On a generalization of the selection theorem of Mahler

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Prépublications de l'Institut Fourier n$^{\text{o}}$ 606 (2003)
http://www-fourier.ujf-grenoble.fr/prepublications.html

Abstract

The set of point sets of $\mathbb{R}^n$, $n \geqslant 1$, having the property that their minimal interpoint distance is greater than a given strictly positive constant is shown to be equippable by a metric for which it is a compact topological space. We also show that its subsets of Delone sets of given constants in $\mathbb{R}^n$, $n \geqslant 1$, are compact. We prove that this result implies the classical selection theorem of Mahler.

1. Introduction

In 1946 Mahler [Ma] obtained important results on star bodies and their critical lattices in $\mathbb{R}^n$ using the following fundamental result called now selection theorem of Mahler or Mahler's compactness theorem.

THEOREM 1.1. — *Let (L_r) be a sequence of lattices of $\mathbb{R}^n$ such that, for all r: (i) $\|x\| \geqslant c$ for all $x \in L_r, x \neq 0$, with c a strictly positive constant independent of r, (ii) the Lebesgue measure $|L_r|$ of the fundamental region of L_r satisfies $|L_r| \leqslant M$ with M a constant $< +\infty$ independent of r. Then one can extract from the sequence (L_r) a subsequence $(L_{r'})$ that converges to a lattice L of $\mathbb{R}^n$ such that $|L| = \lim_{r' \to +\infty} |L_{r'}|$.*

This theorem is considerably efficient in many problems of geometry of numbers [Ca] [GL]. The desirability of extending the main theorems of Geometry of numbers, whose Mahler's compactness theorem, to general algebraic number fields and more was emphasized by Mahler in a seminar at Princeton [RSD]. Several authors revisited this theorem, giving generalizations and analogs for other ambiant spaces than $\mathbb{R}^n$: Chabauty [Ch] with subgroups in locally compact abelian groups, Mumford [Mu] in semi-simple Lie groups without compact factors and moduli spaces of compact Riemann surfaces of given genus, Macbeath and Swierczkowski [MS] in locally compact and σ-compact topological groups (abelian or not) which are compactly generated, McFeat [Mf] in adele spaces of number fields, Rogers and Swinnerton-Dyer [RSD] in

2000 Mathematics Subject Classification : 11B05, 11H99, 52C17, 52C22, 52C23, 54E45.
Keywords : Geometry of numbers, Mahler's compactness theorem

algebraic number fields. Groemer [Groe] gave an elegant proof of this theorem by showing that it is a consequence of the selection theorem of Blaschke [Ca], by noticing the bicontinuous one-to-one correspondence between lattices and their Voronoi domains.

The way that Chabauty [Ch] proved the theorem 1.1 is extremely instructive. A careful attention to his ”elementary” proof reveals the very important following fact that the $\mathbb{Z}$-additive structure of the lattices L_r is not necessary to obtain the convergence of a subsequence. From this essential remark, Chabauty proposed in [Ch] a possible extension of Mahler’s compactness theorem to locally compact abelian groups as ambiant spaces with a suitable topology, method which was controversed and refined by Mumford [Mu] for further generalizations. This essential remark opens the way to deal with sequences of non-periodic point sets, that is without any additional algebraic structure, instead of only lattices or subgroups, suggesting that the selection theorem of Mahler should exist in more general situations.

In this paper we will exploit these ideas to develop a new version of the theorem of selection of Mahler adapted to point sets (i.e. not only lattice or subgroup point sets) sticking to the ”elementary” approach of Chabauty. This can be formulated as follows. We will be interested in sets of point sets, say $\mathcal{UD}(H,\delta)_r$, of a metric space (H,δ), where δ is a metric on H, which have the property that the minimal interpoint distance is greater than a given strictly positive constant, say r. Concerning the point (i) of the theorem 1.1, the fundamental question is now the following:

(q1) for which metric spaces (H,δ) can the set $\mathcal{UD}(H,\delta)_r$ be endowed with a topology such that it is compact, and for which values of r ?

In the scope of generalizing the assertion (ii) of the theorem 1.1, let us recall the (Besicovitch) concept of relative denseness [MVG] : we will say that a subset Λ of (H,δ) is relatively dense (for δ) in H if there exists $R > 0$ such that for all $z \in H$ there exists $\lambda \in \Lambda$ such that $\delta(z,\lambda) \leqslant R$. We will say that Λ is relatively dense *of constant* R if R is chosen minimal for that property. Then we can formulate the fundamental question:

(q2) for which metric spaces (H,δ) can the subset $X(H,\delta)_{r,R}$ of $\mathcal{UD}(H,\delta)_r$ of the relatively dense subsets of H of given constant $R > 0$ be endowed with a topology such that it is compact, and for which values of R ?

We will say that a subset Λ of (H,δ) is a Delone set if there exist $r > 0$ and $R > 0$ such that its minimal interpoint distance is $\geqslant r$ and that it is relatively dense of constant $R > 0$. In this case we will say that Λ is a Delone set of constants (r,R) (see [MVG] for possible values of r and R when $H = \mathbb{R}^n$). For instance, observe that a lattice in $(\mathbb{R}^n, \|\cdot\|)$ is already a Delone set, where $\|\cdot\|$ is the standard euclidean metric. The main theorem of this paper is the following. It provides answers to (q1) and (q2) when $H = \mathbb{R}^n$ and $\delta = \|\cdot\|$. For short, in this case, let us denote by $\mathcal{UD}_r$ the set $\mathcal{UD}(\mathbb{R}^n,\delta)_r$ and by $X_{r,R}$ the set $X(\mathbb{R}^n,\delta)_{r,R}$.

THEOREM 1.2. — *For all $r > 0$, the set $\mathcal{UD}_r$ can be endowed with a metric d such that the topological space $(\mathcal{UD}_r, d)$ is compact. For all $R > 0$, the subspace $X_{r,R}$ of $(\mathcal{UD}_r, d)$ of the Delone sets of constants (r,R) is closed.*

In section 2 we will give an "elementary" proof of the theorem 1.2 in the same spirit as in Chabauty's approach. For this we will construct a suitable metric whose properties will be studied. In section 3 we will show that the theorem 1.2 implies the selection theorem of Mahler 1.1 and will comment about sufficient conditions on the space H to provide positive answers to the problems (q1) and (q2) from the present proof. Some applications of the theorem 1.2 will be reported in [MVG1] [MVG2] [MVG3].

2. Proof of the theorem 1.2

It is clear that it suffices to prove the theorem 1.2 for $r = 1$ to obtain it for all $r > 0$ by the properties of the euclidean norm on $\mathbb{R}^n$. We will consider $r = 1$ in the sequel. Let us simplify again the notations and put $\mathscr{UD}$ for $\mathscr{UD}_1$ and X or X_R instead of $X_{1,R}$. We will speak generically of $\mathscr{UD}$-sets for the elements of $\mathscr{UD}$. Namely, a $\mathscr{UD}$-set will be either the "empty set" element $\emptyset$, a point set $\{x\}$ reduced to one point with $x \in \mathbb{R}^n$, or a point set $\Lambda \subset \mathbb{R}^n$ which contains at least two points such that $x, y \in \Lambda, x \neq y \Rightarrow \|x - y\| \geqslant 1$. The sequel will be organized as follows: in the subsection 2.1 we will construct a collection of pseudo-metrics on $\mathscr{UD}$, each pseudo-metric being a kind of counting system normalized by a suitable distance function. Then we will show that the supremum of these pseudo-metrics is a metric which possesses nice properties and leads to the same pointwise behaviour of $\mathscr{UD}$-sets as that reported by Chabauty in the classical lattice case [Ch] (subsection 2.2). Then, in the subsections 2.3 and 2.4 we will prove the completeness and the precompactness of the topological space $\mathscr{UD}$ hence its compactness. In the subsection 2.5 we will prove that the subspaces X_R for all $R > 0$ are closed in $(\mathscr{UD}, d)$, hence compact.

2.1. Construction of a metric and properties

Denote by $B(c, \epsilon)$ the closed ball of $\mathbb{R}^n$ of center c and radius $\epsilon \geqslant 0$ and $\overset{o}{B}(c, \epsilon)$ its interior. Since any $\mathscr{UD}$-set Λ is countable, we denote by Λ_i its i-th element. Let $\mathscr{E} = \{(D, E) \mid D$ countable point set in $\mathbb{R}^n, E$ countable point set in $(0, 1/2)\}$ and $f : \mathbb{R}^n \to [0, 1]$ a continuous function with compact support in $B(0, 1)$ which satisfies $f(0) = 1$ and $f(t) \leqslant \frac{1/2+\|\lambda - t/2\|}{1/2+\|\lambda\|}$ for all $t \in B(0, 1)$ and $\lambda \in \mathbb{R}^n$ (for technical reasons which will appear below). Recall that a pseudo-metric δ on a space satisfies all the axioms of a distance except that $\delta(u, v) = 0$ does not necessarily imply $u = v$.

With each element $(D, E) \in \mathscr{E}$ and origin α of the affine euclidean space $\mathbb{R}^n$ we associate a real-valued function $d_{\alpha,(D,E)}$ on $\mathscr{UD} \times \mathscr{UD}$ in the following way. Let $\mathscr{B}_{(D,E)} = \{\mathscr{B}_m\}$ denote the countable set of all possible finite collections $\mathscr{B}_m = \{\overset{o}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{o}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \ldots, \overset{o}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}$ (with i_m the number of elements $\#\mathscr{B}_m$ of $\mathscr{B}_m$) of open balls such that $c_q^{(m)} \in D$ and $\epsilon_q^{(m)} \in E$ for all $q \in \{1, 2, \ldots, i_m\}$, and such that for all m and any two balls in $\mathscr{B}_m$ of respective centers $c_q^{(m)}$ and $c_k^{(m)}$, we have $\|c_q^{(m)} - c_k^{(m)}\| \geqslant 1$. Then we define the following function, with $\Lambda, \Lambda' \in \mathscr{UD}$,

$$d_{\alpha,(D,E)}(\Lambda, \Lambda') := \sup_{\mathscr{B}_m \in \mathscr{B}_{(D,E)}} \frac{|\phi_{\mathscr{B}_m}(\Lambda) - \phi_{\mathscr{B}_m}(\Lambda')|}{(1/2 + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{i_m}^{(m)}\|)} \qquad (1)$$

where the real-valued function $\phi_{\mathscr{B}_m}$ is given by $\phi_{\mathscr{B}_m}(\Lambda) := \sum_{\overset{o}{B}(c,\epsilon)\in\mathscr{B}_m} \sum_i \epsilon f\left(\frac{\Lambda_i - c}{\epsilon}\right)$. By convention we put $\phi_{\mathscr{B}_m}(\emptyset) = 0$ for all $\mathscr{B}_m \in \mathscr{B}_{(D,E)}$ and all $(D,E) \in \mathscr{E}$. It is clear that, for all m and $\Lambda \in \mathscr{UD}$, inside each ball $\overset{o}{B}(c,\epsilon) \in \mathscr{B}_m$, there is at most one point of Λ and therefore the summation $\sum_i \epsilon f\left(\frac{\Lambda_i - c}{\epsilon}\right)$ is reduced to at most one non-zero term. Therefore the sum $\phi_{\mathscr{B}_m}(\Lambda)$ is finite.

LEMMA 2.1. — *For all* $(\alpha,(D,E))$ *in* $\mathbb{R}^n \times \mathscr{E}$, $d_{\alpha,(D,E)}$ *is a pseudo-metric valued in* $[0,1]$.

Proof. — Let $\alpha \in \mathbb{R}^n$ and $(D,E) \in \mathscr{E}$. It is easy to check that $d_{\alpha,(D,E)}$ is a pseudo-metric on $\mathscr{UD}$. Let us show it is valued in $[0,1]$. Let us consider $\mathscr{B}_m \in \mathscr{B}_{(D,E)}$ for which the centers of its constitutive balls are denoted by $c_1, c_2, \ldots, c_{i_m}$. Then we have $\frac{i_m}{2} \leqslant 1/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\|$. Indeed, if there exists $j \in \{1,2,\ldots,i_m\}$ such that $\|c_j - \alpha\| \leqslant 1/2$, then for all $k \neq j$, $\|c_k - \alpha\| \geqslant 1/2$. Hence $1/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\| \geqslant 1/2 + \|\alpha\| + \frac{i_m - 1}{2} \geqslant \frac{i_m}{2}$. If $\|c_k - \alpha\| \geqslant 1/2$ for all $k \in \{1,2,\ldots,i_m\}$, then $1/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\| \geqslant 1/2 + \|\alpha\| + \frac{i_m}{2} \geqslant \frac{i_m}{2}$. On the other hand, since the radii of the balls $\overset{o}{B}(c_j,\epsilon_j) \in \mathscr{B}_m$ are less than $1/2$ by construction, we have $0 \leqslant \phi_{\mathscr{B}_m}(\Lambda) \leqslant \frac{i_m}{2}$ for all $\mathscr{UD}$-set Λ. Therefore $|\phi_{\mathscr{B}_m}(\Lambda) - \phi_{\mathscr{B}_m}(\Lambda')| \leqslant 1/2 + \|\alpha\| + \|\alpha - c_1\| + \|\alpha - c_2\| + \cdots + \|\alpha - c_{i_m}\|$, for all $\mathscr{B}_m \in \mathscr{B}_{(D,E)}$ and all $\mathscr{UD}$-sets Λ, Λ'. We deduce the claim. □

The uniform topology on $\mathscr{UD}$ given by the pseudo-metrics $d_{\alpha,(D,E)}$ is generated by the open sets $\{\Lambda \in \mathscr{UD} \mid d_{\alpha,(D,E)}(u,\Lambda) < \epsilon\}$, $u \in \mathscr{UD}$ (Weil [We]). In order to get rid of a peculiar choice of the origin α and of the element (D,E) of $\mathscr{E}$, we now take the supremum over all choices $(\alpha,(D,E))$ in $\mathbb{R}^n \times \mathscr{E}$.

THEOREM 2.2. — *The supremum* $d := \sup_{\substack{\alpha\in\mathbb{R}^n \\ (D,E)\in\mathscr{E}}} d_{\alpha,(D,E)}$ *is a metric on* $\mathscr{UD}$, *valued in* $[0,1]$.

Proof. — The supremum of the family of pseudo-metrics $d_{\alpha,(D,E)}$ is obviously a pseudo-metric which takes its values in $[0,1]$. We have only to show that d is a metric. Assume Λ, Λ' are $\mathscr{UD}$-sets which are not empty such that $d(\Lambda,\Lambda') = 0$ and let us show that $\Lambda = \Lambda'$. We will show that $\Lambda \not\subset \Lambda'$ and $\Lambda' \not\subset \Lambda$ are impossible. Assume that $\Lambda \neq \Lambda'$ and that $\Lambda \not\subset \Lambda'$. Then there exists $\lambda \in \Lambda$ such that $\lambda \notin \Lambda'$. Denote by $\epsilon := \frac{1}{2}\min\{\frac{1}{2}, \min\{\|\lambda - u\| \mid u \in \Lambda'\}\}$. We have $\epsilon > 0$ since Λ' is a $\mathscr{UD}$-set. The ball $\overset{o}{B}(\lambda,\epsilon)$ contains no point of Λ' and only the point λ of Λ. Take $\alpha = \lambda$, $D = \{\lambda\}$, $E = \{\epsilon\}$. We have $d_{\lambda,(D,E)}(\Lambda,\Lambda') = \frac{\epsilon}{1/2 + \|\lambda\|} > 0$. Hence $d(\Lambda,\Lambda')$ would be strictly positive. Contradiction. Therefore $\Lambda \subset \Lambda'$. Then, exchanging Λ and Λ', we have $\Lambda' \subset \Lambda$. We deduce the equality $\Lambda = \Lambda'$. If we assume that one of the $\mathscr{UD}$-sets Λ or Λ' is the empty set, we see that the above proof is still valid. □

The metric d can be applied to $\mathscr{UD}$-sets of $\mathbb{R}^n$ which may have very different $\mathbb{R}$-spans, with distinct dimensions possibly strictly less than n. Therefore, it is important to understand the behaviour of the restriction of d to subspaces of $\mathbb{R}^n$.

Descent to lower dimensions.– Once the dimension n and the function f are fixed, the construction of the distance d on the space of $\mathscr{UD}$-sets of $\mathbb{R}^n$ generates a distance constructed in a similar way on the space of $\mathscr{UD}$-sets of $\mathbb{E}$, for all affine subspace $\mathbb{E} \subset \mathbb{R}^n$ containing the origin: it suffices to take the restriction $f_{|\mathbb{E}}$ instead of f, α in $\mathbb{E}$ and collections of balls $\mathscr{B}_m$ in $\mathscr{B}_{(D,E)}$ with $(D,E) \in \mathscr{E}_{|\mathbb{E}} := \{(D,E) \mid D$ countable point set in $\mathbb{E}, E$ countable point set in $(0,1/2)\}$. This process allows to define a new collection of pseudo-metrics $d_{\alpha,(D,E)\,|_{\mathbb{E}}}$ relatively to $\mathbb{E}$ itself. Let us denote $d_{\mathbb{E}} := \sup d_{\alpha,(D,E)\,|_{\mathbb{E}}}$ the corresponding distance. The following lemma is then obvious from the definitions.

Lemma 2.3. — *For all affine subspaces $\mathbb{E}, \mathbb{F}$ of $\mathbb{R}^n$ containing the origin such that $\mathbb{E} \subset \mathbb{F}$ and for any $\mathscr{UD}$-set Γ of $\mathbb{E}$ we have $d_{\mathbb{E}}(\Gamma, \emptyset) \leqslant d_{\mathbb{F}}(\Gamma, \emptyset)$. In particular, if Λ is an arbitrary $\mathscr{UD}$-set of $\mathbb{R}^n$ and $\mathbb{E} \subset \mathbb{R}^n$ denotes its $\mathbb{R}$-span, then $d_{\mathbb{E}}(\Lambda, \emptyset) \leqslant d(\Lambda, \emptyset)$.*

Let us now give the general properties of the metric d.

Proposition 2.4. — *For all $A, B, C \in \mathscr{UD}$ such that $A \cup B \in \mathscr{UD}$ and all $(D,E) \in \mathscr{E}$ and $\mathscr{B}_m \in \mathscr{B}_{(D,E)}$, the following assertions hold: (i) $\phi_{\mathscr{B}_m}(A\cup B)+\phi_{\mathscr{B}_m}(A\cap B) = \phi_{\mathscr{B}_m}(A)+\phi_{\mathscr{B}_m}(B)$; (ii) $d(A \cup B, C) \leqslant d(A, C) + d(B, A \cap B)$; (iii) $d(A \cap B, C) \leqslant d(A, C) + d(B, A \cup B)$. In particular: (iv) $d(A \cup B, \emptyset) \leqslant d(A, \emptyset) + d(B, \emptyset)$ as soon as $A \cap B = \emptyset$; (v) $d(A \cup B, A \cap B) \leqslant \min\{d(A, A\cap B) + d(B, A\cap B), d(A, A\cup B) + d(B, A\cup B)\}$; (vi) if B is reduced to one point, say $\{\lambda\}$, such that $\lambda \notin A$, we have: $d(A\cup\{\lambda\}, C) \leqslant \min\{d(A,C) + d(\{\lambda\}, \emptyset), d(\{\lambda\}, C) + d(A, \emptyset)\}$.*

Proof. — The assertion (i) can easily be checked from the definition of $\phi_{\mathscr{B}_m}$. The assertion (ii) is a consequence of (i) and of the inequality $|\phi_{\mathscr{B}_m}(A \cup B) - \phi_{\mathscr{B}_m}(C)| = |\phi_{\mathscr{B}_m}(A) + \phi_{\mathscr{B}_m}(B) - \phi_{\mathscr{B}_m}(A\cap B) - \phi_{\mathscr{B}_m}(C)| \leqslant |\phi_{\mathscr{B}_m}(A) - \phi_{\mathscr{B}_m}(C)| + |\phi_{\mathscr{B}_m}(B) - \phi_{\mathscr{B}_m}(A\cap B)|$. The assertion (iii) follows from (ii) by exchanging "$\cup$" and "$\cap$". The assertions (iv) to (vi) can be deduced from (i), (ii) and (iii). □

Let us remark that the assertions (iv) and (vi) show the special role played by the "empty set" element $\emptyset$ in the set-theoretic processes of "point addition" and "point removal". A fundamental question is now whether the "point removal process" of the points of a $\mathscr{UD}$-set is continuous. We will precise this question below and will answer to it.

Lemma 2.5. — *The following equalities hold: (i) $d(\{\lambda\}, \emptyset) = \frac{1}{1+2\|\lambda\|}$, for all $\lambda \in \mathbb{R}^n$ (remarkably this value does not depend upon $f(x)$), (ii) $d(\Lambda - \{\lambda\}, \Lambda) = \frac{1}{1+2\|\lambda\|}$ for all non-empty $\mathscr{UD}$-set Λ and all $\lambda \in \Lambda$.*

Proof. — (i) First, let us show that $d(\{\lambda\}, \emptyset) \leqslant \frac{1}{1+2\|\lambda\|}$. By definition we have $d(\{\lambda\}, \emptyset) = \sup_{\substack{\alpha\in\mathbb{R}^n \\ (D,E)\in\mathscr{E}}} \sup_{\mathscr{B}_m\in\mathscr{B}_{(D,E)}} \phi_{\mathscr{B}_m}(\Lambda) \left(1/2 + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{j_m}^{(m)}\|\right)^{-1}$. Whatever $(D,E) \in \mathscr{E}$, $\mathscr{B}_m \in \mathscr{B}_{(D,E)}$, a maximum of one ball of $\mathscr{B}_m$ may contain λ. Denote by $\overset{o}{B}(c, \epsilon)$ this variable generic ball and say that $c = c_1^{(m)}$. The other balls of $\mathscr{B}_m$ have a zero contribution to the numerator $\phi_{\mathscr{B}_m}(\Lambda)$ in the expression of $d(\{\lambda\}, \emptyset)$. The denominator is such that: $1/2 + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{j_m}^{(m)}\| \geqslant 1/2 + \|\alpha\| + \|\alpha - c\|$. But $1/2 + \|\alpha\| + \|\alpha - c\| \geqslant 1/2 + \|c\|$, this minimum being reached on the segment $[0, c]$. Therefore, by definition of the function f, we have $d_{\alpha,(D,E)}(\{\lambda\}, \emptyset) \leqslant \frac{\epsilon f(\frac{\lambda-c}{\epsilon})}{1/2+\|c\|} \leqslant \frac{\epsilon}{1/2+\|\lambda\|} \leqslant \frac{1/2}{1/2+\|\lambda\|} = \frac{1}{1+2\|\lambda\|}$. Conversely, if we take $\alpha = \lambda$, $D = \{\lambda\}$ and E a dense subset in $(0, 1/2)$, we see that $d(\{\lambda\}, \emptyset) \geqslant d_{\alpha=\lambda,(D=\{\lambda\},E)}(\{\lambda\}, \emptyset) = \frac{1/2}{1/2+\|\lambda\|}$. We deduce the equality and the assertion (i); (ii)

The proof is similar as in (i) since Λ and $\Lambda - \{\lambda\}$ differ by only one element λ which belongs to at most one ball in a collection $\mathscr{B}_m$ for any $(D,E) \in \mathscr{E}$ and any $\mathscr{B}_m \in \mathscr{B}_{(D,E)}$. The details are left to the reader. □

Corollary 2.6. — *For all $\mathscr{UD}$-set $\Lambda \neq \emptyset$ and all $\lambda \in \Lambda$, the inequality holds: $|d(\Lambda,\emptyset) - d(\Lambda - \{\lambda\},\emptyset)| \leqslant \frac{1}{1+2\|\lambda\|}$.*

Proof. — From (ii) in the proposition 2.4, we deduce $d(\Lambda,\emptyset) \leqslant d(\Lambda-\{\lambda\},\emptyset)+d(\{\lambda\},\emptyset)$. From (iii) in proposition 2.4, we obtain $d(\Lambda - \{\lambda\},\emptyset) \leqslant d(\Lambda,\emptyset) + d(\Lambda - \{\lambda\},\Lambda)$ but $d(\{\lambda\},\emptyset) = d(\Lambda - \{\lambda\},\Lambda) = \frac{1}{1+2\|\lambda\|}$ by the lemma 2.5. We deduce the claim. □

Let us now turn to the "point removal process" of subcollections of points of $\mathscr{UD}$-sets. In the following, for all $\Lambda \in \mathscr{UD}$ and $R > 0$, we will denote by Λ_R the new $\mathscr{UD}$-set $\Lambda \cap \overset{o}{B}(0,R)$.

Lemma 2.7. — *Let $\Lambda, \Lambda' \in \mathscr{UD}$ and C be an arbitrary subset of $\Lambda \cap \Lambda'$. Then $d(\Lambda,\Lambda') = d(\Lambda \setminus C, \Lambda' \setminus C)$. In particular, $d(\Lambda,\Lambda') = d(\Lambda\setminus(\Lambda\cap\Lambda'),\Lambda'\setminus(\Lambda\cap\Lambda'))$ and $d(\Lambda\setminus\Lambda_R,\emptyset) = d(\Lambda_R,\Lambda)$ for all $R > 0$.*

Proof. — These results follow from the definition of d. □

The following proposition is fundamental.

Proposition 2.8. — *Let $\Lambda \in \mathscr{UD}$. Then $\lim_{R\to\infty} d(\Lambda,\Lambda_R) = \lim_{R\to\infty} d(\Lambda \setminus \Lambda_R,\emptyset) = 0$. Moreover the convergence is uniform in the following sense: for all $\epsilon \in (0,1)$, there exists $R > 0$ such that $d(\Lambda,\emptyset) < \epsilon$ for all $\Lambda \in \mathscr{UD}$ such that $\Lambda \subset \mathbb{R}^n \setminus B(0,R)$.*

Proof. — If Λ is finite, the limit is obviously zero. Therefore we will assume that Λ is infinite in the sequel. To prove this result we will use the inequality of Stolarsky [St] (recalled in proposition 2.9) which will provide an upper bound of $d(\Lambda,\Lambda_R)$. Then we will explicitely compute this upper bound by means of representations of integers as sums of squares (of integers) (Grosswald [Gr]) (steps 1 and 2). This type of computation will provide the uniform convergence property.

Proposition 2.9. — *(Stolarsky [St]) Let u, v rational integers such that $u \geqslant v \geqslant 1$. Let $\{x_1, x_2, \ldots, x_u\}$ be a finite set of u points of $\mathbb{R}^n$ and $\{y_1, y_2, \ldots, y_v\}$ be another finite set of v points of $\mathbb{R}^n$. Let us define $h(u,v) = 1$ if $u = v$, $h(u,v) = \frac{u-1}{v}$ if $u > v$. Then*

$$\sum_{1\leqslant i<j\leqslant u} \|x_i - x_j\| + \sum_{1\leqslant i<j\leqslant v} \|y_i - y_j\| \leqslant h(u,v) \sum_{i=1}^{u}\sum_{j=1}^{v} \|x_i - y_j\| \tag{2}$$

where the constant $h(u,v)$ is best possible.

Take $v = 1$ and $u = i_m + 1 \geqslant 2$ with $x_1 = 0$ and $\|x_i\| \geqslant R$ for all $i = 2,3,\ldots,u$; then put $y_1 = \alpha \in \mathbb{R}^n$ arbitrary. The inequality (2) gives $\sum_{j=2}^{i_m+1} \|x_j\| + \sum_{2\leqslant i<j\leqslant i_m+1} \|x_i - x_j\| \leqslant h(i_m+1,1)\left(\|\alpha\| + \sum_{i=2}^{i_m+1} \|\alpha - x_i\|\right)$. Consequently, setting $c_{i-1} = x_i$ for all $i = 2,3,\ldots,i_m+1$ for

keeping the notations as close as possible to the definition of $d_{\alpha,(D,E)}$, the following inequality holds:

$$\frac{i_m}{1/2 + \|\alpha\| + \sum_{i=1}^{i_m} \|\alpha - c_i\|} \leqslant \frac{i_m}{\frac{1}{2} + \frac{1}{i_m}\left(\sum_{j=1}^{i_m} \|c_j\| + \sum_{1\leqslant i<j\leqslant i_m} \|c_i - c_j\|\right)}. \tag{3}$$

The supremum of the right-hand side expression, over all possible configurations of balls in $\mathscr{B}_{(D,E)}$ and $(D,E) \in \mathscr{E}$ such that their centres c_i satisfy $\|c_i\| \geqslant R$, is greater than $2d(\Lambda\setminus\Lambda_R, \emptyset)$. We will show that it goes to zero when R tends to infinity. For this, we will compute explicitely a lower bound of $\eta(R, i_m) := \frac{1}{2i_m} + \frac{1}{i_m^2}\sum_{j=1}^{i_m} \|c_j\| + \frac{1}{i_m^2}\sum_{1\leqslant i<j\leqslant i_m} \|c_i - c_j\|$ as a function of R and i_m, where $\eta(R, i_m)$ is the inverse of the right-hand side term in the inequality (3). In order to simplify the notations, we will study the quantity $\eta(R, m)$, what amounts merely to replace m by i_m in the rest of the proof for coming back to the inequality (3).

We will proceed as follows, in three steps. The first step (step 1) will consist in making this computation explicit when the points c_i are on the lattice $\mathbb{Z}^n$ with $n \geqslant 5$. The second step (step 2) will describe how to provide a lower bound of $\eta_\Lambda(R, m)$ (see definition in step 2) from $\eta(R, m)$ when the points c_i are in a $\mathscr{UD}$-set $\Lambda \subset \mathbb{R}^n$ which is not $\mathbb{Z}^n$ with still $n \geqslant 5$ for which the dimension of the $\mathbb{R}$-span of Λ is n or less than n. The final step 3 will conclude when $n \in \{1,2,3,4\}$ making use of lemma 2.3 for the descent to lower dimensions.

step 1 .- Let us recall the assumptions: $R > \sqrt{2}$ (for technical reasons) and $c_i \in \mathbb{Z}^n$, $\|c_i\| \geqslant R$, for all $i = 1,2,\ldots,m$ with $i \neq j \Rightarrow c_i \neq c_j$. Minimizing $\eta(R, m)$ corresponds to finding expressions of lower bounds of $m^{-2}\sum_{1\leqslant i<j\leqslant m} \|c_i - c_j\|$ and of $m^{-2}\sum_{j=1}^{m} \|c_j\|$ as a function of R and m, then to studying their sum when R tends to infinity.

Let us compute a lower bound of $m^{-2}\sum_{1\leqslant i<j\leqslant m} \|c_i - c_j\|$. Let s be a positive integer and consider the equation $s = \sum_{i=1}^{n} c_{q,i}^2$ with $c_{q,i} \in \mathbb{Z}$ for all $i = 1,2,\ldots,n$. Any n-tuple $(c_{q,1}, c_{q,2}, \ldots, c_{q,n})$ which satisfies this equation is called a solution of this equation. This solution represents the vector $c_q = {}^t(c_{q,1}, c_{q,2}, \ldots, c_{q,n})$ in $\mathbb{Z}^n$ of norm $s^{1/2}$. Given s, denote by $r_n(s)$ the number of solutions of the above equation; it is the number of elements of $\mathbb{Z}^n$ which lie on the sphere $S(0, \sqrt{s})$ of centre the origin and radius $\sqrt{s}$. Obviously $r_n(0) = 1$, $r_n(1) = 2^n$. Now, for all integer $m > 1$, there exists a unique integer k such that

$$r_n(0) + r_n(1) + \cdots + r_n(k) < m \leqslant r_n(0) + r_n(1) + \cdots + r_n(k) + r_n(k+1) \tag{4}$$

with $r_n(k)r_n(k+1) \neq 0$. From Grosswald [Gr], chapters 9, 12 and 13, we know the behaviour of $r_n(s)$ when $n \geqslant 5$: there exists two strictly positive constants $\widehat{K_1}(n)$ and $\widehat{K_2}(n)$ such that $r_n(s) = \rho_n(s) + O\left(s^{n/4}\right)$ with $\widehat{K_1}(n)s^{n/2-1} \leqslant \rho_n(s) \leqslant \widehat{K_2}(n)s^{n/2-1}$ for all integer $s > 0$. Therefore, there exists two strictly positive constants K_1, K_2, which depend upon n, such that $K_2 \geqslant 1$ and $K_1 s^{n/2-1} \leqslant r_n(s) \leqslant K_2 s^{n/2-1}$ for all integer $s > 0$. By saturating all the spheres $S(c_1, \sqrt{l}) \cap \mathbb{Z}^n$ for $l = 0,1,2,\ldots,k$ we deduce $\sum_{j=2}^{m} \|c_j - c_1\| \geqslant \sum_{j=2}^{r_n(0)+r_n(1)+\cdots+r_n(k)+1} \|c_j - c_1\| \geqslant \sum_{l=0}^{k} r_n(l)\sqrt{l}$. Let us consider that m is equal to $r_n(0) + r_n(1) + \cdots + r_n(k) + r_n(k+1)$. We now proceed with the other sums $\sum_{j=i+1}^{m} \|c_j - c_i\|$, $i \geqslant 2$. For all $i = 1,2,\ldots,r_n(k+1)$, the difference $m - i$ is greater than $r_n(0) + r_n(1) + \cdots + r_n(k)$ and this implies $\sum_{j=i+1}^{m} \|c_j - c_i\| \geqslant \sum_{l=0}^{k} r_n(l)\sqrt{l}$. Hence $\sum_{i=1}^{r_n(k+1)} \sum_{j=i+1}^{m} \|c_j - c_i\| \geqslant r_n(k+1)\left(\sum_{l=0}^{k} r_n(l)\sqrt{l}\right)$. Since $\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \|c_j - c_i\| = \sum_{i=1}^{r_n(k+1)}\sum_{j=i+1}^{m} \|c_j - c_i\| + \sum_{i=r_n(k+1)+1}^{r_n(k+1)+r_n(k)}\sum_{j=i+1}^{m} \|c_j - c_i\| +$

$\ldots + \sum_{i=r_n(k+1)+r_n(k)+\ldots+r_n(2)+1}^{r_n(k+1)+r_n(k)+\ldots+r_n(1)} \sum_{j=i+1}^{m} \|c_j - c_i\|$, by reproducing the same computation term by term, we deduce

$$\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\|c_j - c_i\| \geqslant r_n(k+1)\left(\sum_{l=0}^{k} r_n(l)\sqrt{l}\right) + r_n(k)\left(\sum_{l=0}^{k-1} r_n(l)\sqrt{l}\right) + \cdots + r_n(2)r_n(1) + 2^n$$

$$\geqslant \sum_{p=1}^{k+1} r_n(p)\left(\sum_{l=0}^{p-1} r_n(l)\sqrt{l}\right) \geqslant K_1^2 \sum_{p=1}^{k+1} p^{\frac{n}{2}-1}\left(\sum_{l=0}^{p-1} l^{\frac{n-1}{2}}\right) \tag{5}$$

Now make use of the following classical inequalities: for all $\beta > 0$ and integer $r \geqslant 1$, $0 + 1^\beta + 2^\beta + \cdots + (r-1)^\beta \leqslant \int_0^r x^\beta dx = \frac{r^{\beta+1}}{\beta+1} \leqslant 1^\beta + 2^\beta + \cdots + (r-1)^\beta + r^\beta$. We deduce the following inequalities

$$\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\|c_j - c_i\| \geqslant \frac{2K_1^2}{n+1} \sum_{p=1}^{k+1} p^{\frac{n}{2}-1}(p-1)^{\frac{n+1}{2}} \geqslant \frac{2K_1^2}{n+1} \sum_{p=1}^{k+1}(p-1)^{\frac{n}{2}-1}(p-1)^{\frac{n+1}{2}}$$

$\geqslant \dfrac{2K_1^2}{(n+1)} \displaystyle\sum_{p=1}^{k+1}(p-1)^{n-1/2} \geqslant \dfrac{4K_1^2}{(n+1)(2n+1)} k^{n+1/2}$ and $m = r_n(0) + r_n(1) + \cdots + r_n(k) + r_n(k+1) \leqslant K_2\left(1 + \sum_{l=1}^{k+1} l^{\frac{n}{2}-1}\right) \leqslant \frac{2K_2}{n}\left[\frac{n}{2} + (k+2)^{\frac{n}{2}}\right]$. Hence $m^{-2}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\|c_j - c_i\| \geqslant \frac{K_1^2 n^2 k^{n+1/2}}{K_2^2(n+1)(2n+1)(k+2)^n}\left(1 + \frac{n}{2(k+2)^{\frac{n}{2}}}\right)^{-2}$. Putting $K_3 := \frac{K_1^2 n^2 2^{n+2}}{K_2^2(n+1)(2n+1)3^n(n+2^{\frac{n}{2}+1})^2}$, we deduce

$$m^{-2}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\|c_j - c_i\| \geqslant K_3\sqrt{k}. \tag{6}$$

It is easy to check that the above computation is still valid when m lies strictly between $r_n(0) + r_n(1) + \cdots + r_n(k)$ and $r_n(0) + r_n(1) + \cdots + r_n(k+1)$. Therefore $\lim_{m\to+\infty} \frac{1}{m^2}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\|c_j - c_i\| = +\infty$. Let us observe that this minimal averaged growth to infinity is in "$\sqrt{k}$", which is extremely slow as compared to the growth of m to infinity.

Let us now compute a lower bound of the sum $m^{-2}\sum_{j=1}^{m}\|c_j\|$. Take for R the square root of an integer, say $R = \sqrt{t}$, $t \geqslant 2$. Let us consider that m is equal to $m = r_n(0)+r_n(1)+\cdots+r_n(k+1)$ and let us write it as: $m = r_n(t) + r_n(t+1) + \cdots + r_n(t+u) + w$ for a certain $u \geqslant 0$ and $0 \leqslant w < r_n(t+u+1)$. Then $\sum_{j=1}^{m}\|c_j\| \geqslant \sum_{t}^{t+u} r_n(l)\sqrt{l} \geqslant K_1\sum_{t}^{t+u} l^{\frac{n-1}{2}}$. As above we will make use of the following classical inequalities: for all positive integers s and $r \geqslant s+1$ and for all real number $\beta > 0$, $s^\beta + (s+1)^\beta + \cdots + (r-1)^\beta \leqslant \int_s^r x^\beta dx = \frac{r^{\beta+1} - s^{\beta+1}}{\beta+1} \leqslant (s+1)^\beta + (s+2)^\beta + \cdots + (r-1)^\beta + r^\beta$. We obtain the following inequalities: $\sum_{j=1}^{m}\|c_j\| \geqslant \frac{2K_1}{n+1}\left[(t+u)^{\frac{n+1}{2}} - (t-1)^{\frac{n+1}{2}}\right]$ and $\frac{2K_1}{n}\left[(t+u)^{n/2} - (t-1)^{n/2}\right] \leqslant m \leqslant r_n(t) + r_n(t+1) + \cdots + r_n(t+u) + r_n(t+u+1) \leqslant \frac{2K_2}{n}\left[(t+u+2)^{n/2} - t^{n/2}\right]$. From them we deduce

$$\frac{1}{m}\sum_{j=1}^{m}\|c_j\| \geqslant \frac{K_1 n\sqrt{u}}{K_2(n+1)}\left((1+t/u)^{\frac{n+1}{2}} - \left(\frac{t-1}{u}\right)^{\frac{n+1}{2}}\right)\left(\left(1+\frac{2+t}{u}\right)^{n/2} - \left(\frac{t}{u}\right)^{n/2}\right)^{-1}.$$

Dividing the above inequality by m once again and changing t into $t-1$ and t into $t+2$ in the corresponding factors gives

$$\frac{1}{m^2}\sum_{j=1}^{m}\|c_j\| \geqslant \frac{K_1 n^2 u^{\frac{1-n}{2}}}{2K_2^2(n+1)}\left((1+(t-1)/u)^{\frac{n+1}{2}}-\left(\frac{t-1}{u}\right)^{\frac{n+1}{2}}\right)\left(\left(1+\frac{2+t}{u}\right)^{n/2}-\left(\frac{2+t}{u}\right)^{n/2}\right)^{-2}$$

so that, using first-order developments in $(t-1)u^{-1}$, resp. in $(2+t)u^{-1}$, for u^{-1} close to zero, we obtain

$$\frac{1}{m^2}\sum_{j=1}^{m}\|c_j\| \geqslant \frac{K_1(t-1)^{\frac{n-1}{2}}}{K_2^2}\frac{1}{u(u+2+t)^{n-2}} \tag{7}$$

This lower bound, as a function of u on $[1,+\infty)$, goes to zero at infinity.

Let us now compute a lower bound of $m^{-2}\sum_{j=1}^{m}\|c_j\| + m^{-2}\sum_{1\leqslant i<j\leqslant m}\|c_i - c_j\|$. The lower bound given by eq.(6) is a function of k and that given by eq.(7) a function of u. In order to study their sum, we will deduce from the above a relation between u and $\sqrt{k}$ and replace $\sqrt{k}$ by a lower bound of $\sqrt{k}$ in eq.(6) which will only depend upon u. From the above, with $m = r_n(0)+r_n(1)+\cdots+r_n(k+1)$, the following inequalities hold

$$\frac{2K_1}{n}\left[(t+u)^{n/2}-(t-1)^{n/2}\right] \leqslant m \leqslant \frac{2K_2}{n}\left[\frac{n}{2}+(k+2)^{\frac{n}{2}}\right]. \tag{8}$$

Let $h(x) = (t+x)^{n/2}$. Then $h(u)-h(-1) = (u+1)h'(\xi)$ for a certain $\xi \in [-1,u]$. We deduce $h(u)-h(-1) \geqslant \frac{n}{2}u(t-1)^{\frac{n}{2}-1}$ since the derivative $h'(x)$ is increasing on the interval $[-1,u]$. This last inequality and eq.(8) imply

$$u^{1/n}\left[\left(\frac{n}{2}\left(\frac{K_1}{K_2}(t-1)^{\frac{n}{2}-1}-1\right)\right)^{2/n}-2\right]^{1/2} \leqslant \sqrt{k} \quad \text{for all } k\geqslant 1, u\geqslant 1, t\geqslant 2. \tag{9}$$

From eq.(6) in which $\sqrt{k}$ is replaced by the above lower bound and from eq.(7), we deduce

$$\eta(\sqrt{t},m) \;\geqslant\; g(t,u) := C_1(t)\frac{1}{u(u+2+t)^{n-2}} + C_2(t)u^{1/n}$$

where $C_1(t) = K_1K_2^{-2}(t-1)^{\frac{n-1}{2}}$ and $C_2(t) = K_3\left[\left(\frac{n}{2}\left(\frac{K_1}{K_2}(t-1)^{\frac{n}{2}-1}-1\right)\right)^{2/n}-2\right]^{1/2}$. It is now routine to compute the value $u_{min}(t)$ at which the function $u \to g(t,u)$ is minimal and the value $g(t,u_{min}(t))$ of its minimum. The equation satisfied by $u_{min}(t)$ is $nC_1(t)(u+2+t)^{1-n}[(n-1)u+2+t] = C_2(t)u^{1+1/n}$ and $g(t,u_{min}(t)) = C_2(t)\left[\frac{1}{n}\frac{u_{min}(t)+2+t}{(n-1)u_{min}(t)+2+t}+1\right](u_{min}(t))^{1/n}$. Since obviously $u_{min}(t)\geqslant 1$, $\frac{1}{n}\frac{u+2+t}{(n-1)u+2+t}+1 \geqslant \frac{1}{n(n-1)}+1$ for $t\geqslant 2$, $u\geqslant 1$ and $\lim_{t\to+\infty}C_2(t) = +\infty$, we obtain: $\lim_{t\to+\infty}g(t,u_{min}(t)) = +\infty$. We deduce that for all integer m of the form $r_n(0)+r_n(1)+\cdots+r_n(k+1)$ the limit $\lim_{R\to+\infty}\eta(R,m) = +\infty$ holds. It is easy to check that it is so even when m is an arbitrary integer which is not of this form. This implies, after eq.(3), that $\lim_{R\to+\infty}d(\mathbb{Z}^n,\mathbb{Z}^n_R) = 0$, for all $n\geqslant 5$.

step 2 .- We will make use of the results of step 1 and of the following three lemmas. The assumption $n\geqslant 5$ holds. Let us fix the notations: if Γ is a $\mathscr{UD}$-set which contains the origin, then, for all $k\in\mathbb{N}$, denote $\Gamma^{(k)} := \{x\in\Gamma \mid \sqrt{k}\leqslant\|x\|<\sqrt{k+1}\}$, $r_\Gamma(\sqrt{k})$ its number of elements and $s(\sqrt{k}) := \max_{\Gamma\in\mathscr{UD}}\{r_\Gamma(\sqrt{k})\} < \infty$. Since all the functions $\Gamma\to r_\Gamma(\sqrt{k})$, $k\in\mathbb{N}$, on $\mathscr{UD}$ are valued in $\mathbb{N}$, the maximum $s(\sqrt{k})$ is reached. Since, in particular, $r_{\mathbb{Z}^n}(\sqrt{k}) = r_n(k)$, for any positive integer k, the following lemma is obvious.

LEMMA 2.10. — *For all positive integer k the inequality $s(\sqrt{k}) \geqslant r_n(k)$ holds.*

In the following, we will number the elements x_i of a $\mathscr{UD}$-set Λ in such a way that $\|x_j\| \geqslant \|x_i\|$ as soon as $j \geqslant i \geqslant 1$ (with $x_1 = 0$ if Λ contains the origin). The following lemmas show that the sequence $\{s(\sqrt{k}) \mid k \in \mathbb{N}\}$ is universal for splitting up any $\mathscr{UD}$-set into layers of points with the objective of making use of the inequality of Stolarsky (proposition 2.9) in a suitable way.

LEMMA 2.11. — *Let Λ be an infinite $\mathscr{UD}$-set which contains the origin. For all positive integers $M, m \in \mathbb{N}$ such that $\sum_{k=0}^{M} s(\sqrt{k}) < m \leqslant \sum_{k=0}^{M+1} s(\sqrt{k})$, any point $x_m \in \Lambda$ indexed by such an integer m satisfies $\|x_m\| \geqslant \sqrt{M+1}$.*

Proof. — This fact comes from the way we have numbered the elements of Λ. Obviously, any point $x_m \in \Lambda$ indexed by such an integer m is such that $\sum_{k=0}^{M} r_\Lambda(\sqrt{k}) \leqslant \sum_{k=0}^{M} s(\sqrt{k}) < m$. By definition of the function r_Λ we obtain the inequality. □

LEMMA 2.12. — *Let Λ be an infinite $\mathscr{UD}$-set which contains the origin. There exists a subset Λ^* of Λ, with $0 \in \Lambda^*$, and a surjective mapping $\psi_\Lambda : \Lambda \longrightarrow \mathbb{Z}^n$ such that: (i) $\psi_\Lambda(0) = 0$, $\|\psi_\Lambda(x)\| \leqslant \|x\|$ for all $x \in \Lambda$; (ii) for all integers $M, m \in \mathbb{N}$ such that $\sum_{k=0}^{M} s(\sqrt{k}) < m \leqslant \sum_{k=0}^{M+1} s(\sqrt{k})$ the following equalities hold: $\|\psi_\Lambda(x_m)\| = \sqrt{M+1}$ for all $x_m \in \Lambda \setminus \Lambda^*$, $\|\psi_\Lambda(x_m)\| = 0$ for all $x_m \in \Lambda^*$, (iii) the restriction of ψ_Λ to $\{0\} \cup \Lambda \setminus \Lambda^*$ is a bijection from $\{0\} \cup \Lambda \setminus \Lambda^*$ to $\mathbb{Z}^n$; (iv) when $\Lambda = \mathbb{Z}^n$, then $\Lambda^* = \{0\}$ and ψ_Λ is the identity map up to a renumbering of the elements of the layer $(\mathbb{Z}^n)^{(k)}$ of $\mathbb{Z}^n$ for all $k \in \mathbb{N}$.*

Proof. — Let us construct the function ψ_Λ. For all $M \in \mathbb{N}$, denote $s^{(M)} := \sum_{k=0}^{M} s(\sqrt{k})$. The following $s(\sqrt{M+1})$-tuple of points: $(x_{s^{(M)}+1}, x_{s^{(M)}+2}, \ldots, x_{s^{(M)}+r_n(M+1)}, x_{s^{(M)}+r_n(M+1)+1}, x_{s^{(M)}+r_n(M+1)+2}, \ldots, x_{s^{(M+1)}})$ of Λ will be splitted up into two parts. Let $\Lambda^{*(M)} = \{x_{s^{(M)}+r_n(M+1)+1}, x_{s^{(M)}+r_n(M+1)+2}, \ldots, x_{s^{(M+1)}}\}$ and $\Lambda^* = \cup_{M\in\mathbb{N}} \Lambda^{*(M)}$. Let us put $\psi_\Lambda(z) = 0$ for all $z \in \Lambda^*$, and, for all $M \in \mathbb{N}$ and for all $i = s^{(M)}+1, s^{(M)}+2, \ldots, s^{(M)}+r_n(M+1)$, let us put $\psi_\Lambda(x_i) \in S(0, \sqrt{M+1}) \cap \mathbb{Z}^n$ such that the restriction of ψ_Λ to $\Lambda \setminus \Lambda^*$ is injective. In other terms, the first $r_n(M+1)$ points of the above $s(\sqrt{M+1})$-tuple of points are sent injectively by ψ_Λ to the $r_n(M+1)$ elements of $\mathbb{Z}^n$ of norm $\sqrt{M+1}$ which lie on the sphere $S(0, \sqrt{M+1})$, the remaining points $x_{s^{(M)}+r_n(M+1)+1}, x_{s^{(M)}+r_n(M+1)+2}, \ldots, x_{s^{(M+1)}}$ going to the origin of $\mathbb{Z}^n$. There is no uniqueness of such a mapping ψ_Λ: given Λ^*, any renumbering e of the elements of $\mathbb{Z}^n$ conserving the norm provides another suitable mapping $e \circ \psi_\Lambda : \Lambda \to \mathbb{Z}^n$. The properties (i) to (iv) of ψ_Λ are easy consequences of its definition. □

Let us now consider an infinite $\mathscr{UD}$-set Λ which contains the origin and let us continue the proof of the proposition (if Λ does not contain the origin we modify slightly a few points close to the origin for having this property). In a similar way as in step 1 with eq.(3), we are looking for a lower bound of the quantity (with $c_i, c_j \in \Lambda$ and $\|c_i\| \geqslant R$, $\|c_j\| \geqslant R$)

$$\eta_\Lambda(R, m) := \frac{1}{2m} + \frac{1}{m^2}\sum_{j=1}^{m} \|c_j\| + \frac{1}{m^2}\sum_{1\leqslant i<j\leqslant m} \|c_j - c_i\|$$

as a function of R and m. Let us observe that the differences $c_j - c_i$ belong to the translated $\mathscr{UD}$-sets $\Lambda - c_i = \{\lambda - c_i \mid \lambda \in \Lambda\}$ of Λ which all contain the origin. Let us now compute a lower bound of $m^{-2}\sum_{1\leqslant i<j\leqslant m}\|c_j - c_i\|$. For integers $M, m \in \mathbb{N}$ that satisfy $\sum_{k=0}^{M} s(\sqrt{k}) < m \leqslant \sum_{k=0}^{M+1} s(\sqrt{k})$, we deduce the following inequality: $\sum_{j=2}^{m}\|c_j - c_1\| \geqslant \sum_{j=2}^{m}\|\psi_{\Lambda-c_1}(c_j)\| \geqslant \sum_{l=0}^{M} r_n(l)\sqrt{l}$ from the lemmas 2.10, 2.11 and 2.12. We now proceed with the other sums $\sum_{j=i+1}^{m}\|c_j - c_i\|$, $i \geqslant 2$. Let us assume that $m = \sum_{q=0}^{M+1} s(\sqrt{q})$. For all $i = 1, 2, \ldots, s(\sqrt{M})$, the difference $m - i$ is greater than $\sum_{q=0}^{M} s(\sqrt{q})$ and this implies $\sum_{j=i+1}^{m}\|c_j - c_i\| \geqslant \sum_{j=i+1}^{m}\|\psi_{\Lambda-c_i}(c_j)\| \geqslant \sum_{l=0}^{M} r_n(l)\sqrt{l}$. We deduce the inequality $\sum_{i=1}^{s(\sqrt{M+1})}\sum_{j=i+1}^{m}\|c_j - c_i\| \geqslant s(\sqrt{M+1})\left(\sum_{l=0}^{M} r_n(l)\sqrt{l}\right) \geqslant r_n(M+1)\left(\sum_{l=0}^{M} r_n(l)\sqrt{l}\right)$. Since for all i, j the inequality holds: $\|c_j - c_i\| \geqslant \|\psi_{\Lambda-c_i}(c_j)\|$ and that $\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\|c_j - c_i\| = \sum_{i=1}^{s(\sqrt{M+1})}\sum_{j=i+1}^{m}\|c_j - c_i\| + \sum_{i=s(\sqrt{M+1})+1}^{s(\sqrt{M+1})+s(\sqrt{M})}\sum_{j=i+1}^{m}\|c_j - c_i\| + \ldots + \sum_{i=s(\sqrt{M+1})+s(\sqrt{M})+\ldots+s(\sqrt{2})+1}^{s(\sqrt{M+1})+s(\sqrt{M})+\ldots+s(\sqrt{2})+s(\sqrt{1})}\sum_{j=i+1}^{m}\|c_j - c_i\|$, by reproducing the same computation term by term, we deduce

$$\sum_{1\leqslant i<j\leqslant m}\|c_j - c_i\| \geqslant r_n(M+1)\left(\sum_{l=0}^{M} r_n(l)\sqrt{l}\right) + r_n(M)\left(\sum_{l=0}^{M-1} r_n(l)\sqrt{l}\right) + \cdots + r_n(2)r_n(1) + 2^n.$$

This leads to the same inequality as in eq.(6), with $m = \sum_{q=0}^{M+1} s(\sqrt{q})$, except that "$k$" has to be replaced by "M". Therefore, we obtain

$$m^{-2}\sum_{1\leqslant i<j\leqslant m}\|c_j - c_i\| \geqslant K_3\sqrt{M} \tag{10}$$

Let us now compute a lower bound of $m^{-2}\sum_{j=1}^{m}\|c_j\|$. Consider that $m = \sum_{q=0}^{M+1} s(\sqrt{q})$ and take $R = \sqrt{t}$ with $t \geqslant 2$ an integer. This lower bound corresponds to a distribution by layers of the points $c_1, c_2, \ldots, c_m$ on Λ so that they are located as close as possible to the sphere $S(0, R)$. Let us write m as the following sum: $m = s(\sqrt{t}) + s(\sqrt{t+1}) + \cdots + s(\sqrt{t+U}) + W$ for certain integers $U \geqslant 0$ and $0 < W \leqslant s(\sqrt{Tt+U+1})$. Then, by lemma 2.12, $\sum_{j=1}^{m}\|c_j\| \geqslant \sum_{j=1}^{m}\|\psi_\Lambda(c_j)\| \geqslant \sum_{l=t}^{t+U} r_n(l)\sqrt{l}$. Hence, by the same type of computation as in step 1, and by replacing only "u" by "U", we deduce

$$\frac{1}{m^2}\sum_{j=1}^{m}\|c_j\| \geqslant \frac{K_1(t-1)^{\frac{n-1}{2}}}{K_2^2}\,\frac{1}{U(U+2+t)^{n-2}} \tag{11}$$

In order to compute a lower bound of the sum $m^{-2}\sum_{j=1}^{m}\|c_j\| + m^{-2}\sum_{1\leqslant i<j\leqslant m}\|c_j - c_i\|$ as a function of U only from eq.(10) and eq.(11), it remains to give explicitely a relation between M and U. This relation comes from the computation of a lower bound of m which will be a function of M only and an upper bound of m which will be a function of U only. Let us compute these bounds. First, since $\sum_{k=0}^{M+1} r_n(k) \leqslant \sum_{k=0}^{M+1} s(\sqrt{k}) = m$ we deduce, by the same type of computation as in step 1 (with "U" instead of "u"),

$$\frac{2K_1}{n}\left[(t+U)^{n/2} - (t-1)^{n/2}\right] \leqslant \sum_{q=0}^{M+1} s(\sqrt{q}) = m. \tag{12}$$

Second, if $\mathrm{Vol}(B(0,x))$ denotes the volume of the ball $B(0,x)$, by counting the maximal possible number of points in the annulus $\{\sqrt{k} \leqslant \|x\| < \sqrt{k+1}\}$ (in this annulus any point should be at a distance from another one greater than unity), we deduce that the term $s(\sqrt{k}), k \geqslant 1,$ is smaller than $\left(\mathrm{Vol}(B(0,\sqrt{k+1}+1/2)) - \mathrm{Vol}(B(0,\sqrt{k}-1/2))\right)(\mathrm{Vol}(B(0,1/2)))^{-1}$. Therefore $m = \sum_{k=0}^{M+1} s(\sqrt{k}) \leqslant 1+2^n \sum_{k=1}^{M+1}\left[\left(\sqrt{k+1}+1/2\right)^n - \left(\sqrt{k}-1/2\right)^n\right]$. By a first-order development of each term, we deduce $m \leqslant 1+2^n \sum_{k=1}^{M+1}\left[\sqrt{k+1}-\sqrt{k}+1\right] n\left(\sqrt{k+1}+1/2\right)^{n-1}$. Since $\sqrt{k+1}-\sqrt{k}+1 \leqslant 2$ we obtain that m is certainly exceeded by $n2^{n+1}\sum_{k=1}^{M+1}\left(\sqrt{k+1}+1/2\right)^{n-1}$. Now, for all $1 \leqslant k \leqslant M+1$, we have $\sqrt{k+1}+1/2 \leqslant \sqrt{k+3\sqrt{M+1}}$. We deduce $m \leqslant n2^{n+1}\sum_{k=1}^{M+1}\left(k+3\sqrt{M+1}\right)^{\frac{n-1}{2}} \leqslant \frac{n2^{n+2}}{n+1}\left[(M+2+3\sqrt{M+1})^{\frac{n+1}{2}} - (1+3\sqrt{M+1})^{\frac{n+1}{2}}\right]$. Denote $l(x) = \left(x + \frac{1+3\sqrt{M+1}}{M+1}\right)^{\frac{n+1}{2}}$ and $\omega = \sup_{M\geqslant 1}\left(\sup_{x\in[0,1]} l'(x)\right)$. Then it is easy to check, by factorizing $(M+1)^{(n+1)/2}$ and applying a first-order development to the factors in the right-hand side term of the last inequality that this term is smaller than $n2^{n+2}\omega(n+1)^{-1}(M+1)^{\frac{n+1}{2}}$. Hence

$$m \leqslant 2^{n+2}\omega(M+1)^{\frac{n+1}{2}} \tag{13}$$

From eq.(12) and eq.(13) (as for eq.(8) and eq.(9)) we deduce the following inequality

$$U^{\frac{1}{n+1}}\left[\frac{1}{4}\left(\frac{K_1}{2\omega}\right)^{2/(n+1)}(t-1)^{\frac{n-2}{n+1}} - 1\right]^{1/2} \leqslant \sqrt{M}. \tag{14}$$

Let $g_\Lambda(t,U) := \frac{C_1(t)}{U(U+2+t)^{n-2}} + C_3(t)U^{\frac{1}{n+1}}$, where $C_3(t) := K_3\left[\frac{1}{4}\left(\frac{K_1}{2\omega}\right)^{\frac{2}{n+1}}(t-1)^{\frac{n-2}{n+1}} - 1\right]^{1/2}$. Then (as in step 1) $\eta_\Lambda(\sqrt{t}, m) \geqslant g_\Lambda(t, U_{min}(t))$, for all $m = \sum_{k=0}^{M+1} s(\sqrt{k})$, where $U_{min}(t)$ is the value at which the function $U \to g_\Lambda(t,U)$ is minimal. The proof of $\lim_{t\to+\infty} g_\Lambda(t, U_{min}(t)) = +\infty$ is similar as in step 1, for all integer m, and left to the reader. This implies, after eq.(3), that $\lim_{R\to+\infty} d(\Lambda, \Lambda_R) = 0$ for all $\mathcal{UD}$-set Λ and all $n \geqslant 5$. This convergence is obviously uniform in the sense stated in the proposition since the sequence $s(\sqrt{k})$ is universal and optimal for splitting up any $\mathcal{UD}$-set Λ.

step 3.– If Λ is a $\mathcal{UD}$-set in $\mathbb{R}^{n'}$ with $n' = 1,2,3$ or 4, then it can be viewed as a $\mathcal{UD}$-set in $\mathbb{R}^5$. Since the proposition is true for $n = 5$ by the step 2, the lemma 2.3 implies that it is also true in lower dimensions by descent. □

2.2 Pointwise behaviour of $\mathcal{UD}$-sets, proximity and pairing property

In the following, we will denote by $\mathrm{dist}(A,B)$ the distance $\inf\{\|a-b\| \mid a\in A, b\in B\}$ between two non-empty subsets A and B of $\mathbb{R}^n$. The two following lemmas will be used in the proofs of the completeness and the precompactness of $(\mathcal{UD}, d)$.

LEMMA 2.13. — *Let Λ, Λ' be two non-empty $\mathcal{UD}$-sets, $l = dist(\{0\},\Lambda) < +\infty$, and $\epsilon \in (0, \frac{1}{1+2l})$. Assume that $d(\Lambda,\Lambda') < \epsilon$. Then, for all $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$, (i) there exists a unique $\lambda' \in \Lambda'$ such that $\|\lambda'-\lambda\| < 1/2$, (ii) this pairing (λ,λ') satisfies the inequality: $\|\lambda'-\lambda\| \leqslant (1/2+\|\lambda\|)\epsilon$. In particular, for all $\lambda,\lambda' \in \mathbb{R}^n$, the distance $d(\{\lambda\},\{\lambda'\})$ tends to zero if and only if $\|\lambda-\lambda'\|$ tends to zero.*

Proof. — (i) Let us assume that for all $\lambda' \in \Lambda'$ and all $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$ the inequality $\|\lambda' - \lambda\| \geqslant 1/2$ holds. This will lead to a contradiction. Assume the existence of an element $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$ and take $D = \{\lambda\}$ and let E be a countable dense subset in $(0, 1/2)$. Each $\mathscr{B}_m$ in $\mathscr{B}_{(D,E)}$ is a set constituted by only one element: the ball (say) $\overset{o}{B}(\lambda, e_m)$ with $e_m \in E$. We deduce that $\phi_{\mathscr{B}_m}(\Lambda) = e_m$ and $\phi_{\mathscr{B}_m}(\Lambda') = 0$. Hence

$$d_{\lambda,(D,E)}(\Lambda, \Lambda') = \sup_m \frac{e_m}{1/2 + \|\lambda\|} = \frac{1/2}{1/2 + \|\lambda\|} \leqslant d(\Lambda, \Lambda')$$

But $\epsilon < \frac{1}{1+2\|\lambda\|}$ is equivalent to $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$. Since we have assumed $d(\Lambda, \Lambda') < \epsilon$, we should obtain $\epsilon < d_{\lambda,(D,E)}(\Lambda, \Lambda') \leqslant d(\Lambda, \Lambda') < \epsilon$. Contradiction. The uniqueness of λ' comes from the fact that Λ' is a $\mathscr{UD}$-set allowing only one element λ' close to λ. (ii) Let us assume that $\lambda \neq \lambda'$ for all $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$, with $\lambda' \in \Lambda'$ that satisfies $\|\lambda' - \lambda\| < 1/2$ (if the equality $\lambda = \lambda'$ holds, there is nothing to prove). Then, for all $\lambda \in \Lambda$ such that $\|\lambda\| < \frac{1-\epsilon}{2\epsilon}$, let us take $\alpha = \lambda$ as base point, $D = \{\lambda\}$ and E a dense subset in $(0, \|\lambda - \lambda'\|] \subset (0, 1/2)$. Then $\phi_{\mathscr{B}_m}(\Lambda) - \phi_{\mathscr{B}_m}(\Lambda') = e_m \left(1 - f\left(\frac{\lambda'-\lambda}{e_m}\right)\right)$. The restriction of the function $z \to z(1 - f(\frac{\lambda'-\lambda}{z}))$ to $(0, \|\lambda - \lambda'\|]$ is the identity function and is bounded above by $\|\lambda' - \lambda\|$. Therefore, $d_{\lambda,(D,E)}(\Lambda, \Lambda') = \sup_{\mathscr{B}_m} \frac{|\phi_{\mathscr{B}_m}(\Lambda) - \phi_{\mathscr{B}_m}(\Lambda')|}{1/2+\|\lambda\|} = \frac{\|\lambda'-\lambda\|}{1/2+\|\lambda\|}$. Since $d_{\lambda,(D,E)}(\Lambda, \Lambda') \leqslant d(\Lambda, \Lambda') < \epsilon$, we obtain $\|\lambda' - \lambda\| \leqslant (1/2 + \|\lambda\|)\epsilon$ as claimed. The last assertion in (ii) can easily be deduced from the above and from the continuity of the function f. ☐

In other terms, each time a $\mathscr{UD}$-set Λ is sufficiently close to another one Λ' for the metric d, every element of Λ lying in a large ball centred at the origin in $\mathbb{R}^n$, is automatically associated with a unique element of Λ' which is close to it within distance less than $1/2$. Such pairings of elements occur over larger and larger distances from the origin when Λ' tends to Λ. From (ii), we see that the proximity in the pairings (λ, λ') is much better for the elements $\lambda \in \Lambda$ which are the closest to the origin.

LEMMA 2.14. — *Let $\epsilon \in (0,1)$ and $\Lambda \in \mathscr{UD}, \Lambda \neq \emptyset$. Then the condition $d(\Lambda, \emptyset) < \epsilon$ implies $\Lambda \subset \mathbb{R}^n \setminus B(0, \frac{1-\epsilon}{2\epsilon})$.*

Proof. — Let us assume the existence of $\lambda \in \Lambda$ such that $\|\lambda\| \leqslant \frac{1-\epsilon}{2\epsilon}$ and let us show that this hypothesis implies that the assertion $d(\Lambda, \emptyset) < \epsilon$ is wrong. Take $D = \{\lambda\}$ and E a dense subset in $(0, 1/2)$. Each $\mathscr{B}_m$ in $\mathscr{B}_{(D,E)}$ is a set constituted by only one ball: say the ball $\overset{o}{B}(\lambda, e_m)$ with $e_m \in E$. We deduce that $\phi_{\mathscr{B}_m}(\Lambda) = e_m$. Since $\phi_{\mathscr{B}_m}(\emptyset) = 0$, the following inequality holds: $d_{\lambda,(D,E)}(\Lambda, \emptyset) = \sup_m \frac{e_m}{1/2+\|\lambda\|} = \frac{1/2}{1/2+\|\lambda\|} \leqslant d(\Lambda, \emptyset)$. But $\epsilon \leqslant \frac{1}{1+2\|\lambda\|}$ is equivalent to $\|\lambda\| \leqslant \frac{1-\epsilon}{2\epsilon}$. Hence, $\epsilon \leqslant d_{\lambda,(D,E)}(\Lambda, \emptyset)$. We deduce $d(\Lambda, \emptyset) \geqslant \epsilon$ as claimed. ☐

2.3 Completeness of $(\mathscr{UD}, d)$

Let $(\Lambda^{(i)})_{i \geqslant 0}$ be a non-stationary Cauchy sequence in $\mathscr{UD}$. We will show that it admits a convergent subsequence. Since this sequence is not stationary at the "empty set" element $\emptyset$ in particular, for all $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $d(\Lambda^{(m)}, \Lambda^{(q)}) < \epsilon$ for all $m, q \geqslant N(\epsilon)$, with $\Lambda^{(N(\epsilon))} \neq \emptyset$. Let $l_{N(\epsilon)} = \mathrm{dist}(\{0\}, \Lambda^{(N(\epsilon))}) < +\infty$. There are two cases: either (i) the function $\epsilon \to l_{N(\epsilon)}$ goes very fast to infinity when ϵ goes to zero in the sense that the inequality $\epsilon \leqslant \frac{1}{1+2l_{N(\epsilon)}}$ never holds on $(0,1)$, or (ii) there exists $\epsilon \in (0,1)$ such that $\epsilon \leqslant \frac{1}{1+2l_{N(\epsilon)}}$ holds. In case (i) the sequence $(\Lambda^{(i)})_{i\geqslant 0}$ admits as limit point set a $\mathscr{UD}$-set which has no point at finite arbitrary distance from the origin: this comes from the fact that the condition $\epsilon > \frac{1}{1+2l_{N(\epsilon)}}$ is equivalent to $l_{N(\epsilon)} > \frac{1-\epsilon}{2\epsilon}$. In other words, the sequence $(\Lambda^{(i)})_{i\geqslant 0}$ converges to the "empty set" element and we have proved the assertion in this case. In case (ii), denote by ϵ_0 the largest value of $\epsilon \in (0,1)$ such that the inequality $\epsilon \leqslant \frac{1}{1+2l_{N(\epsilon)}}$ holds. Denote $l_{N(\epsilon_0)} = \text{dist}(\{0\}, \Lambda^{(N(\epsilon_0))})$. Then $\epsilon_0 \leqslant (1+2l_{N(\epsilon_0)})^{-1}$ or equivalently $l_{N(\epsilon_0)} \leqslant \frac{1-\epsilon_0}{2\epsilon_0}$. Let us use the pairings, in the sense of lemma 2.13, between the points of $\Lambda^{(N(\epsilon_0))}$ and $\Lambda^{(N(\epsilon_0)+j)}$, for $j \geqslant 1$. This will give way to the construction, pointwise in $\mathbb{R}^n$, of a limit point set of the given Cauchy sequence $\Lambda^{(i)}$ by a diagonalization process. Indeed, from lemma 2.13, all $\lambda \in \Lambda^{(N(\epsilon_0))}$ such that $l_{N(\epsilon_0)} \leqslant \|\lambda\| < \frac{1-\epsilon_0}{2\epsilon_0}$ (such elements λ are in finite number, say that they are $\lambda_1, \lambda_2, \ldots, \lambda_{i_0}$) are such that there exists $\lambda^{(j)} \in \Lambda^{(N(\epsilon_0)+j)}$, $j \geqslant 1$, such that $\|\lambda^{(j)} - \lambda\| \leqslant (1/2 + \|\lambda\|)\epsilon_0$. Let us use the indexation by $i = 1,2,\ldots,i_0$. Then, for all $i = 1,2,\ldots,i_0$, the compact ball $B(\lambda_i, (1/2+\|\lambda_i\|)\epsilon_0)$ contains the infinite point set $\{\lambda_i^{(j)}\}_{j\geqslant 1}$. The elements of $\Lambda^{(N(\epsilon_0)+j)}$, $j \geqslant 1$, located in $\mathbb{R}^n$ at a distance less than $\sup_{i\in\{1,2,\ldots,i_0\}} (\|\lambda_i\| + (1/2+\|\lambda_i\|)\epsilon_0)$ $\left(< \frac{1-\epsilon_0}{2\epsilon_0} + \frac{1}{2\epsilon_0} \times \epsilon_0 = \frac{1}{2\epsilon_0}\right)$ from the origin, are all within $\bigcup_{i=1}^{i_0} B(\lambda_i, (1/2+\|\lambda_i\|)\epsilon_0)$. From the sequence of points $(\lambda_1^{(j)})_{j\geqslant 1}$ in the first ball $B(\lambda_1, (1/2+\|\lambda_1\|)\epsilon_0)$, which is compact, we extract a convergent subsequence $(\lambda_1^{(j)})_{j\in J_1}$, with J_1 an infinite subset of $J_0 := \mathbb{N}\setminus\{0\}$. Then, from the sequence of points $\{\lambda_2^{(j)}\}_{j\in J_1}$ in the second ball $B(\lambda_2, (1/2+\|\lambda_2\|)\epsilon_0)$, which is compact, we extract a convergent subsequence $\{\lambda_2^{(j)}\}_{j\in J_2}$, with $J_2 \subset J_1$, an infinite subset of J_1. And so on up till $i = i_0$ with $J_{i_0} \subset J_{i_0-1} \subset \cdots \subset J_2 \subset J_1 \subset J_0$ and J_{i_0} an infinite subset of J_{i_0-1}. Denote by $\widetilde{\lambda_i} := \lim_{\substack{j\in J_i\\ j\to+\infty}} \lambda_i^{(j)} = \lim_{\substack{j\in J_{i_0}\\ j\to+\infty}} \lambda_i^{(j)}$ for $i \in \{1,2,\ldots,i_0\}$ the respective limit points, one per closed ball $B(\lambda_i, (1/2+\|\lambda_i\|)\epsilon_0)$. There are two cases: either $(c_0.i)$ $\Lambda^{(N(\epsilon_0))}$ is finite with a number of elements, $\#(\Lambda^{(N(\epsilon_0))})$, equal to i_0, or $(c_0.ii)$ $\#(\Lambda^{(N(\epsilon_0))}) > i_0$, possibly infinite. In the first case $(c_0.i)$, we deduce that any element $\Lambda^{(N(\epsilon_0)+j)}$, $j \geqslant 0$, of the sequence has exactly i_0 elements and that a limit point set for the given Cauchy sequence is exactly the set $\{\widetilde{\lambda_1}, \widetilde{\lambda_2}, \ldots, \widetilde{\lambda_{i_0}}\}$. If $i_0 = 1$, it is clear that it is a $\mathscr{UD}$-set. If $i_0 > 1$, we will prove below that it is a $\mathscr{UD}$-set. In the second case $(c_0.ii)$, we reiterate the process: we take $\epsilon_1 \in (0,\epsilon_0)$. Then there exists $N(\epsilon_1) > N(\epsilon_0)$ such that $d(\Lambda^{(m)}, \Lambda^{(q)}) < \epsilon_1$ for all $m, q \geqslant N(\epsilon_1)$ and $m, q \in J_{i_0}$, with $\Lambda^{(N(\epsilon_1))} \neq \emptyset$. The following inequality $\epsilon_1 \leqslant \frac{1}{1+2l_{N(\epsilon_1)}}$ is satisfied and is equivalent to $l_{N(\epsilon_1)} \leqslant \frac{1-\epsilon_1}{2\epsilon_1}$. The pairing of the elements of $\Lambda^{(N(\epsilon_0))}$ with the elements of $\Lambda^{(N(\epsilon_0)+j)}$, $j \in J_{i_0}$, goes now over greater distances in $\mathbb{R}^n$, namely for all $\lambda \in \Lambda^{(N(\epsilon_0))}$ such that $l_{N(\epsilon_0)} \leqslant \|\lambda\| < \frac{1-\epsilon_1}{2\epsilon_1}$. The number of elements λ which satisfy this last inequality is finite. Assume that such elements λ can be indexed by $\{1,2,\ldots,i_0,i_0+1,\ldots,i_1\}$ with $i_1 > i_0$ (it is always possible to obtain a strict inequality by taking ϵ_1 small enough). We now consider the new series of closed balls $B(\lambda_i, (1/2+\|\lambda_i\|)\epsilon_1)$, $i \in \{i_0+1,\ldots,i_1\}$. Then, from the sequence of points $\{\lambda_{i_0+1}^{(j)}\}_{j\in J_{i_0}}$ in the first ball of the new series $B(\lambda_{i_0+1}, (1/2+\|\lambda_{i_0+1}\|)\epsilon_1)$, which is compact, we extract a convergent subsequence $(\lambda_{i_0+1}^{(j)})_{j\in J_{i_0+1}}$ with J_{i_0+1} an infinite subset of J_{i_0}. We now reiterate the extraction process for all $i = i_0+2,\ldots,i_1$. Denote the points at the limit by $\widetilde{\lambda_i} := \lim_{\substack{j\in J_i\\ j\to+\infty}} \lambda_i^{(j)} = \lim_{\substack{j\in J_{i_1}\\ j\to+\infty}} \lambda_i^{(j)}$ for all $i \in \{i_0+1, i_0+2,\ldots,i_1\}$, one per ball in the new series of balls. Again, there are two cases: either $(c_1.i)$ the number of elements of $\Lambda^{(N(\epsilon_0))}$, $\#(\Lambda^{(N(\epsilon_0))})$, is equal to i_1, or $(c_1.ii)$ $\#(\Lambda^{(N(\epsilon_0))}) > i_1$. In the first case $(c_1.i)$, we deduce that any element $\Lambda^{(N(\epsilon_0)+j)}$, $j \geqslant 0$ of the sequence has exactly i_1 elements and that a limit point set for the given Cauchy sequence is exactly the $\mathscr{UD}$-set $\{\widetilde{\lambda_1}, \widetilde{\lambda_2}, \ldots, \widetilde{\lambda_{i_1}}\}$. We will prove below that it is a $\mathscr{UD}$-set. In the second case $(c_1.ii)$, we reiterate the process: we take $\epsilon_2 \in (0, \epsilon_1)$. And so on. We obtain sequences $\{\epsilon_s\}, \{i_s\}, \{J_{i_s}\}$, $s = 0, 1, 2, \ldots$ and $\{J_t \mid t = 1, 2, \ldots\}$ such that for all $t \geqslant 1$, $J_t \subset J_{t-1}$ and is an infinite set, $0 < \ldots \epsilon_s < \epsilon_{s-1} < \cdots < \epsilon_1 < \epsilon_0$, $1 \leqslant i_0 < i_1 < i_2 < \ldots$ with $\widetilde{\lambda_i} := \lim_{\substack{j \in J_i \\ j \to +\infty}} \lambda_i^{(j)} = \lim_{\substack{j \in J_{i_s} \\ j \to +\infty}} \lambda_i^{(j)}$ for all $i \in \{i_{s-1}+1, i_{s-1}+2, \ldots, i_s\}$ and all $s \geqslant 1$. Let us call $\widetilde{\Lambda} := \{\widetilde{\lambda_i} \mid i \text{ integer } \geqslant 1\}$ the limit point set constructed in such a way. The sequence $\{i_s\}$ is finite when the $\mathscr{UD}$-set $\Lambda^{N(\epsilon_0)}$ is finite. In such a case, $\#(\widetilde{\Lambda}) = \#(\Lambda^{N(\epsilon_0)})$. If not $\#(\widetilde{\Lambda})$ is infinite.

Let us show that $\widetilde{\Lambda}$, finite (not reduced to one point) or not, is a $\mathscr{UD}$-set. Indeed, take two arbitrary elements $\widetilde{\lambda_i}, \widetilde{\lambda_j} \in \widetilde{\Lambda}$, with $i, j \geqslant 1, i \neq j$. We want to show that $\|\widetilde{\lambda_i} - \widetilde{\lambda_j}\| \geqslant 1$. Assume the contrary, that is $\|\widetilde{\lambda_i} - \widetilde{\lambda_j}\| < 1$. Let $i_t, t \geqslant 1$, the smallest integer such that $i \leqslant i_t$, $j \leqslant i_t$. We have $\widetilde{\lambda_i} = \lim_{\substack{k \in J_{i_t} \\ k \to +\infty}} \lambda_i^{(k)}$, resp. $\widetilde{\lambda_j} = \lim_{\substack{k \in J_{i_t} \\ k \to +\infty}} \lambda_j^{(k)}$. Therefore, there exists $q \in J_{i_t}$ such that $\|\lambda_i^{(q)} - \lambda_j^{(q)}\| < 1$. But $\Lambda^{(q)}$ is a $\mathscr{UD}$-set. Contradiction.

We have proved that $(\mathscr{UD}, d)$ is complete by an application of the Bolzano-Weierstrass property infinitely many times. It is the analogue of the proof of Chabauty [Ch] for lattices, but there, only finitely many times sufficed.

2.4. Precompactness of $(\mathscr{UD}, d)$

Recall that, to show that the metric space $(\mathscr{UD}, d)$ is precompact, we have to show that, for all $\epsilon > 0$, there are finitely many point sets $\Lambda_{(1)}, \Lambda_{(2)}, \ldots, \Lambda_{(s)}$ $(s = s(\epsilon))$ of $\mathscr{UD}$ such that the open d-balls $\{z \in \mathscr{UD} \mid d(z, \Lambda_{(i)}) < \epsilon\}$ with $i = 1, 2, \ldots, s$ cover $\mathscr{UD}$. Let $\epsilon \in (0, 1)$. We will explicitely exhibit such finite chains of open d-balls of radius ϵ.

The property of uniform convergence towards the "empty set" element $\emptyset$ (proposition 2.8) implies that there exists $\eta(\epsilon) > 0$ such that, for all $\Lambda \in \mathscr{UD}$ such that $\Lambda \subset \mathbb{R}^n \setminus B(0, \eta(\epsilon))$, the following inequality holds $d(\Lambda, \emptyset) < \epsilon$. Hence, all $\Lambda \in \mathscr{UD}$ such that $\mathrm{dist}(\{0\}, \Lambda) > \eta(\epsilon)$ is such that $d(\Lambda, \emptyset) < \epsilon$, i.e. will belong to the open d-ball $\{z \in \mathscr{UD} \mid d(z, \emptyset) < \epsilon\}$. Put $\Lambda_{(1)} := \emptyset$. Let us now prove that the collection of $\mathscr{UD}$-sets Λ which satisfy $\mathrm{dist}(\{0\}, \Lambda) \leqslant \eta(\epsilon)$ is covered by a finite chain of open d-balls of radius ϵ. Let us consider the lattice $L_\mu = \frac{\mu}{\sqrt{n}} \mathbb{Z}^n$ with $0 < \mu \leqslant 1$. Each point in space is within distance less than $\mu/2$ from L_μ. We will chose μ small enough so that the following continuity arguments hold: (i) any element $\lambda \in \Lambda \cap B(0, \eta(\epsilon))$ can be associated (non uniquely) with an element $l_\lambda \in L_\mu \cap B(0, \eta(\epsilon))$ such that $\|\lambda - l_\lambda\| \leqslant \mu/2$, (ii) denoting by $A(\Lambda) := \{l_\lambda \mid \lambda \in \Lambda \cap B(0, \eta(\epsilon))\} \subset L_\mu \cap B(0, \eta(\epsilon))$ the set of these close-neighbour points, we have $d(\Lambda, A(\Lambda)) < \epsilon$, (iii) $A(\Lambda)$ is a $\mathscr{UD}$-set. These items (i), (ii) and (iii) express merely the possibility of the finite $\mathscr{UD}$-set $\Lambda \cap B(0, \eta(\epsilon))$ to be slightly distorted and "put" on the subset $L_\mu \cap B(0, \eta)$ of the lattice L_μ for μ small enough.

Denote by $\mathscr{S}$ the finite set of non-empty subsets of $L_\mu \cap B(0, \eta(\epsilon))$ which are $\mathscr{UD}$-sets: let $\mathscr{S} = \{\Lambda_{(2)}, \Lambda_{(3)}, \ldots, \Lambda_{(s)}\}$ assuming the number of elements of $\mathscr{S}$ is $s-1$ (obviously $s \geqslant 2$). By the assertions (ii) and (iii), the set $A(\Lambda)$ is one of the elements of $\mathscr{S}$. We deduce that the open d-balls $\{x \in \mathscr{UD} \mid d(x, \Lambda_{(i)}) < \epsilon\}$, $i = 2, 3, \ldots, s$, cover $\mathscr{UD} \setminus \{\Lambda \in \mathscr{UD} \mid \Lambda \subset \mathbb{R}^n \setminus B(0, \eta(\epsilon))\}$. Hence, the open d-balls $\{x \in \mathscr{UD} \mid d(x, \Lambda_{(i)}) < \epsilon\}$, $i = 1, 2, \ldots, s$, cover $\mathscr{UD}$.

We deduce the precompactness of the topological space $(\mathscr{UD}, d)$.

2.5. Compactness of the sets of Delone sets

Proposition 2.15. — *For all $R > 0$ the subspace $X_R = \{\Lambda \in \mathscr{UD} \mid \forall z \in \mathbb{R}^n, \exists \lambda \in \Lambda$ such that $\|z - \lambda\| \leqslant R\}$ of the Delone sets of constant R is closed in $(\mathscr{UD}, d)$.*

Proof. — Let $\Lambda \in \mathscr{UD} \setminus X_R$. We will show that it is contained in an open subset disjoint from X_R that will prove that X_R is closed. Since $\Lambda \notin X_R$, there exists $z \in \mathbb{R}^n$ such that $\|z - \lambda\| > R$ for all $\lambda \in \Lambda$. Let $l = \text{dist}(\{z\}, \Lambda) > R$ and denote $\Lambda - z := \{\lambda - z \mid \lambda \in \Lambda\}$ the translated set. For $\epsilon > 0$ small enough and all Γ in the open d-ball $\{\Omega \in \mathscr{UD} \mid d(\Omega, \Lambda - z) < \epsilon\}$ the elements γ of Γ satisfy all the inequality: $\|\gamma\| \geqslant R + \frac{l-R}{2} > R$ by the property of the pairing (proposition 2.13); all these point sets Γ are outside X_R. Since the translation by z is bicontinuous, the $\mathscr{UD}$-set Λ is contained in the open subset $z + \{\Omega \in \mathscr{UD} \mid d(\Omega, \Lambda - z) < \epsilon\}$ which is disjoint of X_R. □

3. Theorem 1.2 implies theorem 1.1 and comments

Let $\mathscr{L}_n$ be the space of lattices in $\mathbb{R}^n$, identified with the locally compact homogeneous space $GL(n, \mathbb{R})/GL(n, \mathbb{Z})$ [GL] [Ca] (Recall that a lattice in $\mathbb{R}^n$ is a discrete $\mathbb{Z}$-module of maximal rank of $\mathbb{R}^n$, equivalently a discrete subgroup of the group of translations of $\mathbb{R}^n$ with compact fundamental region). The following proposition is a key result for proving the theorem 1.1 from the theorem 1.2.

Proposition 3.1. — *The restriction of the metric d to $\mathscr{L}_n \cap \mathscr{UD} \subset \mathscr{UD}$ is compatible with the topology on $\mathscr{L}_n \cap \mathscr{UD}$ induced by the quotient topology of $\mathscr{L}_n = GL(n, \mathbb{R})/GL(n, \mathbb{Z})$.*

Proof. — This proposition is a reformulation of the following proposition. □

Proposition 3.2. — *Let $L \in \mathscr{L}_n \cap \mathscr{UD}$. Denote by $\{e_1, e_2, \ldots, e_n\}$ a basis of L. Then (i) for all $\epsilon > 0$ small enough there exists $\eta > 0$ such that all $\mathbb{Z}$-module $L' \in \mathscr{UD}$ contained in the open ball $\{\Lambda \in \mathscr{UD} \mid d(L, \Lambda) < \eta\}$ is of rank n and admits a basis $\{e'_1, e'_2, \ldots, e'_n\}$ which satisfies the property: $\max_{i=1,2,\ldots,n} \|e_i - e'_i\| < \epsilon$; (ii) for all $0 < \eta < 1$ there exists $\epsilon > 0$ such that all lattice $L' \in \mathscr{UD}$ of $\mathbb{R}^n$ admitting a basis $\{e'_1, e'_2, \ldots, e'_n\}$ which satisfies $\max_{i=1,2,\ldots,n} \|e_i - e'_i\| < \epsilon$ is such that $d(L, L') < \eta$.*

Proof. — (i) First let us chose $\epsilon_0 > 0$ small enough such that all n-tuple $\{a_1, a_2, \ldots, a_n\}$ of points of $\mathbb{R}^n$ with $a_i \in B(e_i, \epsilon_0)$, $i = 1, 2, \ldots, n$, is such that the vectors $\{Oa_1, Oa_2, \ldots, Oa_n\}$ are $\mathbb{Z}$-linearly independant (as usual we identify the point a_i with the vector Oa_i, $i = 1, 2, \ldots, n$).

For instance, if Vect_i, $i = 1, 2, \ldots, n$, denotes the $\mathbb{R}$-span generated by the vectors $Oe_1, Oe_2, \ldots, Oe_{i-1}, Oe_{i+1}, \ldots, Oe_n$, let us take $\epsilon_0 = \frac{1}{3} \min_{i=1,2,\ldots,n} \{\mathrm{dist}(\{e_i\}, \mathrm{Vect}_i)\}$. Let $\epsilon \in (0, \epsilon_0)$.

Assume that Λ is a $\mathscr{UD}$-set such that $d(L, \Lambda) < \eta$ with η small enough. By lemma 2.13 a pairing between the points of L and Λ occurs over a certain distance, which is $\frac{1-\eta}{2\eta}$, from the origin. Let us take η_1 small enough in order to have $\frac{1-\eta_1}{2\eta_1} \geqslant \max_{i=1,2,\ldots,n} \|e_i\|$. From the lemma 2.13 the condition $0 < \eta < \eta_1$ implies the existence of n points $e'_1, e'_2, \ldots, e'_n$ in Λ, the respective close-neighbours of the points $e_1, e_2, \ldots, e_n$ of L, which satisfy $\|e'_i - e_i\| \leqslant (1/2 + \|e_i\|)\eta$ for $i = 1, 2, \ldots, n$. Take $\eta < \eta_1$ such that $(1/2 + \max_{i=1,2,\ldots,n} \|e_i\|)\eta < \epsilon$. Since $\epsilon < \epsilon_0$, the vectors $Oe'_1, Oe'_2, \ldots, Oe'_n$ are $\mathbb{Z}$-linearly independant. This means that if $\Lambda \in \mathscr{UD}$ is a $\mathbb{Z}$-module of $\mathbb{R}^n$ (necessarily discrete) which satisfies $d(L, \Lambda) < \eta$, Λ is necessarily of rank n and contains the lattice $\sum_{i=1}^n \mathbb{Z}e'_i$. Let us show that there is equality. Denote by $\mathscr{V}' = \{\sum_{i=1}^n \theta_i e'_i \mid 0 \leqslant \theta_i < 1 \text{ for all } i = 1, 2, \ldots, n\}$. The adherence $\overline{\mathscr{V}'}$ of $\mathscr{V}'$ contains only the points $\sum_{i=1}^n j_i e'_i$ of Λ, with $j_i = 0$ or 1, by the property of the pairing (proposition 2.13). Therefore the free system $\{Oe'_1, Oe'_2, \ldots, Oe'_n\}$ is a basis of Λ. (ii) Conversely, let $0 < \eta < 1$ and $L' \in \mathscr{UD} \cap \mathscr{L}_n$. For all $R > 0$ the inequality $d(L, L') \leqslant d(L, L_R) + d(L_R, L'_R) + d(L'_R, L')$ holds. By the proposition 2.8 let us take R large enough such that $d(L, L_R) < \eta/3$ and $d(L', L'_R) < \eta/3$. Let us now show that, if L' admits a basis $\{e'_1, e'_2, \ldots, e'_n\}$ which satisfies $\max_{i=1,2,\ldots,n} \|e_i - e'_i\| < \epsilon$, then ϵ can be taken small enough to have $d(L_R, L'_R) < \eta/3$ (R kept fixed). Indeed, L_R and L'_R are finite $\mathscr{UD}$-sets. Denote $N := \#L_R$. For all $\alpha \in \mathbb{R}^n$, all $(D, E) \in \mathscr{E}$ and all $\mathscr{B}_m \in \mathscr{B}_{(D,E)}$, by continuity of the function f, the mapping $(x_1, x_2, \ldots, x_N) \to \phi_{\mathscr{B}_m}(\{x_1, x_2, \ldots, x_N\}) := \sum_{\overset{0}{B}(c,\omega) \in \mathscr{B}_m} \sum_{i=1}^N \omega f\left(\frac{x_i - c}{\omega}\right)$ is continuous on $B(0, R)^N$ for the standard product topology. Therefore all the mappings $d_{\alpha,(D,E)}(L_R, \cdot)$:

$$(x_1, x_2, \ldots, x_N) \to \sup_{\mathscr{B}_m} \frac{|\phi_{\mathscr{B}_m}(L_R) - \phi_{\mathscr{B}_m}(\{x_1, x_2, \ldots, x_N\})|}{(1/2 + \|\alpha\| + \|\alpha - c_{j_1}\| + \|\alpha - c_{j_2}\| + \cdots + \|\alpha - c_{j_N}\|)}$$

are continuous on $B(0, R)^N$. The continuity of $(x_1, x_2, \ldots, x_N) \to d(L_R, \{x_1, x_2, \ldots, x_N\})$ on $B(0, R)^N$ follows. Take for $\{x_1, x_2, \ldots, x_N\}$ the point set L'_R. Consequently the quantity $d(L_R, L'_R)$ is strictly less than $\eta/3$ as soon as ϵ is small enough. Finally $d(L, L') < 3\eta/3 = \eta$ and we deduce the claim. □

Recall that if L is a lattice in $\mathbb{R}^n$ and A a basis of L, then $|\det(A)|$ is called the determinant of L; we will denote it by $|L|$. It is the volume of its fundamental region.

Proposition 3.3. — *For all* $M > 0$, *the subspace* $\{L \in \mathscr{UD} \cap \mathscr{L}_n \mid 0 < |L| \leqslant M\} \subset \mathscr{L}_n \cap \mathscr{UD}$ *is compact.*

Proof. — By the proposition 3.1 and since $(\mathscr{UD}, d)$ is a compact topological space, we have just to show that $\{L \in \mathscr{UD} \cap \mathscr{L}_n \mid 0 < |L| \leqslant M\}$ is closed. Since the operations $x+y$ and xy are continuous, the determinant function $|\cdot|$ is continuous on $\mathscr{L}_n$. Hence $\{L \in \mathscr{UD} \cap \mathscr{L}_n \mid |L| > M\} = |\cdot|^{-1}((M, +\infty))$ is an open set as reciprocal image of the open interval $(M, +\infty)$ by the continuous application $|\cdot|$. By taking its complement subspace in $\mathscr{UD} \cap \mathscr{L}_n$ we deduce the claim. □

Let us now prove the theorem 1.1. Let us consider a sequence of lattices (L_r) of $\mathbb{R}^n$ such that: (i) $\|x\| \geqslant 1$ for all $x \in L_r, x \neq 0$, (ii) the determinant $|L_r|$ of L_r satisfies $|L_r| \leqslant M$ with

M a constant $< +\infty$ independent of r. Then all $L_r \in \{L \in \mathscr{UD} \cap \mathscr{L}_n \mid 0 < |L| \leqslant M\}$ which is compact by the proposition 3.3. Then, by the Bolzano-Weierstrass property, one can extract from the sequence (L_r) a subsequence $(L_{r'})$ that converges to a lattice L of $\mathbb{R}^n$. By continuity of the determinant function $|\cdot|$ and the proposition 3.1, we obtain: $|L| = \lim_{r' \to +\infty} |L_{r'}|$. This concludes the proof.

In final, let us make some comments about the topology on $(\mathscr{UD}, d)$. The topological space $(\mathscr{UD}, d)$ is obviously a Polish space [Bo]. But this topology is not classical. It is routine to compare it with the topologies reviewed by Kelley [Ke] and Michael [Mi] on spaces of non-empty closed subsets of $\mathbb{R}^n$ (here adapted to point sets) and to conclude that it is none of them. Nevertheless, if we denote $\mathscr{UD}_f := \{\Lambda \in \mathscr{UD} \mid \Lambda \text{ finite}\}$ the subspace of $\mathscr{UD}$ of the finite $\mathscr{UD}$-sets and $\mathscr{UD}_{f,t} := \{\Lambda \in \mathscr{UD} \mid \Lambda \subset B(0,t)\} \subset \mathscr{UD}_f$ for $t > 0$, the following obvious results hold [Bo].

Proposition 3.4. — *The subset $\mathscr{UD}_f$ is dense in $\mathscr{UD}$. For all $t > 0$ the Hausdorff metric Δ on $\mathscr{UD}_{f,t}$ is compatible with the topology on $\mathscr{UD}_{f,t}$ induced by that of $(\mathscr{UD}, d)$ and the topological space $(\mathscr{UD}_{f,t}, \Delta) = (\mathscr{UD}_{f,t}, d)$ is compact.*

Let us mention that the theorem 1.2 seems to become an important ingredient in the topological dynamics and the spectral approach in the ergodic theory of tilings [Ga]. At least, the classical selection theorem of Mahler is already as important as the Ascoli-Arzela theorem in analysis. Probably the present generalization will also play a basic role, not only in geometry of numbers.

If we look carefully at the proof we have given in the section 2, particularly the proposition 2.8 which has for consequence the precompactness, we observe that it is based on three ingredients: (i) the use of the standard metric on $\mathbb{R}^n$ to which is added an "origin", (ii) a detailed counting of the points of $\mathbb{Z}^n$ in spheres in order to obtain uniform bounds of the function η at infinity, (iii) a universal function $s(\sqrt{k})$ allowing to propagate this counting information to general $\mathscr{UD}$-sets. Therefore, to answer in general to the questions (q1) and (q2) means a priori that the expected compactness theorem on $\mathscr{UD}(H,\delta)_r$ will depend on the choice of the metric, of the chosen "origin", of the reference discrete subspace chosen for the counting, on the behaviour of the function η at infinity, of the possibility of universalizing the counting to general elements of $\mathscr{UD}(H,\delta)_r$. The present approach can be pursued to general hyperbolic manifolds as ambiant spaces [MR].

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