Generalised Einstein equations on Kähler manifolds and prescribed cohomology for the first Chern Weil form

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Résumé: Dans le cadre Kählérien les équations d'Einstein généralisées (équations d'Einstein "avec sources" pour les physiciens) peuvent être regardées comme des équations cohomologiques à l'intérieur de la première classe de Chern. En introduisant une classe secondaire (ou source) à deux paramètres pour prescrire cette relation cohomologique, on caractérise la région dans l'espace de ces paramètres pour laquelle l'équation associée admet au moins une solution. Quand la première classe de Chern est positive, la constante de Aubin-Tian et les bornes pour la concavité et convexité pluriharmonique de la source caractérisent les bornes de cette région. En tenant compte de la régularité minimale de la classe secondaire pour assurer l'existence des solutions classiques, on observe, en particulier, une amélioration des résultats cités dans la littérature dans le contexte de la conjecture de E. Calabi.

Abstract: Generalised Einstein equations (Einstein equations with sources in the physicist's grammar) can, in the Kähler setup, be seen as cohomological equations within the first Chern class. Introducing a two parameter secondary class (or source term) to prescribe such a cohomological relation, we characterise regions for those parameters to ensure that the associated equation admits at least one solution. When the first Chern class is positive, the Aubin Tian constant and the bounds for the pluriharmonic concavity and convexity of the source term characterise the bounds of that region. Taking into account the minimal regularity of the secondary class to ensure the existence of classical solutions, we observe, in particular, an improvement of the results quoted in the literature in the context of E. Calabi's conjecture.

1 Introduction

A. Einstein's equations of *General Relativity* are, in the Riemannian context, written as $Ric(g)-gS(g)/2=\vartheta T$, where ϑ is a suitable constant, T denotes the energy momentum tensor (a function of nongravitational effects, say), while $Ric(g)$ and $S(g)$ denote the (symmetric) Ricci tensor and the scalar curvature of the underlying metric background or gravitational field g, respectively. That equation has many variants.

Let V be a closed Kähler manifold of complex dimension n, j a complex structure compatible with some Kähler form ω. Assume that the first Chern class $c^1(V,j)$ of the anticanonical bundle is represented by $\lambda\omega/2\pi$, where λ is either $-1,0$ or 1: one says that the anticanonical bundle is negatively curved, flat, or positively curved, respectively (see

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the Section Preliminaires for more details). Let X denote a Banach space where the (real) test functions on V (namely $\mathcal{D}(V)$) are dense. We will study the equation

$$\sqrt{-1}Ric(\omega_x) - \lambda\omega_x = \sqrt{-1}d^{10}d^{01}((t-\lambda)x - sz) \tag{1}$$

on V. The eventual solution to the equation is x, a real valued function on V that belongs to a certain X. In the equation z denotes a given real valued function on V, t and s are real numbers, and ω_x is an abbreviation for $\omega + \sqrt{-1}d^{10}d^{01}x$. It is important to note that if $t = \lambda$ and $z = 0$ (or $s = 0$), then we have the equation associated to the problem of Kähler Einstein forms extensively treated in the literature ([1], [7], [13], [28], [32]). If also $\lambda = 0$, then the equation is equivalent to the equation associated to E. Calabi's conjecture. Recall that the case when λ is positive still remains open ([2], [8], [13], [15], [28], [29]).

That family of equations also has a natural interpretation in terms of prescribed (or directed) cohomology for the first Chern Weil form: if F is a $GL(n,\mathbb{C})$ invariant polynomial of degree p on the set of n by n hermitean matrices, and if $R(\omega)$ denotes the curvature form associated to the Kähler form ω (seen as a $End(TV)$ valued form of type $(1,1)$), then $F(\sqrt{-1}R(\omega)/2\pi)$ is a closed form of type (p,p) on V, an example of a Chern Weil form. That form depends on ω. If ω_x is another Kähler form, then $F(\sqrt{-1}R(\omega_x)/2\pi)$ and $F(\sqrt{-1}R(\omega)/2\pi)$ differ by the $d^{10}d^{01}$ of a form on V ([20]). The simplest example of that generality is when F is the usual trace: then $F(\sqrt{-1}R(\omega)/2\pi)$ coincides with $\sqrt{-1}Ric(\omega)/2\pi$, and represents the first Chern class of (V,j). Under our hypothesis on ω one has that (1) is equivalent to

$$\sqrt{-1}Ric(\omega_x) - \sqrt{-1}Ric(\omega) = \sqrt{-1}d^{10}d^{01}(tx - y_\omega - sz), \tag{2}$$

where $\sqrt{-1}Ric(\omega) - \lambda\omega = \sqrt{-1}d^{10}d^{01}y_\omega$.

Important: The reader should imagine himself moving in the t versus s plane in front of a screen. On the screen he probably sees, as he moves in certain directions and within certain zones, deformations of metrics that have a certain regularity, symmetry and further properties (that could eventually be read from the equation). We assert that if he moves within certain zones (and in certain directions), he will certainly see something. That is the naive content of

Theorem 1.1. *The equation (1) or (2) has at least one solution in $C^{k+2,\alpha}(V)$ provided that z belongs to $C^{k,\alpha}(V)$ for $k \geq 2$ and $\alpha \in]0,1[$ and:*

i) s is any real number and t is smaller than 0. The solution will be unique.

ii) s is any real number and t is equal to 0. The solution will be unique up to a constant.

iii) λ is equal to 1, t is bigger than 0 but less or equal than $min(1,\kappa)$, where $\kappa < (n+1)\alpha_{(V,j)}/n$ and: $t + sK_- \leq 1$ if $s \leq 0$, $t + sK_+ \leq 1$ if $0 \leq s$. Here $\alpha_{(\mathbf{V},\mathbf{j})}$ is what is known in the literature as the Aubin Tian constant, while K_- and K_+ are the bounds for the (pluriharmonic) convexity and concavity of z with respect to ω.

The proof of Theorem 1.1 will be completed in Section 4, where the definitions for $\alpha_{(\mathbf{V},\mathbf{j})}$, K_- and K_+ will be found. In fact i) can be considered as a Corollary of

Theorem 1.2. *Assume that the anticanonical bundle is negatively curved, that the sign of $c^1(V,j)$ is identified with λ (so λ is equal to -1). Then whenever z belongs to $C^{k,\alpha}(V)$ for $k \geq 2$ and $\alpha \in]0,1[$, the equation (2) admits a unique solution in $C^{k+2,\alpha}(V)$ provided that $t = \lambda$ and $s = -1$.*

Note that (1) when $t = \lambda$ and $s = -1$ becomes

$$E_\lambda(x) = \sqrt{-1} d^{10} d^{01} z, \tag{3}$$

where $E_\lambda(x)$ is an abbreviation for $\sqrt{-1}Ric(\omega_x) - \lambda\omega_x$. That is a natural generalisation of the problem of Kähler Einstein forms.

We observe that the regularity needed for the term z is lower than the $k \geq 3$ that appears explicitly in the work of T. Aubin ([1], [3]) and implicitly in other works ([7], [32]) in the context of E. Calabi's conjecture. The $k = 2$ could be considered as being implicit in the work of Y.T. Siu ([28]) that modifies, in a paticular way, results from the theory of second order fully nonlinear elliptic equations in real domains. Those results were obtained, among other people, by L. Evans ([16]), N. Trudinger ([19]), and N. Krylov ([9]). Y. T. Siu performs a possible adaptation of those results to the complex setup. As far as we know, however, the minimal requirements for the regularity of z that we propose have not been mentioned before (see [3] or [22] for statements about the state of the subject). This is relevant in (1): the eventual solution parametrises connections, curvatures and geodesics.

The regularity characterisation of the solutions follows from the chain of estimates proposed in Section 3, where the proof of Theorem 1.2 is given. That chain of estimates (and the subtleties involved) is obtained following (and clarifying) the work of many people, among them T. Aubin ([1]), J.P. Bourguignon ([7]), E. Calabi, L. Evans ([16]), Y.T. Siu ([28]), N. Trudinger ([19]), and S.T. Yau ([32]). That chain of estimates is essentially the same for i), ii), and iii) in Theorem 1.1, modulo different uses of the implicit and inverse function theorem within B manifolds [2] to proceed in the methods of continuity proposed, where a bound for the oscillation is needed. The estimate for the oscillation required in statement ii) is the same as the one proposed by S.T. Yau ([32]) and later clarified by T. Aubin ([1]), J.P. Bourguignon and his collaborators ([7]), and then further by Y.T. Siu ([28]) and G.Tian ([31]), so it will not be written here. The reader interested in details on the history of the subject can consult [28]. To ensure a bound for the oscillation in iii), we proceed via a constructive generalisation of Aubin's contant ([2], [5], [27]), its relation with Tian's constant (or invariant) ([30]), and their relation with comparative estimates, functionals and concavity constants.

The proof of Theorem 1.2 could be read in a sequential order, noting that each $\bullet$ is associated to a subroutine that assumes the following subroutine (and so on). Equivalently, the reader can begin from the last $\bullet$ and then read the preceeding subroutine (that assumes the one preceeding it), and so on (noting that the symbol $\circ$ denotes only a subsubroutine). In fact it is better to read the proof in both directions.

The proof of the bound for the oscillation needed in iii), Theorem 1.1, should, in principle, be read following the sequential order in the script. As one proceeds, definitions and results that are needed will require to go forward and backwards, but everything is numbered in a clear way.

We will be concerned in proving existence of solutions to Monge Ampère type equations, leaving for a future work relations and estimates to complete the picture.

The reader interested in other Generalised Einstein equations and their relation with ideas that could be relevant for physicists (and geometers) can consult [26].

[2] The letter B is an abbreviation for Banach, thus B space means Banach space, for example.

2 Preliminaires and Notation

Let V be a Kähler manifold of (complex) dimension n. One chooses a complex structure j therein compatible with some Kähler form ω. Identifying $\mathcal{A}$ and $\mathcal{O}$ with the sheaf of germs of real and complex analytic functions on V respectively, one notes that $\mathcal{O}$ is a subsheaf of $\mathcal{A}_{\mathbb{C}} := \mathcal{A} \oplus \sqrt{-1}\mathcal{A}$. Related to ω and j are the hermitean, riemannian and symplectic forms h, g and f in [3]

$$Hom_{C^\infty}(\otimes_2 TV, C^\infty_{\mathbb{C}}), \quad Hom_{C^\infty}(\odot_2 TV, C^\infty) \quad \text{and} \quad Hom_{C^\infty}(\wedge_2 TV, C^\infty)$$

respectively, that one introduces, recursively, through

$$h := \frac{g + \sqrt{-1}f}{\sqrt{2}}, \quad g := f \cdot (j, 1), \text{and} \quad f := -\omega.$$

Whenever O is the domain of a local parametrisation on V, $(Z^i | i \in (1...n))$ denotes the set of (canonical) generators of the $(C^\infty_{\mathbb{C}})_O$ free module $\wedge^1_{C^\infty_{\mathbb{C}}}(TV, C^\infty_{\mathbb{C}})_O$.[4] Therefore h is on O expressed through $\sum_{i,j=1}^n Z^i \otimes \overline{Z^j} h_{ij}$, if and only if ω on O is given by $\sum_{i,j=1}^n \sqrt{-1} Z^i \wedge \overline{Z^j} h_{ij}$. Decomposing $\wedge^1_{C^\infty}(TV, C^\infty_{\mathbb{C}})$ into the direct sum $\wedge^{10}_{C^\infty}(TV, C^\infty_{\mathbb{C}}) \oplus \wedge^{01}_{C^\infty}(TV, C^\infty_{\mathbb{C}})$ making use of the complex structure, one then separates the mapping $d : C^\infty_{\mathbb{C}} \to \wedge^1_{C^\infty}(TV, C^\infty_{\mathbb{C}})$ in $d^{10} + d^{01}$, obviously ruled by matching the 10 and 01 indexes.

If $\mathbf{h} = \mathbf{h}^+ := \overline{\mathbf{h}}^\top$ denotes the element $(h_{ij} | i, j \in (1...n))$ in $\mathbf{M}(n, \mathbb{C})$, and $\mathbf{\Gamma}$ denotes the local section in $\wedge^1_{C^\infty}(End(TV))$ associated to the hermitean connection ∇ for (ω, j) (see [20] for example), one has the identity $(d^{10}\mathbf{h})\mathbf{h}^{-1} = \mathbf{\Gamma}$. Identifying $R(\omega) := \nabla\nabla$ locally with $\mathbf{R}(\omega)$ and after deducing that $\mathbf{R}(\omega) = d\mathbf{\Gamma} - \mathbf{\Gamma} \wedge \mathbf{\Gamma}$, one readily infers the equality $\mathbf{R}(\omega) = d^{01}((d^{10}\mathbf{h})\mathbf{h}^{-1})$ between local sections in $\wedge^{1,1}_{C^\infty}(End(TV))$.

The trace of $R(\omega)$ with respect to TV is a section in $\wedge^{1,1}_{C^\infty} TV$, usually denoted by $Ric(\omega)$ (because in the Kähler setup different traces of the curvature tensor are equivalent), and known as the Ricci tensor of the Kähler form ω. Expressing the $2n$ vectorfield describing the orientation of V, say $\vec{\nu}$, locally as $\wedge_{i=1}^n(-\sqrt{-1}Z_i \wedge \overline{Z_i})$ (where $(Z_i | i \in (1...n)) \cup (\overline{Z_i} | i \in (1...n))$ denotes the set of generators of $(TV \otimes \mathbb{C})_O$), one deduces that $Ric(\omega)$ is equal to $d^{01}d^{10}log\langle \vec{\nu}, \frac{\omega^n}{n!}\rangle$ (recall that $\langle \vec{\nu}, \frac{\omega^n}{n!}\rangle = \det \mathbf{h}$), always regarding the local parametrisation. One verifies that $\sqrt{-1}Ric(\omega)/2\pi$ is a closed real form of type $(1,1)$, defining thus a class in $H^{1,1}(V, \mathbb{R})$, known as the first Chern class for (V, j), and usually denoted by $c^1(V, j)$. If V is closed, the $d^{10}d^{01}$ Lemma (see [20]) enables us to conclude that such a class is independent of the connection and of the Kähler form.

From a different perspective: if $\varphi : O \to \mathbb{C}^n$ denotes a representative of an atlas or parametrisation, then we have an associated cocycle $\Psi \equiv \varphi \cdot \varphi^{-1}$, and therefore the (inductive limiting) cocycle $D\Psi^{-1}$ denotes the element in $\check{H}^1(V, GL(n, \mathbb{C}))$ that represents TV. Thus $\wedge_n D\Psi^{-1}$ denotes the element in $\check{H}^1(V, \mathcal{O}^*)$ that represents K_V^{-1}, the anti-canonical bundle for (V, j) (sometimes written as $\det TV$). Any Kähler form on (V, j), say ω, induces a metric on K_V^{-1} in the usual way: the curvature form of that metric is also $Ric(\omega)$ (see [20], [12]). Moreover, if V is closed, then different Kähler forms on (V, j) give,

[3] The reader not familiar with this notation should compare with the nomenclature introduced in Chapter 1 of [17].

[4] Some authors write dz^i instead of Z^i.

thanks to the $d^{10}d^{01}$ Lemma, cohomologous curvature forms on K_V^{-1}. That leads to the equality $c^1(V,j) = c^1(K_V^{-1})$ in $H^{1,1}(V,\mathbb{R})$, enabling us to endow K_V^{-1} with some curvature properties of (V,j).

Let X denote a B space where $\mathcal{D}(V)$ is dense. Then we identify $P_X(V,\omega)$ with $X \cap (x \mid \omega_x > 0)$, where ω_x is an abbreviation for $\omega + \sqrt{-1}d^{10}d^{01}x$. Therefore if $\mathbf{h}$ is the hermitean matrix associated locally with ω, then the hermitean matrix associated with ω_x is $\mathbf{h} + D^2_{\mathbb{C}}x =: \mathbf{h_x}$, where $D^2_{\mathbb{C}}x$ denotes the complex hessian of x. If $\mathcal{M}(x)$ (the complex Monge Ampère operator) is given by $\det \mathbf{h_x}/\det \mathbf{h}$, one deduces, assuming that $\lambda\omega/2\pi$ represents $c^1(V,j)$, that $E_\lambda(0) = \sqrt{-1}d^{10}d^{01}y_\omega$ together with $E_\lambda(x) = \sqrt{-1}d^{10}d^{01}((t-\lambda)x - sz)$ are equivalent to

$$\mathcal{M}(x) = Ce^{y_\omega - tx + sz} \tag{4}$$

provided that X is contained in $C^4(V)$, where $E_\lambda(x) := \sqrt{-1}Ric(\omega_x) - \lambda\omega_x$, and C is a constant related to the distortion in the total volume induced by the factor in the exponential.

3 Proof of Theorem 1.2

The equation (1) when $t = \lambda$ and $s = -1$ becomes

$$\omega_x^n = Ce^{y_\omega - z - \lambda x}\omega^n \tag{5}$$

whenever x is in $C^4(V)$, where C is explicitly given by

$$\int_V \frac{\omega^n}{n!} = |V| = \int_V \frac{\omega_x^n}{n!} =: C\int_V e^{y_\omega - z - \lambda x}\frac{\omega^n}{n!}.$$

By a well known argument that uses the convexity of $P_X(V,\omega)$ and that finally appeals to the classical maximum principle (see [26] for example), one deduces that the equation (5) has at most one solution.

• One then considers, as t varies within $[0,1]$, the family of equations

$$\mathcal{M}(x) = C_t e^{t(y_\omega - z) - \lambda x},$$

where C_t is a normalising constant that varies with t. With the help of the map

$$\begin{aligned} F_\lambda \quad : \quad & P_{C^{k+2,\alpha}}(V,\omega) \longrightarrow C^{k,\alpha}(V) \\ & x \to \log \mathcal{M}(x) + \lambda x, \end{aligned}$$

the family of equations can then be rewritten as

$$F_\lambda(x) = t(y_\omega - z) + \log C_t. \tag{6}$$

∘ In what follows $\langle \tau, D\varphi \rangle$ will denote the Fréchet derivative of φ in the direction of τ whenever φ belongs to $C^1(U,Z)$, Y and Z are complex B spaces, U is an open subset in Y, and τ is an arbitrary direction in Y ([17]). This notation will be used in various contexts.

The linearisation of F_λ at x (that we denote by $DF_\lambda(x)$) is

$$\begin{aligned} DF_\lambda(x) \quad : \quad & C^{k+2,\alpha}(V) \longrightarrow C^{k,\alpha}(V) \\ & b \to \langle b, DF_\lambda(x) \rangle = (\triangle_x + \lambda)b, \end{aligned}$$

where $\triangle_x$ is the (negative and divergence free) Laplacian with respect to the form ω_x. If λ is smaller than zero, the linearised map is invertible whenever x is in $P_{C^{k+2,\alpha}}(V,\omega)$. Thus, by the Inverse Function Theorem, F_λ is an invertible map from an open neighborhood of x in $P_{C^{k+2,\alpha}}(V,\omega)$ to an open neighborhood of $F_\lambda(x)$ in $C^{k,\alpha}(V)$. This is valid for all values of k and α.

$\circ$ But (6) has 0 as a (unique) solution when t is equal to zero (since C_0 is equal to 1). Due to the last arguments, there exists an ε bigger than zero such that (6) has a unique solution in $C^{k+2,\alpha}(V)$ for all t in $[0,\varepsilon[$ provided that z belongs to $C^{k,\alpha}(V)$: this is still valid for all values of k and α. We choose ε as big as possible.

Let $(t(i)|i \in \mathbb{N})$ be an increasing sequence in $[0,\varepsilon[$ so that $\lim_{i\to\infty} t(i) = \varepsilon$, and assume that $(x_{t(i)}|i \in \mathbb{N})$ is a bounded sequence in $C^{k+2,\alpha}(V)$ so that $F_\lambda(x_{t(i)}) = t(i)(y_\omega - z)$ holds whenever i belongs to $\mathbb{N}$. If $\varepsilon > 1$, then x_1 is the solution that we are looking for.

Being the embedding $C^{k+2,\alpha}(V) \hookrightarrow C^{k+2,\beta}(V)$ equibounded and equicontinuous (thus compact) whenever $0 < \beta < \alpha \leq 1$, there exists a subsequence of $(x_{t(i)}|i \in \mathbb{N})$ in $C^{k+2,\beta}(V)$, say $(x_{t(s(i))}|i \in \mathbb{N})$, such that $lim_{i\to\infty} x_{t(s(i))} = x_\varepsilon$ in $C^{k+2,\beta}(V)$, where x_ε is a solution to (6) when $t = \varepsilon$ (that obviously belongs to $C^{k+2,\beta}(V)$). But, considering the arguments given in the next subroutine (or the next $\bullet$), one deduces in particular that x_ε also belongs to $C^{k+2,\alpha}(V)$ (to achieve that, the requirements $k \geq 1$ and $\alpha > 0$ must be satisfied, as will be seen). If $\varepsilon \leq 1$, we arrive at a contradiction, since the Inverse Function Theorem ensures us that there also exists a solution for t in neighborhood of ε.

We conclude, as expected (taking into account the arguments given in the next subroutine), that to ensure the existence of a solution for (3) in $C^{k+2,\alpha}(V)$, it suffices to have a finite upper bound for the set of real numbers (up to now with the yet mentioned restrictions on (k,α))

$$(|x_t|_{(k+2,\alpha,\Omega)}|x_t \text{ solves (6) for all } t \text{ in } [0,1],\ \Omega \text{ is any open ball in a local chart}).$$

By a local chart we understand any chart in a system of charts (or an atlas) for V satisfyng the conditions described in the Section Preliminaires, and: the distance d_ω induced by the Kähler form ω on a local chart is equivalent to the standard distance $|\ \ |_{\mathbb{C}^n}$ in $\mathbb{C}^n$, i.e. if $\varphi : O \to \mathbb{C}^n$ is a local parametrisation, then there exists a constant $C > 0$ such that whenever z_1 and z_2 are in the open subset of $\mathbb{C}^n$ that corresponds to O, one has

$$C^{-1}|z_1 - z_2|_{\mathbb{C}^n} \leq d_\omega(\varphi^{-1}(z_1), \varphi^{-1}(z_2)) \leq C|z_1 - z_2|_{\mathbb{C}^n}.$$

$\bullet$ Let the pairing $[\cdot|\cdot]$ denote the euclidean (not the hermitean!) inner product in $\mathbf{M}(n,\mathbb{C})$. Let $\mathbf{h}_x := \mathbf{h} + D^2_{\mathbb{C}}x$, where $D^2_{\mathbb{C}}x$ denotes the element in $\mathbf{M}(n,\mathbb{C})^\Omega$ associated to the complex hessian of x. Let $\mathbf{h}^{-1}$ denote the one associated to the inverse of $\mathbf{h}$. Let $\mathtt{H} := M(n,\mathbb{C}) \cap (\mathbf{h}|\mathbf{h} = \mathbf{h}^+)$, and let $\mathtt{H}[\gamma,\Lambda]$ denote the subset of $\mathtt{H}$ whose elements have all their eigenvalues in the interval $[\gamma,\Lambda]$.

Remarks: Within this framework:
i) x belongs to $P(V,\omega)$ if and only if $\mathbf{h}_x(v)$ belongs to $\mathtt{H}_+ := \cup_{\epsilon>0}\mathtt{H}[\epsilon,+\infty]$ for each v in V.
ii) The ellipticity constants for $\triangle_x$ on Ω are γ and Λ if and only if $\mathbf{h}_x^{-1}$ belongs to $(\mathtt{H}[\gamma,\Lambda])^\Omega$.
iii) Some of the results of the theory of fully nonlinear elliptic equations can be naturally used in complex geometry, and viceversa.

Note: In the following the index Ω will be sometimes omitted. If there is no dependence on t in the estimates, that means that we set t to be equal to 1.

Noting then that the differentiation in any direction in $\mathbb{C}^n$ (that we omit) of $F_\lambda(x)$ is

$$D\log\mathcal{M}(x) + D\lambda x = [\mathbf{h}_x^{-1}|D\mathbf{h}_x] - [\mathbf{h}^{-1}|D\mathbf{h}] + \lambda Dx,$$

we then differentiate the equation (6) once, to get the equality

$$[\mathbf{h}_x^{-1}|D_{\mathbb{C}}^2 Dx] + \lambda Dx = [\mathbf{h}^{-1} - \mathbf{h}_x^{-1}|D\mathbf{h}] + tD(y_\omega - z), \tag{7}$$

where $[\mathbf{h}_x^{-1}|D_{\mathbb{C}}^2 Dx]$ should indeed be identified with $\triangle_x Dx$.

Assume now that a bound for $(|x|_{2,\alpha}|x$ solves (6) for $t=1)$ has been obtained. If $|y_\omega - z|_0$ is bounded, then (6) ensures that we have a bound for $|(h_x^{-1})_{ij}|_{0,\alpha}$ for each $(i,j) \in (1...n) \times (1...n)$, thus a bound Θ for $|[\mathbf{h}^{-1} - \mathbf{h}_x^{-1}|D\mathbf{h}]|_{0,\alpha}$, of course for the ellipticity constants γ and Λ of $\triangle_x$, and obviously for $|Dx|_0$.

If also $|D(y_\omega - z)|_{0,\alpha}$ is bounded, one has, due to the equality (7) and E. Schauder's estimate (see Chapter 6 of [19] for example), the inequality

$$|Dx|_{2,\alpha} \le C(n,\alpha,\gamma,\Lambda,\Theta,\lambda)(|Dx|_0 + |D(y_\omega - z)|_{0,\alpha}),$$

hence a bound for $|x|_{3,\alpha}$.

If the bound for $|x|_{3,\alpha}$ is obtained, then one has a bound for $|(h_x^{-1})_{ij}|_{1,\alpha}$ for each $(i,j) \in (1...n) \times (1...n)$. If also $|z|_{2,\alpha}$ (thus $|D(y_\omega - z)|_{1,\alpha}$) is bounded, then one has (due to E. Schauder's estimates and (7)) a bound for $|Dx|_{3,\alpha}$, hence for $|x|_{4,\alpha}$, and so on.

One concludes that if $k \ge 1$ and $\alpha > 0$, x solves (6) at $t=1$ and z belongs to $C^{k,\alpha}(V)$, then if one has a bound for $|x|_{2,\alpha}$, a bound for $|x|_{k+2,\alpha}$ is obtained.

• Assume now that $|D_{\tau\overline{\tau}}x(v)|$ is bounded whenever τ is a vector in $\mathbb{C}^n$ and v is in V (where $D_\tau x := \langle \tau, Dx\rangle$, for example), that z belongs to $C^2(V)$. Rewriting (6) at $t=1$ on Ω as

$$\log\det\mathbf{h}_x = \phi(z) - \lambda x, \tag{8}$$

where $\phi(z)$ is an abbreviation for $y_\omega - z - \log\det\mathbf{h} - \log C$, we choose a vector τ in $\mathbb{C}^n$, to then differentiate (8) twice, first in the direction of $\overline{\tau}$, then in the direction of τ, to get the equality

$$[D_\tau \mathbf{h}_x^{-1}|D_{\overline{\tau}}\mathbf{h}_x] + [\mathbf{h}_x^{-1}|D_{\tau\overline{\tau}}\mathbf{h}_x] = D_{\tau\overline{\tau}}\phi(z) - \lambda D_{\tau\overline{\tau}}x.$$

But $[\mathbf{h}_x^{-1}|D_{\tau\overline{\tau}}\mathbf{h}_x]$ is equal to $[\mathbf{h}_x^{-1}|D_{\tau\overline{\tau}}\mathbf{h}] + [\mathbf{h}_x^{-1}|D_{\mathbb{C}}^2 D_{\tau\overline{\tau}}x]$, and $[D_{\overline{\tau}}\mathbf{h}_x^{-1}|D_\tau\mathbf{h}_x]$ is less or equal than zero (to check the last inequality one can choose coordinates so that $(h_x)_{ij}$ is equal to zero unless $i=j$). We obtain thus the inequality

$$[\mathbf{h}_x^{-1}|D_{\mathbb{C}}^2 D_{\tau\overline{\tau}}x] \ge D_{\tau\overline{\tau}}\phi(z) - \lambda D_{\tau\overline{\tau}}x - [\mathbf{h}_x^{-1}|D_{\tau\overline{\tau}}\mathbf{h}]. \tag{9}$$

∘ Being $\log\det$ a concave function on $\mathtt{H}_+$ (and being $(M(n,\mathbb{C}), [\cdot|\overline{\cdot}])$ a Hilbert space), from the graph of $\log\det$ one infers that

$$\log\det\mathbf{h}_x(v_2) \le \log\det\mathbf{h}_x(v_1) + \langle \mathbf{h}_x(v_2) - \mathbf{h}_x(v_1), D\log\det(\mathbf{h}_x(v_1))\rangle$$

whenever v_1 and v_2 belong to Ω.

Since $\langle \mathbf{h}_x(v_2) - \mathbf{h}_x(v_1), D\log\det(\mathbf{h}_x(v_1))\rangle$ is equal to $[\mathbf{h}_x^{-1}(v_1)|\mathbf{h}_x(v_2) - \mathbf{h}_x(v_1)]$, from the previous expression and (8) we deduce

$$[\mathbf{h}_x^{-1}(v_1)|D_{\mathbb{C}}^2 x(v_1) - D_{\mathbb{C}}^2 x(v_2)] \le$$

$$\phi(z)(v_1) - \phi(z)(v_2) - \lambda(x(v_1) - x(v_2)) - [\mathbf{h}_x^{-1}(v_1)|\mathbf{h}(v_1) - \mathbf{h}(v_2)]. \tag{10}$$

∘ Having the inequalities (9) and (10), we can make the natural link with the theory of real Fully Nonlinear Elliptic equations to obtain the Hölder estimate for the second derivatives (the similitudes between Nonlinear Elliptic equations in real and complex domains seem to indicate that a *traduction* of all the known results would be unnecessary). We briefly explain how to do that in a slightly different form as done by Y. T. Siu in [28] (using the $d^{10}d^{01}$ Poincaré Lemma (see [20])) and by N. Trudinger ([19]) (in real domains) with the purpose of clarifying the regularity issues (**that otherwise would have remained in the shadows**). The reader should consult [28] and [19] for a complete exposition of some subroutines of what follows.

Remark: Y. T. Siu considers essentially the Fully Nonlinear operator $F(D^2_{\mathbb{C}}x) = \det(D^2_{\mathbb{C}}x)$, as done by N. Trudinger in real domains. We are essentially considering $F(D^2_{\mathbb{C}}x) = \det(1 + D^2_{\mathbb{C}}x)$ instead. Although the real analog of this case is contained in a much more general result of N. Trudinger, our argument, when restricted to this case, is much simpler. That might be of value for some real analysts and real differential geometers ([11]).

∘ Fixing $R \in \mathbb{R}$ small enough, setting $M(sR,\tau) := \sup_{B(sR)} D_{\tau\overline{\tau}}x$ whenever $B(sR)$ (the open ball of radius sR centered in the origin of the local parametrisation) is contained in Ω (that contains $B(2R)$ at least), we observe that (9) is equivalent to

$$[\mathbf{h}_x^{-1}|D^2_{\mathbb{C}}(M(sR,\tau) - D_{\tau\overline{\tau}}x)] \leq -D_{\tau\overline{\tau}}(\phi(z) - \lambda x) + [\mathbf{h}_x^{-1}|D_{\tau\overline{\tau}}\mathbf{h}], \tag{11}$$

to then note that the $|\ \ |_{0,B(2R)}$ norm of the right hand side is bounded since both

$$|D_{\tau\overline{\tau}}\phi(z) - \lambda D_{\tau\overline{\tau}}x|_{0,B(2R)}$$

and

$$|\ [\mathbf{h}_x^{-1}|D_{\tau\overline{\tau}}\mathbf{h}]\ |_{0,B(2R)} \leq \sqrt{n}\Lambda|\ [D_{\tau\overline{\tau}}\mathbf{h}|D_{\tau\overline{\tau}}\overline{\mathbf{h}}]^{\frac{1}{2}}\ |_{0,B(2R)}$$

are bounded by assumption.

After those observations one can invoke the Harnack type estimate of N. Krylov and M. Safanov for non negative supersolutions (see Chapter 9 in [19]) of Elliptic Equations, and use it on (11) for fixed τ in $\mathbb{C}^n$ and for $s = 2$, yielding

$$(R^{-2n}\int_{B(R)}(M(2R,\tau) - D_{\tau\overline{\tau}}x)^p dv)^{\frac{1}{p}} \leq$$

$$C_1\{M(2R,\tau) - M(R,\tau) + R^2\Upsilon(D^2_{\mathbb{C}}x, D^2_{\mathbb{C}}z, D^2_{\mathbb{C}}\mathbf{h})(\tau)\}, \tag{12}$$

where dv is the measure induced by ω in the local parametrisation, p and C_1 are positive constants that depend only on n and Λ/γ, and where $|\ D_{\tau\overline{\tau}}(\phi(z)-\lambda x)+[\mathbf{h}_x^{-1}|D_{\tau\overline{\tau}}\mathbf{h}]\ |_{0,B(2R)}$ has been abbreviated by $\Upsilon(D^2_{\mathbb{C}}x, D^2_{\mathbb{C}}z, D^2_{\mathbb{C}}\mathbf{h})(\tau)$.

∘ Now something should be done with (10) to couple it afterwards with (12). Since $\mathbf{h}_x^{-1}$ belongs to $\mathtt{H}$, we can certainly express it in the form $\sum_{i,j=1}^n (h_x^{-1})_{ij} Z_i \otimes \overline{Z_j}$ with $(h_x^{-1})_{ij} = \overline{(h_x^{-1})_{ji}}$. If $\mathbf{h}_x^{-1}$ belongs also to $\mathtt{H}[\gamma,\Lambda]$ for some $0 < \gamma < \Lambda < \infty$ (something that happens under our actual assumptions), then in virtue of the result in Linear Algebra of T. Motskin and W. Wascow (that was obtained to approximate Real Linear Elliptic equations by Difference equations ([25])) we can choose a finite set of unit vectors in

$\mathbb{C}^n$, say $(\tau_k|k \in (1...N))$, so that $\mathbf{h}_x^{-1}$ can be written also as $\sum_{k=1}^N (h_x^{-1})_k \tau_k \otimes \overline{\tau_k}$, where $\tau_k = \sum_{i=1}^n \tau_k^i Z_i$ and $0 < \frac{\gamma}{2N} \leq (h_x^{-1})_k \leq \Lambda$ for every k in $(1...N)$. Identifying $D_{\mathbb{C}}^2 x$ with $\sum_{i,j=1}^n Z^i \otimes \overline{Z^j} D_{Z_i \overline{Z_j}} x$, we conclude that $[\mathbf{h}_x^{-1}|D_{\mathbb{C}}^2 x]$ is equal to $\sum_{k=1}^N (h_x^{-1})_k D_{\tau_k \overline{\tau_k}} x$.

Therefore (10) gives

$$D_{\tau_l \overline{\tau_l}} x(v_1) - D_{\tau_l \overline{\tau_l}} x(v_2) \leq \frac{2N}{\gamma} \{ \Lambda \sum_{k \neq l} (M(2R, \tau_k) - D_{\tau_k \overline{\tau_k}} x(v_1)) +$$

$$3R(\ |\lambda||Dx|_{0,B(2R)} + |D\phi(z)|_{0,B(2R)} + \sqrt{n}\Lambda ||D\mathbf{h}||_{0,B(2R)} \ )\}$$

for every l in $(1...N)$, since

$$[\mathbf{h}_x^{-1}(v_1)|\mathbf{h}(v_1) - \mathbf{h}(v_2)] \leq \sqrt{n}\Lambda| \ [\mathbf{h}(v_2) - \mathbf{h}(v_1)|\overline{\mathbf{h}}(v_2) - \overline{\mathbf{h}}(v_1)]^{\frac{1}{2}} \ |_{0,B(2R)}$$

with

$$| \ [\mathbf{h}(v_2) - \mathbf{h}(v_1)|\overline{\mathbf{h}}(v_2) - \overline{\mathbf{h}}(v_1)]^{\frac{1}{2}} \ |_{0,B(2R)} = ||\mathbf{h}(v_2) - \mathbf{h}(v_1)||_{0,B(2R)} \leq 3R||D\mathbf{h}||_{0,B(2R)},$$

for example.

Abbreviating now

$$(\ |\lambda||Dx|_{0,B(2R)} + |D\phi(z)|_{0,B(2R)} + \sqrt{n}\Lambda ||D\mathbf{h}||_{0,B(2R)} \ )$$

by $\Phi(Dx, Dz, D\mathbf{h})$, introducing $m(sR, \tau) := \inf_{B(sR)} D_{\tau\overline{\tau}} x$ and choosing v_2 so that $m(2R, \tau_l) = D_{\tau_l \overline{\tau_l}} x(v_2)$, one gets from the previous inequality

$$D_{\tau_l \overline{\tau_l}} x(v_1) - m(sR, \tau_l) \leq \frac{2N}{\gamma} \{ \Lambda \sum_{k \neq l} (M(2R, \tau_k) - D_{\tau_k \overline{\tau_k}} x(v_1)) + 3R\Phi(Dx, Dz, D\mathbf{h}) \}. \quad (13)$$

∘ Defining finally $w(sR) := \sum_{k=1}^N (M(sR, \tau_k) - m(sR, \tau_k))$, integrating (13) over $B(R)$, putting that together with the sum over $k \neq l$ of the inequalities (12) (one for each τ_k), adding then over l and after a little algebra

$$w(2R) \leq C_2 \{ w(2R) - w(R) + R\Phi(Dx, Dz, D\mathbf{h}) + R^2 \sup_{\tau \in \mathbb{C}^n} \Upsilon(D_{\mathbb{C}}^2 x, D_{\mathbb{C}}^2 z, D_{\mathbb{C}}^2 \mathbf{h})(\tau) \}$$

is obtained. Then an iteration procedure *a la J. Moser* begins, to yield (after some recursive and logarithmic relations) the desired Hölder estimate with an exponent α that depends only on n, γ and Λ.

One concludes that if one has a bound for $|D_{\tau\overline{\tau}} x(v)|$ as v varies in V and τ within $\mathbb{C}^n$, and z belongs to $C^2(V)$ (in fact slightly less is needed), then a bound for $|x|_{2,\alpha}$ is obtained, where $\alpha > 0$.

• Assume now that a bound for $\mathrm{osc}_V x := \sup_V x - \inf_V x$ has been obtained.

Differentiating the equality (6) twice, using normal coordinates at the point in V where the estimate that one will obtain is valid, computing $\triangle_x(e^{-cx}(n + \triangle_0 x))$ for any c in $\mathbb{R}_+$ that satisfies

$$c + \inf_{v \in V} \inf_{\eta_1 \neq \eta_2 \in Gr_{\mathbb{C}}((1,1), T_v(V))} K(\eta_1, \eta_2)(v) > 0$$

(where $K(\eta_1,\eta_2)(v)$ denotes the bisectional curvature associated to the planes at v defined by the (simple) elements η_1 and η_2 in the Grassmann bundle of type $(1,1)$ vectorfields (see [24] for other definitions of curvature operators)), using some arithmetic inequalities and putting all that together as S.T. Yau did ([32], [7], [31]), one gets for each v in V

$$\triangle_x(e^{-cx(v)}(n+\triangle_0 x(v))) \geq$$
$$e^{-cx(v)}\ (\ A(v)+B(n+\triangle_0 x(v))+C(v)(n+\triangle_0 x(v))^{\frac{n}{n-1}}\ ), \tag{14}$$

where

$$A(v) := \lambda n - n^2 \inf_{\eta_1\neq\eta_2\in Gr_{\mathbb{C}}((1,1),T_v(V))} K(\eta_1,\eta_2)(v) + \triangle_0(y_\omega - z)(v),$$
$$B := -(\lambda + cn),$$

and

$$C(v) := (c + \inf_{\eta_1\neq\eta_2\in Gr_{\mathbb{C}}((1,1),T_v(V))} K(\eta_1,\eta_2)(v))e^{-\frac{(y_\omega - z - \lambda x)(v)}{n-1}} > 0.$$

Assume now that $e^{-cx}(n+\triangle_0 x)$ achieves a maximum at a certain point in V, say v_0. Then by one side one gets, whenever v belongs to V, the obvious comparisons

$$0 < (n+\triangle_0 x(v)) \leq e^{c(x(v)-x(v_0))}(n+\triangle_0 x(v_0)) \leq e^{c\cdot \mathrm{osc}_V x}(n+\triangle_0 x(v_0)),$$

(the first inequality is due to the fact that $0<[\mathbf{h}^{-1}|\mathbf{h}_x] = [\mathbf{h}^{-1}|\mathbf{h}] + \triangle_0 x$ whenever x is in $P_{C^2}(V,\omega)$). By other side the estimate (14) gives at that maximal point, the inequality

$$0 \geq A(v_0) + Bs + C(v_0)s^{\frac{n}{n-1}},$$

where

$$s := n + \triangle_0 x(v_0).$$

Hence everywhere $0 < n+\triangle_0 x \leq e^{c\cdot \mathrm{osc}_V x}K_1$, where $K_1 := (n + \sup_{v\in V}\triangle_0 x(v))$ is bounded in virtue of the polynomial inequality. Considering that x is in $P(V,\omega)$, we obtain the (pointwise) uniform bounds $-1 < D_{Z_i\overline{Z_i}}x \leq K_2 < \infty$ for each $i\in(1...n)$ (recall that we are using a special normal coordinate system at every point), where K_2 depends only on K_1 and n. As an outset the desired bound for $\langle\eta, d^{10}d^{01}x\rangle$ as η varies within the bounded sections in $Gr_{\mathbb{C}}((1,1),TV)$ follows.

One concludes that if z belongs to $C^2(V)$ (in fact slightly less is needed) and if one has a bound for $\mathrm{osc}_V x$, then a bound for $|D_{\tau\overline{\tau}}x(v)|$ for all $v\in V$ and τ in $\mathbb{C}^n$ is obtained.

• Thus a bound on $\mathrm{osc}_V x$ is finally required. It is very easy to obtain this bound: assume that x solves (6) at t, that it achieves a minimum at a certain point in V, say v_0. Being then $\mathcal{M}(x)(v_0)$ bigger or equal than 1, one deduces that

$$\inf_V x \geq -(y_w - z)(v_0) - \log C.$$

An analogous argument enables us to obtain an upper bound for $\sup_V x$. Therefore $\mathrm{osc}_V\ x$ is bounded provided that so is $y_w - z$. **q.e.d.**

The modifications for the case when λ is equal to 0 lead to

Corollary 3.1. *Assume that the canonical bundle is flat, that the sign of $c^1(V,j)$ is identified with λ (so λ is equal to 0). Then whenever z belongs to $C^{k,\alpha}(V)$ for $k\geq 2$ and α is in $]0,1[$, the equation (3) admits a unique (up to a constant) solution in $C^{k+2,\alpha}(V)$.*

The main differences between the negative and the flat case (the uniqueness is similar as for the negative case) is the estimate for $\mathrm{osc}_V\, x$. But, as seen, when λ is negative, the term z plays a passive role there, say: something similar happens when λ is zero, as one can verify ([32], [1], [7], [28], [31]).

4 Proof of iii), Theorem 1.1

Observe first that the equation (4) could be written as

$$F_t(x) = F(x,t) = sz + y, \tag{15}$$

where $y = y_\omega + \log C$ and $F_t(x) = \log \mathcal{M}(x) + tx$, C being the usual normalising constant.

When t is less than zero (equal to zero), we have proved in Section 3 that there exists a unique (up to a constant) solution x in $C^{k+2,\alpha}(V)$ for (15) whenever z is in $C^{k,\alpha}(V)$ for any s in $\mathbb{R}$, provided that $k \geq 2$ and $\alpha \in]0,1[$. Those are the statements i) (and ii)) in Theorem 1.1.

The statement concerning the Hölder regularity in iii) follows from the chain of estimates explained in Section 3. Therefore, without any danger of confussion, we will identify the B space to which z belongs with Z, while the one to which x belongs with X. The additional features that appear in this Section are the difficulties that arise in the Method of Continuity and the longer procedure to obtain a bound for the oscillation of the eventual solution. Roughly speaking, z does not play a passive role when t is positive.

We will see that if λ is equal to one and $\alpha_{(V,j)}$ is greater than $n/(n+1)$, then the equation (15) has at least one solution whenever $t \leq 1$ and $s = 0$ ([30]). This implies that (V,j) admits at least one Einstein form. All those forms are then in a single orbit under the natural action on functions of the identity component $Aut^0(V,j)$ of $Aut(V,j)$, the group of j holomorphic automorphisms of V ([4]). Therefore if ω is an Einstein form and $Stab(\omega)$ denotes the isotropy subgroup of $Aut^0(V,j)$ at ω, one certainly has that the set of Einstein forms for (V,j) is isomorphic to $Aut^0(V,j)/Stab(\omega)$. Moreover, if (V,j) carries an Einstein form, then the relation between Lie algebras $aut(V,j) = stab(\omega) + \sqrt{-1}stab(\omega)$ holds ([23]) (note that $aut(V,j)$ is the Lie algebra of j holomorphic vector fields on V). Those results seem to be useless to prove the existence of Einstein forms, but in some (Fano) models they enable us to assert: (V,j) *cannot support an Einstein form, therefore* $\alpha_{(\mathbf{V},\mathbf{j})}$ *is less or equal than* $n/(n+1)$ (see [18] for that and other criteria). If (V,j) is not Einstein, one can characterise the singularities of the divisor associated to the anticanonical bundle through its Lelong numbers, also through the complex singularity exponents and Arnold multiplicity of the associated Kähler potential, and relate it with $\alpha_{(\mathbf{V},\mathbf{j})}$. Once those relations are estabilished, one can give new conditions to ensure the existence of Einstein forms: that is a Corollary of what J.-P. Demailly and J. Kollár did in [13].

In Subsection 4.1 we describe how to proceed in the Continuity Methods (two are needed for t that couple with another for s). In Subsection 4.2 we explain how to obtain a bound for $\mathrm{osc}_V\, x = \sup_V x - \inf_V x$ in terms of $\alpha_{(V,j)}$, t, s and z, concluding the proof in Subsection 4.3, where K_- and K_+ are defined.

4.1 Methods of Continuity

Consider the equation (15), namely $F(x,t) = y + sz$, where $F : P_X(V,\omega) \times \mathbb{R} \to Z$ is considered as a map between open sets in B spaces. We first note that if $t = 0 = s$, then we have the equation $\mathcal{M}(x) = e^y$ that has a unique (up to a constant) solution in a certain set (or Sub Category) of functions on V (see the previous Section).

Consider a path (t,s) in $\mathbb{R}^2$ together with the associated path of equations. Consider the subfibration of $P_X(V,\omega) \times Z \to \mathbb{R}^2$ whose fibre over (t,s) is $Sol(F)(t,s) := ((x,z)|F(t,x) = y + sz)$. In the region where $t < 0$ we have seen that each fibre is isomorphic to $Z \times Z$, i.e. $Sol(F)(t,s)$ can there be seen as a single valued graph over Z, matching thus $P_X(V,\omega)$ with Z. Along the line $t = 0$ fibres are isomorphic to $(Z + \mathbb{R}) \times Z$, i.e. for all s in $\mathbb{R}$ one can then identify Z with $P_X(V,\omega)/\mathbb{R}$.

If we follow the path between $(0,0)$ and $(0,s)$ along the s axis the associated equations have always a unique (up to a constant) solution. However if we move in the positive direction from $(0,s)$ to (t,s) a natural obstruction appears: the associated linearised map might not be invertible. To avoid that obstruction we propose a particular route $\gamma : \mathbb{R}_+ \times \mathbb{R} \to Sol(F)$ between $(0,0)$ and (t,s) as follows:

Step 4.1. *Starting from $(0,0)$, we follow the t axis up to the point $(\varepsilon, 0)$, where ε is in a neighborhood of 0. This step is described in Subroutine 4.4.*

Step 4.2. *Starting from $(\varepsilon, 0)$, we continue along the t axis up to the point $(t,0)$. This step (and the obstructions that appear in it) are described in Subroutine 4.5.*

Step 4.3. *Starting from $(t,0)$, we now move along the s axis up to the point (t,s). This step (and the corresponding obstructions) are described in Subroutine 4.6.*

It might be possible to proceed along another route.

Subroutine 4.4. *P. Delanoë showed how to proceed in Step 4.1 during his study of the real Monge Ampère equation ([11], [2]). Consider the map*

$$\begin{aligned} H \ &: \ P_X(V,\omega) \times \mathbb{R} \longrightarrow Z \\ &(x,t) \to \log \mathcal{M}(x) + tx + \langle x \rangle_0, \end{aligned}$$

where $\langle x \rangle_0$ is an abbreviation for the average of x with respect to ω (see the Definition 4.7 in Subsection 4.2), together with its linearisation at (x,t)

$$\begin{aligned} DH(x,t) \ &: \ X \times \mathbb{R} \longrightarrow Z \\ &(b,r) \to \langle (b,r), DH(x,t) \rangle = \langle b, D_1 H(x,t) \rangle + \langle r, D_2 H(x,t) \rangle, \end{aligned}$$

where $\langle b, D_1 H(x,t) \rangle = \triangle_x b + tb + \langle b \rangle_0$ is the differential of H in the direction of b when evaluated at (x,t). One verifies that $D_1 H(x,0)$ is an invertible map in $L(X,Z)$ whenever x is in $P_X(V,\omega)$. Then, by the Implicit Function Theorem there exists, for every x, a neighborhood of $(x,0)$ so that the map $(x,t) \to (H(x,t),t)$ is a homeomorphism from there to some open set in $Z \times \mathbb{R}$.

Consider the equation $H(x,0) = y$, i.e. $\mathcal{M}(x) = e^{y - \langle x \rangle_0}$. Since $\mathcal{M}(x) = e^y$ has a unique up to a constant solution, we infer that $H(x,0) = y$ has a unique solution (choosing the constant so that $\langle x \rangle_0 = 0$): let that solution be $\mathring{x}_0$. Returning to the previous paragraph, we *conclude that there is implicit in H a homeomorphism $nbhd(\grave{x}_0, 0) \cong nbhd(y, 0)$, therefore an $\varepsilon > 0$ so that $H(x,t) = y$ has a unique solution, say $\grave{x}_t$, for every t in $[0, \varepsilon]$. This is the ε mentioned in Step 4.1... indeed, setting $x_t = \grave{x}_t + \langle \grave{x}_t \rangle_0 / t$ one notices that x_t is a solution for $F(x_t, t) = y$.*

Remark: One can use the same argument to show that for every s there exists a neighborhood of $t = 0$ such that $H(x,t) = y + sz$ has a unique solution whenever t is in such a neighborhood. Since we do not know how to proceed further along the t axis in all those cases, we do not consider them in statement iii) in Theorem 1.1.

Subroutine 4.5. *We now move from $(\varepsilon, 0)$ further along the t axis in the positive direction. We note that $D_1F(x,t) = \triangle_x + t$ is invertible unless t is an eigenvalue of $\triangle_x$. If $t < 1$ and x satisfies (15) at $(t, 0)$ this is not possible (see Subsection 4.3). Appealing again to the Implicit Function Theorem, we conclude that $F(x,t) = y$ has a solution for t in some open neighborhood of ε. That neighborhood is very important. It will be shown in the next Section that such a neighborhood is at least of a certain size (given by $\alpha_{(\mathbf{V},\mathbf{j})}$). We observe that F is of type C^1.*

Subroutine 4.6. *Assume now that $F(x,t) = y$ has a solution, also that $D_1F(x,t)$ is invertible. By the Inverse Function Theorem $F_t : X \to Z$ induces a homeomorphism $nbhd(x) \cong nbhd(y)$. We choose s so that $y + sz$ is within $nbhd(y)$. One continues with this procedure along the s axis until $F(x,t) = y + sz$ has a solution (thanks to the a Priori Estimates) but the linearised map might not be invertble anymore (see Subsection 4.3).*

4.2 Estimates for the Oscillation

Definition 4.7. *To abbreviate, whenever ϕ is a scalar valued function on V and x belongs to $P_X(V, \omega)$, we identify "the x weighted average of ϕ" with $\langle \phi \rangle_x := \int_V \phi \omega_x^n / (n! |V|) = \langle \nu, \phi \frac{\omega_x^n}{n!} \rangle / |V|$, where ν is the locally rectifiable current of bidimension (n, n) associated to the fundamental cycle for V (see [17]).*

Definition 4.8. *The functionals $I_\omega(x)$ and $J_\omega(x)$ are defined, whenever x is in $P_X(V, \omega)$, through $I_\omega(x) := \langle x \rangle_0 - \langle x \rangle_x$ and $J_\omega(x) := \int_{[0,1]} I_\omega(sx)/s$.*

Lemma 4.9. *Whenever x belongs to $P_X(V, \omega)$ one has that $I_\omega(x) \leq (n+1) J_\omega(x) \leq n I_\omega(x)$.*

Lemma 4.10. *Assume that x is in $C^1([0,1], P_X(V, \omega))$. Then for each t in $]0,1[$ one has the identity $\frac{d}{dt} J_\omega(x) = \langle \frac{dx}{dt} \rangle_0 - \langle \frac{dx}{dt} \rangle_x$.*

Those statements are well known. They follow from direct computations that can be found in different form in either [2], [28], [30] or [26].

4.2.1 The Aubin Tian constant

We now give sufficient conditions to ensure that x solves (15), i.e. to obtain a bound for osc x. This is done *via* a generalisation of what is known in the literature as the Aubin Tian constant.

Rewriting the equation (15) as $\omega_x^n / n! = e^{y+sz-tx} \omega^n / n!$, one readily deduces, after an integration by parts, the simple equalities

$$|V| := \int_V \frac{\omega^n}{n!} = \frac{(2\pi)^n}{n!} \langle \nu, c^1(V)^n \rangle = \int_V e^{y+sz-tx} \frac{\omega^n}{n!},$$

thus the inequality

$$|V| \le e^{\sup_V (y+sz)} \langle \nu, e^{-tx} \frac{\omega^n}{n!} \rangle.$$

The previous inequality then becomes $1 \le c_0(s) e^{-t\langle x \rangle_0} \langle e^{-t(x - \langle x \rangle_0)} \rangle_0$, where $0 < c_0(s) := e^{\sup_V (y+sz)} < \infty$ under our assumptions.

Therefore $c_0(s)^{-1} e^{t\langle x \rangle_0} \le \langle e^{-t(x-\langle x \rangle_0)} \rangle_0$ follows, where $\langle e^{-t(x-\langle x \rangle_0)} \rangle_0 \le \langle e^{-tp(x-\langle x \rangle_0)} \rangle_0^{\frac{1}{p}}$ whenever $1 \le p$ in virtue of Hölder's inequality.

Fix t and κ so that $0 < t \le \kappa$. Then $\frac{\kappa}{t} \ge 1$. We obtain

$$c_0(s)^{-1} e^{t\langle x \rangle_0} \le \langle e^{-\kappa(x-\langle x \rangle_0)} \rangle_0^{\frac{t}{\kappa}}. \tag{16}$$

We note that $\langle x \rangle_0 \le \sup x$ and $\inf x \le x$, whence $\langle x \rangle_0 - x \le \sup x - \inf x = \operatorname{osc} x$. Thus $\langle e^{-b(x-\langle x \rangle_0)} \rangle_0 \le e^{b \cdot \operatorname{osc} x}$ whenever $b > 0$ (independent of the equation (15)).

But from Lemma 4.15 we have the bound $\operatorname{osc} x \le I_\omega(x) + D(t,s)$ provided that x solves (15). Under those assumptions for every $b > 0$ one has the estimate

$$\langle e^{-b(x-\langle x \rangle_0)} \rangle_0 \le e^{bD(t,s)} e^{bI_\omega(x)}.$$

After those remarks, from (16) we infer

$$c_0(s)^{-\frac{\kappa}{t}} e^{\kappa\langle x \rangle_0} \le \langle e^{-\kappa(x-\langle x \rangle_0)} \rangle_0 \le e^{\kappa D(t,s)} e^{\kappa I_\omega(x)} \tag{17}$$

whenever $0 < t \le \kappa$ and x solves (15).

One is naturally led to

Definition 4.11. *The real number $\Upsilon(\kappa, s)$ is defined as*

$$\inf(\ \Upsilon > 0 \mid (\exists\, \mathcal{C} \in \mathbb{R})\ \{\langle e^{-\kappa(x-\langle x \rangle_0)} \rangle_0 \le \mathcal{C} e^{\Upsilon I_\omega(x)}, x \text{ solves (15) for } t \le \kappa \text{ at } s\}\ ).$$

The number $\Upsilon(\kappa, s)$ is a generalisation of what is known in the literature as Aubin's constant ([2]). In particular, when $s = 0$, a different definition has been proposed by A. Ben Abdesselem ([5]). The importance of this Definition is the following: it leads to Proposition 4.12, that in turn leads, thanks to the the Definition 4.13, to Corollary 4.14, as will be seen.

Important: *It is fundamental to observe that if the quantified constant $\mathcal{C}$ in the Definition of $\Upsilon(\kappa, s)$ does not exist, then $\Upsilon(\kappa, s) = \infty := \inf \emptyset$. That is the case if the equation (15) has no solution.*

From (17) and the Definition of $\Upsilon(\kappa, s)$ we deduce the inequality

$$\langle x \rangle_0 \le \frac{\log c_0(s)}{t} + \frac{1}{\kappa}(\ \Upsilon(\kappa, s) I_\omega(x) + \log \mathcal{C}\ ).$$

Setting $H(\kappa, s, t) := \frac{\log c_0(s)}{t} + \frac{\log \mathcal{C}}{\kappa}$ the last inequality becomes

$$-I_\omega(x) \le \frac{\kappa}{\Upsilon(\kappa, s)}(\ H(\kappa, s, t) - \langle x \rangle_0\ ). \tag{18}$$

Along a different perspective, we differentiate the equation (15) with respect to t (since x has a continuous derivative with respect to t as observed in Subroutine 4.5, this can be done) to note that

$$\frac{\partial}{\partial t}|V| = 0 = \frac{\partial}{\partial t}\langle \nu, \frac{\omega_x^n}{n!} \rangle = \langle \nu, \frac{\partial}{\partial t}\frac{\omega_x^n}{n!} \rangle = \langle \nu, (-x - t\frac{\partial}{\partial t}x)\frac{\omega_x^n}{n!} \rangle,$$

thus $-t\langle\frac{\partial}{\partial t}x\rangle_x = \langle x\rangle_x$.

From the Definitions of $I_\omega(x)$ and $J_\omega(x)$, Lemma 4.10 and the last equality one deduces the expression

$$\frac{\partial}{\partial t}(J_\omega(x) - \langle x\rangle_0) = \frac{1}{t}(\langle x\rangle_0 - I_\omega(x)) \tag{19}$$

whenever $0 < t$ and x solves $\log \mathcal{M}(x) + tx = \Phi$ for every function Φ that does not depend on x.

Using (18) and (19) one gets

$$\frac{\partial}{\partial t}(J_\omega(x) - \langle x\rangle_0) \leq \frac{1}{t}(\frac{\kappa}{\Upsilon(\kappa,s)}H(\kappa,s,t) + (1 - \frac{\kappa}{\Upsilon(\kappa,s)})\langle x\rangle_0).$$

We recall that $H(\kappa,s,t) = \log(c_0(s)^{\frac{1}{t}}\mathcal{C}^{\frac{1}{\kappa}})$, to define

$$H(\kappa,s) := \sup(0, \sup(H(\kappa,s,t)|0 < \varepsilon \leq t \leq \kappa))$$

for a fixed ε given by the Subroutine 4.4 in Subsection 4.1. This enables us to conclude from the previous inequality the useful estimate

$$\frac{\partial}{\partial t}(J_\omega(x) - \langle x\rangle_0) \leq \frac{\kappa}{\varepsilon\Upsilon(\kappa,s)}H(\kappa,s) + \frac{1}{t}(1 - \frac{\kappa}{\Upsilon(\kappa,s)})\langle x\rangle_0 \tag{20}$$

valid whenever $0 < \varepsilon \leq t \leq \kappa$ if x solves (15).

Invoking Jensen's inequality one deduces $e^{\langle y-tx+sz\rangle_0} \leq \langle e^{y-tx+sz}\rangle_0 = 1$ if x solves (15), to obtain the lower bound $\langle x\rangle_0 \geq \langle y+sz\rangle_0/t$.

In virtue of the inequality that lead us to the Definition of $\Upsilon(\kappa,s)$ one observes that a sufficient condition for the existence of the *quantified* constant $\mathcal{C}$ is the achievement of a bound for $D(t,s)$. If such a bound is achieved, one also observes that $\Upsilon(\kappa,s)$ must be smaller or equal than κ. If that is the case, then $\kappa H(\kappa,s)/\Upsilon(\kappa,s)$ is bounded. Taking into account the bound for $\langle x\rangle_0$ (and also that $1 - \kappa/\Upsilon(\kappa,s) \leq 0$ if the bound for $D(t,s)$ is obtained), we get a bound for the right hand side of (20), say $K(\kappa,s) > 0$, to infer

$$J_\omega(x) \leq tK(\kappa,s) + \langle x\rangle_0 \leq \kappa K(\kappa,s) + \langle x\rangle_0$$

after an integration from 0 to t, for $0 < \varepsilon \leq t \leq \kappa$.

Plugging (18) in the previous inequality we note that

$$J_\omega(x) \leq T(\kappa,s) + \frac{\Upsilon(\kappa,s)}{\kappa}I_\omega(x),$$

where $\kappa K(\kappa,s) + H(\kappa,s,t) \leq \kappa K(\kappa,s) + H(\kappa,s) =: T(\kappa,s) < \infty$ has been used. But from Lemma 4.9 we know that $\frac{1}{n+1}I_\omega(x) \leq J_\omega(x)$. As a conclusion the relation

$$(\frac{1}{n+1} - \frac{\Upsilon(\kappa,s)}{\kappa})I_\omega(x) \leq T(\kappa,s) \tag{21}$$

is obtained.

We should state

Proposition 4.12. *For fixed s, let $t(s)$ be given by*

$$\sup(\ \kappa \mid \Upsilon(\kappa, s) < \frac{\kappa}{n+1}\ ).$$

Then $\log \mathcal{M}(x) + tx = y + sz$ has a solution whenever $t \leq t(s)$ if $D(t,s)$ is bounded.

Proof: Given $0 < \kappa$, from (21) we infer that if $\Upsilon(\kappa, s) < \frac{\kappa}{n+1}$, then $I_\omega(x)$ is bounded whenever $t \leq \kappa$. If moreover $D(t,s)$ is bounded, then so is osc x in virtue of Lemma 4.15. **q.e.d.**

Through a reasoning similar to the one that led us to the definition of $\Upsilon(\kappa, s)$, we observe that whenever $\kappa > 0$ and $\alpha \leq \kappa$ one has

$$\langle e^{-\kappa(x - \langle x \rangle_0)} \rangle_0 \leq \langle e^{-\alpha(x-\langle x \rangle_0)} \rangle_0 \ e^{(\kappa-\alpha)(I_\omega(x)+D(t,s))}$$

if x satisfies (15).

One arrives to

Definition 4.13. *The real number $\alpha_{(\mathbf{V},\mathbf{j})}$ is defined as*

$$\sup(\ \alpha > 0 \mid (\ \exists \mathcal{C}_\alpha \in \mathbb{R}\ )\ (\ \forall x \in P_X(V,\omega)\ )\ (\ \langle\ e^{-\alpha\ (x - \langle x \rangle_0)}\ \rangle_0 \leq \mathcal{C}_\alpha\ )\ ).$$

The constant $\alpha_{(\mathbf{V},\mathbf{j})}$ is known in the literature as Tian's invariant. The importance of this definition is the following: the set on which the supremum is searched is not empty in virtue of G. Tian's result (Proposition 1.2 in [30]) that appeals to results of L. Hörmander (Theorem 4.4.5 in [21]).

Considering the Subroutines 4.5 and 4.6 in Subsection 4.1, and of course the definitions of $\Upsilon(\kappa,s)$ and $\alpha_{(\mathbf{V},\mathbf{j})}$ (and the observations that drove us to those definitions), from Proposition 4.12 one deduces

Corollary 4.14. *Assume that $\alpha_{(\mathbf{V},\mathbf{j})} > \frac{\kappa\, n}{n+1}$. Then $\log \mathcal{M}(x) + tx = y + sz$ has a solution whenever $0 < t \leq \kappa$ if $D(t,s)$ is bounded.*

Proof: If $\kappa > 0$, $\kappa - \alpha \leq 0$, $\alpha \leq \alpha_{(\mathbf{V},\mathbf{j})}$ and $D(t,s)$ is bounded, the inequality that led us to the definition of $\alpha_{(\mathbf{V},\mathbf{j})}$ shows that the inequality $\langle e^{-\kappa(x-\langle x \rangle_0)} \rangle_0 \leq \mathcal{C} e^{(\kappa-\alpha) I_\omega(x)}$ holds, where $\mathcal{C} < \infty$. Therefore (by construction) $\Upsilon(\kappa,s) \leq \kappa - \alpha$. Moreover, if $\alpha_{(\mathbf{V},\mathbf{j})} > (\kappa\ n)/(n+1)$, then the conditions for $\Upsilon(\kappa,s)$ given in Proposition 4.12 are fulfilled. **q.e.d.**

4.2.2 The estimate *osc* $x \leq I_\omega(x) + D(t,s)$ and the lower bound for the Green's function.

Let x belong to $P_X(V,j)$. Let dv_x denote the probability measure on V that makes $\triangle_x$ self adjoint on $L_2(V, dv_x)$. Let G_x denote the Green's function for x, namely the mapping

$$\begin{aligned} G_x \quad : \quad & V^2 \setminus \textit{diagonal} \longrightarrow \mathbb{R} \\ & (v_1, v_2) \to G_x(v_1,v_2) = G_x(v_2,v_1) \end{aligned}$$

that satisfies, whenever Ψ is in $\mathcal{D}(V)$, the equality $\Psi = \langle \Psi \rangle_x - \langle G_x \triangle_x \Psi \rangle_x$. One observes that G_x is determined uniquely up to a constant.

Let x belong to $P_X(V,\omega)$, and assume that there exists a real number $-c_x$ so that $\inf_{v_1\neq v_2} G_x(v_1,v_2) \geq -c_x > -\infty$. Then $\Psi = \langle\Psi\rangle_x - \langle (G_x + c_x) \bigtriangleup_x \Psi\rangle_x$ for any Ψ in $\mathcal{D}(V)$.

Recalling now that $-n \leq \bigtriangleup_0 x$ (since $[\ \mathbf{h^{-1}}|\mathbf{h_x}\ ] > 0$), and noting that $n \geq \bigtriangleup_x x$ whenever x is in $P_X(V,\omega)$ (since $[\ \mathbf{h}_x^{-1}|\mathbf{h}_x\ ] = n$), one obtains, assuming that both c_0 and c_x exist, the useful inequalities (that are useless if either c_0 or c_x cannot be found) $x \leq \langle x\rangle_0 + nc_0$ and $-x \leq -\langle x\rangle_x + nc_x$ (where we have chosen $\langle G_x\rangle_x = 0 = \langle G_0\rangle_0$).

One concludes, if both c_0 and c_x exist, that (nontrivially)

$$\sup x - \inf x = \text{osc } x \leq \langle x\rangle_0 - \langle x\rangle_x + n(c_0 + c_x).$$

But $\langle x\rangle_0 - \langle x\rangle_x$ is equal to $I_\omega(x)$. We summarise what has being said in

Lemma 4.15. *Assume that x satisfies (15) at (t,s). Then one has the estimate*

$$osc\ x \leq I_\omega(x) + D(t,s),$$

where $D(t,s)$ is an abbreviation for

$$-n\ (\inf_{V^2\setminus diagonal} G_0 + \inf_{V^2\setminus diagonal} G_x\ ).$$

Using standard relations between the Green's function and the heat kernel S. Bando and T. Mabuchi (see also [10] and [28] and the references therein) obtained the inequality

$$\inf_{V^2\setminus diagonal} G_x \geq -B(n,K_x)\ \frac{diam_x(V)^2}{|V|}$$

valid whenever x is in $P_X(V,\omega)$, where $B(n,K_x)$ is a positive constant that depends only on the dimension of V and on a lower bound for $\sqrt{-1}Ric(\omega_x)diam_x(V)^2$ ([4]). That inequality was also obtained by S. Gallot (unpublished) as a Corollary of his work with P. Bérard and G. Besson ([6]) on isoperimetric profiles ([3]). Therefore to conclude the proof of Theorem 1.1 we must give conditions for the pairs (t,s) to ensure both a lower bound for $\sqrt{-1}Ric(\omega_x)$ and an upper bound for $diam_x(V)$ when x satisfies (15), taking into account:

i) The critical points in the continuity method, namely those values of (t,s) mentioned in the Subroutines 4.5 and 4.6 in Subsection 4.1.

ii) The restrictions on t obtained in Corollary 4.14.

iii) That the complex structure remains fixed in our discussion.

4.3 Final Step

We first observe that (15) with $\lambda = 1$ is equivalent to

$$\sqrt{-1}Ric(\omega_x) = t\omega_x + (1-t)\omega - s\sqrt{-1}d^{10}d^{01}z. \tag{22}$$

We understand that ω and z in (15) or in (22) are given. Thus there exist a unique pair of (best) constants $K_- \leq 0 \leq K_+$ for which the inequality

$$K_- \leq \langle -\sqrt{-1}\tau\wedge\overline{\tau}, \sqrt{-1}d^{10}d^{01}z\rangle / \langle -\sqrt{-1}\tau\wedge\overline{\tau},\omega\rangle \leq K_+$$

holds as τ varies within $\wedge_{10}T_vV$ while v over V. It is important to choose the best possible constants: K_- and K_+ are the bounds for the (pluriharmonic) concavity and convexity of z with respect to ω, respectively.

After those remarks we can investigate the range of values for (t, s) mentioned in the Subroutines 4.5 and 4.6 in Subsection 4.1, considering also the restrictions on t obtained in Corollary 4.14.

Recall that the critical points in the continuity method, say, are those pairs (t, s) where $\triangle_x + t$ is not invertible anymore (where x solves (15) at (t, s)): they are indeed related with the first eigenvalue (or spectral gap) of $-\triangle_x$, that is usually denoted by $\lambda_1(-\triangle_x)$.

Before travelling further in our continuity method, we recall two Basic Results that will allow us to do that.

Basic Result 4.16. *(Bochner Weitzenböck Lichnerowicz') (see [7] for example)*

$$\textit{If} \inf_{v\in V} \inf_{\tau\in\wedge_{10}T_vV} \frac{\langle -\sqrt{-1}\tau\wedge\overline{\tau}, \sqrt{-1}Ric(\omega_x)\rangle}{\langle -\sqrt{-1}\tau\wedge\overline{\tau}, \omega_x\rangle} \geq t, \ \textit{then}\ \lambda_1(-\triangle_x) \geq t.$$

Basic Result 4.17. *(Bonnet Myers') (see [14] for example)*

$$\textit{If} \inf_{v\in V} \inf_{\tau\in\bigwedge_{10}T_vV} \frac{\langle -\sqrt{-1}\tau\wedge\overline{\tau}, \sqrt{-1}Ric(\omega_x)\rangle}{\langle -\sqrt{-1}\tau\wedge\overline{\tau}, \omega_x\rangle} \geq t > 0, \textit{then}\ diam_x(V) \leq \pi(\frac{2n-1}{t})^{\frac{1}{2}}.$$

Hence whenever $0 < t < min(1, (n+1)\alpha_{(\mathbf{V},\mathbf{j})}/n)$ and $s = 0$ we can proceed via Subroutine 4.5, because then for every τ in $section(\wedge_{10}TV)$ we have from (22) and the Basic Result 4.16

$$\langle -\sqrt{-1}\tau\wedge\overline{\tau}, \sqrt{-1}Ric(\omega_x)\rangle \ > \ t\ \langle -\sqrt{-1}\tau\wedge\overline{\tau}, \omega_x\rangle, \quad \text{thus} \quad \lambda_1(-\triangle_x) > t,$$

(showing that the associated linearised map is at this stage invertible), and because then we have found, also from (22) and the Basic Result 4.17, a bound for $diam_x(V)$, thus for $D(t, 0)$.

If $\alpha_{(\mathbf{V},\mathbf{j})} > n/n+1$, then we can proceed along t up to when t is equal to 1 but not further, since then (22) and the Basic Result 4.16 show that $\lambda_1(-\triangle_x) \geq 1$, and therefore the associated linearised map might not be invertible anymore.

So whenever $0 < t < min(1, (n+1)\alpha_{(\mathbf{V},\mathbf{j})}/n)$ and $s = 0$ we have that (15) has at least one solution, and also what is fundamental for the next developments, that $\triangle_x + t$ is then invertible. This last observation enables us to continue with the Subroutine 4.6 in our multiple Method of Continuity.

For fixed t in $]0, min(1, (n+1)\alpha_{(\mathbf{V},\mathbf{j})}/n)[$ we consider the following values of s:

i) if s is positive, we require that $t + s\ K_+ \leq 1$;

ii) if s is negative, we require that $t + s\ K_- \leq 1$.

Consider the case i): the other is similar. Let $((t, r)|r \in [0, s])$ be path from $(t, 0)$ to (t, s).

If s satisfies the conditions yet mentioned, then whenever $0 \leq r < s$ we have the inequality

$$(1-t)\ \langle -\sqrt{-1}\tau\wedge\overline{\tau}, \omega\rangle - r\ \langle -\sqrt{-1}\tau\wedge\overline{\tau}, \sqrt{-1}d^{10}d^{01}z\rangle > 0$$

as τ varies within $\wedge_{10}T_vV$ while v in V. Evaluating (22) at $-\sqrt{-1}\tau \wedge \overline{\tau}$ and inserting the last inequality therein, we conclude, with the help of the Theorem of Bonnet and Myers, that $D(t,r)$ is indeed bounded, i.e. that (15) has at least one solution at (t,r) in virtue of the Subroutines at the end of Subsection 4.2.2. We also conclude, with the additional help of A. Lichnerowicz' comparison, that if x denotes the solution to (15), then $\lambda_1(-\triangle_x) > t$, i.e. that the linearised map associated to the equation at (t,r) is invertible.

The continuity method concludes when r is equal to s. We have that $D(t,s)$ is indeed bounded, but we have an inequality with the possibility of equality for the spectral gap for $-\triangle_x$, and the linearised map might not be invertible anymore.

Finally, considering all the allowed values of t and the sequences of r's converging from 0 towards the corresponding s, one should, without any need of further explanations, say that the proof of Theorem 1.1 is complete. **q.e.d.**

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