Boundary Hölder and L^p Estimates for local solutions of the tangential Cauchy-Riemann equation

Christine LAURENT-THIÉBAUT and Mei-Chi SHAW*

Prépublication de l'Institut Fourier n° 604 (2003)
`http://www-fourier.ujf-grenoble.fr/prepublications.html`

1 Introduction

In this paper we study the local solvability of the tangential Cauchy-Riemann equation $\overline{\partial}_b$ on an open neighborhood ω of a point $z_0 \in M$ when M is a generic CR manifold of real codimension k in $\mathbb{C}^n$, where $1 \leq k \leq n-1$. We assume that M is q-concave near z_0 (see Definition 2.2.1). Our method is to first derive an homotopy formula for $\overline{\partial}_b$ in ω when ω is the intersection of M with a strongly pseudoconvex domain. The homotopy formula gives a local solution operator for any $\overline{\partial}_b$-closed form on ω without shrinking. We obtain Hölder and L^p estimates up to the boundary for the solution operator.

Let $\mathcal{C}^\alpha(\overline{\omega})$, $0 < \alpha < 1$, be the space of Hölder continuous functions of order α in $\overline{\omega}$. We use $\mathcal{C}^\alpha_{n,s}(\overline{\omega})$ to denote the space of (n,s)-forms with $C^\alpha(\overline{\omega})$ coefficients The norm in $\mathcal{C}^\alpha_{n,s}(\overline{\omega})$ is defined to be the sum of $\mathcal{C}^\alpha(\overline{\omega})$ norm of each coefficient. We also denote by $L^p_{(n,s)}(\omega)$ the space of (n,s)-forms with $L^p(\omega)$ coefficients, $1 \leq p \leq \infty$. The norm in $L^p_{(n,s)}(\omega)$ is denoted by $\| \ \|_{L^p}$ for (n,s)-forms. Our main results are the following:

Theorem 1.0.1. (Homotopy formula for $\overline{\partial}_b$.) *Let M be a strictly q-concave generic CR manifold in $\mathbb{C}^n$ and $z_0 \in M$. Let Ω be a strictly pseudoconvex domain containing z_0 in $\mathbb{C}^n$ with C^3 boundary and $\omega = M \cap \Omega$. For any s, $n-k-q+1 \leq s \leq n-k$, there exists a continuous operator T_{s-1} from $\mathcal{C}_{n,s}(\overline{\omega})$ into $\mathcal{C}^{\frac{1}{2}-\epsilon}_{n,s-1}(\overline{\omega})$ such that for any $f \in \mathcal{C}_{n,s}(\overline{\omega})$ with $\overline{\partial} f \in \mathcal{C}_{n,s+1}(\overline{\omega})$,*

$$f = \overline{\partial}_b T_{s-1} f + T_s \overline{\partial} f.$$

Theorem 1.0.2. (Hölder and L^p estimates for $\overline{\partial}_b$.) *Let M be a strictly q-concave generic CR manifold in $\mathbb{C}^n$ and $z_0 \in M$. Let Ω be a strictly pseudoconvex domain containing z_0 in $\mathbb{C}^n$ with C^3 boundary and $\omega = M \cap \Omega$. For any $f \in L^p_{(n,s)}(\omega)$ with $\overline{\partial}_b f = 0$ in ω, $1 \leq p \leq \infty$ and $n-k-q+1 \leq s \leq n-k$, there exists an operator $\widetilde{T}_{s-1}$ satisfying $\overline{\partial}_b \widetilde{T}_{s-1} f = f$ in ω and the following estimates hold:*

*NSF grant DMS01-00492.
A.M.S. Classification : 32F20, 32F10, 32F4.
Key words : CR manifolds, Hölder estimates, L^p-estimates, Tangential Cauchy Riemann equation.

(1) $\|\widetilde{T}_{s-1}f\|_{L^{\frac{2n+2}{2n+1}-\epsilon}} \leq C\|f\|_{L^1}$, *for any small* $\epsilon > 0$.

(2) $\|\widetilde{T}_{s-1}f\|_{L^{p'}} \leq C\|f\|_{L^p}$, *where* $\frac{1}{p'} = \frac{1}{p} - \frac{1}{2n+2}$ *and* $1 < p < 2n+2$.

(3) $\|\widetilde{T}_{s-1}f\|_{L^{p'}} \leq C\|f\|_{L^p}$, *where* $p = 2n+2$ *and* $p < p' < \infty$.

(4) $\|\widetilde{T}_{s-1}f\|_{\mathcal{C}^{\alpha-\epsilon}} \leq C\|f\|_{L^p}$, *where* $2n+2 < p < \infty$, $\alpha = \frac{1}{2} - \frac{n+1}{p}$ *and* $\epsilon > 0$.

(5) $\|\widetilde{T}_{s-1}f\|_{\mathcal{C}^{\frac{1}{2}-\epsilon}} \leq C\|f\|_{L^\infty}$, *for any* $\epsilon > 0$.

Corollary 1.0.3. *Under the same assumption as in Theorem 1.0.2, the range of* $\overline{\partial}_b$ *is closed in* $L^p_{(n,s)}(\omega)$ *spaces for* $1 \leq p \leq \infty$.

The L^2 estimates will give the Hodge decomposition theorem for $\overline{\partial}_b$ and the existence of the $\overline{\partial}_b$-Neumann operators.

Corollary 1.0.4. *Under the same assumption as in Theorem 1.0.2, the following strong Hodge decomposition theorem holds: For* $n-k-q+1 < s < n-k$, *there exists a linear operator* $N_b : L^2_{(n,s)}(\omega) \to L^2_{(n,s)}(\omega)$ *such that*

(1) N_b *is bounded and Range*$(N_b) \subset$ *Dom*$(\Box_b)$.

(2) For any $f \in L^2_{(n,s)}(\omega)$, *we have*

$$f = \overline{\partial}_b\overline{\partial}_b^* N_b f \oplus \overline{\partial}_b^*\overline{\partial}_b N_b f.$$

(3) If $f \in L^2_{(n,s)}(\omega)$ *with* $\overline{\partial}_b f = 0$, *then* $f = \overline{\partial}_b\overline{\partial}_b^* N_b f$. *The solution* $u = \overline{\partial}_b^* N_b f$ *is called the canonical solution, i.e., the unique solution orthogonal to Ker*$(\overline{\partial}_b)$.

Though our theorems are stated for (n,s)-forms, it is clear that they can be extended to any (r,s)-forms for $0 \leq r \leq n$.

It is well known (see [7]) that on a hypersurface, if the Levi form satisfies Kohn's condition $Y(s)$ at one point, then the Poincaré Lemma holds for (r,s)-forms in a neighborhood of the point. Local solvability for $\overline{\partial}_b$ on hypersurfaces has also been investigated in earlier works of Andreotti-Hill [2], Treves [21], Boggess-Shaw [5] and Laurent-Thiébaut-Leiterer [11]. When M is strongly pseudoconvex, homotopy formulas were first obtained by Henkin [8] using integral kernels. Local solvability was also studied in Laurent-Thiébaut-Leiterer [13] and Shaw [19] for CR hypersurfaces with mixed Levi signatures.

In this paper we obtain an homotopy formula for $\overline{\partial}_b$ on ω with Hölder and L^p estimates on CR manifolds with higher codimension. The q-concavity assumption can be viewed as a generalization of condition $Y(s)$ for appropriate degree s to higher codimension case. The local solvability of the $\overline{\partial}_b$ equation in q-concave CR manifolds goes back to Naruki [15], Henkin [9], Airapetyan-Henkin [1] and Nacinovich [14]. Homotopy formula for $\overline{\partial}_b$ for forms with compact support on q-concave manifolds was constructed earlier by Barkatou [3] and Barkatou-Laurent-Thiébaut [4]. A microlocal version of the local homotopy formula for $\overline{\partial}_b$ on q-concave manifolds was studied by Polyakov [16]. Optimal Hölder and L^p estimates for $\Box_b$ have been proved using Campanato spaces in Shaw-Wang [20]. All these are results on the interior regularity for $\overline{\partial}_b$ and $\Box_b$.

The previously known results for the boundary regularity for $\overline{\partial}_b$ are for strongly pseudoconvex or q-concave hypersurfaces. If M is a strongly pseudoconvex hypersurface and ω is a domain in M such that $b\omega$ is the intersection of M with a Levi-flat hypersurface, then one can construct a solution operator which is bounded in L^p. It was proved in Shaw [18] that, in this setting, L^p estimates for the local solutions for $\overline{\partial}_b$ up to the boundary are best possible. If M is a q-concave hypersurface and ω is the intersection of M with a bounded strictly pseudoconvex domain, a solution operator is constructed in Laurent-Thiébaut-Leiterer [13] and Hölder $\mathcal{C}^{\frac{1}{2}-\epsilon}$, $\epsilon > 0$ estimates up to the boundary are obtained.

For q-concave CR manifolds of higher codimension, it follows from the the results in Barkatou-Laurent-Thiébaut [4] that for any given continuous form, the regularity of the solution inside the domain is actually $\mathcal{C}^{\frac{1}{2}}$. The regularity up to the boundary proved in the Theorem 1.0.1 is ϵ less than the interior regularity. It is not known if one can remove the ϵ for the boundary regularity. This phenomenon is similar to the case of the $\overline{\partial}$ equation in domains with piecewise strictly pseudoconvex or q-convex boundary (see [17] and [12]).

In contrast to the Hölder regularity discussed above, the solution operator we constructed in the Theorem 1.0.2 has also a gain of regularity in L^p spaces, but the gain is strictly less than the interior regularity. Actually the interior regularity is given by an operator of weak type $\frac{2n}{2n-1}$ and the boundary regularity is given by an operator of weak type $\frac{2n+2}{2n+1}$. This phenomenon is new and has not been observed before.

The plan of this paper is as follows: in section 2 we construct the homotopy formula for the local solution of $\overline{\partial}_b$ on ω for smooth $\overline{\partial}_b$-closed forms. In section 3, Hölder estimates are obtained. We also obtain better Hölder regularity in the complex tangential directions. In section 4.1, a new homotopy formula for the kernel which involves only integration on ω is derived to facilitate the estimation of the kernels in L^p spaces. The estimation for smooth $\overline{\partial}_b$-closed forms and the approximation argument necessary to pass from *a priori* estimates to actual estimates are carried out in subsection 4.2.

Part of the work was completed while the first author was visiting the University of Notre Dame and the second author was visiting the Institut Fourier, Université de Grenoble, France. The authors would like to thank both institutes for their hospitality during their visit.

2 Homotopy formula on CR manifolds

2.1 Kernels attached to a generic CR manifold

Let M be a generic CR manifold of class $\mathcal{C}^3$ in $\mathbb{C}^n$, U an open subset in $\mathbb{C}^n$ and $\widehat{\rho}_1, \dots, \widehat{\rho}_k$ some functions of class $\mathcal{C}^3$ from U into $\mathbb{R}$ such that

$$M \cap U = \{z \in U | \widehat{\rho}_1(z) = \cdots = \widehat{\rho}_k(z) = 0\}$$

and satisfying $\overline{\partial}\widehat{\rho}_1(z) \wedge \cdots \wedge \overline{\partial}\widehat{\rho}_k(z) \neq 0$ for $z \in M \cap U$.

Let $C>0$ be a fixed constant, we set, for $j=1,\dots,k$,

$$\begin{aligned} \rho_j &= \widehat{\rho}_j + C\sum_{\nu=1}^{k}\widehat{\rho}_\nu^2 \\ \rho_{-j} &= -\widehat{\rho}_j + C\sum_{\nu=1}^{k}\widehat{\rho}_\nu^2. \end{aligned} \tag{2.1.1}$$

We define $\mathcal{I}$ as the set of all subsets $I\subset\{\pm1,\dots,\pm k\}$ such that $|i|\neq|j|$ for all $i,j\in I$ with $i\neq j$. For $I\in\mathcal{I}$, $|I|$ denotes the number of elements in I, then $\mathcal{I}(l)$, $1\leq l\leq k$, is the set of all $I\in\mathcal{I}$ with $|I|=l$ and $\mathcal{I}'(l)$, $1\leq l\leq k$, is the set of all $I\in\mathcal{I}$ of the form $I=(i_1,\dots,i_l)$ with $|i_\nu|=\nu$ for $\nu=1,\dots,l$.

If $I\in\mathcal{I}$ and $\nu\in\{1,\dots,|I|\}$, then i_ν is the element with number ν in I after ordering I by modulus. We set $I(\widehat{\nu})=I\setminus\{i_\nu\}$.

If $I\in\mathcal{I}$, then

sgn$I=1$ if the number of negative elements in I is even
sgn$I=-1$ if the number of negative elements in I is odd.

Let $(e_1,\dots,e_k)$ be the canonical basis of $\mathbb{R}^k$, set $e_{-j}=-e_j$ for every $1\leq j\leq k$. Let $I=(i_1,\dots,i_l)$ be in $\mathcal{I}(l)$, $1\leq l\leq k$, set

$$\Delta_I=\{\sum_{j=1}^{l}\lambda_j e_{i_j}|\lambda_i\geq 0, 1\leq i\leq l, \sum_{i=1}^{l}\lambda_i=1\}.$$

For each $\lambda\in\Delta_I$, we denote by ρ_λ a *defining function of M in the direction* λ,

$$\rho_\lambda=\lambda_1\rho_{i_1}+\dots+\lambda_k\rho_{i_k}.$$

A $\mathcal{C}^2$-map $\psi_\lambda\ :\ U\times U\to\mathbb{C}^n$ such that $\langle\psi_\lambda(\zeta,z),\zeta-z\rangle=1$ is called a *Leray section in the direction* λ.

From now on, we assume that ψ_λ depends smoothly on λ.

We denote by D a relatively compact open subset of U and for $I\in\mathcal{I}$, $I=(i_1,\dots,i_{|I|})$, we define

$$\begin{aligned} D_I &= \{\rho_{i_1}<0\}\cap\dots\cap\{\rho_{i_{|I|}}<0\}\cap D \\ D_I^* &= \{\rho_{i_1}>0\}\cap\dots\cap\{\rho_{i_{|I|}}>0\}\cap D \\ S_I &= \{\rho_{i_1}=0,\dots,\rho_{i_{|I|}}=0\}\cap D \end{aligned}$$

These manifolds are oriented as follows : D_I and D_I^* as $\mathbb{C}^n$ for all $I\in\mathcal{I}$, $S_{\{j\}}$ as the boundary of $D_{\{j\}}$ for $j=\pm1,\dots,\pm k$, S_I as the boundary of $S_{I(\widehat{|I|})}\cap\overline{D}_{\{i_{|I|}\}}$ for all $I\in\mathcal{I}$, $|I|\geq 2$, and $M\cap D$ as S_I with $I=\{1,\dots,k\}$.

If $I\in\mathcal{I}(l)$, $l\leq k$, we set for $z\in\overline{D}_I$, $\zeta\in\overline{D}_I^*$ with $z\neq\zeta$ and $\lambda\in\Delta_I$

$$\psi_I(\zeta,z,\lambda)=\psi_\lambda(\zeta,z).$$

We denote by $\overset{\circ}{\chi}$ a $\mathcal{C}^\infty$-function from $[0,1]$ into $[0,1]$, which satisfy $\overset{\circ}{\chi}(\lambda)=0$, if $0\leq\lambda\leq 1/4$, and $\overset{\circ}{\chi}(\lambda)=1$, if $1/2\leq\lambda\leq 1$.

If $I \in \mathcal{I}(l)$, $1 \leq l \leq k$, for $\lambda \in \Delta_{0I}$ with $\lambda_0 \neq 1$, let $\overset{\circ}{\lambda}$ be the point in Δ_I defined by

$$\overset{\circ}{\lambda}_{i_\nu} = \frac{\lambda_{i_\nu}}{1-\lambda_0} \quad (\nu = 1, \ldots, l).$$

We set

$$\psi_{0I}(\zeta, z, \lambda) = \overset{\circ}{\chi}(\lambda_0) \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^2} + (1 - \overset{\circ}{\chi}(\lambda_0)) \psi_I(\zeta, z, \overset{\circ}{\lambda}) \tag{2.1.2}$$

for every $I \in \mathcal{I}(l)$, $1 \leq l \leq k$, $z \in \overline{D}_I$, $\zeta \in \overline{D}_I^*$ with $z \neq \zeta$ and $\lambda \in \Delta_{0I}$. One may notice that ψ_{0I} is a function of class $\mathcal{C}^2$.

We can now define the kernels $K_{0I}(\zeta, z, \lambda)$, for $z \in \overline{D}_I$, $\zeta \in \overline{D}_I^*$ with $z \neq \zeta$ and $\lambda \in \Delta_{0I}$, by

$$\begin{aligned} K_{0I}(\zeta, z, \lambda) = \frac{(-1)^{n(n-1)/2}}{(2i\pi)^n} \langle \psi_{0I}, d\zeta \rangle \wedge \langle (\overline{\partial}_{\zeta,z} + d_\lambda) \psi_{0I}, d\zeta \rangle^{n-1} \\ \wedge d(\zeta_1 - z_1) \wedge \cdots \wedge d(\zeta_n - z_n), \end{aligned} \tag{2.1.3}$$

and the kernels $K_I(\zeta, z, \lambda)$ by

$$\begin{aligned} K_I(\zeta, z, \lambda) = \frac{(-1)^{n(n-1)/2}}{(2i\pi)^n} \langle \psi_I, d\zeta \rangle \wedge \langle (\overline{\partial}_{\zeta,z} + d_\lambda) \psi_I, d\zeta \rangle^{n-1} \\ \wedge d(\zeta_1 - z_1) \wedge \cdots \wedge d(\zeta_n - z_n). \end{aligned} \tag{2.1.4}$$

The kernels K_{0I} and K_I are differential forms of class $\mathcal{C}^1$ and degree $2n-1$ and, from Proposition 3.9 in [10], we have

$$(\overline{\partial}_{\zeta,z} + d_\lambda) K_{0I}(\zeta, z, \lambda) = 0. \tag{2.1.5}$$

Finally we set, for $z \in \overline{D}_I$, $\zeta \in \overline{D}_I^*$ with $z \neq \zeta$,

$$C_{0I}(\zeta, z) = \int_{\lambda \in \Delta_{0I}} K_{0I}(\zeta, z, \lambda)$$

$$C_I(\zeta, z) = \int_{\lambda \in \Delta_I} K_I(\zeta, z, \lambda)$$

Proposition 2.1.1. *The kernels $C_{0I}(\zeta, z)$ and $C_I(\zeta, z)$ are differential forms of degree $2n - |I| - 1$ and $2n - |I|$, respectively, of class $\mathcal{C}^1$ for $z \in \overline{D}_I$ and $\zeta \in \overline{D}_I^*$ with $z \neq \zeta$, which satisfy the partial differential equation*

$$\overline{\partial}_z C_{0I} + \overline{\partial}_\zeta C_{0I} = C_{0\delta(I)} - C_I,$$

with $C_{0\delta(I)} = \sum_{\nu=1}^{|I|} (-1)^{\nu+1} C_{0I(\widehat{\nu})}$.

The next lemma is proved in [4].

Lemma 2.1.2. *Let f be an (n, r)-form of class $\mathcal{C}^1$ with compact support in $D \cap M$. Then $\int_{\zeta \in S_I} f(\zeta) \wedge C_{0I}(\zeta, z)$ defines an $(n, r-1)$-form of class $\mathcal{C}^{\frac{1}{2}-\varepsilon}$ on $\overline{D}_I$.*

Now set

$$B_M(\zeta, z) = \sum_{I \in \mathcal{I}'(k)} \operatorname{sgn}(I) C_{0I}(\zeta, z) \tag{2.1.6}$$

for $\zeta, z \in M \cap D$ with $\zeta \neq z$, and denote by $[B_M]_{p,s}$ the part of B_M, which is of bidegree (p, s) in z.

2.2 Fundamental solution for the tangential Cauchy-Riemann operator on q-concave generic CR manifolds

In this part we assume that the generic CR manifold M is q-concave.

Definition 2.2.1. A generic CR manifold M in $\mathbb{C}^n$ of real codimension k is *q-concave*, $1 \leq q \leq \frac{n-k}{2}$, if for all $z \in M$ and all $\lambda \in \mathbb{R}^k$ the restriction of the Levi form of the defining function ρ_λ in the direction λ to the complex tangent space $T_z^{\mathbb{C}}M$ of M at z admits at least q negative eigenvalues.

It follows from Lemma 3.1.1 in [1] that we can choose the constant C in (2.1.1) such that the functions ρ_j, $-k \leq j \leq k$, $j \neq 0$, have the following property : for each $I \in \mathcal{I}'(k)$ and every $\lambda \in \Delta_I$, the Levi form of the defining function ρ_λ of M in the direction λ has at least $q+k$ positive eigenvalues on $U' \subset\subset U$. Then using the method developed in section 3 of [12], we can construct for each λ a Leray section in the direction λ, which has some holomorphy properties and depends smoothly on λ. Let us recall the main steps of the construction.

Denote by $F_\lambda(\zeta, .)$ the Levi polynomial of ρ_λ at $\zeta \in U$. For $\zeta \in U$, $z \in \mathbb{C}^n$,

$$F_\lambda(\zeta, z) = 2\sum_{j=1}^{n} \frac{\partial \rho_\lambda}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2 \rho_\lambda}{\partial \zeta_j \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k).$$

Let $G(n, q+k)$ be the grassmannian of all subspaces of $\mathbb{C}^n$ of dimension $q+k$, we consider for all $I \in \mathcal{I}(k)$, a smooth map

$$T_I \ : \ \Delta_I \to G(n, q+k)$$

such that the Levi form of the defining function ρ_λ of M in the direction λ is positive definite on $T(\lambda)$ for all $\lambda \in \Delta_I$.

Denote by P^λ the orthogonal projection from $\mathbb{C}^n$ onto $T_I(\lambda)$ and set $Q^\lambda = Id - P^\lambda$. Taylor's theorem implies that there exist a domain $D \subset\subset U'$ and two positive constants α and A such that

$$\mathrm{Re} F_\lambda(\zeta, z) \geq \rho_\lambda(\zeta) - \rho_\lambda(z) + \alpha|\zeta - z|^2 - A|Q^\lambda(\zeta - z)|^2 \tag{2.2.1}$$

for $\zeta, z \in D$.

Since ρ_λ is of class $\mathcal{C}^2$ on U, we can find $\mathcal{C}^\infty$ functions a_{jk}, $j, k = 1, \ldots, n$, on U' such that for all $\zeta \in U'$

$$|a_{jk} - \frac{\partial^2 \rho_\lambda}{\partial \zeta_j \partial \zeta_k}(\zeta)| < \frac{\alpha}{2n^2}. \tag{2.2.2}$$

Then setting

$$\widetilde{F}_\lambda(\zeta, z) = 2\sum_{j=1}^{n} \frac{\partial \rho_\lambda}{\partial \zeta_j}(\zeta)(\zeta_j - z_j) - \sum_{j,k=1}^{n} a_{jk}(\zeta_j - z_j)(\zeta_k - z_k),$$

it follows from (2.2.1) and (2.2.2) that

$$\mathrm{Re}\widetilde{F}_\lambda(\zeta, z) \geq \rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2}|\zeta - z|^2 - A|Q^\lambda(\zeta - z)|^2 \tag{2.2.3}$$

for $\zeta, z \in D$.

Denote by $(Q^\lambda_{jk})^n_{j,k=1}$ the entries of the matrix Q^λ, and set for $(\zeta,z)\in\mathbb{C}^n\times U'$

$$\begin{aligned}
w^\lambda_j(\zeta,z) &= 2\frac{\partial\rho_\lambda}{\partial\zeta_j} - \sum_{k=1}^n a_{jk}(\zeta_k - z_k) + A\sum_{k=1}^n \overline{Q^\lambda_{jk}(\zeta_k - z_k)}\\
w_\lambda(\zeta,z) &= (w^\lambda_1(\zeta,z),\dots,w^\lambda_n(\zeta,z))\\
\Phi_\lambda(\zeta,z) &= \langle w_\lambda(\zeta,z),\zeta - z\rangle\\
\psi_\lambda(\zeta,z) &= \frac{w_\lambda(\zeta,z)}{\Phi_\lambda(\zeta,z)}
\end{aligned}$$

Since Q^λ is an orthogonal projection, we have

$$\Phi_\lambda(\zeta,z) = \widetilde{F}_\lambda(\zeta,z) + A|Q^\lambda(\zeta - z)|^2$$

and it follows from (2.2.3) that

$$\mathrm{Re}\Phi_\lambda(\zeta,z) \geq \rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\alpha}{2}|\zeta - z|^2 \tag{2.2.4}$$

for $\zeta, z \in D$.

We shall say that a map f defined on some complex manifold X of complex dimension n is *l-holomorphic* if, for each point $\xi\in X$, there exist holomorphic coordinates $h_1,\dots,h_n$ in a neighborhood of ξ such that f is holomorphic with respect to $h_1,\dots,h_l$.

Lemma 2.2.2. *For every $\zeta\in U'$, the map $w_\lambda(\zeta,z)$ and the function $\Phi_\lambda(\zeta,z)$ defined above are $(q+k)$-holomorphic in z.*

This holomorphy condition implies the following vanishing properties of the kernels C_I.

Lemma 2.2.3. *We assume that for $I\in\mathcal{I}(l)$, $1\leq l\leq k$, the functions ψ_I, are $(q+k)$-holomorphic with respect to the variable z, then for each fixed $\zeta\in\overline{D}^*_I$*

$$\begin{aligned}
&[C_I(\zeta,z)]_{p,r} = 0 \quad \text{si} \quad 0\leq p\leq n \quad \text{et} \quad n-k-q+1\leq r\leq n-k\\
&\overline{\partial}_z[C_I(\zeta,z)]_{p,n-k-q} = 0 \quad \text{si} \quad 0\leq p\leq n,
\end{aligned}$$

on $\overline{D}_I\setminus\{\zeta\}$, where $[C_I(\zeta,z)]_{p,r}$ denotes the part of bidegree (p,r) in z of C_I.

It is proved in [4] that the kernel B_M defined by (2.1.6) is a fundamental solution for the $\overline{\partial}_b$ operator on M, i.e.

$$\overline{\partial}_z[B_M]_{p,r-1} + \overline{\partial}_\zeta[B_M]_{p,r} = (-1)^{\frac{k(k+1)}{2}}[\Delta(U')], \tag{2.2.5}$$

for $0\leq p\leq n$ and $n-k-q+1\leq r\leq n-k$, if $[\Delta(U')]$ denotes the integration current on the diagonal of $U'\times U'$.

For all $I\in\mathcal{I}'(k)$, we denote by $I\bullet$ the multi-index $(i_1,\dots,i_k,\bullet)$, where $I=(i_1,\dots,i_k)$, and by $\mathcal{I}'(k,\bullet)$ the set of all multi-indexes $I\bullet$, with $I\in\mathcal{I}'(k)$. We set $\rho_\bullet = \frac{1}{k}(\rho_1+\dots+\rho_k)$ and $\rho_\lambda = \lambda_1\rho_1+\dots+\lambda_k\rho_k+\lambda_\bullet\rho_\bullet$ for $\lambda=(\lambda_1,\dots,\lambda_k,\lambda_\bullet)\in\Delta_{I\bullet}$.

Let $E_\bullet$ be the larger linear subspace in $\mathbb{C}^n$ on which the Levi form of $\rho_\bullet$ on U is positive definite. It follows from the q-concavity of M and the choice of the defining functions $\rho_1,\dots,\rho_k$ that $\dim E_\bullet \geq q+k$.

We get some functions $w^\bullet$ and $\Phi_\bullet$ associated to the function $\rho_\bullet$ by setting

$$\begin{aligned}
w^\bullet_j(\zeta,z) &= 2\frac{\partial\rho_\bullet}{\partial\zeta_j}(\zeta) - \sum_{k=1}^n a^\bullet_{jk}(\zeta)(\zeta_k-z_k) + B\sum_{k=1}^n \overline{Q^\bullet_{jk}(\zeta_k-z_k)}\\
w^\bullet(\zeta,z) &= (w^\bullet_1(\zeta,z),\dots,w^\bullet_n(\zeta,z))\\
\Phi_\bullet(\zeta,z) &= \langle w^\bullet(\zeta,z),\zeta-z\rangle,
\end{aligned}$$

where the function $a^\bullet_{jk}$, $j,k=1,\dots,n$, is of class $\mathcal{C}^\infty$ on U and satisfies for all $\zeta\in U$

$$|a^\bullet_{jk}(\zeta) - \frac{\partial^2\rho_\bullet}{\partial\zeta_j\partial\zeta_k}(\zeta)| < \frac{\alpha^\bullet}{2n^2}$$

and $Q^\bullet$ is the orthogonal projection on the orthocomplement of the subspace $E_\bullet$.

We set

$$\widetilde{F}_\bullet(\zeta,z) = 2\sum_{j=1}^n \frac{\partial\rho_\bullet}{\partial\zeta_j}(\zeta)(\zeta_j-z_j) - \sum_{j,k=1}^n a^\bullet_{jk}(\zeta)(\zeta_j-z_j)(\zeta_k-z_k),$$

then

$$\Phi_\bullet(z,\zeta) = \widetilde{F}_\bullet(\zeta,z) + B|Q^\bullet(\zeta-z)|^2$$

and consequently

$$\mathrm{Re}\ \Phi_\bullet(z,\zeta) \geq \rho_\bullet(\zeta) - \rho_\bullet(z) + \frac{\alpha^\bullet}{2}|\zeta-z|^2.$$

If $\lambda=(\lambda_1,\dots,\lambda_k,\lambda_\bullet)\in\Delta_{I\bullet}$, is such that $\lambda_\bullet\neq 1$, we denote by λ' the point in Δ_I defined by

$$\lambda'_{i_\nu} = \frac{\lambda_{i_\nu}}{1-\lambda_\bullet} \quad (\nu=1,\dots,l).$$

Let us consider a function χ_ε of class $\mathcal{C}^\infty$ from $[0,1]$ into $[0,1]$, which vanishes in a neighborhood of 0, is equal to 1 in a neighborhood of 1, and moreover satisfies $|\chi_\varepsilon(t)-t|<\varepsilon$ for all $t\in[0,1]$. For $\lambda\in\Delta_{I\bullet}$, we set

$$\begin{aligned}
w^{I\bullet}(\zeta,z,\lambda) &= (1-\lambda_\bullet)(2\sum_{j=1}^n \frac{\partial\rho_{\lambda'}}{\partial\zeta_j}(\zeta)(\zeta_j-z_j) - \sum_{j,k=1}^n a_{jk}(\zeta)(\zeta_j-z_j)(\zeta_k-z_k))\\
&+ (1-\chi_\varepsilon(\lambda_\bullet))A\sum_{k=1}^n \overline{Q^{\lambda'}_{jk}(\zeta_k-z_k)}\\
&+ \lambda_\bullet(2\sum_{j=1}^n \frac{\partial\rho_\bullet}{\partial\zeta_j}(\zeta)(\zeta_j-z_j) - \sum_{j,k=1}^n a^\bullet_{jk}(\zeta)(\zeta_j-z_j)(\zeta_k-z_k))\\
&+ \chi_\varepsilon(\lambda_\bullet)B\sum_{k=1}^n \overline{Q^\bullet_{jk}(\zeta_k-z_k)}\\
\Phi_{I\bullet}(\zeta,z,\lambda) &= \langle w^{I\bullet}(\zeta,z,\lambda),\zeta-z\rangle.
\end{aligned}$$

The function $\Phi_{I\bullet}$ has the following expression

$$\Phi_{I\bullet}(\zeta,z,\lambda) = \widetilde{F}_\lambda(\zeta,z) + \langle P^\lambda(\zeta-z), \overline{\zeta}-\overline{z}\rangle,$$

where P^λ is the linear operator defined by $(1-\chi_\varepsilon(\lambda_\bullet))AQ^{\lambda'}+\chi_\varepsilon(\lambda_\bullet)BQ^\bullet$. If ε is sufficiently small, then there exists $\gamma > 0$ such that

$$\text{Re } \Phi_{I\bullet}(\zeta,z,\lambda) \geq \rho_\lambda(\zeta) - \rho_\lambda(z) + \frac{\gamma}{2}|\zeta - z|^2. \tag{2.2.6}$$

We define $(\psi_J)_{J\in\mathcal{I}'(k,\bullet)}$ in U' by setting, for $J = I\bullet$,

$$\psi_J(\zeta,z,\lambda) = \frac{w^{I\bullet}(\zeta,z,\lambda)}{\Phi_{I\bullet}(\zeta,z,\lambda)}.$$

Notice that $\psi_J|_{U'\times U'\setminus\Delta(U')\times\Delta_I} = \psi_I$. To these maps, we associate the kernels $K_{0I\bullet}(\zeta,z,\lambda)$ and $K_{I\bullet}(\zeta,z,\lambda)$, for $(\zeta,z,\lambda)\in U'\times U'\setminus\Delta(U')\times\Delta_{0I\bullet}$, defined by

$$\begin{aligned} K_{0I\bullet}(\zeta,z,\lambda) = \frac{(-1)^{n(n-1)/2}}{(2i\pi)^n} &\langle\psi_{0I\bullet}, d\zeta\rangle \wedge \langle(\overline{\partial}_{\zeta,z}+d_\lambda)\psi_{0I\bullet}, d\zeta\rangle^{n-1} \\ &\wedge d(\zeta_1-z_1)\wedge\cdots\wedge d(\zeta_n-z_n), \end{aligned} \tag{2.2.7}$$

and by

$$\begin{aligned} K_{I\bullet}(\zeta,z,\lambda) = \frac{(-1)^{n(n-1)/2}}{(2i\pi)^n} &\langle\psi_{I\bullet}, d\zeta\rangle \wedge \langle(\overline{\partial}_{\zeta,z}+d_\lambda)\psi_{I\bullet}, d\zeta\rangle^{n-1} \\ &\wedge d(\zeta_1-z_1)\wedge\cdots\wedge d(\zeta_n-z_n). \end{aligned} \tag{2.2.8}$$

We set also for $(\zeta,z)\in U'\times U'\setminus\Delta(U')$,

$$C_{0I\bullet}(\zeta,z) = \int_{\lambda\in\Delta_{0I\bullet}} K_{0I\bullet}(\zeta,z,\lambda),$$
$$C_{I\bullet}(\zeta,z) = \int_{\lambda\in\Delta_{I\bullet}} K_{I\bullet}(\zeta,z,\lambda).$$

As in Proposition 2.1.1 we have

$$\overline{\partial}_{\zeta,z} C_{0I\bullet} = C_{0\delta(I\bullet)} - C_{I\bullet}. \tag{2.2.9}$$

We set

$$E_M = \sum_{I\in\mathcal{I}'(k)} \text{sgn}(I) C_{0I\bullet} \quad \text{and} \quad R_M = \sum_{I\in\mathcal{I}'(k)} \text{sgn}(I) C_{I\bullet}.$$

In [4], it is proved that

$$\overline{\partial}_{\zeta,z} E_M(\zeta,z) = (-1)^k B_M(\zeta,z) - R_M(\zeta,z) \tag{2.2.10}$$

holds in the sense of currents on $U'\times U'$. The relation (2.2.5) associated to (2.2.10) shows then, that the kernel R_M is also a fundamental solution for the $\overline{\partial}_b$ operator on M.

This implies immediately the following integral representation formulas :

Theorem 2.2.4. *Let $\omega \subset\subset M\cap U'$ with piecewise smooth $\mathcal{C}^1$ boundary and f a (n,s)-form of class $\mathcal{C}^1$ on $\overline{\omega}$, then*

1) For $n-k-q+1 \le s \le n-k$,

$$(-1)^{(n+s)(k+1)+\frac{k(k+1)}{2}} f(z) = (-1)^k \int_{\zeta\in b\omega} f(\zeta)\wedge [R_M]_{n,s}(\zeta,z) + \int_{\zeta\in\omega} \overline{\partial}_b f(\zeta)\wedge [R_M]_{n,s}(\zeta,z)$$
$$+(-1)^{k+1}\overline{\partial}_b \int_{\zeta\in\omega} f(\zeta)\wedge [R_M]_{n,s-1}(\zeta,z).$$

2) For $0\le s\le q-1$,

$$(-1)^{(n+s)(k+1)+\frac{k(k+1)}{2}} f(\zeta) = (-1)^k \int_{z\in b\omega} f(z)\wedge [R_M]_{0,n-k-s-1}(\zeta,z)$$
$$+ \int_{z\in\omega} \overline{\partial}_b f(z)\wedge [R_M]_{0,n-k-s-1}(\zeta,z) + (-1)^{k+1}\overline{\partial}_b \int_{z\in\omega} f(z)\wedge [R_M]_{0,n-k-s}(\zeta,z).$$

We can describe the singularity of the kernel R_M in the following way.

A form of type O_s (or of type $O_s(\zeta,z,\lambda)$) on $\overline{D}_I\times\overline{D}_I^*\times\Delta_{I\bullet}$ is, by definition, a continuous differential form $f(\zeta,z,\lambda)$ defined for all $(\zeta,z,\lambda)\in\overline{D}_I^*\times\overline{D}_I\times\Delta_{I\bullet}$ with $z\neq\zeta$ such that the following conditions are fulfilled :

1. All derivatives of the coefficients of f which are of order 0 in ζ, and of order ≤ 1 in z and of arbitrary order in λ are continuous for all $(\zeta,z,\lambda)\in\overline{D}_I^*\times\overline{D}_I\times\Delta_{I\bullet}$ with $z\neq\zeta$.
2. Let ∇_z^κ, $\kappa=0,1$, be a differential operator with constant coefficients, which is of order 0 in ζ, of order κ in z and of arbitrary order in λ. Then there is a constant $C>0$ such that, for each coefficient $\varphi(\zeta,z,\lambda)$ of the form $f(\zeta,z,\lambda)$,
$$|\nabla_z^\kappa\varphi(\zeta,z,\lambda)|\le C|\zeta-z|^{s-\kappa}$$
for all $(\zeta,z,\lambda)\in\overline{D}_I^*\times\overline{D}_I\times\Delta_{I\bullet}$ with $z\neq\zeta$.

Assume σ is a monomial in $d\zeta_1,\dots,d\zeta_n,d\overline{\zeta}_1,\dots,d\overline{\zeta}_n$, then

$$\begin{aligned}\sigma\wedge R_M(\zeta,z) &= \sum_{I\in\mathcal{I}'(k)} \mathrm{sgn}(I)\int_{\lambda\in\Delta_{I\bullet}} [\sigma\wedge K_{I\bullet}(z,\zeta,\lambda)]_{\deg\lambda=|\mathrm{I}|}\\ &= \sum_{I\in\mathcal{I}'(k)}\sum_{\substack{0\le m\le k\\ i_1,\dots,i_m\in I}}\int_{\lambda\in\Delta_{I\bullet}}\frac{O_{k+1-m}}{\Phi^n}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta).\end{aligned} \tag{2.2.11}$$

As the manifold M is supposed to be q-concave with $q\ge1$ and consequently $n>k+1$. The integration with respect to λ allows us to control $|\sigma\wedge R_M(z,\zeta)|$, by a finite sum of terms of the form :

$$K=\frac{|\sigma\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)|}{\Pi_{\nu=1}^{k+1}|\Phi(z,\zeta,\lambda^\nu)||\zeta-z|^{2n-3k+m-3}}, \tag{2.2.12}$$

where $\lambda^1,\dots,\lambda^{k+1}$ are points in Δ_{I*}, $I\in\mathcal{I}'(k)$, which define a system of independent vectors of $\mathbb{R}^{k+1}$.

2.3 Homotopy formula for the tangential Cauchy-Riemann operator on q-concave CR generic manifolds

Let Ω be a domain in U with $\mathcal{C}^3$ boundary such that the intersection of M with the boundary $b\Omega$ of Ω is transversal and that $\omega = M \cap \Omega$ is relatively compact in $M \cap U'$. We assume also that Ω admits a Leray section $\psi_*(\zeta, z)$, which is holomorphic in the variable z. For example if Ω is convex and defined by $\{\zeta \in U |\ \rho_*(\zeta) = 0\}$, one may take $w^*(\zeta, z) = (\frac{\partial \rho_*}{\partial \zeta_1}(\zeta), \dots, \frac{\partial \rho_*}{\partial \zeta_n}(\zeta))$, $\Phi_*(\zeta, z) = \langle w^*(\zeta, z), \zeta - z\rangle$ and $\psi_*(\zeta, z) = \frac{w^*(\zeta,z)}{\Phi_*(\zeta,z)}$.

For each $I \in \mathcal{I}'(k)$, we denote by $I*$ the multi-index $(i_1, \dots, i_k, *)$, where $I = (i_1, \dots, i_k)$, and by $\mathcal{I}'(k, *)$ the set of all multi-indexes $I*$, when I describe $\mathcal{I}'(k)$. Let ρ_* be a defining function for Ω in U, we assume that $d\rho_{i_1} \wedge \dots \wedge d\rho_{i_k} \wedge d\rho_* \neq 0$ on $\overline{\Omega}$.

Let ψ_* be a Leray map for the function ρ_*. If $\lambda = (\lambda_1, \dots, \lambda_k, \lambda_*) \in \Delta_{I*}$ is such that $\lambda_* \neq 1$, we denote by λ' the point in Δ_I defined by

$$\lambda'_{i_\nu} = \frac{\lambda_{i_\nu}}{1 - \lambda_*} \quad (\nu = 1, \dots, l).$$

Let $\overset{\circ}{\chi}$ be a $\mathcal{C}^\infty$-function as in section 2.1, then we set for $\lambda \in \Delta_{I*}$

$$\psi_{I*}(\zeta, z, \lambda) = \overset{\circ}{\chi} (\lambda_*)\psi_*(\zeta, z) + (1 - \overset{\circ}{\chi} (\lambda_*))\psi_I(\zeta, z, \lambda'),$$

To these maps, we associate the kernels $K_{0I*}(\zeta, z, \lambda)$ and $K_{I*}(\zeta, z, \lambda)$, for $(\zeta, z, \lambda) \in U' \times U' \setminus \Delta(U') \times \Delta_{0I*}$, defined by

$$K_{0I*}(\zeta, z, \lambda) = \frac{(-1)^{n(n-1)/2}}{(2i\pi)^n} \langle \psi_{0I*}, d\zeta\rangle \wedge \langle (\overline{\partial}_{\zeta,z} + d_\lambda)\psi_{0I*}, d\zeta\rangle^{n-1} \wedge d(\zeta_1 - z_1) \wedge \dots \wedge d(\zeta_n - z_n), \tag{2.3.1}$$

and by

$$K_{I*}(\zeta, z, \lambda) = \frac{(-1)^{n(n-1)/2}}{(2i\pi)^n} \langle \psi_{I*}, d\zeta\rangle \wedge \langle (\overline{\partial}_{\zeta,z} + d_\lambda)\psi_{I*}, d\zeta\rangle^{n-1} \wedge d(\zeta_1 - z_1) \wedge \dots \wedge d(\zeta_n - z_n). \tag{2.3.2}$$

We set also for $(\zeta, z) \in U' \times U' \setminus \Delta(U')$,

$$C_{0I*}(\zeta, z) = \int_{\lambda \in \Delta_{0I*}} K_{0I*}(\zeta, z, \lambda)$$
$$C_{I*}(\zeta, z) = \int_{\lambda \in \Delta_{I*}} K_{I*}(\zeta, z, \lambda).$$

It follows from Proposition 2.1.1 that

$$\overline{\partial}_{\zeta,z} C_{0I*} = C_{0\delta(I*)} - C_{I*}. \tag{2.3.3}$$

We set

$$F_M = \sum_{I \in \mathcal{I}'(k)} \operatorname{sgn}(I) C_{0I*} \quad \text{and} \quad S_M = \sum_{I \in \mathcal{I}'(k)} \operatorname{sgn}(I) C_{I*},$$

then we get that if $\zeta, z \in U'$, with $z \neq \zeta$

$$\overline{\partial}_{\zeta,z} F_M(\zeta, z) = (-1)^k B_M(\zeta, z) - S_M(\zeta, z). \tag{2.3.4}$$

Replacing I by $J = I\bullet$, we can define in the same way as before the kernels $C_{0I\bullet *}$ and by Proposition 2.1.1 we get

$$\overline{\partial}_{\zeta,z} C_{0I\bullet *} = C_{0\delta(I)\bullet *} + (-1)^k C_{0I*} + (-1)^{k+1} C_{0I\bullet} - C_{I\bullet *}. \tag{2.3.5}$$

Let us introduce the kernel $G_M = \sum_{I\in\mathcal{I}'(k)} \mathrm{sgn}(I) C_{I\bullet *}$, then

$$\overline{\partial}_{\zeta,z} G_M(\zeta, z) = (-1)^k \overline{\partial}_{\zeta,z}(F_M(\zeta, z) - E_M(\zeta, z)),$$

which implies, using (2.2.10) and (2.3.4), the relation

$$\overline{\partial}_{\zeta,z} G_M(\zeta, z) = (-1)^k (R_M(\zeta, z) - S_M(\zeta, z)). \tag{2.3.6}$$

Theorem 2.3.1. *For $n - k - q + 1 \leq s \leq n - k$, there exist bounded operators T_s from $\mathcal{C}_{n,s+1}(\overline{\omega})$ into $\mathcal{C}_{n,s}(\omega)$ such that for each $(0, s)$-form f of class $\mathcal{C}^1$ on $\overline{\omega}$ we have*

$$f = \overline{\partial}_b T_{s-1} f + T_s \overline{\partial}_b f.$$

The operator T_s is the integral operator

$$T_s g = (-1)^{(n+s)(k+1)+\frac{k(k-1)}{2}} [\int_{\zeta\in\omega} g(\zeta)\wedge[R_M]_{n,s}(\zeta, .) + (-1)^{n+s+1} \int_{\zeta\in b\omega} g(\zeta)\wedge[G_M]_{n,s}(\zeta, .)].$$

Proof. Using (2.3.6), we get for $z \in \omega$

$$\begin{aligned}(-1)^k \int_{\zeta\in b\omega} f(\zeta) \wedge [R_M]_{n,s}(\zeta, z) &= (-1)^k \int_{\zeta\in b\omega} f(\zeta) \wedge [S_M]_{n,s}(\zeta, z)\\ &+ \int_{\zeta\in b\omega} f(\zeta) \wedge [\overline{\partial}_\zeta [G_M]_{n,s}(\zeta, z) + \overline{\partial}_z [G_M]_{n,s-1}(\zeta, z)].\end{aligned}$$

Since $\psi_*(\zeta, z)$ is holomorphic in z, the Leray maps ψ_{I*} are $(q + k)$-holomorphic in z in $\mathbb{C}^n$, then $[S_M]_{n,s}$, the part of bidegree (n, s) in z of $[S_M]$, vanishes if $s \geq n - k - q + 1$. Moreover we have

$$\begin{aligned}\int_{\zeta\in b\omega} f(\zeta) \wedge \overline{\partial}_\zeta [G_M]_{n,s}(\zeta, z) &= (-1)^{n+s} \int_{\zeta\in b\omega} \overline{\partial}_\zeta (f(\zeta) \wedge [G_M]_{n,s}(\zeta, z))\\ &- (-1)^{n+s} \int_{\zeta\in b\omega} \overline{\partial}_b f(\zeta) \wedge [G_M]_{n,s}(\zeta, z)\end{aligned}$$

and by Stokes' formula

$$\int_{\zeta\in b\omega} \overline{\partial}_\zeta (f(\zeta) \wedge [G_M]_{n,s}(\zeta, z)) = 0,$$

which proves the homotopy formula using part 1) of Theorem 2.2.4.

The continuity on ω of the integral $\int_{\zeta\in\omega} f(\zeta)\wedge[R_M]_{n,s}(\zeta, .)$ follows from the integrability of the kernel R_M, moreover as the kernels G_M are of class $\mathcal{C}^1$ on $U'\times U'\setminus\Delta(U')$ the integral $\int_{\zeta\in b\omega} f(\zeta) \wedge [G_M]_{n,s}(\zeta, .)$ is of class $\mathcal{C}^1$ in ω, which proves the regulatity of the operator T_s. □

3 Hölder estimates up to the boundary

3.1 A first description of the singularities of the kernel G_M

In this section we will describe the singularities of the kernel G_M in the case when the domain Ω is strictly pseudoconvex. We use the notation of the previous section. Let us recall that $G_M = \sum_{I\in\mathcal{I}'(k)} \operatorname{sgn}(I) C_{I\bullet *}$, with

$$C_{I\bullet *}(\zeta,z) = \int_{\lambda\in\Delta_{I\bullet *}} K_{I\bullet *}(\zeta,z,\lambda)$$

and

$$K_{I\bullet *}(\zeta,z,\lambda) = \frac{(-1)^{n(n-1)/2}}{(2i\pi)^n}\langle\psi_{I\bullet *},d\zeta\rangle\wedge\langle(\overline{\partial}_{\zeta,z}+d_\lambda)\psi_{I\bullet *},d\zeta\rangle^{n-1} \wedge d(\zeta_1-z_1)\wedge\cdots\wedge d(\zeta_n-z_n). \tag{3.1.1}$$

Let ρ_* be a strictly plurisubharmonic defining function for Ω. Let $F_*(\zeta,.)$ be the Levi polynomial of ρ_* at a point ζ in a neighborhood of $b\Omega$. It follows from the strict plurisubharmonicity of ρ_* that there exists a positive constant β such that

$$\mathrm{Re}F_*(\zeta,z)\geq\rho_*(\zeta)-\rho_*(z)+\beta|\zeta-z|^2 \tag{3.1.2}$$

for $(\zeta,z)\in b\Omega\times\Omega$.

We set

$$\begin{aligned} w_j^*(\zeta,z) &= 2\frac{\partial\rho_*}{\partial\zeta_j}(\zeta)-\sum_{k=1}^n a_{jk}^*(\zeta)(\zeta_k-z_k)\\ w^*(\zeta,z) &= (w_1^*(\zeta,z),\dots,w_n^*(\zeta,z))\\ \Phi_*(\zeta,z) &= \langle w^*(\zeta,z),\zeta-z\rangle, \end{aligned}$$

where the functions a_{jk}^*, $j,k=1,\dots,n$, are of class $\mathcal{C}^\infty$ on U and satisfy for all $\zeta\in U$

$$|a_{jk}^*(\zeta)-\frac{\partial^2\rho_*}{\partial\zeta_j\partial\zeta_k}(\zeta)|<\frac{\beta^*}{2n^2}.$$

We have

$$\mathrm{Re}\ \Phi_*(z,\zeta)\geq\rho_*(\zeta)-\rho_*(z)+\frac{\beta}{2}|\zeta-z|^2. \tag{3.1.3}$$

The map $\psi_*=\frac{w^*}{\Phi_*}$ defines a Leray map for the function ρ_*, which is holomorphic in the variable z.

If $\lambda=(\lambda_1,\dots,\lambda_k,\lambda_\bullet,\lambda_*)\in\Delta_{I\bullet *}$ is such that $\lambda_*\neq 1$, we denote by λ' the point in $\Delta_{I\bullet}$ defined by

$$\lambda'_{i_\nu}=\frac{\lambda_{i_\nu}}{1-\lambda_*}\quad(\nu=1,\dots,k,\bullet).$$

Let $\overset{\circ}{\chi}$ be a $\mathcal{C}^\infty$-function as in section 2.1, then we set for $\lambda\in\Delta_{I\bullet *}$

$$\psi_{I\bullet *}(\zeta,z,\lambda)=\overset{\circ}{\chi}(\lambda_*)\psi_*(\zeta,z)+(1-\overset{\circ}{\chi}(\lambda_*))\psi_{I\bullet}(\zeta,z,\lambda'), \tag{3.1.4}$$

We use the following notation

$$W = W(\zeta, z, \lambda') = \langle w_{I\bullet}(\zeta, z, \lambda'), d\zeta\rangle, \quad \Phi = \Phi_{I\bullet}(\zeta, z, \lambda')$$

and

$$N = N(\zeta, z) = \langle \psi_*(\zeta, z), d\zeta\rangle$$

for $\zeta \in b\omega$ and $z \in \overline{\omega}$ with $z \neq \zeta$ and $\lambda \in \Delta_{I\bullet*} \setminus \Delta_*$.

Let f be an (n,r)-form on $\overline{\omega}$, we set

$$f(\zeta) = \widetilde{f}(\zeta) d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

It follows from (3.1.4), that

$$\langle \psi_{I\bullet*}, d\zeta\rangle = \overset{\circ}{\chi} N + (1 - \overset{\circ}{\chi})\frac{W}{\Phi}$$

$$\langle(\overline{\partial}_{z,\zeta} + d_\lambda)\psi_{I\bullet*}, d\zeta\rangle = (\frac{W}{\Phi} - N) \wedge d\,\overset{\circ}{\chi} + \overset{\circ}{\chi}\,\overline{\partial}_{z,\zeta} N + (1 - \overset{\circ}{\chi})\frac{(\overline{\partial}_{z,\zeta} + d_\lambda)W}{\Phi} + (1 - \overset{\circ}{\chi})\frac{W}{\Phi^2}(\overline{\partial}_{z,\zeta} + d_\lambda)\Phi.$$

The kernels $C_{I\bullet*}$ are obtained after integration on $\Delta_{I\bullet*}$, though we have only to consider the part of bidegree $k+1$ in λ of the kernel $K_{I\bullet*}$. The differential forms $(\overline{\partial}_{z,\zeta} + d_\lambda)\Phi$ and $(\overline{\partial}_{z,\zeta} + d_\lambda)W$ are pullback of differential forms on $\overline{\omega} \times \overline{\omega} \times \Delta_{I\bullet}$ by the map $(z, \zeta, \lambda) \mapsto (z, \zeta, \overset{\circ}{\lambda})$; consequently since $\Delta_{I\bullet}$ is of real dimension k, for all $s = 1, 2, \ldots,$ we have $[((\overline{\partial}_{z,\zeta} + d_\lambda)W)^s]_{\deg\lambda=k+1} = 0$ and $[((\overline{\partial}_{z,\zeta} + d_\lambda)W)^s \wedge (\overline{\partial}_{z,\zeta} + d_\lambda)\Phi]_{\deg\lambda=k+1} = 0$, which implies

$$\begin{aligned}
&[\langle \psi_{I\bullet*}, d\zeta\rangle \wedge \langle(\overline{\partial}_{z,\zeta} + d_\lambda)\psi_{I\bullet*}, d\zeta\rangle^{n-1}]_{\deg\lambda=k+1}\\
&= (\overset{\circ}{\chi} N + (1 - \overset{\circ}{\chi})\frac{W}{\Phi}) \wedge (n-1)(\frac{W}{\Phi} - N) \wedge d\,\overset{\circ}{\chi}\\
&\wedge [(\overset{\circ}{\chi}\,\overline{\partial}_{z,\zeta} N + (1 - \overset{\circ}{\chi})\frac{(\overline{\partial}_{z,\zeta} + d_\lambda)W}{\Phi} + (1 - \overset{\circ}{\chi})\frac{W}{\Phi^2}(\overline{\partial}_{z,\zeta} + d_\lambda)\Phi)^{n-2}]_{\deg\lambda=k}.
\end{aligned}$$

Noting that $W \wedge W = 0$ and $N \wedge N = 0$, because W and N are 1-forms, we get

$$\begin{aligned}
[f(\zeta) \wedge K_{I\bullet*}(z, \zeta, \lambda)]_{\deg\lambda=k+1} &= a\widetilde{f}(\zeta)\frac{N \wedge W}{\Phi} \wedge d\,\overset{\circ}{\chi}\\
&\wedge (\overset{\circ}{\chi}\,\overline{\partial}_{z,\zeta} N + (1 - \overset{\circ}{\chi})\frac{\overline{\partial}_{z,\zeta} W}{\Phi})^{n-2-k} \wedge ((1 - \overset{\circ}{\chi})\frac{d_\lambda W}{\Phi})^k,
\end{aligned}$$

where a is a constant.

By the definition of differential forms of type O_s, we have $d\overset{\circ}{\chi}= O_0$, $O_0\wedge\overline{\partial}_{z,\zeta}W = O_0$, $O_0\wedge\overline{\partial}_{z,\zeta}\Phi_* = O_1$ and also

$$O_0\wedge W = \sum_{j\in I\bullet} O_0\wedge\partial\rho_j(\zeta) + O_1$$

$$O_0\wedge d_\lambda W = \sum_{j\in I} O_0\wedge\partial\rho_j(\zeta) + O_1$$

$$O_0\wedge W\wedge(d_\lambda W)^k = \sum_{\substack{0\leq m\leq k\\ i_1,\ldots,i_m\in I}} O_{k+1-m}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)$$

$$O_0\wedge N = \frac{O_0\wedge\partial\rho_*(\zeta)+O_1}{\Phi_*}$$

$$O_0\wedge\overline{\partial}_{z,\zeta}N = \frac{O_0}{\Phi_*} + \frac{O_1\wedge\partial\rho_*(\zeta)+O_2}{\Phi_*^2}$$

and consequently

$$[f(\zeta)\wedge K_{I\bullet*}(z,\zeta,\lambda)]_{\deg\lambda=\mathrm{k}+1} = \sum_{\substack{0\leq s\leq n-2-k\\ 0\leq m\leq k}} \frac{1}{\Phi_*^{n-1-k-s}\Phi^{k+s+1}}(O_0+\frac{O_1\wedge\partial\rho_*(\zeta)+O_2}{\Phi_*})^{n-2-k-s}$$

$$\wedge\,(O_{k+1-m}\wedge\partial\rho_*(\zeta)\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)+O_{k+2-m}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)).$$

Using that $|\Phi_*(\zeta,z)|\geq|\zeta-z|^2$, we get

$$[f(\zeta)\wedge K_{I\bullet*}(z,\zeta,\lambda)]_{\deg\lambda=\mathrm{k}+1} \leq \sum_{\substack{0\leq s\leq n-2-k\\ i_1,\ldots,i_m\in I*,\ 0\leq m\leq k+1}} \frac{O_{k+2-m}}{\Phi_*^{n-1-k-s}\Phi^{k+s+1}}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta). \tag{3.1.5}$$

It follows from section 6 and Lemma 7.4 in [12] that, after a partial integration in λ, we can control $f(\zeta)\wedge C_{I\bullet*}(z,\zeta)$ by a finite sum of terms of the form :

$$\frac{|\sigma\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)|}{\Phi_*(\zeta,z)\Pi_{\nu=1}^k|\Phi(\zeta,z,\lambda^\nu)||\zeta-z|^{2n-3(k+1)+m-1}}, \tag{3.1.6}$$

where $\lambda^1,\ldots,\lambda^k$ are points in $\Delta_{I\bullet}$, $I\in\mathcal{I}'(k)$, which define a system of independent vectors of $\mathbb{R}^{k+1}$, and $i_1,\ldots,i_m\in I*$.

Let σ be a monomial in $d\zeta_1,\ldots,d\zeta_n,d\overline{\zeta}_1,\ldots,d\overline{\zeta}_n$, $\lambda^1,\ldots,\lambda^k$ some points in $\Delta_{I\bullet}$, which are linearly independent as vectors in $\mathbb{R}^{k+1}$, $t_\nu=\mathrm{Im}\,\Phi(\zeta,z,\lambda^\nu)$ and $dt_\nu=d_\zeta\mathrm{Im}\,\Phi(\zeta,z,\lambda^\nu)$. By the definition of Φ, we have

$$dt_\nu(\zeta,z)=i(\overline{\partial}\rho_{\lambda^\nu}(\zeta)-\partial\rho_{\lambda^\nu}(\zeta))+O_1,$$

and consequently

$$\partial\rho_{\lambda^\nu}(\zeta)=\frac{1}{2}d\rho_{\lambda^\nu}(\zeta)+\frac{i}{2}dt_\nu(\zeta,z)+O_1.$$

As $d\rho_i|_M=0$ for $i=\pm1,\ldots,\pm k$, there exists some constant C and some monomials σ_L in $d\zeta_1,\ldots,d\zeta_n,d\overline{\zeta}_1,\ldots,d\overline{\zeta}_n$ such that for all $i_1,\ldots,i_m\in I$, $m\leq k$,

$$|(\sigma\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m})|_M|\leq C\sum_{|L|\leq m}|\sigma_L\wedge_{l\in L}dt_l|\;|\zeta-z|^{m-|L|}.$$

Set $t_{k+1} = \text{Im } \Phi_*(\zeta, z)$ and $dt_{k+1} = d_\zeta \text{Im } \Phi_*(\zeta, z)$. By the definition of Φ_*, we have

$$dt_{k+1}(\zeta, z) = i(\overline{\partial}\rho_*(\zeta) - \partial\rho_*(\zeta)) + O_1,$$

and consequently

$$\partial\rho_*(\zeta) = \frac{1}{2}d\rho_*(\zeta) + \frac{i}{2}dt_{k+1}(\zeta, z) + O_1.$$

As $d\rho_*|_{b\omega} = 0$, there exists some constant C_* and some monomials σ_L in $d\zeta_1, \dots, d\zeta_n$, $d\overline{\zeta}_1, \dots, d\overline{\zeta}_n$ such that for all $i_1, \dots, i_m \in I*$, $m \leq k+1$,

$$|(\sigma \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m})|_{b\omega}| \leq C_* \sum_{|L| \leq m} |\sigma_L \wedge_{l \in L} dt_l| \; |\zeta - z|^{m-|L|}.$$

We deduce that $|f(\zeta) \wedge G_M(\zeta, z)|$ is dominated by a finite sum of differential forms of the type :

$$\frac{|\sigma_s \wedge_{\nu=1}^s dt_\nu|}{\Pi_{\nu=1}^s(|t_\nu| + |\zeta - z|^2)|\zeta - z|^{2n-(k+1)-s-1}}, \tag{3.1.7}$$

where $1 \leq s \leq k+1$.

Let Σ denote the set of the characteristic points of $b\omega$, i.e., points where $\partial\rho_1 \wedge \partial\rho_2 \wedge \cdots \wedge \partial\rho_k \wedge \partial\rho_* = 0$ on $b\omega$.

Lemma 3.1.1. *For any continuous (n, r)-form f on $\overline{\omega}$ in $\mathbb{C}^n$, we have for all $z \in \overline{\omega} \setminus \Sigma$ and ε such that $\Sigma \cap \{\zeta \in b\omega \mid |\zeta - z| < \varepsilon\} = \emptyset$*

$$\int_{\substack{\zeta \in b\omega \\ |\zeta - z| < \varepsilon}} |f(\zeta) \wedge [G_M]_{n,r}(\zeta, z)| \leq C\varepsilon(1 + |\log\varepsilon|)^{k+1} \tag{3.1.8}$$

with a constant C, which does not depend on z.

Proof. If $\zeta \in b\omega \setminus \Sigma$,

$$\begin{aligned} &d_\zeta \text{Im}\Phi(\zeta, z, \lambda^{\nu_1}) \wedge \dots \wedge \text{Im}\Phi(\zeta, z, \lambda^{\nu_k}) \wedge d_\zeta \text{Im}\Phi_*(\zeta, z)|_{\zeta = z} \\ &\quad = i^{k+1}\partial_\zeta \rho_{\lambda^{\nu_1}}(\zeta) \wedge \dots \wedge \partial_\zeta \rho_{\lambda^{\nu_k}}(\zeta) \wedge \partial_\zeta \rho_*(\zeta) \\ &\quad \neq 0. \end{aligned}$$

We can choose coordinates on $\{\zeta \in b\omega \mid |\zeta - z| < \varepsilon\}$ such that $t_i = \text{Im}\Phi(\zeta, z, \lambda^{\nu_i})$, $i = 1, ..., k$, and $t_{k+1} = \text{Im}\Phi_*$.

Then the assertion follows from the estimate (3.1.7) of the singularity at $\zeta = z$ of the differential form $f(\zeta) \wedge [G_M]_{n,r}(\zeta, z)$. □

3.2 Hölder estimates up to the boundary

We are now ready to prove some regularity up to the boundary for the integral operator $T_r f = (-1)^{(n+r)(k+1)+\frac{k(k-1)}{2}}[\int_{\zeta\in\omega} f(\zeta) \wedge [R_M]_{n,r}(\zeta, .) + (-1)^{n+r+1} \int_{\zeta \in b\omega} f(\zeta) \wedge [G_M]_{n,r}(\zeta, .)]$.

Let f be a continuous $(n, r+1)$-form on $\overline{\omega}$. Let us notice that by (2.2.11) and (3.1.5) the integrals $\int_{\zeta\in\omega} f(\zeta) \wedge [R_M]_{n,r}(\zeta, z)$ and $\int_{\zeta \in b\omega} f(\zeta) \wedge [G_M]_{n,r}(\zeta, z)$ are of the same type.

Since ω and $b\omega$ are respectively of dimension $2n-k$ and $2n-(k+1)$, and I and $I*$ respectively of length k and $k+1$, we can deduce the regularity of one of the integrals from the other by exchanging k and $k+1$. As $b\omega$ may have characteristic points we will study $\int_{\zeta\in b\omega} f(\zeta)\wedge[G_M]_{n,r}(\zeta,z)$. As before we will denote by Σ the set of characteristic points in $b\omega$.

Lemma 3.2.1. *Let $I\in\mathcal{I}'(k)$, a and b two integers such that $a+b=n+\alpha$ with $\alpha=0$ or 1, $\beta\in\mathbb{Z}$ and $\varepsilon>0$, then*

$$J_{\alpha,\beta}=\int_{\substack{\zeta\in b\omega\\ \varepsilon\leq|\zeta-z|\leq C}}\int_{\lambda\in\Delta_{I\bullet*}}\frac{O_{k+2-m+\beta}}{\Phi_*^a\Phi^b}\partial\rho_{i_1}\wedge\cdots\wedge\partial\rho_{i_m}\leq C_1(\varepsilon^{\beta+1-2\alpha}+C_2)(1+|\log\varepsilon|)^{k+1}$$

for all $i_1,\ldots,i_m\in I*$, $0\leq m\leq k+1$.

Proof. Outside Σ, we can choose $t_i=\mathrm{Im}\Phi(\zeta,z,\lambda^{\nu_i})$, $i=1,...,k$, and $t_{k+1}=\mathrm{Im}\Phi_*$ as coordinates. This is not possible nearby the characteristic points. However, following a device used in Range-Siu [17], one can replace these functions by second-order polynomial approximation.

It follows from [12] that, after integration in λ, $J_{\alpha,\beta}$ is bounded by some integrals of the type :

$$\begin{aligned}&\sum_{1\leq s\leq k+1}\int_{\substack{X\in\mathbb{R}^{2n-(k+1)}\\ \varepsilon\leq|X|\leq C}}\frac{dX}{\Pi_{\nu=1}^s(|X_\nu|+|X|^2)\ |X|^{2n-(k+1)-s-1+2\alpha-\beta}}\\
&\leq\sum_{1\leq s\leq k+1}\int_{\substack{X\in\mathbb{R}^{2n-(k+1)}\\ \varepsilon\leq|X'|\leq C,\ |X|\leq C}}\frac{dX}{\Pi_{\nu=1}^s(|X_\nu|+|X'|^2)\ |X'|^{2n-(k+1)-s-1+2\alpha-\beta}}\\
&+\leq\sum_{1\leq s\leq k+1}\frac{1}{\varepsilon^\mu}\int_{\substack{X\in\mathbb{R}^{2n-(k+1)}\\ |X'|\leq\varepsilon}}\frac{dX}{\Pi_{\nu=1}^s(|X_\nu|+|X'|^2)\ |X'|^{2n-(k+1)-s-1+2\alpha-\beta-\mu}},\end{aligned}$$

with $X=(X_1,\ldots,X_s,X')$ and μ such that $\frac{1}{\Pi_{\nu=1}^s(|X_\nu|+|X'|^2)\ |X'|^{2n-(k+1)-s-1+2\alpha-\beta-\mu}}$ is integrable at zero.

Then we get

$$\begin{aligned}&\int_{\substack{X\in\mathbb{R}^{2n-(k+1)}\\ \varepsilon\leq|X'|\leq C,\ |X|\leq C}}\frac{dX}{\Pi_{\nu=1}^s(|X_\nu|+|X'|^2)\ |X'|^{2n-(k+1)-s-1+2\alpha-\beta}}\\
&\leq\int_\varepsilon^C\frac{(\log(C+r^2)-\log r^2)^s dr}{r^{2\alpha-\beta}}\\
&\leq C'(\varepsilon^{\beta+1-2\alpha}-C^{\beta+1-2\alpha})(\log(C+C^2)-\log\varepsilon^2)^s\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\varepsilon^\mu}\int_{\substack{X\in\mathbb{R}^{2n-(k+1)}\\ |X'|\leq\varepsilon}}\frac{dX}{\Pi_{\nu=1}^s(|X_\nu|+|X'|^2)\ |X'|^{2n-(k+1)-s-1+2\alpha-\beta-\mu}}\\
&\leq\frac{1}{\varepsilon^\mu}\int_0^\varepsilon\frac{(\log(C+r^2)-\log r^2)^s dr}{r^{2\alpha-\beta-\mu}}\\
&\leq C'\varepsilon^{\beta+1-2\alpha}(1+|\log\varepsilon|)^s.\end{aligned}$$

□

Theorem 3.2.2. *The integral operators T_r, $1 \le r \le n-k$, are continuous operators from $\mathcal{C}_{n,r}(\overline{\omega})$ into $\mathcal{C}^{\frac{1}{2}-\varepsilon}_{n,r-1}(\omega)$.*

Proof. Let z_1 and z_2 be two points in $\overline{\omega}$. We have

$$\widetilde{G}_M f(z_1) - \widetilde{G}_M f(z_2) = \int_{\zeta\in\Omega} f(\zeta) \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta)).$$

and consequently

$$\begin{aligned}|\widetilde{G}_M f(z_1) - \widetilde{G}_M f(z_2)| \le &\int_{\substack{\zeta\in\omega \\ |\zeta-z_1|\le 2|z_1-z_2|^{\frac{1}{2}}}} |f(\zeta) \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))| \\ &+ \int_{\substack{\zeta\in\omega \\ |\zeta-z_1\ge 2|z_1-z_2|^{\frac{1}{2}}}} |f(\zeta) \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))|.\end{aligned}$$

As G_M is a linear operator, we may assume that f is of the form $f = \widetilde{f}\sigma$, with $\widetilde{f}$ a continuous function and σ a monomial in $d\zeta_1, \dots, d\zeta_n, d\overline{\zeta}_1, \dots, d\overline{\zeta}_n$. Then we get

$$\begin{aligned}|\widetilde{G}_M f(z_1) - \widetilde{G}_M f(z_2)| \le \|f\|_\infty &\int_{\substack{\zeta\in\omega \\ |\zeta-z_1|\le 2|z_1-z_2|^{\frac{1}{2}}}} |\sigma \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))| \\ + \|f\|_\infty &\int_{\substack{\zeta\in\omega \\ |\zeta-z_1|\ge 2|z_1-z_2|^{\frac{1}{2}}}} |\sigma \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))|.\end{aligned}$$

Thus we have to estimate the integrals

$$\begin{aligned}J_1 &= \int_{\substack{\zeta\in\omega \\ |\zeta-z_1|\le 2|z_1-z_2|^{\frac{1}{2}}}} |\sigma \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))| \\ J_2 &= \int_{\substack{\zeta\in\omega \\ |\zeta-z_1|\ge 2|z_1-z_2|^{\frac{1}{2}}}} |\sigma \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))|.\end{aligned}$$

Without loss of generality we may assume that $|z_1 - z_2| \le 1$. Note that

$$J_1 \le \int_{\substack{\zeta\in\omega \\ |\zeta-z_1|\le 2|z_1-z_2|^{\frac{1}{2}}}} |\sigma \wedge G_M(z_1,\zeta)| + \int_{\substack{\zeta\in\omega \\ |\zeta-z_2|\le 3|z_1-z_2|^{\frac{1}{2}}}} |\sigma \wedge G_M(z_2,\zeta)|.$$

It follows from Lemma 3.1.1 that, away from the characteristic points of $b\omega$, we have

$$J_1 \le C|z_1 - z_2|(1 + \log|z_1 - z_2|)^{k+1}.$$

Near the characteristic points, one again use the Range-Siu's trick to prove the estimates.

We deduce from the definition of J_2 and from (3.1.5) that

$$\begin{aligned}J_2 = \sum_{I\in\mathcal{I}'(k)} \int_{\substack{(\zeta,\lambda)\in\omega\times\Delta_{I\bullet} \\ |\zeta-z_1|\ge 2|z_1-z_2|^{\frac{1}{2}}}} &\Big|\frac{A(z_1,\zeta,\lambda)}{\Phi_*^{n-1-k-s}(z_1,\zeta,\lambda)\Phi^{k+s+1}(z_1,\zeta,\lambda)} \\ &- \frac{A(z_2,\zeta,\lambda)}{\Phi_*^{n-1-k-s}(z_2,\zeta,\lambda)\Phi^{k+s+1}(z_2,\zeta,\lambda)}\Big| |\sigma \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta)|,\end{aligned}$$

where $A(z,\zeta,\lambda)$ is a smooth function in z, which is O_{k+2-m}, $i_1,\dots,i_m \in I*$, $0 \le m \le k+1$ and $0 \le s \le n-2-k$. We may write

$$\begin{aligned}&\frac{A(z_1,\zeta,\lambda)}{\Phi_*^{n-1-k-s}(z_1,\zeta,\lambda)\Phi^{k+s+1}(z_1,\zeta,\lambda)} - \frac{A(z_2,\zeta,\lambda)}{\Phi_*^{n-1-k-s}(z_2,\zeta,\lambda)\Phi^{k+s+1}(z_2,\zeta,\lambda)}\\ &= \frac{A(z_1,\zeta,\lambda)-A(z_2,\zeta,\lambda)}{\Phi_*^{n-1-k-s}(z_1,\zeta,\lambda)\Phi^{k+s+1}(z_1,\zeta,\lambda)}\\ &+ A(z_2,\zeta,\lambda)[\frac{1}{\Phi_*^{n-1-k-s}(z_1,\zeta,\lambda)\Phi^{k+s+1}(z_1,\zeta,\lambda)} - \frac{1}{\Phi_*^{n-1-k-s}(z_2,\zeta,\lambda)\Phi^{k+s+1}(z_2,\zeta,\lambda)}].\end{aligned}$$

Using Lemma 3.2.1 with $\alpha = 0$, $\beta = -1$ and $\varepsilon = 2|z_1 - z_2|^{\frac{1}{2}}$, we get

$$\begin{aligned}J_2' &= \sum_{I\in\mathcal{I}'(k)} \int_{\substack{(\zeta,\lambda)\in\omega\times\Delta_{I*\bullet}\\ |\zeta-z_1|\ge 2|z_1-z_2|^{\frac{1}{2}}}} |\frac{A(z_1,\zeta,\lambda)-A(z_2,\zeta,\lambda)}{\Phi_*^{n-1-k-s}(z_1,\zeta,\lambda)\Phi^{k+s+1}(z_1,\zeta,\lambda)}|\\ &\le C|z_1-z_2|^{\frac{1}{2}}(1+|\log|z_1-z_2||)^{k+1},\end{aligned}$$

since $|A(z_1,\zeta,\lambda) - A(z_2,\zeta,\lambda)| \le |z_1 - z_2| O_{k+1-m}$.

The function $\Phi(z,\zeta,\lambda)$ and $\Phi_*(z,\zeta,\lambda)$ are of class $\mathcal{C}^\infty$ in z and consequently

$$|\Phi(z_1,\zeta,\lambda) - \Phi(z_2,\zeta,\lambda)| \le c|z_1 - z_2|,$$

moreover noting that if $|\zeta - z_1| \ge 2|z_1 - z_2|^{\frac{1}{2}}$, then $\frac{1}{2} \le \frac{|\zeta-z_1|}{|\zeta-z_2|} \le 2$, we get

$$\begin{aligned}&\frac{1}{\Phi_*^{n-1-k-s}(z_1,\zeta,\lambda)\Phi^{k+s+1}(z_1,\zeta,\lambda)} - \frac{1}{\Phi_*^{n-1-k-s}(z_2,\zeta,\lambda)\Phi^{k+s+1}(z_2,\zeta,\lambda)}\\ &\le C \sum_{a+b=n+1} \frac{|z_1-z_2|}{\Phi_*^a(z_2,\zeta,\lambda)\Phi^b(z_2,\zeta,\lambda)}.\end{aligned}$$

Using Lemma 3.2.1 with $\alpha = 1$, $\beta = 0$ and $\varepsilon = 2|z_1 - z_2|^{\frac{1}{2}}$, after integration in λ, we have

$$\begin{aligned}J"_2 &= \sum_{I\in\mathcal{I}'(k)} \int_{\substack{(\zeta,\lambda)\in\omega\times\Delta_{I\bullet}\\ |\zeta-z_1|\ge 2|z_1-z_2|^{\frac{1}{2}}}} A(z_2,\zeta,\lambda)[\frac{1}{\Phi_*^{n-1-k-s}(z_1,\zeta,\lambda)\Phi^{k+s+1}(z_1,\zeta,\lambda)}\\ &- \frac{1}{\Phi_*^{n-1-k-s}(z_2,\zeta,\lambda)\Phi^{k+s+1}(z_2,\zeta,\lambda)}]\\ &\le C|z_1-z_2||z_1-z_2|^{-\frac{1}{2}}(1+|\log|z_1-z_2||)^{k+1}\\ &\le C|z_1-z_2|^{\frac{1}{2}}(1+|\log|z_1-z_2||)^{k+1}.\end{aligned}$$

It follows then $J_2 \le C|z_1 - z_2|^{\frac{1}{2}}(1 + |\log|z_1 - z_2||)^{k+1}$, which finishes the proof of the theorem. □

Proposition 3.2.3. *Let f be a continuous $(n, r+1)$-form on $\overline{\omega}$ and $\gamma \subset b\omega$ a complex tangent curve in $b\omega$, then $\int_{\zeta\in b\omega} f(\zeta) \wedge [G_M]_{n,r}(\zeta,z)|_\gamma$ defines a form of class $\mathcal{C}^{1-\varepsilon}$, $0 < \varepsilon < 1$, on γ.*

Proof. The proof is analogous to the proof of Theorem 3.2.2. We will cut the integrals in the following way

$$|\widetilde{G}_M f(z_1) - \widetilde{G}_M f(z_2)| \leq \int_{\substack{\zeta\in\Omega \\ |\zeta - z_1 \leq 2|z_1 - z_2|}} |f(\zeta) \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))| + \int_{\substack{\zeta\in\Omega \\ |\zeta - z_1 \geq 2|z_1 - z_2|}} |f(\zeta) \wedge (G_M(z_1,\zeta) - G_M(z_2,\zeta))|.$$

To estimate the first part we use Lemma 3.1.1 with $\varepsilon = 2|z_1 - z_2|$. To study the second part we notice that the function $\Phi(z,\zeta,\lambda)$ and $\Phi_*(z,\zeta,\lambda)$ are of class $\mathcal{C}^\infty$ in z and moreover their gradient vanishes to order 1 in $z = \zeta$ along the complex tangent curve γ; consequently

$$|\Phi(z_1,\zeta,\lambda) - \Phi(z_2,\zeta,\lambda)| \leq |z_1 - z_2| O_1$$
$$|\Phi_*(z_1,\zeta,\lambda) - \Phi_*(z_2,\zeta,\lambda)| \leq |z_1 - z_2| O_1.$$

Then using Lemma 3.2.1 with $\alpha = 1$, $\beta = 1$ and $\varepsilon = 2|z_1 - z_2|$ we get the estimate of the second part. □

4 L^p estimates of the solution

4.1 A new solution kernel

In this section we assume, as in the previous one, that ω is the intersection of M with a strictly pseudoconvex domain Ω with C^3 boundary. Let ρ_* be a C^3 strictly plurisubharmonic defining function for Ω such that the Hessian of ρ_* is positive definite on $\overline{\omega}$.

For any $f \in C_{(n,s)}(\overline{\omega})$, we let

$$I_1 f = \int_{\zeta\in\omega} f(\zeta) \wedge [R_M]_{n,s-1}(\zeta, .)$$

and

$$I_2 f = (-1)^{n+s} \int_{\zeta\in b\omega} f(\zeta) \wedge [G_M]_{n,s-1}(\zeta, .)$$

be the operators constructed in Section 2.3. In order to facilitate the estimates, we shall derive another solution operator for $\bar\partial_b$. The integral I_1 has integrable kernel and can be estimated easily. We shall rewrite $I_2 f$ as an integral on ω to facilitate the L^p estimates. To do this, it is necessary to modify the kernel $[G_M]_{n,s-1}$ so that Stokes' theorem can be applied.

As in section 3.1, we associate to ρ_* a Leray map $\psi_* = \frac{w^*}{\Phi_*}$, where the support function Φ_* satisfies

$$\text{Re } \Phi_*(\zeta,z) \geq \rho_*(\zeta) - \rho_*(z) + \frac{\beta}{2}|\zeta - z|^2. \tag{4.1.1}$$

We define a new support function $\widetilde{\Phi}_*$ for ρ_*, by setting

$$\widetilde{\Phi}_*(\zeta,z) = \Phi_*(\zeta,z) - 2\rho_*(\zeta)$$

It follows from (4.1.1) that

$$\text{Re } \widetilde{\Phi}_*(\zeta, z) \geq -\rho_*(\zeta) - \rho_*(z) + \frac{\beta}{2}|\zeta - z|^2. \tag{4.1.2}$$

for all $\zeta, z \in \overline{\omega}$. Thus Re $\widetilde{\Phi}_*(\zeta, z)$ vanishes only when $\zeta = z$ and $\zeta,\ z$ are both in $b\omega$. Also we have

$$\widetilde{\Phi}_*(\zeta, z) = \Phi_*(\zeta, z), \quad \text{when } \zeta \in b\omega \text{ and } z \in \omega.$$

Define the new kernel $[\widetilde{G}_M]_{n,s-1}(\zeta, z)$ by modifying $[G_M]_{n,s-1}(\zeta, z)$ with $\widetilde{\Phi}_*$ substituting for Φ_*. More explicitly, a typical term in $[\widetilde{G}_M]_{n,s-1}(\zeta, z)$ is of the following form

$$\int_{\lambda \in \Delta_{I \bullet *}} \sum_{\substack{0 \leq s \leq n-2-k \\ 0 \leq m \leq k}} \frac{1}{\widetilde{\Phi}_*^{n-1-k-s} \Phi^{k+s+1}} (O_0 + \frac{O_1 \wedge \partial\rho_*(\zeta) + O_2}{\widetilde{\Phi}_*})^{n-2-k-s}$$
$$\wedge (O_{k+1-m} \wedge \partial\rho_*(\zeta) \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta) + O_{k+2-m} \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}(\zeta)). \tag{4.1.3}$$

where $1 \leq s \leq k$.

Since

$$[\widetilde{G}_M]_{n,s-1}(\zeta, z) = [G_M]_{n,s-1}(\zeta, z), \quad \text{when } \zeta \in b\omega \text{ and } z \in \omega,$$

we shall substitute $[\widetilde{G}_M]_{n,s-1}(\zeta, z)$ in $I_2 f$ for $[G_M]_{n,s-1}(\zeta, z)$. The advantage is that $[\widetilde{G}_M]_{n,s-1}(\zeta, z)$ and its first derivatives are integrable for each fixed $z \in \omega$ since $\widetilde{\Phi}_*$ satisfies (4.1.2). Thus for any $z \in \omega$, by Stokes' theorem and a limiting argument (substituting $\Phi^\epsilon = \Phi + \epsilon$ for Φ and letting $\epsilon \searrow 0$), we can write, for any $f \in C_{(n,s)}(\overline{\omega})$ such that $\bar{\partial}_b f = 0$ on ω,

$$\begin{aligned} I_2 f(z) = (-1)^{n+s} \int_{\zeta \in b\omega} f(\zeta) \wedge [\widetilde{G}_M]_{n,s-1}(\zeta, z) &= (-1)^{n+s} \int_{\zeta \in \omega} \overline{\partial}_\zeta (f(\zeta) \wedge [\widetilde{G}_M]_{n,s-1}(\zeta, z) \\ &= \int_{\zeta \in \omega} f(\zeta) \wedge \overline{\partial}_\zeta [\widetilde{G}_M]_{n,s-1}(\zeta, z). \end{aligned}$$

Define

$$\widetilde{T}_{s-1} f = (-1)^{(n+s)(k+1) + \frac{k(k-1)}{2}} [\int_{\zeta \in \omega} f(\zeta) \wedge [R_M]_{n,s-1}(\zeta, .) + \int_{\zeta \in \omega} f(\zeta) \wedge \overline{\partial}_\zeta [\widetilde{G}_M]_{n,s-1}(\zeta, .)]. \tag{4.1.4}$$

We have derived the following:

Proposition 4.1.1. (Second solution operator for $\bar{\partial}_b$.) *Let M be a strictly q-concave manifold in $\mathbb{C}^n$ and $0 \in M$. Let Ω be a sufficiently small strictly pseudoconvex domain containing 0 in $\mathbb{C}^n$ with C^2 boundary and $\omega = M \cap \Omega$. For any $f \in C_{(n,s)}(\overline{\omega})$, where $n - k - q + 1 \leq s \leq n - k$ such that $\bar{\partial}_b f = 0$ on ω, we have*

$$\begin{aligned} f(z) &= (-1)^{(n+s)(k+1) + \frac{k(k-1)}{2}} \overline{\partial}_b [\textstyle\int_{\zeta \in \omega} f(\zeta) \wedge [R_M]_{n,s-1}(\zeta, .) + \int_{\zeta \in \omega} f(\zeta) \wedge \overline{\partial}_\zeta [\widetilde{G}_M]_{n,s-1}(\zeta, z)] \\ &= \bar{\partial}_b \widetilde{T}_{s-1} f. \end{aligned} \tag{4.1.5}$$

Moreover $\widetilde{T}_{s-1} f$ is continuous on ω.

Notice that $\bar{\partial}_\zeta\Phi = O(|\zeta - z|)$ and $\bar{\partial}_\zeta\widetilde{\Phi}_* = O(|\zeta - z|) - \bar{\partial}_\zeta\rho_*$. From (4.1.3), it follows that $\overline{\partial}_\zeta[\widetilde{G}_M]_{n,s-1}(\zeta, z)$ can be bounded by sums of terms of the form

$$\int_{\lambda\in\Delta_{I\bullet*}} \sum_{\substack{0\leq s\leq n-2-k\\0\leq m\leq k}} \frac{1}{\widetilde{\Phi}_*^{n-1-k-s}\Phi^{k+s+1}}(O_0 + \frac{O_1\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*}) (O_0 + \frac{O_1\wedge\partial\rho_*(\zeta)}{\widetilde{\Phi}_*} + \frac{O_1\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*} + \frac{O_0\wedge\partial\rho_*(\zeta)\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*}) \wedge (O_{k-m}\wedge\partial\rho_*(\zeta)\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta) + O_{k+1-m}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)). \tag{4.1.6}$$

or

$$\int_{\lambda\in\Delta_{I\bullet*}} \sum_{\substack{0\leq s\leq n-3-k\\0\leq m\leq k}} \frac{1}{\widetilde{\Phi}_*^{n-2-k-s}\Phi^{k+s+1}}(\frac{O_0}{\widetilde{\Phi}_*} + \frac{O_1}{\widetilde{\Phi}_*^2} + \frac{O_0}{\Phi} + \frac{O_1}{\Phi^2} + \frac{O_0\wedge\partial\rho_*(\zeta)}{\widetilde{\Phi}_*^2} + \frac{O_0\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*^2} + \frac{O_{-1}\wedge\partial\rho_*(\zeta)\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*^2})(O_0 + \frac{O_1\wedge\partial\rho_*(\zeta)}{\widetilde{\Phi}_*} + \frac{O_1\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*} + \frac{O_0\wedge\partial\rho_*(\zeta)\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*}) \wedge (O_{k+1-m}\wedge\partial\rho_*(\zeta)\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta) + O_{k+2-m}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)). \tag{4.1.7}$$

or

$$\int_{\lambda\in\Delta_{I\bullet*}} \sum_{\substack{0\leq s\leq n-2-k\\0\leq m\leq k}} \frac{1}{\widetilde{\Phi}_*^{n-1-k-s}\Phi^{k+s+1}} (O_0 + \frac{O_1\wedge\partial\rho_*(\zeta)}{\widetilde{\Phi}_*} + \frac{O_1\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*} + \frac{O_0\wedge\partial\rho_*(\zeta)\wedge\overline{\partial}\rho_*(\zeta)}{\widetilde{\Phi}_*}) \wedge (O_{k-m}\wedge\partial\rho_*(\zeta)\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta) + O_{k+1-m}\wedge\partial\rho_{i_1}(\zeta)\wedge\cdots\wedge\partial\rho_{i_m}(\zeta)). \tag{4.1.8}$$

where $1 \leq s \leq k$.

Setting $\alpha = 1$ or 2 and using that $|\widetilde{\Phi}_*(\zeta, z)| \geq |\zeta - z|^2$, we can control $\overline{\partial}_\zeta[\widetilde{G}_M]_{n,s-1}(\zeta, z)$ by finite sums of terms of the following type

$$\int_{\lambda\in\Delta_{I\bullet*}} \frac{O_{k+1-m}}{\widetilde{\Phi}_*^{n-\alpha-k-s}\Phi^{k+s+\alpha}} \wedge \Theta_m,$$

where Θ_m is a monomial of length m in $\partial\rho_{i_1}, \ldots, \partial\rho_{i_m}, \partial\rho_*(\zeta), \overline{\partial}\rho_*(\zeta)$, $1 \leq m \leq k+2$.

Let σ be a monomial in $d\zeta_1, \ldots, d\zeta_n, d\overline{\zeta}_1, \ldots, d\overline{\zeta}_n$, $\lambda^1, \ldots, \lambda^k$ some points in $\Delta_{I\bullet}$, which are linearly independent as vectors in $\mathbb{R}^{k+1}$. After integration in λ, $|\sigma\wedge\overline{\partial}_\zeta G_M|$ is dominated by

$$J = \frac{|\sigma\wedge\Theta_m|}{|\widetilde{\Phi}_*|\Pi_{\nu=1}^k|\Phi(z,\zeta,\lambda^\nu)||\zeta - z|^{2n-3k+m-3}}, \tag{4.1.9}$$

with $1 \leq m \leq k+2$.

4.2 L^p estimates for the kernel

Next we shall estimate $\widetilde{T}_{s-1}f$ in L^p spaces. To estimate the solution kernel, it suffices to estimate the kernels $K(\zeta, z)$ and $J(\zeta, z)$ defined in (2.2.12) and (4.1.9) respectively.

Since all the kernels only have singularities when $\zeta = z$, we shall estimate the kernels when $U = \{|\zeta - z| < \epsilon\}$ are sufficiently small. To estimate $K(\zeta, z)$, we use the following change of coordinates $\zeta \to t$ such that $t_\nu = \mathrm{Im}\Phi(z, \zeta, \lambda^\nu)$, $\nu = 1, ..., k$, $t = (t_1, ..., t_k, t')$. This is possible since $d_\zeta \Phi(\zeta, z, \lambda^\nu)|_{\zeta=z} = \partial\rho_{\lambda^\nu}(\zeta)$ and $\partial\rho_{\lambda^\nu} = -\overline{\partial}\rho_{\lambda^\nu}$ on ω, it follows that $\partial\rho_{\lambda^\nu}(\zeta) = \frac{1}{2}(\partial\rho_{\lambda^\nu} - \overline{\partial}\rho_{\lambda^\nu}) = id_\zeta \mathrm{Im}\Phi(\zeta, z, \lambda^\nu)|_{\zeta=z}$. Thus,

$$\partial\rho_{\lambda^\nu}(\zeta) = id_\zeta \mathrm{Im}\Phi(\zeta, z, \lambda^\nu) + O(|\zeta - z|).$$

Thus, since M is generic, if $\lambda^1, \ldots, \lambda^k$ are independent vectors in $\mathbb{R}^{k+1}$ and $\zeta \in \omega$,

$$\begin{aligned} & d_\zeta \mathrm{Im}\Phi(\zeta, z, \lambda^1)) \wedge ... \wedge \mathrm{Im}\Phi(\zeta, z, \lambda^k)|_{\zeta=z} \\ & \quad = i^k \partial_\zeta \rho_{\lambda^1}(\zeta) \wedge ... \wedge \partial_\zeta \rho_{\lambda^k}(\zeta) \\ & \quad \neq 0. \end{aligned}$$

We get then

$$|\sigma \wedge \partial\rho_{i_1}(\zeta) \wedge \cdots \wedge \partial\rho_{i_m}| \leq C \sum_{0 \leq |L| \leq m} |\sigma_L \wedge_{l \in L} dt_l| \; |\zeta - z|^{m - |L|},$$

where $L = (l_1, \ldots, l_{|L|})$ is a multi-index of length $|L| \leq k$ contained in $(\nu_1, \ldots, \nu_k)$ and C some constant.

Using these coordinates for $K(\zeta, z)$, it suffices to show that the functions

$$K_0(t) = \frac{1}{|t|^{2n-k-1}}, \; |t| < 1$$

and, for $1 \leq s \leq k$,

$$K_s(t) = \frac{1}{\Pi_{i=1}^s (|t_i| + |t|^2)|t|^{2n-k-s-1}}, \; |t| < 1$$

are of weak type $\frac{2n-(k-s)}{2n-(k-s)-1}$, where $t = (t_1, ..., t_k, ..., t_{2n-k})$.

To estimates J, we note that the kernel is more singular at the boundary point. Thus we assume that $\zeta \in b\omega$ and omit the others.

Let Σ denote the set of the characteristic points, i.e., points where $\partial\rho_1 \wedge \partial\rho_2 ... \wedge \partial\rho_k \wedge$ $\wedge \partial\rho_* = 0$ on $b\omega$. We first assume that ζ is not a characteristic point. We may assume that $U \cap \Sigma = \emptyset$ for sufficiently small U. We choose special coordinates for $\omega \cap U$.

Let $t_\nu = \mathrm{Im}\Phi(\zeta, z, \lambda^\nu)$, $\nu = 1, ..., k+1$ as before and $(\nu_1, \ldots, \nu_k)$ be a multi-index contained in $(1, \ldots, k+1)$. We set $t_* = \mathrm{Im}\Phi_* = \mathrm{Im}\widetilde{\Phi}_*$. For $\zeta \in b\omega$, we have

$$\partial_\zeta \rho_* = id_\zeta \mathrm{Im}\widetilde{\Phi}_*(\zeta, z) + O(|\zeta - z|).$$

Thus, if $\zeta \in b\omega \setminus \Sigma$,

$$\begin{aligned} & d_\zeta \mathrm{Im}\Phi(\zeta, z, \lambda^{\nu_1}) \wedge ... \wedge \mathrm{Im}\Phi(\zeta, z, \lambda^{\nu_k}) \wedge d_\zeta \mathrm{Im}\widetilde{\Phi}_*(\zeta, z)|_{\zeta=z} \\ & \quad = i^{k+1} \partial_\zeta \rho_{\lambda^{\nu_1}}(\zeta) \wedge ... \wedge \partial_\zeta \rho_{\lambda^{\nu_k}}(\zeta) \wedge \partial_\zeta \rho_*(\zeta) \\ & \quad \neq 0. \end{aligned}$$

Also, using $d\rho_1 \wedge ... \wedge d\rho_k \wedge dr \neq 0$ on $b\omega$, we can choose $\rho_*(\zeta)$ as a coordinate function near $b\omega$ in $U \cap \omega$ such that $t_i = \mathrm{Im}\Phi(\zeta, z, \lambda^{\nu_i})$, $i = 1, ..., k$, $t_{k+1} = \mathrm{Im}\Phi_* = \mathrm{Im}\widetilde{\Phi}_*$ and $t_{k+2} = \rho_*(\zeta)$. Under these coordinates, the kernel $J(\zeta, z)$ is bounded by

$$K_s(t) = \frac{1}{\Pi_{i=1}^s(|t_i| + |t|^2)|t|^{2n-k-s-1}}, \ 1 \leq s \leq k+2.$$

Lemma 4.2.1. *For $0 < A < \infty$, we have*

$$\int_{|t|<A} K_s(t) \ dt_1...dt_{2n-k} = \int_{|t|<A} \frac{dt_1...dt_{2n-k}}{\Pi_{i=1}^s(|t_i| + |t|^2)|t|^{2n-k-s-1}} < \infty$$

where $0 \leq s \leq k+2$. Moreover

$$\int_{|t|<A} [K_s(t)]^p \ dt_1...dt_{2n-k} < \infty,$$

for $p < \frac{2n-(k-s)}{2n-(k-s)-1}$, and the function $K_s(t)$, $|t| < A$, is of weak type $\frac{2n-(k-s)}{2n-(k-s)-1}$.

Proof. The first assertion can be verified easily by integrating over t_i variables for $i = 1, ..., s$. Let $t' = (t_{s+1}, ..., t_{2n-k})$. We obtain

$$\begin{aligned} &\int_{|t|<A} \frac{dt_1...dt_{2n-k}}{\Pi_{i=1}^s(|t_i| + |t|^2)|t|^{2n-k-s-1}} \\ &\leq \int_{|t'|<A} \frac{(\log|t'|)^s dt_{s+1}...dt_{2n-k}}{|t'|^{2n-k-s-1}} \\ &< \infty. \end{aligned}$$

In the same way, we get for $1 < p < \frac{2n-(k-s)}{2n-(k-s)-1}$,

$$\begin{aligned} \int_{|t|<A} [K_s(t)]^p \ dt_1...dt_{2n-k} &= \int_{|t|<A} \frac{dt_1...dt_{2n-k}}{\Pi_{i=1}^s(|t_i| + |t|^2)^p|t|^{(2n-k-s-1)p}} \\ &\leq \int_{|t'|<A} \frac{dt_{s+1}...dt_{2n-k}}{|t'|^{2s(p-1)}|t'|^{(2n-k-s-1)p}} \\ &< \infty. \end{aligned}$$

Let S_λ^s be the subset

$$S_\lambda^s = \{t \in \mathbb{R}^{2n-k}, \ |t| < A, \ | \ K_s(t) > \lambda\}, \qquad \lambda > 0,$$

and let m be the Lebesgue measure in $\mathbb{R}^{2n-k}$. We shall show that there exists a constant $\tilde{c} > 0$ such that

$$m(S_\lambda^s) \leq \left(\frac{\tilde{c}}{\lambda}\right)^{\frac{2n-(k-s)}{2n-(k-s)-1}}, \qquad \text{for all} \quad \lambda > 0.$$

Set $t' = (t_{s+1}, \ldots, t_{2n-k})$, then we have

$$m(S_\lambda^s) \leq m(\{t \in \mathbb{R}^{2n-k}, \ |t'|^{2n-k+s-1} \leq \frac{1}{\lambda}, \ |t_i| \leq \frac{1}{\lambda |t'|^{2n-k+s-3}}, \ i = 1, \ldots, s\})$$

Consequently

$$m(S^s_\lambda) \leq c \int_0^{(\frac{1}{\lambda})^{\frac{1}{2n-k+s-1}}} \frac{r^{2n-k-s-1}dr}{\lambda^s r^{s(2n-k+s-3)}} = \left(\frac{\tilde{c}}{\lambda}\right)^{\frac{2n-(k-s)}{2n-(k-s)-1}}.$$

□

Lemma 4.2.2. *The kernels $R_M(\zeta, z)$and $\overline{\partial}_\zeta G_M(\zeta, z)$ are of weak type $\frac{2n}{2n-1}$ and $\frac{2n+2}{2n+1}$ respectively on ω uniformly in ζ and in z.*

Proof. From the previous discussion, the lemma follows from Lemma 4.2.1 near the non-characteristic points when $\zeta \notin \Sigma$. Near the characteristic points, we can apply again the Range-Siu's trick and estimate the integrals J as a finite covering of the integrals discussed in the previous lemma. This proves the lemma. □

To get Hölder estimates for $\widetilde{T}_{s-1}f$, by the Hardy-Littlewood lemma, we need to control the gradient $\nabla_z\widetilde{T}_{s-1}f$.

It follows from the definition of the kernel R_M and $\overline{\partial}_\zeta G_M$ that $\nabla_z R_M$ is controlled by

$$\int_{\lambda\in\Delta_{I\bullet}} (\frac{O_{k-m}}{\Phi^n} + \frac{O_{k+1-m}}{\Phi^{n+1}}) \wedge \Theta_m,$$

where Θ_m is a monomial of length m in $\partial\rho_{i_1}, \dots, \partial\rho_{i_m}$, and that $\nabla_z\widetilde{T}_{s-1}f$ is bounded by

$$\int_{\lambda\in\Delta_{I\bullet*}} (\frac{O_{k-m}}{\widetilde{\Phi}_*^{n-\alpha-k-s}\Phi^{k+s+\alpha}} + \frac{O_{k+1-m}}{\widetilde{\Phi}_*^{n-\alpha-k-s+1}\Phi^{k+s+\alpha}} + \frac{O_{k+1-m}}{\widetilde{\Phi}_*^{n-\alpha-k-s}\Phi^{k+s+\alpha+1}}) \wedge \Theta_m,$$

where Θ_m is a monomial of length m in $\partial\rho_{i_1}, \dots, \partial\rho_{i_m}, \partial\rho_*(\zeta), \overline{\partial}\rho_*(\zeta)$, $1 \leq m \leq k+2$.

Choosing the same coordinate system as before, we have to estimate, for $1 \leq s \leq k+3$, the integrals

$$J_s = \int_{|t|<A} \frac{dt_1 ... dt_{2n-k}}{\Pi_{i=1}^s(|t_i| + d + |t|^2)|t|^{2n-k-s}}$$
$$H_s = \int_{|t|<A} \frac{dt_1 ... dt_{2n-k}}{(\Pi_{i=1}^s(|t_i| + d + |t|^2)|t|^{2n-k-s})^{\frac{2n+2}{2n+1}}}$$

for each $z \in \overline{D}_I \cap \Omega$, with $d = \text{dist}(z, bD_I \cap \Omega)$.

Using estimates similar to p.289 in [6], for $1 \leq s \leq k+3$, we get

$$J_s \leq C\ d^{-\frac{1}{2}-\varepsilon},\ \varepsilon > 0,$$
$$H_s \leq C\ d^{-\frac{4n-(k-s)-1}{4n+2}}.$$

Consequently, from Hölder's inequality,

$$|\nabla_z\widetilde{T}_{s-1}f| \leq C\ \|f\|_\infty \text{dist(z, b}\omega)^{-\frac{1}{2}-\varepsilon},\ \varepsilon > 0$$
$$|\nabla_z\widetilde{T}_{s-1}f| \leq C\ \|f\|_{2n+2} \text{dist(z, b}\omega)^{-1}.$$

Thus by interpolation, we have the following estimates for smooth $\overline{\partial}_b$-closed forms.

Proposition 4.2.3. (L^p estimates of $\bar\partial_b$ for smooth forms.) *Let M be a strictly q-concave generic CR manifold in $\mathbb{C}^n$ and $0 \in M$. Let Ω be a strictly pseudoconvex domain containing 0 in $\mathbb{C}^n$ with C^2 boundary and $\omega = M \cap \Omega$. For any $f \in C_{n,s}(\overline{\omega})$, $1 \le p \le \infty$ and $n-k-q+1 \le s \le n-k$, $\widetilde{T}_{s-1}f$ defined by (4.1.4) satisfies the following estimates:*

(1) $\|\widetilde{T}_{s-1}f\|_{L^{\frac{2n+2}{2n+1}-\epsilon}} \le C\|f\|_{L^1}$, *for any small* $\epsilon > 0$.

(2) $\|\widetilde{T}_{s-1}f\|_{L^{p'}} \le C\|f\|_{L^p}$, *where* $\frac{1}{p'} = \frac{1}{p} - \frac{1}{2n+2}$ *and* $1 < p < 2n+2$.

(3) $\|\widetilde{T}_{s-1}f\|_{L^{p'}} \le C\|f\|_{L^p}$, *where* $p = 2n+2$ *and* $p < p' < \infty$.

(4) $\|\widetilde{T}_{s-1}f\|_{\mathcal{C}^{\alpha-\varepsilon}} \le C\|f\|_{L^p}$, *where* $2n+2 < p < \infty$, $\alpha = \frac{1}{2} - \frac{n+1}{p}$ *and* $\varepsilon > 0$.

(5) $\|\widetilde{T}_{s-1}f\|_{\frac{1}{2}-\varepsilon} \le C\|f\|_{L^\infty}$, $\varepsilon > 0$.

In order to prove Theorem 1.0.2, we need the following density lemma:

Lemma 4.2.4. *Under the same assumption as in Theorem 1.0.1, the set of $\overline{\partial}_b$-closed forms in $C_{(n,s)}(\overline{\omega})$ is dense in the set of $\overline{\partial}_b$-closed $L^p_{(n,s)}(\omega)$ forms in the $L^p_{(n,s)}(\omega)$ norm where $1 \le p < \infty$, $n-k-q+1 \le s \le n-k$.*

Proof. Let $\alpha \in L^p_{(n,s)}(\omega)$ and $\overline{\partial}_b\alpha = 0$ on ω. We approximate α by C^1 smooth $(0,s)$-forms $\alpha_l \in C^1_{(n,s)}(\overline{\omega})$ such that $\alpha_l \to \alpha$ in $L^p_{(n,s)}(\omega)$ and $\overline{\partial}_b\alpha_l \to 0$ in $L^p_{(n,s+1)}(\omega)$. This is possible by Friedrichs' Lemma. When $s = n-k$, the lemma is already proved since every form is $\overline{\partial}_b$-closed. We assume that $s < n-k$. Since $\overline{\partial}_b\alpha_l$ is a continuous $\overline{\partial}_b$-closed form on a slightly larger set $\omega_l \supset \omega$ where $l \to \infty$ and $\cap_l\omega_l = \overline{\omega}$, we can apply Proposition 4.2.3 to $\overline{\partial}_b\alpha_l$ on ω_l (since $n-k-q+1 \le s < n-k$,) to find $(0,s)$-forms v_l such that

$$\begin{cases} \overline{\partial}_b v_l = \overline{\partial}_b\alpha_l & \text{on } \omega_l, \\ \|v_l\|_{L^p_{(n,s)}(\omega_l)} \le c_p\|\overline{\partial}_b\alpha_l\|_{L^p_{(n,s+1)}(\omega_l)}, \end{cases}$$

where c_p is a constant independent of l. This is true since the constant proved in Proposition 4.2.3 is independent of small perturbation of ω. We set

$$\alpha'_l = \alpha_l - v_l,$$

then $\alpha'_l \in C_{(n,s)}(\overline{\omega})$. It follows that α'_l is $\overline{\partial}_b$-closed and $\alpha'_l \to \alpha$ in $L^p_{(n,s)}(\omega)$. This proves the lemma. □

Theorem 1.0.2 can be proved for any $\overline{\partial}_b$-closed α with $L^p(\omega)$ coefficients using an approximation argument.

Using Lemma 4.2.4, there exists a sequence of $\overline{\partial}_b$-closed forms $\alpha'_m \in C_{(n,s)}(\overline{\omega})$ such that $\alpha'_m \to \alpha$ in $L^p_{(n,s)}(\omega)$. We can apply Proposition 4.2.3 to α'_m to find $(0,s-1)$-form u_m such that

$$\overline{\partial}_b u_m = \alpha'_m \quad \text{on } \omega,$$

and

$$\|u_m\|_{L^{p'}_{(n,s-1)}(\omega)} \le c_p\|\alpha'_m\|_{L^p_{(n,s)}(\omega)}.$$

Thus, some subsequence of u_m must converge weakly to some $(0,s-1)$-form u such that u satisfies $\overline{\partial}_b u = \alpha$ on ω and

$$\|u\|_{L^{p'}_{(n,s-1)}(\omega)} \leq c_p \|\alpha\|_{L^p_{(n,s)}(\omega)}.$$

Assertions (1), (2), (3) of Theorem 1.0.2 are proved. In order to prove (4) and (5) of Theorem 1.0.2, the above argument cannot be used. Going back to the definition of the operators $\widetilde{T}_s$, it is easy to prove that for any $f \in \mathcal{C}^1_{n,s}(\overline{\omega})$

$$f = \overline{\partial}_b \widetilde{T}_{s-1} f + \widetilde{T}_s \overline{\partial}_b f + (-1)^{(n+s)k+\frac{k(k-1)}{2}} \overline{\partial}_b \int_{\zeta\in\omega} \overline{\partial}_b f(\zeta) \wedge [\widetilde{G}_M]_{n,s-1}(\zeta, z). \qquad (4.2.1)$$

By Friedrichs' lemma, the relation (4.2.1) extends to any $f \in L^p_{(n,s)}(\omega)$ such that $\overline{\partial}_b f \in L^p_{(n,s+1)}(\omega)$, $p > 2n+2$, since all the kernels involves in the formula are of weak type at least $\frac{2n+2}{2n+1}$. Consequently if $f \in L^p_{(n,s)}(\omega)$, $p > 2n+2$, satisfies $\overline{\partial}_b f = 0$, we still have $f = \overline{\partial}_b \widetilde{T}_{s-1} f$ and the estimates can be done as in Proposition 4.2.3.

Corollary 1.0.3 follows easily. The proof of Corollary 1.0.3 is exactly the same as in Shaw [19] for the strongly pseudoconvex case. As usual, the Hodge decomposition and the existence of the $\overline{\partial}_b$-Neumann operators given in Corollary 1.0.4 can be deduced from the classical Hilbert space theory.

References

[1] R. A. Airapetjan and G. M. Henkin, *Integral representation of differential forms on Cauchy-Riemann manifolds and the theory of CR function*, Russian Math.Survey **39** (1984), 41–118.

[2] A. Andreotti and C. D. Hill, *Convexity and the H. Levi problem. Part I : Reduction to the vanishing theorems*, Ann. Scuola Norm. Sup. Pisa **26** (1972), 325–363.

[3] M. Y. Barkatou, *Formules locales de type Bochner-Martinelli-Koppelman sur des variétés CR*, Math. Nachr. **196** (1998), 5–41.

[4] M. Y. Barkatou and C. Laurent-Thiébaut, *Estimations optimales pour l'opérateur de Cauchy-Riemann tangentiel*, Prépublication de l'Institut Fourier **593** (2003), 1–46.

[5] A. Boggess and M.-C. Shaw, *A kernel approach to local solvability of the tangential Cauchy-Riemann equations*, Trans. Amer. Math. Soc. **289** (1985), 643–659.

[6] S.-C. Chen and M.-C. Shaw, *Partial differential equations in several complex variables*, Studies in Advanced Math., vol. 19, AMS-International Press, 2001.

[7] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. Math. Studies, vol. 75, Princeton University Press, Princeton,N.J., 1972.

[8] G. M. Henkin, *The Hans Lewy equation and analysis on pseudoconvex manifolds*, Math. USSR Sbornik **31** (1977), 59–130.

[9] ———, *Solution des équations de Cauchy-Riemann tangentielles sur des variétés Cauchy-Riemann q-concaves*, Comptes Rendus Acad. Sciences **293** (1981), 27–30.

[10] G. M. Henkin and J. Leiterer, *Andreotti-Grauert theory by integral formulas*, Progress in Math., vol. 74, Birkhaüser, 1988.

[11] C. Laurent-Thiébaut and J. Leiterer, *On the Hartogs-Bochner extension phenomenon for differential forms*, Math. Ann. **284** (1989), 103–119.

[12] ———, *Uniform estimates for the Cauchy-Riemann equation on q-convex wedges*, Ann. Inst. Fourier **43** (1993), 383–436.

[13] ———, *Andreotti-Grauert theory on real hypersurfaces*, Quaderni, Scuola Normale Superiore, Pisa, 1995.

[14] M. Naconovich, *Poincaré lemma for tangential Cauchy Rieman complexes*, Math. Ann. **268** (1984), 449–471.

[15] I. Naruki, *Localisation principle for differential complexes and its applications*, Pub. Res. Inst. Math. Sc. Kyoto Univ. **8** (1972), 43–110.

[16] P. L. Polyakov, *Sharp estimates for operator $\overline{\partial}_M$ on a q-concave CR manifold*, J. Geom. Anal. **6** (1996), 233–276.

[17] R. M. Range and Y. T. Siu, *Uniform estimates for the $\overline{\partial}$ on domains with piecewise smooth strictly pseudoconvex boundaries*, Math. Ann. **206** (1974), 325–354.

[18] M.-C. Shaw, *L^p estimates for local solutions of $\overline{\partial}_b$ on strongly pseudoconvex CR manifolds*, Math. Ann. **288** (1990), 35–62.

[19] ———, *Homotopy formulas for $\overline{\partial}_b$ in CR manifolds with mixed levi signatures*, Math. Zeit. **224** (1997), 113–135.

[20] M.-C. Shaw and L. Wang, *Hölder and L^p estimates for $\Box_b$ on CR manifolds of arbitrary codimension*, (Preprint).

[21] F. Treves, *Homotopy formulas in the tangential Cauchy-Riemann complex*, Memoirs of the Amer. Math. Soc., Providence, Rhode Island, 1990.

Université de Grenoble
Institut Fourier
UMR 5582 CNRS/UJF
BP 74
38402 St Martin d'Hères Cedex
France
Christine.Laurent@ujf-grenoble.fr

Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556
USA
mei-chi.shaw.1@nd.edu