Prépublication de l'institut Fourier n°598 (2003)
http://www-fourier.ujf-grenoble.fr/prepublications.html

UNRAMIFIED COHOMOLOGY OF DEGREE 3 AND NOETHER'S PROBLEM

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Abstract. — Let G be a finite group and W be a faithful representation of G over $\mathbf{C}$. The group G acts on the field of rational functions $\mathbf{C}(W)$. The aim of this paper is to give a description of the unramified cohomology group of degree 3 of the field of invariant functions $\mathbf{C}(W)^G$ in terms of the cohomology of G when G is a group of odd order. This enables us to give an example of a group for which this field is not rational, although its unramified Brauer group is trivial.

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1. Introduction

If G is a finite group and W a faithful representation of G over $\mathbf{C}$, then the field of invariant rational functions $\mathbf{C}(W)^G$ depends only on G, up to stable equivalence. The problem which goes back to Noether is to determine whether this field is rational. A natural obstruction is given by the unramified cohomology groups which are trivial for stably rational fields.

In degree two, this group coincides with the unramified Brauer group which has been used by Saltman in [**Sa1**] to give the first example of a group G for which $\mathbf{C}(W)^G$ is not

1991 ***Mathematics Subject Classification.*** — primary 14E08; secondary 12G05, 14F43.
Keywords: Birational invariants, Noether's problem, unramified cohomology.

rational. Bogomolov then gave a general description of this group in [**Bo1**, theorem 3.1]. More precisely, one may describe this group in terms of the cohomology of G by the formula

$$\mathrm{Br}_{\mathrm{nr}}(\mathbf{C}(W)^G) \xrightarrow{\sim} \bigcap_{B\in\mathscr{B}_G} \mathrm{Ker}(H^2(G,\mathbf{Q}/\mathbf{Z}) \to H^2(B,\mathbf{Q}/\mathbf{Z}))$$

where $\mathscr{B}_G$ denotes the set of bicyclic subgroups of G, that is the set of subgroups of G which are a quotient of $\mathbf{Z}^2$. This result enabled Bogomolov to give other examples of groups for which the unramified Brauer group of $\mathbf{C}(W)^G$ is not trivial.

In higher degree, the unramified cohomology groups have been introduced by Colliot-Thélène and Ojanguren [**CTO**] to give new examples of unirational fields over $\mathbf{C}$ which are not stably rational.

The aim of this text is to describe a computation of the unramified cohomology group of degree 3 in terms of the cohomology of the group G and then to use this description to construct a group G for which $\mathbf{C}(W)^G$ is not rational but has a trivial unramified Brauer group. Saltman has proven in [**Sa2**] that the unramified cohomology group in degree three is contained in the image of the inflation map

$$H^3(G,\mathbf{Q}/\mathbf{Z}) \to H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}).$$

One of the main difficulty is to describe the kernel of this inflation map.

In [**Pe3**], we proved, extending ideas of Saltman [**Sa2**], that there is a natural exact sequence

$$0 \to \mathrm{CH}^2_G(\mathbf{C}) \to H^3(G,\mathbf{Q}/\mathbf{Z}(2)) \to H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}(2))$$

where $\mathrm{CH}^2_G(\mathbf{C})$ denotes the equivariant Chow group of codimension two of a point. The main step of our proof relates the image of $\mathrm{CH}^2_G(\mathbf{C})$ with the permutation negligible classes introduced by Saltman in [**Sa2**].

It is somewhat easier to describe the inverse of the unramified cohomology group in $H^3(G,\mathbf{Q}/\mathbf{Z})$. Such a description had been made by Bogomolov in [**Bo2**]. We give here a description which is purely in terms of the cohomology of G.

In section 2 we introduce the notations used in the rest of this paper, section 3 states the main result and section 4 contains its proof. In section 5 we consider the case of a central extension of an $\mathbf{F}_p$-vector space by another one. The last section is devoted to the explicit construction of an example.

2. Definitions

Let us fix a few notations for the rest of this text.

Notations 2.1. — Let k be a field of characteristic 0, $\overline{k}$ be an algebraic closure of k. For any positive integer n, we denote by μ_n the n-th roots of unity in $\overline{k}$ and for j in $\mathbf{Z}$ we put

$$\mu_n^{\otimes j} = \begin{cases} \mu_n^{\otimes j-1} \otimes \mu_n & \text{if } j > 1, \\ \mathbf{Z}/n\mathbf{Z} & \text{if } j = 0, \\ \mathrm{Hom}(\mu_n^{\otimes j}, \mathbf{Z}/n\mathbf{Z}) & \text{if } j < 0, \end{cases}$$

and we consider the Galois cohomology groups

$$H^i(k,\mu_n^{\otimes j}) = H^i(\mathrm{Gal}(\overline{k}/k),\mu_n^{\otimes j})$$

as well as their direct limits

$$H^i(k,\mathbf{Q}/\mathbf{Z}(j)) = \varinjlim_n H^i(k,\mu_n^{\otimes j}).$$

If V is a variety over k, we also consider the étale sheafs $\mu_n^{\otimes j}$ and $\mathbf{Q}/\mathbf{Z}(j)$.

For any function field over k, that is finitely generated as a field over k, we denote by $\mathscr{P}(K/k)$ the set of discrete valuation rings A of rank one such that $k \subset A \subset K$ and such that the fraction field $\mathrm{Fr}(A)$ of A is K. If A belongs to $\mathscr{P}(K/k)$, then κ_A is its residue field and, for any strictly positive integer i and any j in $\mathbf{Z}$,

$$\partial_A : H^i(K,\mu_n^{\otimes j}) \to H^{i-1}(\kappa_A,\mu_n^{\otimes j-1})$$

is the corresponding residue map (see [**CTO**]). They induce residue maps

$$\partial_A : H^i(K,\mathbf{Q}/\mathbf{Z}(j)) \to H^{i-1}(\kappa_A,\mathbf{Q}/\mathbf{Z}(j-1)).$$

We then consider the unramified cohomology groups defined by

$$H^i_{\mathrm{nr}}(K,\mathbf{Q}/\mathbf{Z}(j)) = \bigcap_{A\in\mathscr{P}(K/k)} \mathrm{Ker}\Big(H^i(K,\mathbf{Q}/\mathbf{Z}(j)) \xrightarrow{\partial_A} H^{i-1}(\kappa_A,\mathbf{Q}/\mathbf{Z}(j-1))\Big).$$

In particular, the unramified Brauer group may be described as

$$\mathrm{Br}_{\mathrm{nr}}(K) = H^2_{\mathrm{nr}}(K,\mathbf{Q}/\mathbf{Z}(1)).$$

Let us also recall that two function fields K and L are said to be stably isomorphic over k if there exist indeterminates $U_1,\dots,U_m,T_1,\dots,T_n$ and an isomorphism from $K(U_1,\dots,U_m)$ to $L(T_1,\dots,T_n)$ over k. By [**CTO**], if K and L are stably isomorphic over k, then

$$H^i_{\mathrm{nr}}(K,\mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{\mathrm{nr}}(L,\mu_n^{\otimes j}).$$

In particular, if k is algebraically closed and K stably rational over k then $H^i_{\mathrm{nr}}(K,\mu_n^{\otimes j})$ is trivial.

We shall also use the equivariant Chow groups as defined by Totaro [**To**] and Edidin and Graham [**EG**, §2.2].

Definition 2.2. — Let G be a finite group and W a faithful representation of G over k. For any strictly positive n, let U_n be the maximal open set in W^n on which G acts freely. We have that $\mathrm{codim}_{W^n}(W^n - U_n) \geqslant n$. If Y is a quasi-projective smooth geometrically integral variety equipped with an action of G over k, the equivarient Chow group of Y is defined by

$$\mathrm{CH}^i_G(Y) = \mathrm{CH}^i(Y \times U_{i+1}/\!\!/G).$$

We put $\mathrm{CH}^i_G(k) = \mathrm{CH}^i_G(\mathrm{Spec}\,k)$, where the action of G on $\mathrm{Spec}\,k$ is trivial, and define $\mathrm{Pic}_G(Y)$ as $\mathrm{CH}^1_G(Y)$.

By [**Pe3**, definition 3.1.3], if k is algebraically closed, the étale cycle map induces a natural cycle map

$$\mathrm{cl}_i : \mathrm{CH}^i_G(k) \to H^{2i-1}(G,\mathbf{Q}/\mathbf{Z}(i))$$

such that, by [**Pe3**, example 3.1.1],

$$\mathrm{cl}_1 : \mathrm{Pic}_G(k) \xrightarrow{\sim} H^1(G, \mathbf{Q}/\mathbf{Z}(1))$$

is an isomorphism.

As indicated in the introduction, one of the main problem to compute the unramified cohomology is to determine the kernel of the inflation map

$$\mathrm{Ker}\big(H^3(G, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2))\big),$$

which by [**Pe3**, corollary 3.1.3] coincides with the image of cl_2. More generally, let us recall the notion of geometrically negligible classes, due to Saltman, which is a variant of the notion introduced by Serre in his lectures at the Collège de France in 1990–91 [**Se1**].

Definition 2.3. — If G is a finite group, M a G-module and k a field, then a class λ in $H^i(G, M)$ is said to be totally k-negligible if and only if for any extension K of k and any morphism

$$\rho : \mathrm{Gal}(K^s/K) \to G$$

where K^s is a separable closure of K, the image of λ by ρ^* is trivial in $H^i(K, M)$. The class λ is said to be geometrically negligible if $k = \mathbf{C}$.

As was proved by Serre (see also [**Bo2**]), the group of geometrically negligible classes in $H^i(G, M)$ coincides with the kernel of the map

$$H^i(G, M) \to H^i(\mathbf{C}(W)^G, M).$$

In the following, we shall be interested by the case where $i = 3$ and $M = \mathbf{Q}/\mathbf{Z}(2)$. We shall also assume that $k = \mathbf{C}$ and fix an isomorphism from $\mathbf{Q}/\mathbf{Z}$ to $\mathbf{Q}/\mathbf{Z}(1)$. In this setting, Saltman introduced the group of permutation negligible classes which is defined by

$$H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) = \mathrm{Ker}(H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3(G, \mathbf{C}(W)^*)).$$

In [**Pe3**, pp. 196–197], we prove that this group may be described in terms of the cohomology of G as

$$H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) = \sum_{H \subset G} \mathrm{Cores}_H^G\Big(\mathrm{Im}\big(H^1(H, \mathbf{Q}/\mathbf{Z})^{\otimes 2} \xrightarrow{\cup} H^3(H, \mathbf{Q}/\mathbf{Z})\big)\Big). \tag{2.1}$$

Finally we shall also need to pull back the residue maps to the cohomology of G.

Definition 2.4. — For any subgroup H of G and any element g of the centralizer $Z_G(H)$ of H in G, we define a map

$$\partial_{H,g} : H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^2(H, \mathbf{Q}/\mathbf{Z})$$

as follows: let I be the seubgroup generated by g. The natural map

$$H \times I \to G$$

induces a map

$$\rho_{H,g} : H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3(H \times I, \mathbf{Q}/\mathbf{Z}).$$

But the pull-back of the projection gives a splitting of the restriction map

$$H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \to H^3(I, \mathbf{Q}/\mathbf{Z}).$$

This yields a morphism

$$H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Ker}(H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \to H^3(I, \mathbf{Q}/\mathbf{Z})).$$

Using Hochschild-Serre spectral sequence and the fact that $H^2(I, \mathbf{Q}/\mathbf{Z}) = 0$ we get a map

$$H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \to H^2(H, H^1(I, \mathbf{Q}/\mathbf{Z})).$$

But g defines an injection

$$H^1(I, \mathbf{Q}/\mathbf{Z}) \hookrightarrow \mathbf{Q}/\mathbf{Z}$$

which yields

$$\partial : H^3(H \times I, \mathbf{Q}/\mathbf{Z}) \to H^2(H, \mathbf{Q}/\mathbf{Z}).$$

The map $\partial_{H,g}$ is then defined as the composite $\partial \circ \rho_{H,g}$. We define

$$H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) = \bigcap_{\substack{H \subset G \\ g \in Z_G(H)}} \mathrm{Ker}(\partial_{H,g}).$$

Remark 2.5. — One may show that this group coincides with the one defined by Bogomolov in a more geometric way in [**Bo2**].

Remark 2.6. — Similarly, one can easily define for any subgroup H of G and any g in $Z_G(H)$ a morphism

$$\partial_{H,g} : H^2(G, \mathbf{Q}/\mathbf{Z}) \to H^1(H, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}(H, \mathbf{Q}/\mathbf{Z})$$

and

$$H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) = \bigcap_{\substack{H \subset G \\ g \in Z_G(H)}} \mathrm{Ker}(\partial_{H,g}).$$

Let us show that

$$H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) = \bigcap_{B \in \mathscr{B}_G} \mathrm{Ker}(H^2(G, \mathbf{Q}/\mathbf{Z}) \to H^2(B, \mathbf{Q}/\mathbf{Z})).$$

If γ belongs to the right hand side, let H be a subgroup of G, let g belong to $Z_G(H)$, and let $x \in H$; $B = \langle g, x \rangle$ is a bicyclic group of G and there is a commutative diagram

$$\begin{array}{ccc} H^2(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\partial_{H,g}} & H^1(H, \mathbf{Q}/\mathbf{Z}) \\ \downarrow{\scriptstyle \mathrm{Res}^H_B} & & \downarrow{\scriptstyle \mathrm{Res}^H_{\langle x \rangle}} \\ H^2(B, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\partial_{\langle x \rangle, g}} & H^1(\langle x \rangle, \mathbf{Q}/\mathbf{Z}). \end{array}$$

Since $\mathrm{Res}^H_B(\gamma) = 0$, for any x in H we have $\mathrm{Res}^H_{\langle x \rangle}(\partial_{H,g}(\gamma)) = 0$. Hence $\partial_{H,g}(\gamma) = 0$.

Conversely, if γ belongs to $H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$ and B is a bicyclic subgroup of G, then $\mathrm{Res}^G_B(\gamma)$ belongs to $H^2_{\mathrm{nr}}(B, \mathbf{Q}/\mathbf{Z})$. But

$$H^2(B, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \Lambda^2 B$$

where $\Lambda^2 B$ is either trivial or cyclic generated by an element of the form $u \wedge v$. In the latter case, one has that $\partial_{\langle u\rangle,v}$ is injective and $\mathrm{Res}_B^G(\gamma) = 0$.

3. Description of the unramified cohomology group

The aim of this paper is to prove and illustrate the following theorem:

Theorem 3.1. — *If G is a finite group and if W is faithful representation of G over $\mathbf{C}$ then the inflation map induces an isomorphism*

$$H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})/H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Z}[1/2] \xrightarrow{\sim} H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Z}[1/2].$$

Remarks 3.2. — (i) If G is of odd order, we may remove the $\otimes\mathbf{Z}[1/2]$ in the above isomorphism. However, in [**Sa2**], Saltman gave an example of a 2-group for which the kernel of the inflation map is strictly bigger than $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$. Therefore one has to consider the prime to 2 part of the groups in general.

(ii) In fact $H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$ is the inverse image of $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ in $H^3(G, \mathbf{Q}/\mathbf{Z})$. The prime 2 does not play a rôle in this part of the statement.

(iii) Using remark 2.6, Bogomolov's theorem may be stated as

$$H^2_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} H^2_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}).$$

(iv) In higher degrees one would have to take into account the negligible classes in order to define $H^i_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$. Moreover the question whether the classes in $H^i_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ come from the cohomology of G is still open.

4. Proof of the main theorem

We shall first recall the result relating the geometrically negligible classes to the equivariant Chow group of codimension 2.

Notations 4.1. — If V is a variety over a field k of characteristic 0, $V^{(p)}$ denotes the set of points of codimension p in V. For any x in $V^{(p)}$, let $\kappa(x)$ be its residue field. We also denote by $\mathscr{H}^i_{\text{ét}}(\mu_n^{\otimes j})$ the Zariski sheaf corresponding to the presheaf mapping U to $H^i_{\text{ét}}(U, \mu_n^{\otimes j})$. We define similarly the sheaf $\mathscr{H}^i_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(j))$ and $\mathscr{K}_j$ the Zariski sheaf corresponding to the presheaf mapping U to $K_i(U)$, the i-th group of Quillen K-theory.

We denote by $|X|$ the cardinal of a set X.

The following proposition follows from theorem 2.3.1 in [**Pe3**], but we shall now give a direct proof of it which is due to Colliot-Thélène.

Proposition 4.2. — *If G is a finite group, W a faithful representation of G over $\mathbf{C}$, Let U be the maximal open subset in W on which G acts freely and assume that $\mathrm{codim}_W W - U$ is bigger than 4. Then there is a canonical exact sequence*

$$O \to \mathrm{CH}^2_G(\mathbf{C}) \to H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^0_{\mathrm{Zar}}(U/G, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to 0.$$

Proof. — Let $X = U/G$. The Bloch-Ogus spectral sequence [**BO**]

$$E_2^{p,q} = H^p_{\text{Zar}}(X, \mathscr{H}^q_{\text{ét}}(\mu_n^{\otimes 2})) \Rightarrow H^{p+q}_{\text{ét}}(X, \mu_n^{\otimes 2})$$

yields an exact sequence

$$\begin{aligned} 0 \to H^1_{\text{Zar}}(X, \mathscr{H}^2_{\text{ét}}(\mu_n^{\otimes 2})) \to H^3_{\text{ét}}(X, \mu_n^{\otimes 2}) \\ \to H^0_{\text{Zar}}(X, \mathscr{H}^3_{\text{ét}}(\mu_n^{\otimes 2})) \to H^2_{\text{Zar}}(X, \mathscr{H}^2_{\text{ét}}(\mu_n^{\otimes 2})) \to H^4_{\text{ét}}(X, \mu_n^{\otimes 2}) \end{aligned}$$

since $E_2^{p,q} = E_1^{p,q} = \{0\}$ if $p > q$. But we have the following diagram with exact lines and columns

$$\begin{array}{ccccccc}
\bigoplus_{x\in X^{(1)}} \kappa(x)^* & \longrightarrow & \bigoplus_{x\in X^{(2)}} \mathbf{Z} & \longrightarrow & \mathrm{CH}^2(X) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \times n} & & \downarrow{\scriptstyle \times n} & & \downarrow & & \\
\bigoplus_{x\in X^{(1)}} \kappa(x)^* & \longrightarrow & \bigoplus_{x\in X^{2}} \mathbf{Z} & \longrightarrow & \mathrm{CH}^2(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\bigoplus_{x\in X^{1}} H^1(\kappa(x), \mu_n) & \longrightarrow & \bigoplus_{x\in X^{(2)}} \mathbf{Z}/n\mathbf{Z} & \longrightarrow & H^2_{\text{Zar}}(X, \mathscr{H}^2_{\text{ét}}(\mu_n^{\otimes 2})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

which gives an isomorphism

$$\mathrm{CH}^2(X)/n \xrightarrow{\sim} H^2_{\text{Zar}}(X, \mathscr{H}^2_{\text{ét}}(\mu_n^{\otimes 2})).$$

By [**CT**, (3.2)], Merkur′ev-Suslin theorem gives an exact sequence

$$0 \to H^1(X, \mathscr{K}_2)/n \to H^1(X, \mathscr{H}^2_{\text{ét}}(\mu_n^{\otimes 2})) \to \mathrm{CH}^2(X)_n \to 0.$$

Since we have $\operatorname{codim}_W W - U \geqslant 4$, we get that

$$CH^2(U) = \mathrm{CH}^2(W) = \{0\},$$
$$H^1(U, \mathscr{K}_2) = H^1(W, \mathscr{K}_2) = \{0\},$$

and

$$H^0_{\text{Zar}}(U, \mathscr{H}^3_{\text{ét}}(\mu_n^{\otimes 2})) = H^0_{\text{Zar}}(W, \mathscr{H}^3_{\text{ét}}(\mu_n^{\otimes 2})) = 0.$$

But using a restriction-corestriction argument (see e.g. [**Ro**]) for the map $\pi : U \to U/G$, we get that the corresponding groups for X are killed by $|G|$. Taking inductive limits we get an exact sequence

$$0 \to \mathrm{CH}^2(X) \to H^3_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(2)) \to H^0_{\text{Zar}}(X, \mathscr{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to 0.$$

By [**Pe3**, Lemma 3.1.1], the Hochschild-Serre spectral sequence yields an isomorphism

$$H^3_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(2)) \xrightarrow{\sim} H^3(G, \mathbf{Q}/\mathbf{Z}(2)). \quad \square$$

To get the group of geometrically negligible classes in $H^3(G, \mathbf{Q}/\mathbf{Z})$, it remains to compute the image of $\mathrm{CH}^2_G(\mathbf{C})$ in that group.

Proposition 4.3. — *If G is a finite group, then the prime to 2 part of the group of geometrically negligible classes in $H^3(G, \mathbf{Q}/\mathbf{Z})$ is contained in the group $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ of permutation negligible classes.*

Remark 4.4. — The fact that the group $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ is contained in the group of negligible classes was proven by Saltman in [**Sa2**].

Proof. — Let p a prime factor of $|G|$ and G_p be a p-Sylow subgroup of G. By the description (2.1) of permutation negligible classes, we have that

$$\mathrm{Cores}^G_{G_p}(H^3_{\mathrm{p}}(G_p, \mathbf{Q}/\mathbf{Z})) \subset H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$$

and we have commutative diagrams

$$\begin{array}{ccc} H^3(G, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}) \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ H^3(G_p, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(\mathbf{C}(W)^{G_p}, \mathbf{Q}/\mathbf{Z}) \end{array}$$

and

$$\begin{array}{ccccc} H^3_{\mathrm{p}}(G_p, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(G_p, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(\mathbf{C}(W)^{G_p}, \mathbf{Q}/\mathbf{Z}) \\ \downarrow{\scriptstyle \mathrm{Cores}^G_{G_p}} & & \downarrow{\scriptstyle \mathrm{Cores}^G_{G_p}} & & \downarrow{\scriptstyle \mathrm{Cores}} \\ H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(G, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}). \end{array}$$

By taking the p-primary part of the group of negligible classes, we are reduced to the case where G is a p-group for p an odd prime.

By [**Pe3**, corollary 3.1.9], the image of

$$\mathrm{CH}^2_G(\mathbf{C}) \to H^3(G, \mathbf{Q}/\mathbf{Z})$$

coincides with the image of the second Chern class

$$\mathscr{R}(G) \xrightarrow{c_2} H^3(G, \mathbf{Q}/\mathbf{Z})$$

where $\mathscr{R}(G)$ denotes the ring of representations of G over $\mathbf{C}$. By Whitney formula, if x and y belong to $\mathscr{R}(G)$, one has

$$c_2(x+y) = c_2(x) + c_1(x)c_1(y) + c_2(y).$$

By (2.1), we have that $c_1(x)c_1(y) \in H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$. Thus the induced map

$$\mathscr{R}(G) \xrightarrow{c_2} H^3(G, \mathbf{Q}/\mathbf{Z})/H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$$

is a morphism of groups. We want to show that this morphism is trivial.

By Brauer's theorem (see [**Se2**, §10.5, theorem 20]), $\mathscr{R}(G)$ is generated as a group by the representations induced from characters of subgroups. It remains to show that for any subgroup H of G and any character χ of H, one has

$$c_2(\operatorname{Ind}_H^G \chi) \in H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}).$$

But Fulton and MacPherson give an expression for such Chern classes (see [**FMP**, corollary 5.3])

$$\begin{aligned} c_2(\operatorname{Ind}_H^G \chi) = \operatorname{Cores}(c_2(\chi)) + \operatorname{Cores}^{(2)}(c_1(\chi)) \\ + c_1(\operatorname{Ind}_H^G 1).\operatorname{Cores}(c_1(\chi)) + c_2(\operatorname{Ind}_H^G 1), \end{aligned}$$

where we denote by $\operatorname{Cores}^{(k)}$ the intermediate transfer maps. By [**FMP**, p. 4], for any z in $H^1(H, \mathbf{Q}/\mathbf{Z})$, one has

$$\operatorname{Cores}(z^2) - \operatorname{Cores}(z)^2 + 2\operatorname{Cores}^{(2)}(z) = 0.$$

Since $p \neq 2$, we get the relation

$$\operatorname{Cores}^{(2)}(z) = \frac{1}{2}(\operatorname{Cores}(z)^2 - \operatorname{Cores}(z^2))$$

and therefore the relation

$$\begin{aligned} c_2(\operatorname{Ind}_H^G \chi) = \frac{1}{2}(\operatorname{Cores}_H^G(c_1(\chi))^2 - \operatorname{Cores}_H^G(c_1(\chi)^2)) \\ + c_1(\operatorname{Ind}_H^G 1).\operatorname{Cores}_H^G(c_1(\chi)) + c_2(\operatorname{Ind}_H^G 1), \end{aligned}$$

Therefore, it remains to show that for any subgroup H of G, one has

$$c_2(\operatorname{Ind}_H^G 1) \in H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}).$$

We shall proceed by induction on $[G : H]$. If $[G : H] = 1$, then $c_2(1) = 0$ and the result is proven. Let us assume the result for subgroups of index strictly smaller than p^m for $m \geqslant 1$. Let H be a subgroup of G with $[G : H] = p^m$. There exists a subgroup H_1 of G such that H is a normal subgroup of H_1 of index p [**Su**, theorem 1.6]. We have

$$c_2(\operatorname{Ind}_H^G 1) = c_2(\operatorname{Ind}_{H_1}^G(\operatorname{Ind}_H^{H_1} 1)).$$

We may choose $\chi \in \operatorname{Hom}(H_1, \mathbf{C}^*)$ such that $H = \operatorname{Ker}\chi$. Then the induced representation is given by $\operatorname{Ind}_H^{H_1} 1 = 1 + \chi + \cdots + \chi^{p-1}$ in $\mathscr{R}(H_1)$. We get

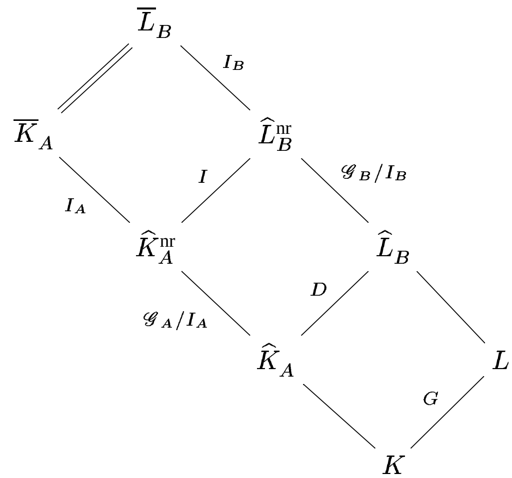
$$\begin{aligned} c_2(\operatorname{Ind}_H^G 1) &= c_2(\operatorname{Ind}_{H_1}^G(1) + \cdots + \operatorname{Ind}_{H_1}^G(\chi^{p-1})) \\ &\equiv c_2(\operatorname{Ind}_{H_1}^G(1)) + \cdots + c_2(\operatorname{Ind}_{H_1}^G(\chi^{p-1})) \bmod H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z}). \end{aligned}$$

By induction, we obtain that $c_2(\operatorname{Ind}_H^G 1)$ belongs to $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$. □

Let us now describe the inverse image in $H^3(G, \mathbf{Q}/\mathbf{Z})$ of the unramified cohomology group of $\mathbf{C}(W)^G$.

Proposition 4.5. — *The group $H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$ is the inverse image in $H^3(G, \mathbf{Q}/\mathbf{Z})$ of the group $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$.*

Proof. — Let γ in $H^3_{\mathrm{nr}}(G, \mathbf{Q}/\mathbf{Z})$. We want to prove that its image in $H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ is unramified. Let $A \in \mathscr{P}(\mathbf{C}(W)^G/\mathbf{C})$ and B be an element of $\mathscr{P}(\mathbf{C}(W)/\mathbf{C})$ above A. We put $K = \mathbf{C}(W)^G$, $L = \mathbf{C}(W)$, $\widehat{L}_B$ the completion of L at B, $\widehat{K}_A$ the completion of K in $\widehat{L}_B$, $\overline{L}_B$ an algebraic closure of $\widehat{L}_B$, $\widehat{K}_A^{\mathrm{nr}}$ (resp. $\widehat{L}_B^{\mathrm{nr}}$) the maximal unramified extension of K_A (resp. L_B) in $\overline{L}_B$. We denote by D the decomposition group of B in G and by I the inertia group. We also put $\mathscr{G}_A = \mathrm{Gal}(\overline{L}_B/\widehat{K}_A)$, $\mathscr{G}_B = \mathrm{Gal}(\overline{L}_B/\widehat{L}_B)$, $I_A = \mathrm{Gal}(\overline{L}_B/\widehat{K}_A^{\mathrm{nr}})$, and $I_B = \mathrm{Gal}(\overline{L}_B/\widehat{L}_B^{\mathrm{nr}})$. We have the following diagram of fields



which yields a commutative diagram of groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_A & \longrightarrow & \mathscr{G}_A & \longrightarrow & \mathscr{G}_A/I_A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_I} & & \downarrow{\scriptstyle f_{\mathscr{G}}} & & \downarrow & & \\ 0 & \longrightarrow & I & \xrightarrow{j} & D & \longrightarrow & D/I & \longrightarrow & 0. \end{array} \tag{4.1}$$

On the other hand the residue map

$$H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}(2)) \xrightarrow{\partial_A} H^2(\kappa_A, \mathbf{Q}/\mathbf{Z}(1))$$

is defined as the composite of the maps

$$\begin{aligned} H^3(K, \mathbf{Q}/\mathbf{Z}(2)) &\to H^3(\widehat{K}_A, \mathbf{Q}/\mathbf{Z}(2)) \\ &\to H^2(\mathscr{G}_A/I_A, H^1(I_A, \mathbf{Q}/\mathbf{Z}(2))) \xrightarrow{\sim} H^2(\kappa_A, \mathbf{Q}/\mathbf{Z}(1)) \end{aligned} \tag{4.2}$$

where the second map is induced be the hochschild-Serre spectral sequence

$$H^p(\mathscr{G}_A/I_A, H^q(I_A, \mathbf{Q}/\mathbf{Z}(2))) \Rightarrow H^{p+q}(\mathscr{G}_A, \mathbf{Q}/\mathbf{Z}(2)).$$

Indeed I_A, which is isomorphic to $\widehat{\mathbf{Z}}(1)$ is of cohomological dimension 1, and the group $H^1(I_A, \mathbf{Q}/\mathbf{Z}(n))$ is canonically isomorphic to $\mathbf{Q}/\mathbf{Z}(n-1)$. The latter fact gives the last morphism in (4.2). Since the roots of unity are in $\mathbf{C}$, we may choose a splitting of the central extension

$$0 \to I_A \to \mathscr{G}_A \to \mathscr{G}_A/I_A \to 0.$$

Using (4.1), we get that I is central in D and the morphism $f_{\mathscr{G}}$ factorizes through $D \times I$: let s be a section of $I_A \to \mathscr{G}_A$, then the following diagram commutes

$$\begin{array}{ccc}
\mathscr{G}_A & \xrightarrow{(\mathrm{Id}-s)\times s} & \mathscr{G}_A \times I_A \\
\downarrow & & \downarrow{\scriptstyle f_{\mathscr{G}} \times f_I} \\
D & \xleftarrow[\mathrm{Id}+j]{} & D \times I
\end{array}$$

where we denote by $(\mathrm{Id}-s) \times s$ the morphism sending g to $(gs(g)^{-1}, s(g))$. Thus we get the commutative diagram

$$(4.3) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & I_A & \longrightarrow & \mathscr{G}_A & \longrightarrow & \mathscr{G}_A/I_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & I \times D & \longrightarrow & D & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{Id}+j} & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & D/I & \longrightarrow & 0.
\end{array}$$

For the cohomology groups we have commutative diagrams

$$\begin{array}{ccccc}
H^3(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\mathrm{Res}} & H^3(D, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(D \times I, \mathbf{Q}/\mathbf{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
H^3(K, \mathbf{Q}/\mathbf{Z}(2)) & \longrightarrow & H^3(\widehat{K}_A, \mathbf{Q}/\mathbf{Z}(2)) & \xrightarrow{\sim} & H^3(\mathscr{G}_A, \mathbf{Q}/\mathbf{Z}(2))
\end{array}$$

and

$$\begin{array}{ccc}
H^3(I, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\mathrm{pr}_2^*} & H^3(D \times I, \mathbf{Q}/\mathbf{Z}) \\
\downarrow & & \downarrow \\
0 = H^3(I_A, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{s^*} & H^3(\mathscr{G}_A, \mathbf{Q}/\mathbf{Z}).
\end{array}$$

Thus we get commutative diagrams

$$\begin{array}{ccc}
H^3(D\times I,\mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^3(\mathscr{G}_A,\mathbf{Q}/\mathbf{Z}(2)) \\
\downarrow & & \| \\
\mathrm{Ker}(H^3(D\times I,\mathbf{Q}/\mathbf{Z})\to H^3(I,\mathbf{Q}/\mathbf{Z})) & \longrightarrow & H^3(\mathscr{G}_A,\mathbf{Q}/\mathbf{Z}(2)) \\
\downarrow & & \downarrow \\
H^2(D,H^1(I,\mathbf{Q}/\mathbf{Z})) & \longrightarrow & H^2(\mathscr{G}_A/I_A,H^1(I_A,\mathbf{Q}/\mathbf{Z}(2)))
\end{array}$$

and we may choose a generator g of I so that the diagram

$$\begin{array}{ccc}
H^3(G,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{\partial_{D,g}} & H^2(D,\mathbf{Q}/\mathbf{Z}) \\
\downarrow & & \downarrow \\
H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}(2)) & \xrightarrow{\partial_A} & H^2(\kappa_A,\mathbf{Q}/\mathbf{Z}(1)).
\end{array}$$

commutes. Therefore $\partial_A(\gamma)=0$ whenever γ belongs to $H^3_{\mathrm{nr}}(G,\mathbf{Q}/\mathbf{Z})$.

We now want to prove the reverse inclusion. For any positive integer i, let $H^i_{\mathrm{gnr}}(G,\mathbf{Q}/\mathbf{Z})$ be the inverse image in $H^i(G,\mathbf{Q}/\mathbf{Z})$ of $H^i_{\mathrm{nr}}(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z})$. For any morphism of group $\pi : H\to G$, we have

$$\pi^*(H^i_{\mathrm{gnr}}(G,\mathbf{Q}/\mathbf{Z}))\subset H^i_{\mathrm{gnr}}(H,\mathbf{Q}/\mathbf{Z}).$$

Indeed let W be a faithful representation of G and V be a faithful representation of H. Then W is a representation of H via π and $V\oplus W$ a faithful representation of H. But we have the following field inclusions

$$\mathbf{C}(W)^G\subset\mathbf{C}(W)^H\subset\mathbf{C}(V\oplus W)^H.$$

Therefore, we get a commutative diagram

$$\begin{array}{ccc}
H^3(G,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{\pi^*} & H^3(H,\mathbf{Q}/\mathbf{Z}) \\
\downarrow & & \downarrow \\
H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{i} & H^3(\mathbf{C}(V\oplus W)^H,\mathbf{Q}/\mathbf{Z})
\end{array}$$

and by [**CTO**] the image by i of $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z})$ is contained in $H^3_{\mathrm{nr}}(\mathbf{C}(V\oplus W)^H,\mathbf{Q}/\mathbf{Z})$. This implies the claim.

We have to show that for any γ in $H^3_{\mathrm{gnr}}(G,\mathbf{Q}/\mathbf{Z})$, for any subgroup H of G, and for any g in $Z_G(H)$ generating a subgroup I of G, we have $\partial_{H,g}(\gamma)=0$. By the last claim and the definition of $\partial_{H,g}$, we can restrict ourselves to the case where $G=H\times I$. In that particular case, let W be a faithful representation of H and χ be the injection $I\hookrightarrow\mathbf{C}^*$ sending g to the chosen generator of $\mu_{|I|}$. Then $W\oplus\chi$ is a faithful representation of G. We may consider

$\mathbf{C}(W \oplus \chi)$ as $\mathbf{C}(W)(X)$ and define $B \in \mathscr{P}(\mathbf{C}(W \oplus \chi)/\mathbf{C})$ as the valuation defined by the divisor $X = 0$. Let A be the induced element of $\mathbf{C}(W \oplus \chi)^G = \mathbf{C}(W)^H(X^{|I|})$. We now are precisely in the situation described in the first part of the proof and we get a commutative diagram

$$\begin{array}{ccc} H^3(G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\partial_{H,g}} & H^2(H, \mathbf{Q}/\mathbf{Z}) \\ \downarrow & & \downarrow \\ H^3(\mathbf{C}(W \oplus \chi)^G, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\partial_A} & H^2(\mathbf{C}(W)^H, \mathbf{Q}/\mathbf{Z}). \end{array}$$

But the group of geometrically negligible classes in degree 2 is trivial (see, for example, [**Sa1**]). Therefore, if γ belongs to $H^3_{\mathrm{gnr}}(G, \mathbf{Q}/\mathbf{Z})$ then $\partial_{H,g}(\gamma) = 0$. □

5. Central extensions of vector spaces

5.1. The setting. — It is well known that if G is abelian and W a faithful representation of G, then $\mathbf{C}(W)^G$ is rational over $\mathbf{C}$. Therefore the first interesting extensions are central extensions of an $\mathbf{F}_p$-vector space by another one. The unramified Brauer group have been computed for these groups by Bogomolov in [**Bo1**] (see also Saltman [**Sa1**]). A few preliminary results in degree 3 have been given in [**Pe2**]. Let us first recall these results, since they will be used later.

Notations 5.1. — Let U and V be two $\mathbf{F}_p$-vector spaces for p an odd prime number and let

$$0 \to V \xrightarrow{j} G \xrightarrow{\pi} U \to 0$$

be a central extension of U by V such that $\exp(G) = p$. For any g in G, we put $\overline{g} = \pi(g)$. Without loss of generality, we may assume that $V = [G, G]$ or in other words, that the map

$$\begin{array}{rccc} \gamma : & \Lambda^2 U & \to & V \\ & \pi(g_1) \wedge \pi(g_2) & \mapsto & [g_1, g_2] \end{array}$$

is surjective. By [**Bro**, §IV.3, exercise 8], this map γ determines this extension up to isomorphism. More precisely, we may choose a set-theoritic section $s : U \to G$ of π such that

$$\forall u_1, u_2 \in U, \quad s(u_2)s(u_1u_2)^{-1}s(u_1) = \frac{1}{2}\gamma(u_1 \wedge u_2).$$

If $Z(G) \neq [G, G]$ then G is isomorphic to a product $E \times H$ where E is the $\mathbf{F}_p$-vector space $Z(G)/[G, G]$. Let W be a faithful representation of H and W' a faithful representation of E. Then $W \oplus W'$ is a faithful representation of G and $\mathbf{C}(W \oplus W')^G$ is rational over $\mathbf{C}(W)^H$. Thus we may assume that $Z(G) = [G, G]$.

For any $\mathbf{F}_p$-vector space E we denote by $E^\vee$ it dual. For any positive integer there is a natural isomorphism

$$\begin{array}{rcl} \Lambda^i(E^\vee) & \to & (\Lambda^i E)^\vee \\ f_1 \wedge \dots \wedge f_i & \mapsto & \Big(v_1 \wedge \dots \wedge v_i \mapsto \sum_{\sigma \in \mathfrak{S}_i} \epsilon(\sigma) f_1(v_{\sigma(1)}) \dots f_i(v_{\sigma(i)})\Big). \end{array}$$

From now on, we identify $\Lambda^i(E^\vee)$ with $(\Lambda^i E)^\vee$ and denote it by $\Lambda^i E^\vee$. For any subgroup F of $\Lambda^i E$ (resp. $\Lambda^i E^\vee$) we denote by $F^\perp$ its orthogonal in $\Lambda^i E^\vee$ (resp. $\Lambda^i E$).

The linear map γ induces an injection

$$\gamma^\vee : V^\vee \to \Lambda^2 U^\vee.$$

We shall identify $V^\vee$ with its image and put

$$K^2 = V^\vee \subset \Lambda^2 U^\vee \quad \text{and} \quad K^3 = V^\vee \wedge U^\vee \subset \Lambda^3 U^\vee.$$

We put $S^i = (K^i)^\perp$ if $i = 2$ or 3. Let S^3_{dec} (resp. S^2_{dec}) be the subgroup of S^3 (resp S^2) generated by the elements of the form $u \wedge v$ for $u \in \Lambda^2 U$ (resp. U) and $v \in U$. We define $K^i_{\max} \supset K^i$ as the orthogonal of S^i_{dec} for $i = 2$ or 3.

Using [**Bro**, p. 60, 126], we get an injection

$$\Lambda^i U^\vee \hookrightarrow H^i(U, \mathbf{Q}/\mathbf{Z})$$

defined as the composite map

$$\Lambda^i U^\vee \xrightarrow{\sim} \Lambda^i H^1(U, \mathbf{F}_p) \xrightarrow{\cup} H^i(U, \mathbf{F}_p) \to H^i(U, \mathbf{Q}/\mathbf{Z}) \qquad (5.1)$$

where $\cup$ is the cup-product (see also [**Pe1**, lemma 7]).

Let us recall the result of Bogomolov in this context: by [**Bo1**, lemma 5.1], one has that

$$K^2_{\max}/K^2 \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(\mathbf{C}(W)^G) = H^2_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}).$$

The results obtained in [**Pe2**] imply the following proposition:

Proposition 5.2. — *The inverse image in $\Lambda^3 U^\vee$ of the group $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$ coincides with $K^3_{\max}$.*

Proof. — By [**Pe2**, lemma 9.3], the kernel of the map $\Lambda^3 U^\vee \to H^3(G, \mathbf{Q}/\mathbf{Z})$ is $U^\vee \wedge V^\vee$. Therefore

$$K^3 \subset \mathrm{Ker}(\Lambda^3 U^\vee \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})).$$

Therefore

$$S^3 \supset \mathrm{Ker}(\Lambda^3 U^\vee \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}))^\perp.$$

Taking the subgroup for both groups generated by elements of the form $u \wedge v$ for $u \in \Lambda^2 U$ and $v \in U$, we get

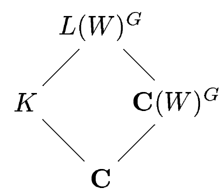
$$S^3_{\text{dec}} \supset \mathrm{Ker}(\Lambda^3 U^\vee \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}))^\perp_{\text{dec}}.$$

Thus for any f in $K^3_{\max}$,

$$f_{|\mathrm{Ker}(\Lambda^3 U^\vee \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}))^\perp_{\text{dec}}} = 0.$$

By [**Pe1**, theorem 2], this implies that $K^3_{\max}$ is contained in the inverse image of the group $H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$.

By [**Pe2**, proposition 9.4 and lemma 9.2], there exists a function field K over $\mathbf{C}$ and a Galois extension L of K with Galois group G such that K^3_{max} is the inverse image of $H^3_{\text{nr}}(K, \mathbf{Q}/\mathbf{Z})$ in $\Lambda^3 U^\vee$. But we have a diagram of fields



By the no-name lemma, the extension $L(W)^G/K$ is rational. Therefore

$$H^3_{\text{nr}}(K, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} H^3_{\text{nr}}(L(W)^G, \mathbf{Q}/\mathbf{Z}).$$

But, by [**CTO**, p. 143], the natural extension map

$$\phi : H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}) \to H^3(L(W)^G, \mathbf{Q}/\mathbf{Z})$$

verifies

$$\phi(H^3_{\text{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})) \subset H^3_{\text{nr}}(L(W)^G, \mathbf{Q}/\mathbf{Z}).$$

Thus if γ in $\Lambda^3 U^\vee$ is in the inverse image of $H^3_{\text{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$, it belongs to the inverse image of $H^3_{\text{nr}}(K, \mathbf{Q}/\mathbf{Z})$ and thus to K^3_{max}. □

5.2. The result. — Our aim in this paragraph is to prove the following result:

Theorem 5.3. — *With notations as above, there is an injection*

$$K^3_{\text{max}}/K^3 \subset H^3_{\text{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}).$$

Remark 5.4. — In [**Pe2**, §9.3], we construct an example of a 2-group where

$$K^3 \neq \text{Ker}(\Lambda^3 U^\vee \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})).$$

This shows that the condition $p \neq 2$ is necessary.

To prove this theorem it remains to prove that

$$K^3 = \text{Ker}(\Lambda^3 U^\vee \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}))$$

or, using theorem 3.1, that K^3 is the inverse image in $\Lambda^3 U^\vee$ of $H^3_{\text{p}}(G; \mathbf{Q}/\mathbf{Z})$. The most technical part to prove this is to be able to deal with the corestriction map. We shall do it step by step.

5.3. Technical lemmata. — To begin with let us recall why the corestriction map is compatible with Hochschild-Serre spectral sequence.

Notation 5.5. — If H is normal subgroup of a group G, we denote by $E_i^{p,q}(G/H)$ the groups pertaining to the Hochschild-Serre spectral sequence

$$E_2^{p,q}(G/H) = H^p(G/H, H^q(H, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(G, \mathbf{Q}/\mathbf{Z}).$$

Lemma 5.6. — *Let G be a group, H be a subgroup of G of finite index and K a normal subgroup of G contained in H. Then the Hochschild-Serre spectral sequences*

$$E_2^{p,q}(G/K) = H^p(G/K, H^q(K, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(G, \mathbf{Q}/\mathbf{Z})$$

and

$$E_2^{p,q}(H/K) = H^p(H/K, H^q(K, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(H, \mathbf{Q}/\mathbf{Z})$$

are compatible with the corestriction maps

$$\mathrm{Cores}_{H/K}^{G/K} : H^p(H/K, H^q(K, \mathbf{Q}/\mathbf{Z})) \to H^p(G/K, H^q(K, \mathbf{Q}/\mathbf{Z}))$$

and

$$\mathrm{Cores}_H^G : H^p(H, \mathbf{Q}/\mathbf{Z}) \to H^p(G, \mathbf{Q}/\mathbf{Z}).$$

Proof. — The proof of this well-known lemma is similar to the one given for lemma 3.1.6 in [**Pe3**]: for any G-module M, we may consider M as an H-module and define the induced G-module $\mathrm{Ind}_H^G M$. There exists a natural trace map $\mathrm{Tr} : \mathrm{Ind}_H^G M \to M$ and the corestriction is the composite of the maps

$$H^p(H, M) \xrightarrow{\sim} H^p(G, \mathrm{Ind}_H^G M) \xrightarrow{\mathrm{Tr}} H^p(G, M)$$

where the first map is Shapiro isomorphism. Both maps are compatible with the Hochschild-Serre spectral sequences. □

We shall now recall a few basic facts about the cohomology groups of an $\mathbf{F}_p$-vector space.

Lemma 5.7. — *If p is a prime number and E an $\mathbf{F}_p$-vector space, then for any strictly positive integer i, one has*

$$pH^i(E, \mathbf{Q}/\mathbf{Z}) = \{0\}.$$

Proof. — We prove it by induction on the dimension n of E. The result is true if $n = 0$. If $n \geqslant 1$, let E' be a subgroup of index p in E. We may write E as $E' \oplus \mathbf{F}_p$. The multiplication by p in $H^i(E, \mathbf{Q}/\mathbf{Z})$ coincides with the composite map $\mathrm{Cores}_{E'}^E \circ \mathrm{Res}_{E'}^E$. But $\mathrm{Cores}_{E'}^E$ is equal to $p.\,\mathrm{pr}_1^*$. Thus $p = \mathrm{pr}_1^* \circ \mathrm{Res}_{E'}^E \circ p$. By induction, we get that $p = 0$. □

Notations 5.8. — Let p be an odd prime number. For any $\mathbf{F}_p$-vector space E of finite dimension, we denote by ϕ_i the natural injection $\Lambda^i E^\vee \hookrightarrow H^i(E, \mathbf{Q}/\mathbf{Z})$ defined as in (5.1) and we consider the map

$$\psi_i : S^i(E^\vee) \hookrightarrow H^{2i-1}(E, \mathbf{Q}/\mathbf{Z})$$

given as the composite map

$$S^i(E^\vee) \xrightarrow{\sim} S^i H^2(E, \mathbf{Z}) \xrightarrow{\cup} H^{2i}(E, \mathbf{Z}) \xrightarrow{\sim} H^{2i-1}(E, \mathbf{Q}/\mathbf{Z}).$$

Lemma 5.9. — *We have the following isomorphisms*

$$\begin{aligned} \mathbf{Q}/\mathbf{Z} &= H^0(E, \mathbf{Q}/\mathbf{Z}), \\ E^\vee &\xrightarrow{\sim} H^1(E, \mathbf{Q}/\mathbf{Z}), \\ \Lambda^2 E^\vee &\xrightarrow[\sim]{\phi_2} H^2(E, \mathbf{Q}/\mathbf{Z}), \end{aligned}$$

and

$$\Lambda^3 E^\vee \oplus S^2(E^\vee) \xrightarrow[\sim]{\phi_3+\psi_2} H^3(E, \mathbf{Q}/\mathbf{Z}).$$

Proof. — This lemma follows from the description of the homology of E given in [**Car**, theorem 1] and the isomorphism

$$H^n(E, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}(H_n(E, \mathbf{Z}), \mathbf{Q}/\mathbf{Z})$$

(see [**Bro**, p. 60]). □

Notation 5.10. — From now on, we fix a group G as in notation 5.1 and we consider the Hochschild-Serre spectral sequence

$$H^p(U, H^q(V, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(G, \mathbf{Q}/\mathbf{Z}).$$

We denote by $F^p H^j(G, \mathbf{Q}/\mathbf{Z})$ the corresponding decreasing filtration on the cohomology of the group G.

Lemma 5.11. — *There is a commutative diagram*

$$\begin{array}{ccccc}
H^2(V,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{d^{0,2}} & H^2(U,H^1(V,\mathbf{Q}/\mathbf{Z})) & \xrightarrow{d^{2,1}} & H^4(U,\mathbf{Q}/\mathbf{Z}) \\
\uparrow \wr & & \uparrow & & \uparrow \\
\Lambda^2 V^\vee & \longrightarrow & \Lambda^2 U^\vee \otimes V^\vee & \longrightarrow & \Lambda^4 U^\vee \\
\rho_1 \wedge \rho_2 & \longmapsto & -\gamma^\vee(\rho_1)\otimes\rho_2 + \gamma^\vee(\rho_2)\otimes\rho_1 & & \\
 & & \lambda\otimes\rho & \longmapsto & -\lambda\wedge\gamma^\vee(\rho)
\end{array}$$

where $d^{0,2}$ and $d^{2,1}$ are the maps defined by the Hochschild-Serre spectral sequence

$$H^p(U, H^q(V, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(G, \mathbf{Q}/\mathbf{Z}).$$

In particular, if we denote by $\mathscr{C}$ the complex of the bottom line we get an injection from the homology group $H(\mathscr{C})$ of $\mathscr{C}$ into $E_\infty^{2,1}(G/V)$.

Proof. — The map $d^{0,2}$ has been computed in [**Pe2**, p. 135]. The description of the map $d^{2,1}$ follows from the fact that there is a commutative diagram

$$\begin{array}{ccc}
H^1(V,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{d^{0,1}} & H^2(U,\mathbf{Q}/\mathbf{Z}) \\
\uparrow \wr & & \uparrow \\
V^\vee & \xrightarrow{-\gamma^\vee} & \Lambda^2 U^\vee
\end{array}$$

(see [**Pe2**, p. 135]) and the compatibility of the Hochschild-Serre spectral sequence with the cup-product. □

Remark 5.12. — Using $\gamma^\vee : V^\vee \hookrightarrow \Lambda^2 U^\vee$, we get a natural map

$$S^2V^\vee \hookrightarrow \Lambda^2U^\vee \otimes V^\vee$$

which maps $\rho_1\rho_2$ to $1/2(\gamma^\vee(\rho_1)\otimes\rho_2 + \gamma^\vee(\rho_2)\otimes\rho_1)$ and therefore an injection

$$\mathrm{Ker}(S^2V^\vee \to \Lambda^4U^\vee) \hookrightarrow E_\infty^{2,1}. \tag{5.2}$$

The strategy for the proof is to construct a subgroup of $H^3(G, \mathbf{Q}/\mathbf{Z})$ which does not intersect the image of $\Lambda^3U^\vee$ and contains $H^3_{\mathrm{p}}(G,\mathbf{Q}/\mathbf{Z})$. In order to do this, we want to construct a map $\tau : \mathrm{Ker}(S^2V^\vee \to \Lambda^4U^\vee) \to F^2H^3(G,\mathbf{Q}/\mathbf{Z})$ which lifts the map (5.2), that is such that the diagram

$$\begin{array}{ccc} H(\mathscr{C}) & \hookrightarrow & E_\infty^{2,1}(G/V) \\ \uparrow & & \uparrow \\ \mathrm{Ker}(S^2U^\vee \to \Lambda^4U^\vee) & \xrightarrow{\tau} & F^2H^3(G,\mathbf{Q}/\mathbf{Z}) \end{array}$$

commutes. We also want this lifting to be compatible with the corestriction in a sense which shall be described later. The road-map for this construction is given by the construction of the Hochschild-Serre spectral sequence [**HS**, §2]: if we take a class γ in $H^2(U, H^1(V,\mathbf{Q}/\mathbf{Z}))$, we can lift it to an element $\tilde\gamma$ of $C^2(U, C^1(V,\mathbf{Q}/\mathbf{Z}))$ which gives a map

$$\begin{array}{rccl} \widehat{\gamma}: & V\times G^2 & \to & \mathbf{Q}/\mathbf{Z} \\ & (v,g_2,g_3) & \mapsto & \tilde\gamma(\overline{g}_2,\overline{g}_3)(v). \end{array}$$

We extend this map in a cochain $f : G^3 \to \mathbf{Q}/\mathbf{Z}$ by

$$f(g_1,g_2,g_3) = \widehat{\gamma}(g_1 s(\overline{g}_1)^{-1}, g_2, g_3).$$

Then $\mathrm{d}f$ factorizes through a cocycle $U^4 \to \mathbf{Q}/\mathbf{Z}$. the class of which in $H^4(U,\mathbf{Q}/\mathbf{Z})$ is $d^{2,1}(\gamma)$. If $d^{2,1}(\gamma)=0$, then there exists an element h in $C^3(U,\mathbf{Q}/\mathbf{Z})$ such that

$$\mathrm{d}f\,(g_1,g_2,g_3,g_4) = \mathrm{d}h\,(\overline{g}_1,\overline{g}_2,\overline{g}_3,\overline{g}_4) \tag{5.3}$$

thus the class of the cocycle

$$(g_1,g_2,g_3) \mapsto f(g_1,g_2,g_3) - h(\overline{g}_1,\overline{g}_2,\overline{g}_3)$$

is a lifting of γ in $F^2H^3(G,\mathbf{Q}/\mathbf{Z})$.

Therefore the first step of this construction is the description of f and $\mathrm{d}f$.

Lemma 5.13. — *For any ρ in $V^\vee$ and any λ in $\Lambda^2U^\vee$, we define a map $f_{\rho,\lambda} : G^3 \to \mathbf{Q}/\mathbf{Z}$ by*

$$f_{\rho,\lambda}(g_1,g_2,g_3) = \frac{1}{2}\rho(g_1 s(g_1)^{-1})\lambda(\overline{g}_2\wedge\overline{g}_3). \tag{5.4}$$

One has

$$\mathrm{d}f_{\rho,\lambda}(g_1,g_2,g_3,g_4) = -\frac{1}{4}\gamma^\vee(\rho)(\overline{g}_1\wedge\overline{g}_2)\lambda(\overline{g}_3\wedge\overline{g}_4).$$

Remark 5.14. — One may notice that $\mathrm{d}f_{\rho,\lambda}$ defines a class in $H^4(U, \mathbf{Q}/\mathbf{Z})$ which coincides with the image of $-\lambda \wedge \gamma^\vee(\rho)$. Thus lemma 5.13 implies the description of $d^{2,1}$ given in lemma 5.11.

Proof. — Since the map $(g_2, g_3) \mapsto \lambda(\overline{g}_2 \wedge \overline{g}_3)$ is a cocycle, it is sufficient to prove that if $h : G \to \mathbf{Q}/\mathbf{Z}$ is defined by

$$h(g) = \rho(gs(\overline{g})^{-1})$$

then

$$\mathrm{d}h\,(g_1, g_2) = -\frac{1}{2}\gamma^\vee(\rho)(\overline{g}_1 \wedge \overline{g}_2).$$

But

$$\begin{aligned}\mathrm{d}h\,(g_1, g_2) &= h(g_2) - h(g_1g_2) + h(g_1)\\ &= \rho(g_2s(\overline{g}_2)^{-1}) - \rho(g_1g_2s(\overline{g}_1\overline{g}_2)^{-1}) + \rho(g_1s(\overline{g}_1)^{-1})\\ &= \rho(g_2s(\overline{g}_2)^{-1}) - \rho(g_1g_2s(\overline{g}_2)^{-1}s(\overline{g}_1)^{-1}) - \frac{1}{2}\rho(\gamma(\overline{g}_1 \wedge \overline{g}_2)) + \rho(g_1s(\overline{g}_1)^{-1}).\end{aligned}$$

The element $g_2s(\overline{g}_2)^{-1}$ belongs to $V = Z(G)$ so that

$$\mathrm{d}h\,(g_1, g_2) = -\frac{1}{2}\gamma^\vee(\rho)(\overline{g}_1 \wedge \overline{g}_2). \quad \square$$

The next step of the construction is to describe the map h in (5.3). This is done in the following two lemmata.

Lemma 5.15. — *The group $\mathfrak{S}_4$ acts on $U^{\vee\otimes 4}$ by permutation of the components. Let*

$$\mathfrak{S}_- = \langle (1\,2), (3\,4)\rangle \subset \mathfrak{S}_4 \quad \textit{and} \quad \mathfrak{S}_+ = \langle (1\,4), (2\,3)\rangle \subset \mathfrak{S}_4$$

and let $S^\square U^\vee$ be the image in $U^{\vee\otimes 4}$ of the map

$$\lambda \mapsto \sum_{\sigma\in\mathfrak{S}_-} \epsilon(\sigma)\sigma\Big(\sum_{\sigma'\in\mathfrak{S}_+} \sigma'\lambda\Big).$$

Then

$$\mathrm{Ker}(S^2(\Lambda^2U^\vee) \to \Lambda^4U^\vee)$$

is isomorphic to $S^\square U^\vee$.

Remark 5.16. — If $p \geqslant 5$, $S^\square U^\vee$ is the irreducible $\mathfrak{S}_4$-submodule of $U^{\vee\otimes 4}$ corresponding to the Young table

1	4
2	3

Proof. — The kernel of the map $U^{\vee\otimes 4} \to \Lambda^4U^\vee$ may be described as the image of

$$S^2U^\vee \otimes U^{\vee\otimes 2} \oplus U^\vee \otimes S^2U^\vee \otimes U^\vee \oplus U^{\vee\otimes 2} \otimes S^2U^\vee$$

in $U^{\vee\otimes 4}$. Therefore the kernel of the map $\Lambda^2U^\vee \otimes \Lambda^2U^\vee \to \Lambda^4U^\vee$ is given as the image of the composite map

$$U^\vee \otimes S^2U^\vee \otimes U^\vee \to U^{\vee\otimes 4} \to \Lambda^2U^\vee \otimes \Lambda^2U^\vee. \tag{5.5}$$

It remains to describe the composite map

$$U^\vee \otimes S^2U^\vee \otimes U^\vee \to \Lambda^2U^\vee \otimes \Lambda^2U^\vee \to S^2(\Lambda^2U^\vee) \to U^{\vee\otimes 4}. \tag{5.6}$$

The image of an element of the form $u \otimes vw \otimes x \in U^\vee \otimes S^2U^\vee \otimes U^\vee$ in $U^{\vee\otimes 4}$ by the map defined in (5.5) is

$$\frac{1}{2}(u \otimes v \otimes w \otimes x + u \otimes w \otimes v \otimes x)$$

its image in $S^2(\Lambda^2U^\vee)$ is

$$\frac{1}{2}(u \wedge v.w \wedge x + u \wedge w.v \wedge x)$$

and its image in $(U^\vee)^{\otimes 4}$ is given as

$$\begin{aligned}\frac{1}{16}(&u\otimes v\otimes w\otimes x + w\otimes x\otimes u\otimes v - v\otimes u\otimes w\otimes x - w\otimes x\otimes v\otimes u\\ &+ v\otimes u\otimes x\otimes w + x\otimes w\otimes v\otimes u - u\otimes v\otimes x\otimes w - x\otimes w\otimes u\otimes v\\ &+ u\otimes w\otimes v\otimes x + v\otimes x\otimes u\otimes w - w\otimes u\otimes v\otimes x - v\otimes x\otimes w\otimes u\\ &+ w\otimes u\otimes x\otimes v + x\otimes v\otimes w\otimes u - u\otimes w\otimes x\otimes v - x\otimes v\otimes u\otimes w).\end{aligned} \tag{5.7}$$

□

Notations 5.17. — We put

$$S_{13}U^\vee = \{\, \lambda \in U^{\vee\otimes 4} \mid (13).\lambda = \lambda\}$$

and

$$S_{23}U^\vee = \{\, \lambda \in U^{\vee\otimes 4} \mid (23).\lambda = \lambda\}$$

and define maps

$$\tau_{13} : S_{13}U^\vee \to C^3(U, \mathbf{Q}/\mathbf{Z})/\operatorname{Im} \mathrm{d} \quad \text{and} \quad \tau_{23} : S_{23}U^\vee \to C^3(U, \mathbf{Q}/\mathbf{Z})/\operatorname{Im} \mathrm{d}$$

as follows $\tau_{23}(u \otimes v \otimes w \otimes x + u \otimes w \otimes v \otimes x)$ is the class of the cochain

$$(g_1, g_2, g_3) \mapsto u(\overline{g}_1)v(\overline{g}_2)w(\overline{g}_2)x(\overline{g}_3) \tag{5.8}$$

and $\tau_{13}(u \otimes v \otimes w \otimes x + w \otimes v \otimes u \otimes x)$ the class of the cochain

$$\begin{aligned}(g_1, g_2, g_3) \mapsto &u(\overline{g}_1)v(\overline{g}_2)w(\overline{g}_2)x(\overline{g}_3)\\ &- u(\overline{g}_1)w(\overline{g}_1)v(\overline{g}_2)x(\overline{g}_3)\\ &+ w(\overline{g}_1)v(\overline{g}_2)u(\overline{g}_2)x(\overline{g}_3).\end{aligned} \tag{5.9}$$

We also consider the natural morphism $U^{\vee\otimes 4} \xrightarrow{\mu} C^4(U, \mathbf{Q}/\mathbf{Z})$ sending $u \otimes v \otimes w \otimes x$ onto

$$(g_1, g_2, g_3, g_4) \mapsto u(\overline{g}_1)v(\overline{g}_2)w(\overline{g}_3)x(\overline{g}_4).$$

Lemma 5.18. — *The following diagrams are commutative*

$$\begin{array}{ccc} S_{13}U^\vee & \hookrightarrow & U^{\vee\otimes 4} \\ \downarrow{\scriptstyle \tau_{13}} & & \downarrow{\scriptstyle \mu} \\ C^3(U,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d} & \xrightarrow{\mathrm{d}} & C^4(U,\mathbf{Q}/\mathbf{Z}) \end{array} \qquad \begin{array}{ccc} S_{23}U^\vee & \hookrightarrow & U^{\vee\otimes 4} \\ \downarrow{\scriptstyle \tau_{23}} & & \downarrow{\scriptstyle \mu} \\ C^3(U,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d} & \xrightarrow{\mathrm{d}} & C^4(U,\mathbf{Q}/\mathbf{Z}). \end{array}$$

Moreover the maps τ_{13} and τ_{23} coincide on $S_{13}U^\vee \cap S_{23}U^\vee$ and define a map

$$\tilde\tau : S_{13}U^\vee + S_{23}U^\vee \to C^3(U,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d}\,.$$

Proof. — We first prove the commutativity of the second diagram. Let h be the cochain (5.8). We get

$$\begin{aligned} \mathrm{d}h\,(g_1,g_2,g_3,g_4) &= u(\overline{g}_2)v(\overline{g}_3)w(\overline{g}_3)x(\overline{g}_4) - u(\overline{g}_1\overline{g}_2)v(\overline{g}_3)w(\overline{g}_3)x(\overline{g}_4) \\ &\quad + u(\overline{g}_1)v(\overline{g}_2\overline{g}_3)w(\overline{g}_2\overline{g}_3)x(\overline{g}_4) - u(\overline{g}_1)v(\overline{g}_2)w(\overline{g}_2)x(\overline{g}_3\overline{g}_4) \\ &\quad + u(\overline{g}_1)v(\overline{g}_2)w(\overline{g}_2)x(\overline{g}_3) \\ &= u(\overline{g}_1)v(\overline{g}_2)w(\overline{g}_3)x(\overline{g}_4) + u(\overline{g}_1)v(\overline{g}_3)w(\overline{g}_2)x(\overline{g}_4). \end{aligned}$$

The commutativity of the first diagram follows from a similar computation with (5.9).

The space $S_{13}U^\vee \cap S_{23}U^\vee$ may be described as

$$S_{123}U^\vee = \{\, \lambda \in U^{\vee\otimes 4} \mid \forall\sigma \in \mathfrak{S}_{\{1,2,3\}},\ \sigma.\lambda = \lambda \,\}.$$

Since $p \neq 2$, it is generated by elements of the form

$$u \otimes u \otimes u \otimes v.$$

The value of $\tau_{13} - \tau_{23}$ on an element of this form is given by the class of the cochain

$$(g_1,g_2,g_3) \mapsto \frac{1}{2}\Big(u(\overline{g}_1)u(\overline{g}_2)^2 - u(\overline{g}_1)^2u(\overline{g}_2)\Big)v(\overline{g}_3).$$

Thus it is sufficient to show that the 2-cochain

$$(g_1,g_2) \mapsto u(\overline{g}_1)u(\overline{g}_2)^2 - u(\overline{g}_1)^2u(\overline{g}_2)$$

is a coboundary. But it is a cocycle and factorizes through $U^\vee/\operatorname{Ker}u$. In other words, it comes by inflation from a cocycle in $C^2(U^\vee/\operatorname{Ker}u,\mathbf{Q}/\mathbf{Z})$. Since $U^\vee/\operatorname{Ker}u$ is an $\mathbf{F}_p$-vector space of dimension 1, one has

$$H^2(U^\vee/\operatorname{Ker}u,\mathbf{Q}/\mathbf{Z}) = \{0\}$$

and the cocycle is a coboundary. □

Remark 5.19. — (i) The generators of $S^\square U^\vee$ given by (5.7) belong to $S_{13}U^\vee + S_{23}U^\vee$. Therefore $\tilde\tau$ gives by restriction a morphism

$$\tilde\tau : S^\square U^\vee \to C^3(U,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d}$$

such that the following diagram commutes

$$\begin{array}{ccc} S^{\square}U^{\vee} & \longrightarrow & U^{\vee\otimes 4} \\ \downarrow{\scriptstyle \tilde\tau} & & \downarrow{\scriptstyle \mu} \\ C^3(U,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d} & \xrightarrow{\mathrm{d}} & C^4(U\mathbf{Q}/\mathbf{Z}). \end{array} \tag{5.10}$$

(ii) We shall also use later the fact that for any u, v in $U^{\vee}$, the cochain defining the class $\tilde\tau(u\wedge v.u\wedge v)$ factorizes through $(U/(\operatorname{Ker}u\cap\operatorname{Ker}v))^3$.

Lemma 5.20. — *There is a group homomorphism*

$$\tau : \operatorname{Ker}(S^2V^{\vee}\to\Lambda^4U^{\vee})\to F^2H^3(G,\mathbf{Q}/\mathbf{Z})$$

which sends $\sum_{i=1}^r \rho_i.\rho'_i$ *to the class of*

$$\frac{1}{2}\sum_{i=1}^r (f_{\rho_i,\gamma^{\vee}(\rho'_i)}+f_{\rho'_i,\gamma^{\vee}(\rho_i)})+\tilde\tau\Big(\sum_{i=1}^r\gamma^{\vee}(\rho_i)\gamma^{\vee}(\rho'_i)\Big), \tag{5.11}$$

and the diagram

$$\begin{array}{ccc} \operatorname{Ker}(S^2V^{\vee}\to\Lambda^4U^{\vee}) & \hookrightarrow & E_{\infty}^{2,1}(G/V) \\ & {\scriptstyle\tau}\searrow \quad \nearrow & \\ & F^2H^3(G,\mathbf{Q}/\mathbf{Z}) & \end{array}$$

commutes.

Proof. — The definition of $f_{\rho,\lambda}$ given in (5.4), shows that (5.11) does not depend on the decomposition $\sum_{i=1}^r \rho_i.\rho'_i$.

There is a commutative diagram

$$\begin{array}{ccccc} \Lambda^2V^{\vee} & \xrightarrow{d^{0,2}} & \Lambda^2U^{\vee}\otimes V^{\vee} & \xrightarrow{d^{2,1}} & \Lambda^4U^{\vee} \\ \big\downarrow{\scriptstyle \gamma^{\vee}\wedge\gamma^{\vee}} & & \big\downarrow{\scriptstyle -\operatorname{Id}\otimes\gamma^{\vee}} & & \| \\ \Lambda^2(\Lambda^2U^{\vee}) & \longrightarrow & \Lambda^2U^{\vee}\otimes\Lambda^2U^{\vee} & \longrightarrow & \Lambda^4U^{\vee} \end{array} \tag{5.12}$$

which yields an injection

$$H(\mathscr{C})\overset{j}{\hookrightarrow}\operatorname{Ker}(S^2(\Lambda^2U^{\vee})\to\Lambda^4U^{\vee}).$$

Let τ_1 be the map

$$\begin{array}{rcl} S^2V^{\vee} & \to & C^3(G,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d} \\ \rho_1\otimes\rho_2 & \mapsto & \frac{1}{2}(f_{\rho_1,\gamma^{\vee}(\rho_2)}+f_{\rho_2,\gamma^{\vee}(\rho_1)}) \end{array}$$

and τ_2 be the composite of the maps

$$H(\mathscr{C})\overset{j}{\hookrightarrow}\operatorname{Ker}(S^2(\Lambda^2U^{\vee})\to\Lambda^4U^{\vee})\xrightarrow{\tilde\tau}C^3(U,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d}\xrightarrow{\operatorname{Inf}}C^3(G,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d}\,.$$

Lemma 5.13 gives a commutative diagram

$$\begin{array}{ccc} S^2V^\vee & \xrightarrow{-\gamma^\vee\wedge\gamma^\vee} & U^{\vee\otimes 4} \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \mu} \\ C^3(G,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d} & \xrightarrow{\mathrm{d}} & C^4(G,\mathbf{Q}/\mathbf{Z}). \end{array}$$

Combining it with the diagram (5.10), we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Ker}(S^2V^\vee \to \Lambda^4U^\vee) & \xrightarrow{\tau_1} & C^3(G,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d} \\ \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \mathrm{d}} \\ C^3(G,\mathbf{Q}/\mathbf{Z})/\operatorname{Im}\mathrm{d} & \xrightarrow{\mathrm{d}} & C^4(G,\mathbf{Q}/\mathbf{Z}). \end{array}$$

Therefore $\tau_1 - \tau_2$ induces a map

$$\tau : \mathrm{Ker}(S^2V^\vee \to \Lambda^4U^\vee) \to H^3(G,\mathbf{Q}/\mathbf{Z})$$

which, considering the signs in (5.12) is the map described in the lemma. Let $\lambda = \sum_{i=1}^r \rho_i\rho'_i$ belong to $\mathrm{Ker}(S^2V^\vee \to \Lambda^4U^\vee)$ then $\tau(\lambda)$ is the class of a cochain f which by the definition of $f_{\rho,\lambda}$ and τ_2 verifies

$$\forall g_1,g_2,g_3 \in G, \quad \forall v_2,v_3 \in V, \quad f(g_1,g_2v_2,g_3v_3) = f(g_1,g_2,g_3).$$

Therefore, using the notations of [**HS**, §II.1, p. 119] f belongs to $A^3 \cap A_2^*$ and its image in $H^2(U,H^1(V,\mathbf{Q}/\mathbf{Z}))$ is obtained by considering the induced element $\tilde f$ of the group $C^2(U,C^1(V,\mathbf{Q}/\mathbf{Z}))$. But this cochain $\tilde f$ is given by

$$\begin{aligned} \forall u_1,u_2 \in U, \quad \forall v \in V, \quad & \tilde f(u_1,u_2)(v) \\ & = f(v,s(u_1),s(u_2)) \\ & = \sum_{i=1}^r \frac{1}{4}(\rho_i(v)\gamma^\vee(\rho'_i)(u_1\wedge u_2) + \rho'_i(v)\gamma^\vee(\rho_i)(u_1\wedge u_2)). \end{aligned}$$

Therefore the image of f in $\Lambda U^\vee \otimes V^\vee \subset H^2(U,H^1(V,\mathbf{Q}/\mathbf{Z}))$ is the image of λ in this group. □

We can now turn to the corestriction itself. If H is a subgroup of G, we have a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [H,H] & \longrightarrow & H & \longrightarrow & H/[H,H] & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & Z(H) & \longrightarrow & H & \longrightarrow & H/Z(H) & \longrightarrow & 0 \end{array}$$

where the groups on the right or the left are $\mathbf{F}_p$-vector spaces. Since $p \neq 2$, the group

$$S^2H^1(H, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} S^2(H/[H,H])^\vee \tag{5.13}$$

is generated by elements of the form $\chi \cup \chi$ for χ in $(H/[H,H])^\vee$. Thus $H^3_{\mathrm{p}}(G, \mathbf{Q}/\mathbf{Z})$ is generated by elements of the form $\mathrm{Cores}_H^G(\chi \cup \chi)$ for H a subgroup of G and χ an element of $(H/[H,H])^\vee$.

Lemma 5.21. — *With notations as above, if H is a subgroup of G such that $Z(G) \not\subset Z(H)$ and if χ belongs to $H^1(H, \mathbf{Q}/\mathbf{Z})$, then*

$$\mathrm{Cores}_H^G(\chi \cup \chi) = 0.$$

Proof. — Let H' be the subgroup of G generated by H and $Z(G)$. Then

$$\mathrm{Cores}_H^G(\chi \cup \chi) = \mathrm{Cores}_{H'}^G \circ \mathrm{Cores}_H^{H'}(\chi \cup \chi).$$

By choosing a decomposition

$$Z(G) = (Z(G) \cap Z(H)) \oplus E$$

we get an isomorphism $H' \xrightarrow{\sim} H \times E$. Then

$$\mathrm{Cores}_H^{H'} = |E| \times \mathrm{pr}_1^*.$$

But $p||E|$ and $p\chi \cup \chi = 0$. Thus $\mathrm{Cores}_H^{H'}(\chi \cup \chi) = 0$. □

Notations 5.22. — By the preceding lemma, it is sufficient to consider the subgroups H such that

$$V = [G,G] = Z(G) \subset Z(H).$$

In particular, $[H,G] \subset H$ and H is normal in G. Moreover, there exists a sequence of normal subgroups of G

$$H = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_r = G.$$

such that H_i/H_{i-1} is cyclic of order p. Using lemma 5.21, we may also assume that $Z(H_i)$ is contained in $Z(H)$. We denote by U_i the quotient H_i/V which may be considered as a subgroup of U.

We consider for each i in $\{0, \dots, r\}$ the Hochschild-Serre spectral sequence

$$E_2^{p,q}(H_i/V) = H^p(U_i, H^q(V, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(H_i, \mathbf{Q}/\mathbf{Z})$$

and we denote by $F^pH^j(H_i, \mathbf{Q}/\mathbf{Z})$ the corresponding decreasing filtration on the cohomology groups of H_i.

Lemma 5.23. — *For any i in $\{1, \dots, r\}$, and any $j > 0$, one has*

$$\mathrm{Cores}_{H_{i-1}}^{H_i} F^pH^j(H_{i-1}, \mathbf{Q}/\mathbf{Z}) \subset F^{p+1}H^j(H_i, \mathbf{Q}/\mathbf{Z}).$$

Proof. — Let ψ_i be the canonical map

$$\psi_i : F^p H^{p+q}(H_i, \mathbf{Q}/\mathbf{Z}) \to E_\infty^{p,q}(H_i/V).$$

By lemma 5.6, one has

$$\psi_i \circ \mathrm{Cores}_{H_{i-1}}^{H_i} = \mathrm{Cores}_{H_{i-1}}^{H_i} \circ \psi_{i-1}.$$

By choosing $u_i \in U_i - U_{i-1}$, we get a decomposition $U_i \xrightarrow{\sim} U_{i-1} \oplus \mathbf{F}_p u_i$, so that $\mathrm{Cores}_{U_{i-1}}^{U_i} = p\,\mathrm{pr}_1^*$. But $E_\infty^{p,q}(H_i/V)$ is a subquotient of the group $H^p(U_i, H^q(V, \mathbf{Q}/\mathbf{Z}))$, which, by lemma 5.7 is killed by p. ☐

In particular we get that $\mathrm{Cores}_H^G(\chi \cup \chi) = 0$ if $[G : H] > p^3$. We shall now improve this result and relate the corestriction for subgroups of index p with the map τ defined in lemma 5.20.

Lemma 5.24. — *With notations as above,*

$$\mathrm{Cores}_{H_1}^H(\chi \cup \chi) \in F^2 H^3(H_1, \mathbf{Q}/\mathbf{Z}).$$

Moreover there exists a constant λ in $\mathbf{F}_p^*$ depending only on p such that if $[G : H] = p$ and if ρ is the restriction of χ to $V = [G, G]$, then the image of $\mathrm{Cores}_H^G(\chi \cup \chi)$ in $E_\infty^{2,1}(G/V)$, which is a subquotient of

$$H^2(U, H^1(V, \mathbf{Q}/\mathbf{Z})) \xleftarrow{\sim} \Lambda^2 U^\vee \otimes V^\vee \oplus U^\vee \otimes V^\vee$$

is given by $\lambda\gamma^\vee(\rho) \otimes \rho$ and up to the image of an element of $S^2U^\vee$ in $H^3(G, \mathbf{Q}/\mathbf{Z})$, this corestriction coincide with $\lambda\tau(\rho^2)$ where τ is the map defined by lemma 5.20.

Proof. — The character χ belongs to $(H/[H, H])^\vee$. If $\chi([H_1, H_1]) = \{0\}$ then χ is the restriction of an element $\tilde\chi$ of $H^1(H_1, \mathbf{Q}/\mathbf{Z})$ and we have a commutative diagram

$$\begin{array}{ccc}
\tilde\chi \cup \tilde\chi \in H^3(H/[H_1, H_1], \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\mathrm{Cores}_{H/[H_1,H_1]}^{H_1/[H_1,H_1]}} & H^3(H_1/[H_1, H_1], \mathbf{Q}/\mathbf{Z}) \\
\downarrow \mathrm{Inf} & & \downarrow \mathrm{Inf} \\
H^3(H, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\mathrm{Cores}_H^{H_1}} & H^3(H_1, \mathbf{Q}/\mathbf{Z}).
\end{array}$$

But, as in the proof of lemma 5.23, $\mathrm{Cores}_{H/[H_1,H_1]}^{H_1/[H_1,H_1]} = 0$ and we get in that case

$$\mathrm{Cores}_H^{H_1}(\chi \cup \chi) = 0$$

which implies the first assertion. If moreover $G = H_1$, the assumption may be written as $\rho = 0$ and the other assertions follow.

Therefore, we may assume that $\chi_{|[H_1,H_1]} \neq 0$. The commutator induces a linear surjective map

$$\gamma_1 : \Lambda_2 U_1 \to [H_1, H_1]$$

and therefore an injection

$$\gamma_1^\vee : [H_1, H_1]^\vee \to \Lambda^2 U_1^\vee.$$

Let $u \in U_1 - U_0$, and let $u^\vee$ be defined by $u^\vee_{|U_0} = 0$ and $u^\vee(u) = 1$. For any h, h' in H, we have

$$\gamma_1^\vee(\chi_{|[H_1,H_1]})(h \wedge h') = \chi([h,h']) = 0.$$

In other words,

$$\gamma_1^\vee(\chi_{|[H_1,H_1]})_{|\Lambda^2 U_0} = 0.$$

This implies that

$$\gamma_1^\vee(\chi_{|[H_1,H_1]}) \in u^\vee \wedge U_1^\vee$$

and there is a unique $v^\vee$ in $U_1^\vee - \{0\}$ such that

$$\gamma_1^\vee(\chi_{|[H_1,H_1]}) = u^\vee \wedge v^\vee \quad \text{and} \quad v^\vee(u) = 0.$$

Let v in U_0 be such that $v^\vee(v) = 1$. We put $\tilde{u} = s(u)$ and $\tilde{v} = s(v)$. By construction, $\tilde{v} \in \mathrm{Ker}(u^\vee) = H$. Let K be the subgroup of H defined as the intersection

$$K = \mathrm{Ker}(\chi) \cap \mathrm{Ker}(v^\vee).$$

The subgroup K is normal in H_1. Indeed, it is normal in H and we only have to show that $\tilde{u}K\tilde{u}^{-1} \subset K$. But $\mathrm{Ker}(v^\vee)$ is normal in H_1 and if h belongs to K, we have

$$\chi(\tilde{u}h\tilde{u}^{-1}) = \chi(\tilde{u}h\tilde{u}^{-1}h^{-1}) = \gamma_1^\vee(\chi_{|[H_1,H_1]})(u \wedge \overline{h}) = (u^\vee \wedge v^\vee)(u \wedge \overline{h}) = v^\vee(h) = 0.$$

The quotient H_1/K is a non-abelian group of order p^3. In fact, if T is the subgroup of H_1 generated bu $\tilde{u}$ and $\tilde{v}$, then we have an isomorphism $T \xrightarrow{\sim} H_1/K$ and we may describe H_1 as a semi-direct product $H_1 \xrightarrow{\sim} K \rtimes T$. Let $I = T \cap [G,G] = [\tilde{u},\tilde{v}]\mathbf{F}_p$ and $Q = T/I$. The Hochschild-Serre spectral sequence

$$H^p(Q, H^q(I, \mathbf{Q}/\mathbf{Z})) \Rightarrow H^{p+q}(T, \mathbf{Q}/\mathbf{Z})$$

defines a decreasing filtration $F^pH^j(T, \mathbf{Q}/\mathbf{Z})$ and the morphism

$$H^p(T, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\mathrm{Inf}} H^p(H_1, \mathbf{Q}/\mathbf{Z})$$

is compatible with the filtrations on the cohomology groups of T and H_1.

Let us first prove the lemma in the case where $H_1 = T$. In other words H_1 is the group generated by two elements $A = \tilde{u}$ and $B = \tilde{v}$ with the relations

$$A^p = B^p = [A,B]^p = [A,[A,B]] = [B,[A,B]] = 1$$

and H is the subgroup of H_1 generated by B and $[A,B]$. Then H is an $\mathbf{F}_p$-vector space with a basis given by $e_1 = B$ and $e_2 = [A,B]$. Let $(e_1^\vee, e_2^\vee)$ be the dual basis. Then $e_1^\vee$ is the restriction to H of the character $v^\vee$ of H_1 and

$$\chi(e_2) = \gamma_1^\vee(\chi_{|[H_1,H_1]})(u \wedge v) = 1.$$

Thus $\chi_{|[T,T]} = e_2^\vee$. We have

$$\begin{aligned}
\mathrm{Cores}_H^T(e_1^\vee \cup e_1^\vee) &= \mathrm{Cores}_H^T(\mathrm{Res}_H^T(v^\vee \cup v^\vee)) = pv^\vee \cup v^\vee = 0,\\
\mathrm{Cores}_H^T(e_1^\vee \cup e_2^\vee) &= \mathrm{Cores}_H^T(\mathrm{Res}_H^T(v^\vee) \cup e_2^\vee) = v^\vee \cup \mathrm{Cores}_H^T(e_2^\vee) = 0
\end{aligned}$$

where the last equality follows from [**Le**, lemma 6.22]. By lemma 5.23

$$\mathrm{Cores}_H^T(e_2^\vee \cup e_2^\vee) \in F^1H^3(T, \mathbf{Q}/\mathbf{Z}).$$

But $E_\infty^{1,2}(T/I)$ is a subquotient of $H^1(Q, H^2(I, \mathbf{Q}/\mathbf{Z}))$ which by lemma 5.9 is trivial. Thus

$$\mathrm{Cores}_H^T(e_2^\vee \cup e_2^\vee) \in F^2H^3(T, \mathbf{Q}/\mathbf{Z})$$

this proves the first assertion of the lemma in that case.

Using [**Le**, p. 517], we get that $E_\infty^{2,1}(T/I)$ is generated by $u^\vee \wedge v^\vee \otimes e_2^\vee$ and by [**Le**, theorem 6.26],

$$\mathrm{Cores}_H^T(e_2^\vee \cup e_2^\vee) \notin F^3(H^3(T, \mathbf{Q}/\mathbf{Z})) = \mathrm{Im}(\mathrm{Inf} : H^3(Q, \mathbf{Q}/\mathbf{Z}) \to H^3(T, \mathbf{Q}/\mathbf{Z})).$$

Therefore there exists a constant λ depending only on p such that the image of the element $\mathrm{Cores}_H^T(e_2^\vee \cup e_2^\vee)$ in $E_\infty^{2,1}(T/I)$ is given by $\lambda u^\vee \wedge v^\vee \otimes e_2^\vee$. But $\chi(e_2) = 1$ implies that $\chi = ae_1^\vee + e_2^\vee$ for some a in $\mathbf{F}_p$. Thus

$$\mathrm{Cores}_H^T(\chi \cup \chi) = \mathrm{Cores}_H^T(e_2^\vee \cup e_2^\vee).$$

which implies the second assertion in the case $T = H_1 = G$. Finally if $T = H_1 = G$, then $\rho = \chi_{|[T,T]} = e_2^\vee$ and ρ^2 belongs to $\mathrm{Ker}(S^2V^\vee \to \Lambda^4 U^\vee)$. By lemma 5.20, $\lambda\tau(\rho^2)$ and $\mathrm{Cores}_H^T(\chi \cup \chi)$ which are both in $F^2H^3(G, \mathbf{Q}/\mathbf{Z})$ have the same image in $E_\infty^{2,1}(T/I)$. Thus

$$\mathrm{Cores}_H^T(\chi \cup \chi) - \lambda\tau(\rho^2) \in \mathrm{Im}(\mathrm{Inf} : H^3(Q, \mathbf{Q}/\mathbf{Z}) \to H^3(T, \mathbf{Q}/\mathbf{Z})).$$

But $\dim_{\mathbf{F}_p} Q = 2$ and $S^2Q^\vee \xrightarrow{\sim} H^3(Q, \mathbf{Q}/\mathbf{Z})$. This implies the third assertion in the case $T = H_1 = G$.

The first two assertions in the general case are obtained using the inflation map from the cohomology of T to that of G. It remains to prove the third. Let h be the map

$$\begin{array}{rccl} h: & G & \to & \mathbf{Q}/\mathbf{Z} \\ & g & \mapsto & \rho(gs(g)^{-1}). \end{array}$$

We have seen that

$$\mathrm{d}h\,(g_1, g_2) = -\frac{1}{2}\gamma^\vee(\rho)(\overline{g}_1 \wedge \overline{g}_2).$$

Let $\chi' = h_{|H}$. The map χ' is a morphism of groups. Indeed, if $h_1, h_2 \in H$

$$\begin{aligned} \chi'(h_1h_2) &= \chi'(h_1) + \chi'(h_2) + \frac{1}{2}\gamma^\vee(\rho)(\overline{h}_1 \wedge \overline{h}_2) \\ &= \chi'(h_1) + \chi'(h_2) + \frac{1}{2}\chi([h_1, h_2]) = \chi'(h_1) + \chi'(h_2). \end{aligned}$$

We have $(\chi - \chi')_{|V} = 0$, thus there exists a character ν of G such that $\chi - \chi' = \nu_{|H}$.

$$\begin{aligned} \mathrm{Cores}_H^G(\chi \cup \chi) &= \mathrm{Cores}_H^G(\chi' \cup \chi') + 2\,\mathrm{Cores}_H^G(\chi' \cup \nu_{|H}) + \mathrm{Cores}_H^G(\nu_{|H} \cup \nu_{|H}) \\ &= \mathrm{Cores}_H^G(\chi' \cup \chi') \end{aligned}$$

and since $\chi_{|V} = \chi'_{|V}$, the value of $\lambda\tau(\rho^2)$ is the same for χ and χ'. Therefore it is sufficient to prove the last assertion in the case where $\chi = \chi'$. Since $\gamma^\vee(\rho) = u^\vee \wedge v^\vee$, we see that the map

$$\begin{array}{rccl} f_{\rho,\gamma^\vee(\rho)}: & G^3 & \to & \mathbf{Q}/\mathbf{Z} \\ & (g_1, g_2, g_3) & \mapsto & \frac{1}{2}\rho(g_1s(g_1)^{-1})\gamma^\vee(\rho)(\overline{g}_2 \wedge \overline{g}_3) \end{array}$$

verifies

$$\forall g_1, g_2, g_3 \in G, \quad \forall k_1, k_2, k_3 \in K, \quad f_{\rho,\gamma^\vee(\rho)}(g_1k_1, g_2k_2, g_3k_3) = f_{\rho,\gamma^\vee(\rho)}(g_1, g_2, g_3).$$

Using remark 5.19 (ii), we get that $\tau(\rho^2)$ comes by inflation from $H^3(T, \mathbf{Q}/\mathbf{Z})$ and the last assertion also reduces to the case where $G = T$. ☐

Lemma 5.24 implies that

$$\mathrm{Cores}_H^G(\chi \cup \chi) = 0$$

if $[G : H] > p^2$. Let us now deal with the subgroups H of index p^2.

Lemma 5.25. — *If* $[G : H] = p^2$, *then* $\mathrm{Cores}_H^G(\chi \cup \chi)$ *belongs to the image of* $S^2U^\vee$ *in* $H^3(G, \mathbf{Q}/\mathbf{Z})$.

Proof. — In that case, we have

$$H/[G,G] = U_0 \subsetneq U_1 \subsetneq U_2 = G/[G,G].$$

We choose $u_1 \in U_1 - U_0$ and $u_2 \in U_2 - U_1$ and define $u_1^\vee$ and $u_2^\vee$ in $U^\vee$ by $u_i^\vee(u_j) = \delta_{i,j}$ and $u_i^\vee(U_0) = 0$. As in the proof of lemma 5.24, we may assume that $\rho = \chi_{|[G,G]} \neq 0$ and we have

$$\gamma^\vee(\rho)_{|\Lambda^2 U_0} = 0$$

which implies that $\gamma^\vee(\rho)$ may be written as

$$\gamma^\vee(\rho) = u_1^\vee \wedge v_1^\vee + u_2^\vee \wedge v_2^\vee. \tag{5.14}$$

Let K be the subgroup of H defined by

$$K = \mathrm{Ker}(\chi) \cap \mathrm{Ker}(v_1^\vee) \cap \mathrm{Ker}(v_2^\vee).$$

Using (5.14), we get as in the proof of lemma 5.24 that K is a normal subgroup of G. Let T be the quotient G/K and I the image of $[G,G]$ in T. The group I is isomorphic to $V/\operatorname{Ker}\rho$. Thus it is a cyclic group. Since $\gamma^\vee(\rho) \neq 0$, T is not abelian and I coincides with the commutator group $[T,T]$. Putting $Q = T/I$, there is a commutative diagram

$$\begin{array}{ccc} S^2Q^\vee & \longrightarrow & H^3(T, \mathbf{Q}/\mathbf{Z}) \\ \downarrow & & \downarrow \mathrm{Inf} \\ S^2U^\vee & \longrightarrow & H^3(G, \mathbf{Q}/\mathbf{Z}) \end{array}$$

Therefore it is sufficient to prove the lemma for $G = T$.

From now on we assume $G = T$. Since $\dim_{\mathbf{F}_p} U \leqslant 4$, any element in $\Lambda^3 U$ may be written as $u \wedge v$ with u in U and v in $\Lambda^2 U$ (see [**Re**, §1.4]). Thus $K^3 = K^3_{\max}$ in that case. Using proposition 5.2, we get that

$$V^\vee \wedge U^\vee = \mathrm{Ker}(\Lambda^3 U^\vee \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})).$$

Therefore, in this case, using the isomorphism of lemma 5.9

$$S^2U^\vee \oplus V^\vee \wedge U^\vee = \mathrm{Ker}(H^3(U, \mathbf{Q}/\mathbf{Z}) \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})). \tag{5.15}$$

Since $\mathrm{Cores}_H^G(\chi \cup \chi)$ belongs to the kernel of the map

$$H^3(G, \mathbf{Q}/\mathbf{Z}) \to H^3(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z})$$

and to $F^3H^3(G,\mathbf{Q}/\mathbf{Z})$, it belongs to the image of the group given by (5.15) in $H^3(G,\mathbf{Q}/\mathbf{Z})$. But by [**Pe2**, lemma 9.3 and p. 135]

$$V^\vee \cup U^\vee = \mathrm{Ker}(H^3(U,\mathbf{Q}/\mathbf{Z}) \to H^3(G,\mathbf{Q}/\mathbf{Z})).$$

Thus $\mathrm{Cores}_H^G(\chi\cup\chi)$ belongs to the image of $S^2U^\vee$. □

5.4. Proof of the result. — Using proposition 5.2, we have an injection

$$K^3_{\mathrm{max}}/\,\mathrm{Ker}(\Lambda^3U^\vee \to H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z})) \hookrightarrow H^3_{\mathrm{nr}}(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z}).$$

So we want to prove that

$$K_3 = \mathrm{Ker}(\Lambda^3U^\vee \to H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z})).$$

But, since $p\neq 2$, by theorem 3.1,

$$H^3_{\mathrm{p}}(G,\mathbf{Q}/\mathbf{Z}) = \mathrm{Ker}(H^3(G,\mathbf{Q}/\mathbf{Z}) \to H^3(\mathbf{C}(W)^G,\mathbf{Q}/\mathbf{Z})).$$

It remains to show that

$$\mathrm{Im}(\Lambda^3U^\vee \to H^3(G,\mathbf{Q}/\mathbf{Z})) \cap H^3_{\mathrm{p}}(G,\mathbf{Q}/\mathbf{Z}) = \{0\}.$$

But, using [**Pe2**, lemma 9.3 and p. 135], we have that

$$K^3 = U^\vee \wedge V^\vee = \mathrm{Ker}(S^2U^\vee \oplus \Lambda^3U^\vee = H^3(U,\mathbf{Q}/\mathbf{Z}) \to H^3(G,\mathbf{Q}/\mathbf{Z})). \tag{5.16}$$

Using lemmas 5.23, 5.24, and 5.25,

$$H^3_{\mathrm{p}}(G,\mathbf{Q}/\mathbf{Z}) \subset \mathrm{Im}(S^2U^\vee \to H^3(G,\mathbf{Q}/\mathbf{Z})) + \mathrm{Im}(\tau).$$

Since $\mathrm{Im}(\tau)\cap F^3H^3(G,\mathbf{Q}/\mathbf{Z}) = \{0\}$, and using (5.16), we have a direct sum

$$\Lambda^3U^\vee/K_3 \oplus S^2U^\vee \oplus \mathrm{Im}(\tau) \subset H^3(G,\mathbf{Q}/\mathbf{Z})$$

and the result is proven. □

6. A particular case

If the dimension of U is less than 5 then any λ in Λ^3U may be written as $\lambda = u\wedge v$ with u in U and v in Λ^2U (see [**Re**]). Therefore $K^3 = K^3_{\mathrm{max}}$ whenever $\dim U \leqslant 5$. Let us give an example with $\dim U = 6$.

Theorem 6.1. — *Let U and V be two $\mathbf{F}_p$-vector spaces of dimension 6 for p an odd prime. We denote by $(u_i)_{1\leqslant i\leqslant 6}$ a basis of U and $(v_i)_{1\leqslant i\leqslant 6}$ a basis of V. We denote by $(u_i^\vee)_{1\leqslant i\leqslant 6}$ the dual basis of $U^\vee$. Let γ be the element of $\Lambda^2U^\vee\otimes V$ defined by*

$$\begin{aligned}\gamma = v_1\otimes(u_1^\vee\wedge u_2^\vee - u_4^\vee\wedge u_5^\vee) + v_2\otimes(u_2^\vee\wedge u_3^\vee - u_5^\vee\wedge u_6^\vee)\\ + v_3\otimes u_1^\vee\wedge u_4^\vee + v_4\otimes u_2^\vee\wedge u_5^\vee + v_5\otimes u_3^\vee\wedge u_6^\vee + v_6\otimes u_4^\vee\wedge u_6^\vee.\end{aligned}$$

This defines a map $\gamma:\Lambda^2U\to V$. Let

$$0\to V\to G\to U\to 0$$

be the corresponding central extension (see notations 5.1), then for any faithful representation W of G one has

$$\mathrm{Br}_{\mathrm{nr}}(\mathbf{C}(W)^G) = \{0\}$$

but

$$H^3_{\text{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}) \neq \{0\}.$$

In particular, $\mathbf{C}(W)^G$ *is not a rational extension of* $\mathbf{C}$.

Proof. — By [**Bo1**, lemma 5.1], one has

$$\mathrm{Br}_{\text{nr}}(\mathbf{C}(W)^G) \xrightarrow{\sim} K^2_{\text{max}}/K^2$$

But

$$K^2 = \langle u_1^\vee \wedge u_2^\vee - u_4^\vee \wedge u_5^\vee, u_2^\vee \wedge u_3^\vee - u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_4^\vee, u_2^\vee \wedge u_5^\vee, u_3^\vee \wedge u_6^\vee, u_4^\vee \wedge u_6^\vee\rangle$$

and

$$K^{2\perp} = \langle u_1 \wedge u_2 + u_4 \wedge u_5, u_2 \wedge u_3 + u_5 \wedge u_6, u_3 \wedge u_4, u_6 \wedge u_1, u_1 \wedge u_3, u_2 \wedge u_4, u_3 \wedge u_5, u_5 \wedge u_1, u_6 \wedge u_2\rangle.$$

Since

$$u_1 \wedge u_2 + u_4 \wedge u_5 = (u_1 + u_4) \wedge (u_2 + u_5) + u_2 \wedge u_4 + u_5 \wedge u_1$$

and

$$u_2 \wedge u_3 + u_5 \wedge u_6 = (u_2 + u_5) \wedge (u_3 + u_6) + u_6 \wedge u_2 + u_3 \wedge u_5,$$

we have

$$K^{2\perp}_{\text{dec}} = K^{2\perp} \quad \text{and} \quad K^2 = K^2_{\text{max}}.$$

This proves the first assertion. We now compute K^3 and K^3_{max}

$$\begin{aligned} K^3 = \langle & u_1^\vee \wedge u_4^\vee \wedge u_5^\vee, u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_1^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_5^\vee, \\ & \quad u_1^\vee \wedge u_3^\vee \wedge u_6^\vee, u_1^\vee \wedge u_4^\vee \wedge u_6^\vee, \\ & u_2^\vee \wedge u_4^\vee \wedge u_5^\vee, u_2^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_4^\vee, u_2^\vee \wedge u_3^\vee \wedge u_6^\vee, u_2^\vee \wedge u_4^\vee \wedge u_6^\vee, \\ & u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_3^\vee \wedge u_4^\vee \wedge u_5^\vee, u_3^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_3^\vee \wedge u_4^\vee, \\ & \quad u_2^\vee \wedge u_3^\vee \wedge u_5^\vee, u_3^\vee \wedge u_4^\vee \wedge u_6^\vee, \\ & u_1^\vee \wedge u_2^\vee \wedge u_4^\vee, u_2^\vee \wedge u_3^\vee \wedge u_4^\vee - u_4^\vee \wedge u_5^\vee \wedge u_6^\vee, u_2^\vee \wedge u_4^\vee \wedge u_5^\vee, u_3^\vee \wedge u_4^\vee \wedge u_6^\vee, \\ & u_1^\vee \wedge u_2^\vee \wedge u_5^\vee, u_2^\vee \wedge u_3^\vee \wedge u_5^\vee, u_1^\vee \wedge u_4^\vee \wedge u_5^\vee, u_3^\vee \wedge u_5^\vee \wedge u_6^\vee, u_4^\vee \wedge u_5^\vee \wedge u_6^\vee, \\ & u_1^\vee \wedge u_2^\vee \wedge u_6^\vee - u_4^\vee \wedge u_5^\vee \wedge u_6^\vee, u_2^\vee \wedge u_3^\vee \wedge u_6^\vee, u_1^\vee \wedge u_4^\vee \wedge u_6^\vee, u_2^\vee \wedge u_5^\vee \wedge u_6^\vee\rangle \\ = \langle & u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_1^\vee \wedge u_5^\vee \wedge u_6^\vee, u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_3^\vee \wedge u_4^\vee \wedge u_5^\vee, \\ & u_1^\vee \wedge u_2^\vee \wedge u_4^\vee, u_1^\vee \wedge u_2^\vee \wedge u_5^\vee, u_1^\vee \wedge u_2^\vee \wedge u_6^\vee, u_1^\vee \wedge u_3^\vee \wedge u_4^\vee, \\ & u_1^\vee \wedge u_3^\vee \wedge u_6^\vee, u_1^\vee \wedge u_4^\vee \wedge u_5^\vee, u_1^\vee \wedge u_4^\vee \wedge u_6^\vee, u_2^\vee \wedge u_3^\vee \wedge u_4^\vee, \\ & u_2^\vee \wedge u_3^\vee \wedge u_5^\vee, u_2^\vee \wedge u_3^\vee \wedge u_6^\vee, u_2^\vee \wedge u_4^\vee \wedge u_5^\vee, u_2^\vee \wedge u_4^\vee \wedge u_6^\vee, \\ & u_2^\vee \wedge u_5^\vee \wedge u_6^\vee, u_3^\vee \wedge u_4^\vee \wedge u_6^\vee, u_3^\vee \wedge u_5^\vee \wedge u_6^\vee, u_4^\vee \wedge u_5^\vee \wedge u_6^\vee\rangle. \end{aligned}$$

Therefore

$$K^{3\perp} = \langle u_1 \wedge u_2 \wedge u_3 + u_3 \wedge u_4 \wedge u_5 + u_5 \wedge u_6 \wedge u_1, u_1 \wedge u_3 \wedge u_5\rangle.$$

By [**Pe1**, p. 264, example 2],

$$K^{3\perp}_{\text{dec}} = \langle u_1 \wedge u_3 \wedge u_5\rangle.$$

Therefore $K^3_{\max}/K^3 \xrightarrow{\sim} \mathbf{F}_p$ and by theorem 5.3, we get that

$$H^3_{\mathrm{nr}}(\mathbf{C}(W)^G, \mathbf{Q}/\mathbf{Z}) \neq \{0\}. \quad \square$$

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4th March 2003

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