Wavelet Methods for Compact Embeddings

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ABSTRACT

In this note, the authors illustrate how compact embeddings between function spaces can be obtained using wavelet methods. They consider weighted Hölder spaces and obtain optimal growth conditions on the wavelet coefficients for functions in these weighted spaces. These conditions lead to continuous embeddings between weighted Hölder spaces and certain weighted l^∞ spaces.

RÉSUMÉ

Dans cette note, les auteurs utilisent des méthodes d'ondelettes pour obtenir des plongements compacts entre des espaces de fonctions. Ils considèrent des espaces de Hölder à poids et obtiennent des conditions de croissances optimales sur les cefficients d'ondelettes pour des fonctions dans ces espaces à poids. Ces conditions donnent des prolongements continus entre les espaces de Hölder à poids et certains espaces l^∞ à poids.

1 Introduction

The purpose of this note is to illustrate how compact embeddings between function spaces can be obtained using growth conditions on wavelet coefficients. While compact embeddings for weighted Hölder spaces are well known , we offer a novel approach for obtaining such embeddings, which may be applied to many other function spaces. We refer the reader to the work of Haroske in [2] on compact embeddings between weighted Besov spaces. Besov spaces are generalizations of the Hölder classes.

We define the weighted Hölder spaces $C^{s,\sigma}$, $s > 0, \sigma \geq 0$, as the intersection of the usual Hölder class C^s with the collection of all functions f such that $(1+|x|^\sigma)f(x)$ is bounded. It is well known that the inclusion $C^{t,\tau} \subset C^{s,\sigma}$ is compact if and only if $s < t$ and $\sigma < \tau$. In this note, we present an interesting approach using wavelet methods for proving these compact embeddings.

Wavelets provide a "universal" decomposition for various classes of functions or distributions f:

$$f \;=\; \sum_{k\in\mathbf{Z}} \langle f, \phi_k\rangle \phi_k + \sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}. \tag{1}$$

Here, the functions $\phi_k(x) = \phi(x-k)$, $k \in \mathbf{Z}$, form an orthonormal set in $L^2(\mathbf{R})$ while the wavelets $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbf{Z}$, form an orthonormal basis for $L^2(\mathbf{R})$. The function ϕ is sometimes called the scaling function while ψ is called the mother wavelet. For functions f and g defined on $\mathbf{R}$,

$$\langle f, g\rangle \;=\; \int_{\mathbf{R}} f(x)\overline{g(x)}\, dx$$

whenever the integral is defined.

In his book [5], Yves Meyer obtains characterizations of various function spaces in terms of growth of wavelet coefficients. For the non-homogeneous Hölder spaces C^s, boundedness of the sequences

$$\{\langle f, \phi_k\rangle : k \in \mathbf{Z}\}, \quad \{2^{-j(1/2+s)}\langle f, \psi_{j,k}\rangle : j, k \in \mathbf{Z},\ j \geq 0\}$$

is equivalent to $f \in C^s$. This is our starting point for obtaining inclusions relating the weighted Hölder classes and certain weighted l^∞ spaces. Compact embeddings for these weighted l^∞ spaces are easily obtained.

For $0 < s \leq 1$ and $\tau \geq 0$, we define $C^{s,\tau}$ as the Banach space of all continuous functions $f : \mathbf{R} \longrightarrow \mathbf{C}$ for which the norm $||f||_{s,\tau} = R_s(f) + S_\tau(f)$ is finite where

$$R_s(f) = \sup \{|f(x) - f(y)| \cdot |x - y|^{-s} : x, y \in \mathbf{R}, x \neq y\}$$

for $0 < s < 1$, while for $s = 1$

$$R_s(f) = \sup \{|f(x + y) + f(x - y) - 2f(x)| \cdot |y|^{-1} : \; x, y \in \mathbf{R}, y \neq 0\}.$$

For all $\tau \geq 0$, we define

$$S_\tau(f) = \sup_{x \in \mathbf{R}} (1 + |x|^\tau)|f(x)|.$$

For $s > 1$, we define $C^{s,\tau}$ as the Banach space of functions f which have derivatives (in the usual sense) up to order $N = \lfloor s \rfloor$ for which the norm

$$||f||_{s,\tau} = R_\theta(f^{(N)}) + S_\tau(f) \tag{2}$$

is finite, where $\theta = s - N$, and $\lfloor s \rfloor$ denotes the largest integer strictly less than s.

2 Wavelet Coefficients of functions in weighted Hölder spaces

In this section, ϕ denotes a scaling function for a multiresolution analysis with regularity $r + 1$ where r is a fixed positive integer. In other words, ϕ has derivatives up to order $r + 1$ which all have rapid decay; i.e., for each $\alpha > 0$, there exists a constant C_α such that

$$|\phi^{(i)}(x)| \leq \frac{C_\alpha}{1 + |x|^\alpha}$$

for $i = 0, 1, \cdots, r + 1$ and for all $x \in \mathbf{R}$. The corresponding mother wavelet denoted by ψ is also $r+1$ times differentiable and all its derivatives also have rapid decay. Moreover, it has the property that

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0 \quad \text{for} \quad k = 0, 1, 2, \cdots, r + 1.$$

We refer the reader to [5] for the details.

Given a non-empty set I, and any strictly positive function w defined on I, we let $l^\infty(I, w) = l^\infty(w)$ denote the Banach space of all functions $s : I \longrightarrow C$ normed by

$$|s|_w \;=\; \sup\{w(i)|s(i)| : i \in I\}$$

We shall take $I = \{(j, k) \in \mathbf{Z} \times \mathbf{Z} : j \geq -1\}$. Given $s > 0$ and $\sigma \geq 0$, we define $w_{s,\sigma} : I \longrightarrow (0, \infty)$ by

$$w_{s,\sigma}(j, k) \;=\; 2^{j/2} \max\{2^{js}, 1 + |k2^{-j}|^\sigma\}, \quad \text{if} \quad j \geq 0 \tag{3}$$

and $w_{s,\sigma}(j, k) \;=\; 1 + |k|^\sigma$ if $j = -1$. In this special case, the norm of c in $l^\infty(w_{s,\sigma})$ will be denoted by $|c|_{s,\sigma}$

For a function f in some weighted Hölder class, we let Sf denote the sequence of wavelet coefficients of f:

$$Sf \;=\; \{f_{j,k} : (j, k) \in I\}$$

where $f_{j,k} = \langle f, \psi_{j,k}\rangle$ if $j \geq 0$ and $f_{j,k} = \langle f, \phi_k\rangle$ if $j = -1$.

On the other hand, for a sequence $c = \{c_{j,k} : (j, k) \in I\}$ in some weighted space $l^\infty(w_{s,\sigma})$, we define Tc to be the wavelet series

$$Tc = \sum_{k=-\infty}^{\infty} c_{-1,k}\phi_k + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k}\psi_{j,k}.$$

Proposition 1 *Let $s > 0, \tau \geq 0$ with $s < r + 1$. Then*

A) S is a bounded operator from $C^{s,\tau}$ into $l^\infty(w_{s,\tau})$.

B) there exists a finite constant C (depending only on s, τ and ϕ) such that whenever $c \in l^\infty(w_{s,\tau})$,

$$R_\theta(D^N(Tc)) \;\leq\; C\; |c|_{s,\tau} \quad \textit{and} \quad |Tc(x)| \;\leq\; C\frac{|c|_{s,\tau}\,(1 + \log|x|)}{1 + |x|^\tau} \tag{4}$$

for $x \in \mathbf{R}$, where $N = \lfloor s \rfloor$ and $\theta = s - N$. In particular, given $\sigma \in (0, \tau)$, T defines a bounded operator from $l^\infty(w_{s,\tau})$ into $C^{s,\sigma}$.

Remark: The second estimate in (4) is optimal as demonstrated by the following example.

Let ψ be a compactly supported mother wavelet with $\psi(0) \neq 0$. Suppose that $\psi(x) = 0$ whenever $|x| \geq L$. We define

$$f(x) = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}(1+2^{n\tau})^{-1}\psi(2^j x - 2^{j+n}).$$

If $0 < s \leq \tau$, then $|Sf|_{s,\tau} \leq 1$.

Observe that if $j \geq 0$, $n \geq 0$ and M is a positive integer with $2^{M-1} \geq L$ and $n \neq M$, then $\psi(2^{j+M} - 2^{j+n}) = 0$. Therefore, for these values of M,

$$f(2^M) = \psi(0)(M+1)(1+2^{M\tau})^{-1}.$$

Finally, letting $x_M = 2^M$, we have

$$f(x_M) = \psi(0)\,\frac{1 + c_0 \log\ x_M}{1 + x_M^\tau}$$

whenever $2^{M-1} \geq L$, where $c_0 = (\log 2)^{-1}$. This shows optimality of the second estimate in (4).

Proof of Proposition 1: We omit the proofs of (A) and the first part of (B) which are both standard. The regularity $r+1$ of ψ is used in the proof of (A).

Let $c \in l^\infty(w_{s,\tau})$. We have $Tc = a + d$ with $a = \sum_{k=-\infty}^{\infty} c_{-1,k}\phi_k$ and $d = \sum_{j=0}^{\infty} h_j$ where

$$h_j = \sum_{k\in Z} c_{j,k}\psi_{j,k}. \tag{5}$$

To prove the second part of (4), it is enough to consider large values of x. We fix $x \in \mathbf{R}$ with $|x| \geq 2$. Choose α such that $\alpha - \tau > 1$. If $|k|2^{-j} < |x|/2$ with $j \geq 0$, then $|2^j x - k| \geq 2^j|x|/2$ and therefore,

$$|\psi(2^j x - k)| \leq \frac{C}{(1+|x|^\tau)(1+|2^j x - k|^{\alpha-\tau})}. \tag{6}$$

Here and in what follows, C denotes a constant depending only on s, τ, α and ϕ, and may be different at each occurrence. We shall also also adopt the notations

$$a_\tau(c) = \sup\{|c_{-1,k}|w_{s,\tau}(-1,k)\ : k \in Z\},$$

$$d_{s,\tau}(c) \;=\; \sup\{|c_{j,k}|w_{s,\tau}(j,k) : (j,k) \in \mathbf{Z} \times \mathbf{Z},\; j \geq 0\}\;,$$

for a sequence $c \in l^\infty(w_{s,\tau})$.

Combining (6) with the rough estimate $|c_{j,k}| \;\leq\; d_{s,\tau}(c)2^{-j/2}$, we obtain, for a fixed $j \geq 0$,

$$|\sum_{|k|2^{-j}<|x|/2} c_{j,k}\psi_{j,k}(x)| \;\leq\; \frac{C \cdot d_{s,\tau}(c)}{1+|x|^\tau}. \tag{7}$$

On the other hand, the estimate $|c_{j,k}| \;\leq\; d_{s,\tau}(c)2^{-j/2}(1+|k2^{-j}|^\tau)^{-1}$ yields

$$|\sum_{|k|2^{-j}\geq|x|/2} c_{j,k}\psi_{j,k}(x)| \;\leq\; \frac{C \cdot d_{s,\tau}(c)}{1+|x|^\tau}. \tag{8}$$

Combining (7) and (8), we conclude that

$$|h_j(x)| \;\leq\; \frac{C \cdot d_{s,\tau}(c)}{1+|x|^\tau} \tag{9}$$

for any non-negative integer j.

Let m denote the positive integer such that

$$2^m \;\leq\; |x|^{\tau/s} < 2^{m+1}.$$

In view of (9), we obtain

$$|\sum_{j=0}^{m} h_j(x)| \;\leq\; \frac{C \cdot d_{s,\tau}(c)\;\log|x|}{1+|x|^\tau}.$$

Meanwhile, from the estimates $|c_{j,k}| \leq d_{s,\tau}(c)2^{-j(1/2+s)}$, we conclude that $|h_j(x)| \;\leq\; C \cdot d_{s,\tau}(c)\; 2^{-js}$, for $j \geq 0$ and obtain

$$|\sum_{j=m+1}^{\infty} h_j(x)| \;\leq\; \frac{C \cdot d_{s,\tau}(c)}{1+|x|^\tau}.$$

Combining these last two inequalities yields the desired estimate for d. For $|x| > 2$,

$$|d(x)| \;\leq\; \frac{C \cdot d_{s,\tau}(c)\;\log|x|}{1+|x|^\tau}. \tag{10}$$

The estimate for a is obtained in the same way that we obtained (9). We have

$$|a(x)| \;\leq\; \frac{C \cdot a_\tau(c)}{1+|x|^\tau} \tag{11}$$

whenever $|x| > 2$. □

3 Compact embeddings between weighted Hölder spaces

Lemma 1 below, applied to the weights defined in (3), implies that the inclusion

$$l^\infty(w_{t,\tau}) \subset l^\infty(w_{s,\sigma})$$

is compact if $t > s > 0$ and $\tau > \sigma \geq 0$.

Lemma 1 *Let w and v be strictly positive functions defined on I such that for some finite constant C,*

$$v(i) \quad \leq \quad C\ w(i), \qquad i \in I.$$

Then the following are equivalent:
(i) the inclusion $l^\infty(w) \longrightarrow l^\infty(v)$ is compact
(ii) for any $\epsilon > 0$, the set $\{i : v(i) > \epsilon w(i)\}$ is finite.

For the reader's convenience, the proof of the lemma is given in the last section of this note. We can now easily obtain compact embeddings between the weighted Hölder spaces.

Theorem 1 *Let $0 < s \leq t < \infty$ and $0 \leq \sigma \leq \tau < \infty$.*
(a) Suppose $s < t$ and $\sigma < \tau$. Then the inclusion $C^{t,\tau} \subset C^{s,\sigma}$ is compact.
(b) The inclusions $C^{t,\sigma} \subset C^{s,\sigma}$ and $C^{s,\tau} \subset C^{s,\sigma}$ are not compact.

Proof of (a): Choose $\rho \in (\sigma, \tau)$. By Proposition 1, the linear mappings

$$S : C^{t,\tau} \longrightarrow l^\infty(w_{t,\tau}) \qquad \text{and} \qquad T : l^\infty(w_{s,\rho}) \longrightarrow C^{s,\sigma}$$

are bounded. Taking the composition of this mappings with the compact inclusion

$$i : l^\infty(w_{t,\tau}) \longrightarrow l^\infty(w_{s,\rho}),$$

we obtain the compactness of

$$T \circ i \circ S : C^{t,\tau} \longrightarrow C^{s,\sigma}.$$

Meanwhile, for bounded continuous functions f, we have pointwise convergence of its wavelet series:

$$f(x) \;=\; \sum_{k\in\mathbf{Z}} \langle f, \phi_k\rangle \phi_k(x) + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}(x),$$

for all $x \in \mathbf{R}$. See, for instance, [4] . Therefore, the composition $T \circ i \circ S$ is precisely the inclusion $C^{t,\tau} \subset C^{s,\sigma}$. □

Proof of (b): We shall only consider the case $0 < s \le t < 1$. We retain the notations

$$R_s(f) \;=\; \sup_{x\neq y} |f(x) - f(y)| \cdot |x-y|^{-s}$$

for $0 < s < 1$, while for any $\tau > 0$,

$$S_\tau(f) \;=\; \sup_{x\in R}(1 + |x|^\tau)|f(x)|.$$

To prove the non-compactness of the first inclusion in (b), we fix $f \in C^{t,\sigma}$, $f \not\equiv 0$ and let $T_n f(x) = f(x-n)$. We let

$$f_n \;=\; \frac{T_n f}{||T_n f||_{t,\sigma}}.$$

If the first inclusion in (2) were compact, then there would exist $g \in C^{s,\sigma}$ and a subsequence f_{n_k} such that $||f_{n_k} - g||_{s,\sigma} \longrightarrow 0$ as $k \longrightarrow \infty$. Since $\lim_{n\to\infty} S_\sigma(T_n f) \;=\; \infty$, we have

$$||f_n||_{s,\sigma} \;\longrightarrow\; 1, \qquad ||f_n||_\infty \;\longrightarrow\; 0.$$

Thus, $g = 0$. This implies $||f_{n_k}||_{s,\sigma} \;\longrightarrow\; 0$, a contradiction.

To prove the non-compactness of the second inclusion in (b), we fix a non-constant function $h \in C^{s,\tau}$ and define

$$h_n \;=\; \frac{D_n h}{||D_n h||_{s,\tau}}$$

where $D_n h(x) \;=\; h(nx)$. Assuming compactness of the second inclusion in (2), there would exist a function $k \in C^{s,\sigma}$ and a subsequence h_{n_i} such that

$$||h_{n_i} - k||_{s,\sigma} \;\longrightarrow\; 0, \quad i \longrightarrow \infty.$$

Since $S_\sigma(D_n h) \le ||h||_\infty + S_\sigma(h)$ for $n \ge 1$ and $R_s(D_n h) = n^s R_s(h)$, we have

$$||h_n||_{s,\sigma} \;\longrightarrow\; 1, \quad ||h_n||_\infty \;\longrightarrow\; 0$$

as $n \longrightarrow \infty$. This implies $k = 0$ and therefore $||h_{n_i}||_{s,\sigma} \;\longrightarrow\; 0$, a contradiction. □

4 Proof of Lemma 1

To prove the implication $(ii) \Rightarrow (i)$, it is sufficient to show that Ball $l^\infty(w)$ is totally bounded in $l^\infty(v)$. We use the notation $Ball\ X = \{x \in X : ||x|| \le 1\}$ if X is a normed space.

Let $\epsilon > 0$. For each $i \in I$, choose complex numbers $a(i,j),\ 1 \le j \le n(i)$, with $|a(i,j)|w(i) < 1$ such that

$$\{z \in C : |z|w(i) \le 1\} \subset \bigcup_{j=1}^{n(i)} \{z \in C : |z - a(i,j)|v(i) < \epsilon.\}.$$

We write $\{i \in I : v(i) > \epsilon w(i)/2\} = \{i_1, i_2, \cdots, i_M\}$ and consider the Cartesian product

$$P = \prod_{k=1}^{M} \{1, 2, \cdots, n(i_k)\}.$$

For $m = (m(1), \cdots, m(M)) \in P$, we define $x_m : I \longrightarrow C$ by $x_m(i_k) = a(i_k, m(k))$ for $1 \le k \le M$ and $x_m(i) = 0$ if $i \ne i_k$ for all $k \in \{1, 2, \cdots, M\}$. Then $x_m \in$ Ball $l^\infty(w)$ and

$$\text{Ball } l^\infty(w) \subset \bigcup_{m \in P} B_v(x_m, \epsilon). \tag{12}$$

To prove the implication $(i) \Rightarrow (ii)$, suppose there is an $\epsilon > 0$ such that $A_\epsilon = \{i \in I : v(i) > \epsilon w(i)\}$ is an infinite set. . Let $\{i_n\}_{n=1}^\infty$ be a sequence of distinct elements in A_ϵ. For $n = 1, 2, 3, \cdots$, define $e_n : I \longrightarrow C$ such that $e_n(j) = w(j)^{-1}$ if $j = i_n$ and $e_n(j) = 0$, otherwise. Then $||e_n||_w = 1$. Assuming that (i) holds, there would exists a subsequence $\{e_{n_k}\}_{k=1}^\infty$ convergent in $l^\infty(v)$, say to e. It is immediate that $e = 0$. Hence

$$w(i_{n_k})^{-1} v(i_{n_k}) = ||e_{n_k}||_v \longrightarrow 0$$

as k tends to infinity. This contradicts the fact that $\{i_{n_k}\}_{k=1}^\infty$ is a sequence of elements of A_ϵ. □

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