CNRS)
BP 74
38402 St MARTIN D'HÈRES Cedex (France)

email: Francoise.Truc@ujf-grenoble.fr