# THE FINITENESS OF CERTAIN SETS OF ATTACHED PRIME IDEALS AND THE LENGTH OF GENERALIZED FRACTIONS


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**Abstract.**[1] We introduce the notion of *strictly f-sequence* and apply this concept to study the finiteness of asymptotic sets of attached prime ideals of local cohomology modules of $M$, to study the polynomial property of the length of generalized fractions as defined by Sharp and Hamieh [SH], and to characterize pseudo generalized Cohen-Macaulay modules defined in [CN].


## 1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a noetherian local ring, $M$ a finitely generated $R-$module with $\dim M = d$. For each artinian $R-$module $A$ and a sequence of elements $(x_1, \dots, x_k)$ of $R$, R. Y. Sharp has proved in [Sh1] that $\bigcup_n \operatorname{Att}(0 :_A (x_1, \dots, x_k)^n R)$ is a finite set. We also have known that the local cohomology module $H^i_{\mathfrak{m}}(M)$ is artinian for all $i = 1, \dots, d$. Therefore, it is natural to ask whether $\bigcup\limits_{n_1,\dots,n_k} \operatorname{Att}_R(0 :_{H^i_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_k^{n_k})R)$ and $\bigcup\limits_{n_1,\dots,n_k} \operatorname{Att}_R(H^i_{\mathfrak{m}}(M/(x_1^{n_1}, \dots, x_k^{n_k})M)$ are finite sets for all $i = 1, \dots, d$, where the unions run through all $k$-tuples of positive integers $(n_1, \dots, n_k)$. However, M. Katzman [K, Corollary 1.3] has constructed a noetherian local ring $(T, \mathfrak{m})$ and two elements $u, v \in \mathfrak{m}$ such that $\dim T = 5$, $\dim T/(u,v)T = 4$ and $\operatorname{Ass} H^2_{(u,v)T}(T)$ is a infinite set. Therefore it is not difficult to see that $\bigcup\limits_n \operatorname{Att}_R(H^i_{\mathfrak{m}}(T/(u^n, v^n)T)$ is a infinite set for some $i \in \{1, 2, 3\}$. It should be note that $(u, v)$ is not a part of a system of parameters of $T$. Therefore we ask the following question.


[1] **Keywords** Strictly f-sequence, attached prime, local cohomology, generalized fraction.
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**Question.** *Let $(x_1, \dots, x_k)$ be a part of a system of parameters of $M$. Are the sets*

$$\bigcup_{n_1,\dots,n_k} \operatorname{Att}_R(0 :_{H^i_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_k^{n_k})R) \text{ and } \bigcup_{n_1,\dots,n_k} \operatorname{Att}_R(H^i_{\mathfrak{m}}(M/(x_1^{n_1}, \dots, x_k^{n_k})M)$$

*finite for all $i = 1, \dots, d$?*

The purpose of this paper is to give a positive answer to this question for a kind of sequences called strictly f-sequences. Then we use strictly f-sequences to study the length of generalized fractions defined by Sharp and Hamieh [SH], and to characterize pseudo generalized Cohen-Macaulay modules defined in [CN].

**Definition.** A sequence $(x_1, \dots, x_k)$ of elements in $\mathfrak{m}$ is called a *strictly f-sequence* if $x_{j+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \bigcup_{i=1}^{d-j} \operatorname{Att}(H^i_{\mathfrak{m}}(M/(x_1, \dots, x_j)M) \setminus \{\mathfrak{m}\}$ for all $j = 0, \dots, k-1$.

Note that each strictly f-sequence is an f-sequence which was defined in [CST]. Therefore each strictly f-sequence of $d$ elements is a system of parameters of $M$. For each positive integer $k$, strictly f-sequences of $k$ elements always exist, and if $(x_1, \dots, x_k)$ is a strictly f-sequence of $M$ then $(x_1^{n_1}, \dots, x_k^{n_k})$ is again a strictly f-sequence of $M$ for all positive integers $n_1, \dots, n_k$ (Lemmas 3.1, 3.4).

The following theorem gives a positive answer to the above question for all strictly f-sequences.

**Theorem 1.1.** *Let $(x_1, \dots, x_k)$ be a strictly f-sequence of $M$ and $\{n_1, \dots, n_k\}$ a set of positive integers. Then for all $i = 1, \dots, d$, both of sets* $\operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_k^{n_k})R) \setminus \{\mathfrak{m}\}$ *and* $\operatorname{Att}(H^i_{\mathfrak{m}}(M/(x_1^{n_1}, \dots, x_k^{n_k})M) \setminus \{\mathfrak{m}\}$ *are independent of $n_1, \dots, n_k$.*

In [SZ1], Sharp and Zakeri introduced the theory of modules of generalized fractions. In this theory, for given positive integer $k$, the subsets so-called *triangular subsets* of $R^k$ play a role as multiplicative close subsets of $R$ in the usual theory of localization of modules. Given a triangular subset $U$ of $R^k$, Sharp and Zakeri constructed an $R-$module $U^{-k}M$ and they call it *the module of generalized fractions* of $M$ with respect to $U$. Especially, the set

$$U(M)_{d+1} = \{(y_1, \dots, y_d, 1) \in R^{d+1} : \exists j, 0 \le j \le d, \text{ such that } (y_1, \dots, y_j) \text{ form a subset of a system of parameters of } M \text{ and } y_{j+1} = \dots = y_d = 1\}$$

is a triangular subset of $R^{d+1}$. Let $\underline{x} = (x_1, \dots, x_d)$ be a system of parameters of $M$ and $\underline{n} = (n_1, \dots, n_d)$ a set of positive integers. Denote by $M(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))$ the submodule $\{m/(x_1^{n_1}, \dots, x_d^{n_d}, 1) : m \in M\}$ of $U(M)_{d+1}^{-d-1}M$. Then the length of the modules $M(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))$ is finite. Set

$$q_{\underline{x};M}(\underline{n}) = \ell(M(1/(x_1^{n_1}, \dots, x_d^{n_d}, 1))).$$

Following Sharp and Hamieh [SH], $q_{\underline{x};M}(\underline{n})$ is called *the length of the generalized fraction* $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$.

Set $J_{\underline{x};M}(\underline{n}) = n_1 \dots n_d e(\underline{x}; M) - q_{\underline{x};M}(\underline{n})$, where $e(\underline{x}; M)$ is the multiplicity of $M$ with respect to the ideal generated by $\underline{x}$. Consider $J_{\underline{x};M}(\underline{n})$ as a function in $n_1, \dots, n_d$. Sharp and Hamieh in [SH, Question 1.2] asked whether $q_{\underline{x};M}(\underline{n})$, or equivalently $J_{\underline{x};M}(\underline{n})$, is a polynomial for $\underline{n}$ large enough. Unfortunately, $J_{\underline{x};M}(\underline{n})$

is in general not a polynomial, but it is a non-negative function and bounded above by polynomials. Especially, the least degree of all polynomials in $\underline{n}$ bounding above $J_{\underline{x};M}(\underline{n})$ does not depend on the choice of $\underline{x}$ (see [CM, Theorem 3.2]). This least degree is denoted by $pf(M)$. When $pf(M) > 0$, it has shown in [CMN, Theorem 1.2] several special cases for which $q_{\underline{x};M}(\underline{n})$ is not a polynomial for $\underline{n}$ large enough, for some system of parameters $\underline{x}$ of $M$. By using strictly f-sequence, we can extend this result to general case.

**Theorem 1.2.** *If $pf(M) > 0$ then there always exists a strictly f-sequence $(x_1, \dots, x_d)$ of $M$ such that the length of generalized fraction $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$ is not a polynomial for $n_1, \dots, n_d$ large enough.*

We stipulate that the degree of the zero polynomial is $-\infty$. Following [CN], $M$ is called *pseudo Cohen-Macaulay* if $pf(M) = -\infty$ and $M$ is called *pseudo generalized Cohen-Macaulay* if $pf(M) \leq 0$. Note that if $d \leq 1$ or $M$ is Cohen-Macaulay then $M$ is pseudo Cohen-Macaulay. Moreover, if $d \leq 2$ or $M$ is generalized Cohen-Macaulay then $M$ is pseudo generalized Cohen-Macaulay. However, there are many pseudo Cohen-Macaulay modules $M$ such that $M$ is neither Cohen-Macaulay nor generalized Cohen-Macaulay. The class of pseudo Cohen-Macaulay and pseudo generalized Cohen-Macaulay modules have been studied in [CN], and it was shown that these modules have many good properties closely related to that of Cohen-Macaulay and generalized Cohen-Macaulay modules respectively. As a consequence of Theorem 1.2, we have the following characterizations for pseudo generalized Cohen-Macaulay modules.

**Corollary 1.3.** *The following statements are equivalents:*

*(i) $M$ is pseudo generalized Cohen-Macaulay.*

*(ii) For each system of parameters $(x_1, \dots, x_d)$ of $M$, the length of generalized fraction $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$ is a polynomial for $n_1, \dots, n_d$ large enough.*

*(iii) For each system of parameters $(x_1, \dots, x_d)$ of $M$ which is a strictly f-sequence, the length of generalized fraction $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$ is a polynomial for $n_1, \dots, n_d$ large enough.*

This paper is divided into 4 sections. In the next section we give some properties on the secondary representation of Artinian modules which are used in the sequel. Theorem 1.1 will be proved in Section 3. Theorem 1.2 and Corollary 1.3 will be proved in Section 4.

## 2. Attached prime ideals of Artinian modules

Following I. G. Macdonald [Mac], any Artinian $R-$module $A$ has a minimal secondary representation $A = A_1 + \ldots + A_n$, where $A_i$ is $\mathfrak{p}_i-$secondary. The set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is independent of the choice of minimal representation of $A$ and is denoted by $\operatorname{Att}_R A$. Note that if $0 < \ell(A) < \infty$ then $\operatorname{Att} A = \{\mathfrak{m}\}$ and if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of Artinian modules then

$$\operatorname{Att} A'' \subseteq \operatorname{Att} A \subseteq \operatorname{Att} A' \cup \operatorname{Att} A''.$$

**Lemma 2.1.** *Let $0 \longrightarrow P \longrightarrow A \xrightarrow{f} B \longrightarrow Q \longrightarrow 0$ be an exact sequence of Artinian modules such that $\ell(P) < \infty$ and $\ell(Q) < \infty$. Then we have*

$$\operatorname{Att} A \setminus \{\mathfrak{m}\} = \operatorname{Att} B \setminus \{\mathfrak{m}\}.$$

*Proof.* We get from the above exact sequence two short exact sequences

$$0 \longrightarrow A/P \longrightarrow B \longrightarrow Q \longrightarrow 0 \text{ and } 0 \longrightarrow P \longrightarrow A \longrightarrow \operatorname{Im} f \longrightarrow 0.$$

Therefore

$$\operatorname{Att} B \subseteq \operatorname{Att}(A/P) \cup \{\mathfrak{m}\} \subseteq \operatorname{Att} A \cup \{\mathfrak{m}\}$$

and

$$\operatorname{Att} A \subseteq \operatorname{Att}(\operatorname{Im} f) \cup \{\mathfrak{m}\}.$$

Let $B = B_1 + \ldots + B_t + C$ be a minimal secondary representation of $B$, where $C$ is zero or $\mathfrak{m}-$secondary. Since $\ell(Q) < \infty$, there exists a positive integer $n$ such that $\mathfrak{m}^n Q = 0$. Since $B/\operatorname{Im} f \cong Q$, we have

$$B_1 + \ldots + B_t \subseteq \mathfrak{m}^n B \subseteq \operatorname{Im} f.$$

Therefore

$$\ell(\operatorname{Im} f/(B_1 + \ldots + B_t)) \leq \ell(B/(B_1 + \ldots + B_t)) < \infty.$$

Hence

$$\operatorname{Att}(\operatorname{Im} f) \subseteq \operatorname{Att}(B_1 + \ldots + B_t) \cup \operatorname{Att}\ (\operatorname{Im} f/(B_1 + \ldots + B_t)) \subseteq \operatorname{Att} B \cup \{\mathfrak{m}\}.$$

It follows that $\operatorname{Att} A \subseteq \operatorname{Att} B \cup \{\mathfrak{m}\}$ and the lemma is proved. □

**Lemma 2.2.** *Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an exact sequence of Artinian $R-$modules with $\ell(A') < \infty$. Then for any elements $x_1, \ldots, x_k \in R$ we have*

$$\operatorname{Att}(0 :_A (x_1, \ldots, x_k)R) \setminus \{\mathfrak{m}\} = \operatorname{Att}(0 :_{A''} (x_1, \ldots, x_k)R) \setminus \{\mathfrak{m}\}.$$

*Proof.* We prove by induction on $k$. Let $k = 1$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle x_1} & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \end{array}$$

with the rows are exact. So, we have by [AM, 2.10] the exact sequence

$$(*) \qquad 0 \longrightarrow 0 :_{A'} x_1 \longrightarrow 0 :_A x_1 \xrightarrow{f} 0 :_{A''} x_1 \longrightarrow A'/x_1 A'.$$

Since $\ell(A') < \infty$, $\ell(0 :_{A'} x_1) < \infty$ and $\ell(A'/x_1 A') < \infty$. Therefore we have by Lemma 2.1 that $\operatorname{Att}(0 :_A x_1) \setminus \{\mathfrak{m}\} = \operatorname{Att}(0 :_{A''} x_1) \setminus \{\mathfrak{m}\}$. Thus, the lemma is proved for the case $k = 1$. Let $k > 1$. We get from (*) the two exact sequences

$$0 \longrightarrow 0 :_{A'} x_1 \longrightarrow 0 :_A x_1 \longrightarrow \operatorname{Im} f \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Im} f \longrightarrow 0 :_{A''} x_1 \longrightarrow A'/x_1A'.$$

Since $\ell(0 :_{A'} x_1) < \infty$, by using the induction hypothesis to the first exact sequence we have

$$\mathrm{Att}(0 :_A (x_1, \ldots, x_k)R) \setminus \{\mathfrak{m}\} = \mathrm{Att}(0 :_{\mathrm{Im} f} (x_2, \ldots, x_k)R) \setminus \{\mathfrak{m}\}.$$

Moreover, by applying the functor $\mathrm{Hom}(R/(x_2, \ldots, x_k)R; -)$ to the second exact sequence, with notice that $\ell(\mathrm{Hom}(R/(x_2, \ldots, x_k)R; A'/x_1A')) < \infty$, we have by Lemma 2.1 that

$$\mathrm{Att}(0 :_{\mathrm{Im} f} (x_2, \ldots, x_k)R) \setminus \{\mathfrak{m}\} = \mathrm{Att}(0 :_{A''} (x_1, \ldots, x_k)R) \setminus \{\mathfrak{m}\}$$

and the lemma follows. □

## 3. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need some auxiliary results as follows.

**Lemma 3.1.** *Let $k$ be a positive integer. Then there always exists a strictly f-sequence of $M$ of $k$ elements.*

*Proof.* We prove by induction on $k$. Let $k = 1$. Set $C_1 = \bigcup_{i=1}^d \mathrm{Att}(H^i_{\mathfrak{m}}(M))$. Let $x_1 \in \mathfrak{m}$ such that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_1 \setminus \{\mathfrak{m}\}$. Then $x_1$ is a strictly f-sequence of $M$. Let $k > 1$ and assume that $(x_1, \ldots, x_{k-1})$ is a strictly f-sequence of $M$. Let $x_k \in \mathfrak{m}$ such that $x_k \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_k \setminus \{\mathfrak{m}\}$, where we set $C_k = \bigcup_{i=1}^d \mathrm{Att}(H^i_{\mathfrak{m}}(M/(x_1, \ldots, x_{k-1})M))$. Then $(x_1, \ldots, x_k)$ is a strictly f-sequence of $M$. □

**Lemma 3.2.** *Let $(x_1, \ldots, x_k)$ be a strictly f-sequence of $M$. Then*

$$\mathrm{Att}(H^i_{\mathfrak{m}}(M/(x_1, \ldots, x_k)M)) \setminus \{\mathfrak{m}\} = \mathrm{Att}(0 :_{H^{i+k}_{\mathfrak{m}}(M)} (x_1, \ldots, x_k)R) \setminus \{\mathfrak{m}\}$$

*for all $i = 1, \ldots, d - k$.*

*Proof.* First it should be noted by the definition of strictly f-sequences that every strictly f-sequence of $M$ is a part of a system of parameters provided $k \leq d = \dim M$. Therefore we need only to prove the statement for the case $k < d$. We prove the lemma now by induction on $k$. Let $k = 1$. We have $\mathrm{Ass}\, M \subseteq \bigcup_{i=0}^d \mathrm{Att}(H^i_{\mathfrak{m}}(M))$ by [BS, 11.3.9]. Hence $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}\, M \setminus \{\mathfrak{m}\}$. So $\ell(0 :_M x_1) < \infty$. Therefore from the exact sequences

$$0 \longrightarrow 0 :_M x_1 \longrightarrow M \longrightarrow M/0 :_M x_1 \longrightarrow 0$$

$$0 \longrightarrow M/0 :_M x_1 \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

we get the exact sequences

$$0 \longrightarrow H^i_{\mathfrak{m}}(M)/x_1H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M/x_1M) \longrightarrow 0 :_{H^{i+1}_{\mathfrak{m}}(M)} x_1 \longrightarrow 0,$$

for all $i = 1, \ldots, d-1$. Note that for each $i \in \{1, \ldots, d\}$, $\ell(H^i_{\mathfrak{m}}(M)/x_1H^i_{\mathfrak{m}}(M)) < \infty$ since $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Att}\, H^i_{\mathfrak{m}}(M) \setminus \{\mathfrak{m}\}$. It follows by Lemma 2.2 that

$$\mathrm{Att}(H^i_{\mathfrak{m}}(M/x_1M)) \setminus \{\mathfrak{m}\} = \mathrm{Att}(0 :_{H^{i+1}_{\mathfrak{m}}(M)} x_1) \setminus \{\mathfrak{m}\}$$

for all $i = 1, \ldots, d-1$. Thus, the lemma is proved for $k = 1$. Let $k > 1$. Set $M_0 = M$ and $M_t = M/(x_1, \ldots, x_t)M$, $t = 1, \ldots, k$. It follows by the hypothesis and by [BS, 11.3.9] that $x_t \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M_{t-1}) \setminus \{\mathfrak{m}\}$ for all $t = 1, \ldots, k$. Hence $\ell(0 :_{M_{t-1}} x_t) < \infty$. Therefore from exact sequences

$$0 \longrightarrow 0 :_{M_{t-1}} x_t \longrightarrow M_{t-1} \longrightarrow M_{t-1}/0 :_{M_{t-1}} x_t \longrightarrow 0$$

$$0 \longrightarrow M_{t-1}/0 :_{M_{t-1}} x_t \xrightarrow{x_t} M_{t-1} \longrightarrow M_t \longrightarrow 0$$

we get the exact sequences

$$H^0_{\mathfrak{m}}(M_t) \longrightarrow 0 :_{H^1_{\mathfrak{m}}(M_{t-1})} x_t \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow H^i_{\mathfrak{m}}(M_{t-1})/x_t H^i_{\mathfrak{m}}(M_{t-1}) \longrightarrow H^i_{\mathfrak{m}}(M_t) \longrightarrow 0 :_{H^{i+1}_{\mathfrak{m}}(M_{t-1})} x_t \longrightarrow 0, \tag{2}$$

for $t = 1, \ldots, k$ and $i \geq 1$. Note that $\ell(H^i_{\mathfrak{m}}(M_{t-1})/x_t H^i_{\mathfrak{m}}(M_{t-1})) < \infty$ by the hypothesis on $x_t$. So by applying Lemma 2.2 to all the exact sequences in (2) for $t = k, \ldots, 1$ we have

$$\begin{aligned}
\operatorname{Att}(H^i_{\mathfrak{m}}(M/(x_1, \ldots, x_k)M)) \setminus \{\mathfrak{m}\} &= \operatorname{Att}(H^i_{\mathfrak{m}}(M_k)) \setminus \{\mathfrak{m}\} \\
&= \operatorname{Att}(0 :_{H^{i+1}_{\mathfrak{m}}(M_{k-1})} x_k) \setminus \{\mathfrak{m}\} \\
&= \operatorname{Att}(0 :_{H^{i+2}_{\mathfrak{m}}(M_{k-2})} (x_{k-1}, x_k)R) \setminus \{\mathfrak{m}\} \\
&= \ldots \\
&= \operatorname{Att}(0 :_{H^{i+k}_{\mathfrak{m}}(M)} (x_1, \ldots, x_k)R) \setminus \{\mathfrak{m}\}
\end{aligned}$$

for all $i = 1, \ldots, d-k$. Thus the lemma is proved. □

**Lemma 3.3.** *Let $(x_1, \ldots, x_k)$ be a strictly f-sequence of $M$. Then for all $j = 0, \ldots, k-1$, $x_{j+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \bigcup_{i=1}^d \operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1, \ldots, x_j)R) \setminus \{\mathfrak{m}\}$.*

*Proof.* Note that $(x_1, \ldots, x_j)$ is a strictly f-sequence for all $j = 1, \ldots, k$. Therefore by Lemma 3.2, the lemma is proved if we can show that

$$\operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1, \ldots, x_j)R) \subseteq \{\mathfrak{m}\},$$

for all $i = 1, \ldots, j$. Let $i = 1$. By replacing $t = 1$ in the exact sequence (1) of the proof of Lemma 3.2, we obtain $\ell(0 :_{H^1_{\mathfrak{m}}(M)} x_1) < \infty$. Hence $\ell(0 :_{H^1_{\mathfrak{m}}(M)} (x_1, \ldots, x_j)R) < \infty$ since $j \geq 1$, and thus $\operatorname{Att}(0 :_{H^1_{\mathfrak{m}}(M)} (x_1, \ldots, x_j)R) \subseteq \{\mathfrak{m}\}$. Let $1 < i \leq j$. By applying Lemma 2.2 to all the exact sequences (2) for $t = 1, \ldots, i-1$ we get

$$\begin{aligned}
\operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1, \ldots, x_i)R) \setminus \{\mathfrak{m}\} &= \operatorname{Att}(0 :_{H^{i-1}_{\mathfrak{m}}(M_1)} (x_2, \ldots, x_i)R) \setminus \{\mathfrak{m}\} \\
&= \operatorname{Att}(0 :_{H^{i-2}_{\mathfrak{m}}(M_2)} (x_3, \ldots, x_i)R) \setminus \{\mathfrak{m}\} \\
&= \ldots \\
&= \operatorname{Att}(0 :_{H^1_{\mathfrak{m}}(M_{i-1})} x_i) \setminus \{\mathfrak{m}\}.
\end{aligned}$$

By replacing $t = i$ in the exact sequence (1), we have $\operatorname{Att}(0 :_{H^1_{\mathfrak{m}}(M_{i-1})} x_i) \subseteq \{\mathfrak{m}\}$. Hence $\operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1, \dots, x_i)R) \subseteq \{\mathfrak{m}\}$. Since $i \leq j$, $\operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1, \dots, x_j)R) \subseteq \{\mathfrak{m}\}$. □

The notion of a filter regular sequence was introduced in [CST]. Recall that a sequence $(x_1, \dots, x_k)$ of elements of $\mathfrak{m}$ is called a *filter regular sequence* (f-sequence for short) of $M$ if for all $j = 0, \dots, k-1$, $x_{j+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \dots, x_j)M) \setminus \{\mathfrak{m}\}$. It follows by [BS, 11.3.9] that $\operatorname{Ass} N \subseteq \bigcup_{i=0}^{\dim N} \operatorname{Att}(H^i_{\mathfrak{m}}(N))$ for a finitely generated $R-$module $N$. Therefore each strictly f-sequence of $M$ is an f-sequence of $M$. Note that $(x_1, \dots, x_k)$ is an f-sequence of $M$ if and only if $(\overline{x}_1, \dots, \overline{x}_k)$ is a regular sequence of $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp} M$ containing $x_1, \dots, x_k$, where we denote by $\overline{x}_i$, $i = 1, \dots, k$, the image of $x_i$ in $R_{\mathfrak{p}}$. Therefore $(x_1, \dots, x_k)$ is an f-sequence of $M$ if and only if so is $(x_1^{n_1}, \dots, x_k^{n_k})$ for all positive integers $n_1, \dots, n_k$.

**Lemma 3.4.** *Let $n_1, \dots, n_k$ be positive integers and $(x_1, \dots, x_k)$ a strictly f-sequence of $M$. Then $(x_1^{n_1}, \dots, x_k^{n_k})$ is again a strictly f-sequence of $M$.*

*Proof.* Let $n_1, \dots, n_k$ be positive integers and $j \in \{1, \dots, k\}$. For each $i \in \{1, \dots, d\}$, we denote by $D^i(M) = \operatorname{Hom}_R(H^i_{\mathfrak{m}}(M), E)$, where E is the injective envelop of the residue field $R/\mathfrak{m}$, the Matlis dual of $H^i_{\mathfrak{m}}(M)$. Note that $D^i(M)$ is a finitely generated $\widehat{R}-$module.

Firstly, it is implied by [Sh2] that

$$\begin{aligned} \operatorname{Att}_R(0 :_{H^i_{\mathfrak{m}}(M)} (x_1, \dots, x_j)R) &= \{\widehat{\mathfrak{p}} \cap R : \ \widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1, \dots, x_j)\widehat{R})\} \\ (3) \qquad\qquad &= \{\widehat{\mathfrak{p}} \cap R : \ \widehat{\mathfrak{p}} \in \operatorname{Ass}_{\widehat{R}}(D^i(M)/(x_1, \dots, x_j)D^i(M)\}. \end{aligned}$$

Therefore we have by Lemma 3.3 and (3) that $(x_1, \dots, x_k)$ is an f-sequence of $D^i(M)$. Hence $(x_1^{n_1}, \dots, x_k^{n_k})$ is also an f-sequence of $D^i(M)$. Therefore we get by (3) that $x_j \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_{j-1}^{n_{j-1}})R \setminus \{\mathfrak{m}\}$.

Next, we claim by induction on $j$, $(j = 1, \dots, k)$ that

$$(4) \quad \operatorname{Att}(H^i_{\mathfrak{m}}(M/(x_1^{n_1}, \dots, x_j^{n_j})M)) \setminus \{\mathfrak{m}\} = \operatorname{Att}(0 :_{H^{i+j}_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_j^{n_j})R) \setminus \{\mathfrak{m}\},$$

for all $i$. In fact, let $j = 1$. Since $x_1^{n_1}$ is also a strictly f-sequence of $M$, the equation (4) follows by Lemma 3.2. Let $j > 1$. Set $M_0 = M$ and $M_{\underline{n},t} = M/(x_1^{n_1}, \dots, x_t^{n_t})M$, $t = 1, \dots, j$. Then we have by induction hypothesis that

$$\operatorname{Att}(H^i_{\mathfrak{m}}(M_{\underline{n},t-1})) \setminus \{\mathfrak{m}\} = \operatorname{Att}(0 :_{H^{i+t-1}_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_{t-1}^{n_{t-1}})R) \setminus \{\mathfrak{m}\},$$

for all $t = 1, \dots, j$ and all $i$. Therefore $x_t \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}(H^i_{\mathfrak{m}}(M_{\underline{n},t-1})) \setminus \{\mathfrak{m}\}$, for all $t = 1, \dots, j$ and all $i$. Then as in the proof of Lemma 3.2, we have the exact sequences

$$0 \longrightarrow H^i_{\mathfrak{m}}(M_{\underline{n},t-1})/x_t H^i_{\mathfrak{m}}(M_{\underline{n},t-1}) \longrightarrow H^i_{\mathfrak{m}}(M_{\underline{n},t}) \longrightarrow 0 :_{H^{i+1}_{\mathfrak{m}}(M_{\underline{n},t-1})} x_t \longrightarrow 0,$$

for $t = 1, \dots, j$ and $i = 1, \dots, d$. By applying Lemma 2.2 to all these exact sequences, the equality (4) follows with the same method that used in the proof of Lemma 3.2. The claim is proved.

Now, since $x_j \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Att}(0 :_{H^i_{\mathfrak{m}}(M)} (x_1^{n_1}, \dots, x_{j-1}^{n_{j-1}})R \setminus \{\mathfrak{m}\}$ for all $j = 1, \dots, k$, it follows by (4) that $(x_1^{n_1}, \dots, x_k^{n_k})$ is a strictly f-sequence of $M$. □

**Proof of Theorem 1.1.** Let $n_1,\dots,n_k$ be positive integers and $i\in\{1,\dots,d\}$. Denote by $D^i(M)$ the Matlis dual of $H^i_{\mathfrak m}(M)$. As in the proof of Lemma 3.4, $(x_1,\dots,x_k)$ is an f-sequence of the $\widehat{R}-$module $D^i(M)$. Now let

$$\widehat{\mathfrak p}\in \operatorname{Ass}_{\widehat{R}}(D^i(M)/(x_1,\dots,x_k)D^i(M))\setminus\{\widehat{\mathfrak m}\}.$$

Then $\widehat{\mathfrak p}\widehat{R}_{\widehat{\mathfrak p}}\in \operatorname{Ass}(D^i(M)/(x_1,\dots,x_k)D^i(M))_{\widehat{\mathfrak p}}$. Therefore $\overline{x}_1,\dots,\overline{x}_k$ is a maximal regular sequence of $D^i(M)_{\widehat{\mathfrak p}}$ and hence so is $\overline{x}_1^{n_1},\dots,\overline{x}_k^{n_k}$. This implies that

$$\widehat{\mathfrak p}\widehat{R}_{\widehat{\mathfrak p}}\in \operatorname{Ass}(D^i(M)/(x_1^{n_1},\dots,x_k^{n_k})D^i(M))_{\widehat{\mathfrak p}}.$$

Hence

$$\widehat{\mathfrak p}\in \operatorname{Ass}_{\widehat{R}}(D^i(M)/(x_1^{n_1},\dots,x_k^{n_k})D^i(M))\setminus\{\widehat{\mathfrak m}\}.$$

Similarly, we have

$$\operatorname{Ass}_{\widehat{R}}(D^i(M)/(x_1^{n_1},\dots,x_k^{n_k})D^i(M))\setminus\{\widehat{\mathfrak m}\}\subseteq \operatorname{Ass}_{\widehat{R}}(D^i(M)/(x_1,\dots,x_k)D^i(M))\setminus\{\widehat{\mathfrak m}\}.$$

Therefore

$$\operatorname{Ass}_{\widehat{R}}(D^i(M)/(x_1^{n_1},\dots,x_k^{n_k})D^i(M))\setminus\{\widehat{\mathfrak m}\}= \operatorname{Ass}_{\widehat{R}}(D^i(M)/(x_1,\dots,x_k)D^i(M))\setminus\{\widehat{\mathfrak m}\}.$$

It follows by the relation (3) in the proof of Lemma 3.4 that

$$\operatorname{Att}_R(0:_{H^i_{\mathfrak m}(M)}(x_1^{n_1},\dots,x_k^{n_k})R)\setminus\{\mathfrak m\}=\operatorname{Att}_R(0:_{H^i_{\mathfrak m}(M)}(x_1,\dots,x_k)R)\setminus\{\mathfrak m\}.$$

Note that $(x_1^{n_1},\dots,x_k^{n_k})$ is a strictly f-sequence of $M$ by Lemma 3.4. Therefore we have by Lemma 3.2 and the above fact that

$$\begin{aligned}\operatorname{Att}(H^i_{\mathfrak m}(M/(x_1^{n_1},\dots,x_k^{n_k})M)\setminus\{\mathfrak m\}&=\operatorname{Att}_R(0:_{H^{i+k}_{\mathfrak m}(M)}(x_1^{n_1},\dots,x_k^{n_k})R)\setminus\{\mathfrak m\}\\&=\operatorname{Att}_R(0:_{H^{i+k}_{\mathfrak m}(M)}(x_1,\dots,x_k)R)\setminus\{\mathfrak m\}\\&=\operatorname{Att}(H^i_{\mathfrak m}(M/(x_1,\dots,x_k)M)\setminus\{\mathfrak m\}.\end{aligned}$$

The theorem is proved. □

The following corollary gives some characterizations of strictly f-sequences.

**Corollary 3.5.** *Let $(x_1,\dots,x_k)$ be a sequence of elements in $\mathfrak m$. Then the following statements are equivalent:*

*(i)* $(x_1,\dots,x_k)$ *is a strictly f-sequence of* $M$.

*(ii)* $(x_1^{n_1},\dots,x_k^{n_k})$ *is a strictly f-sequence of* $M$ *for all positive integers* $n_1,\dots,n_k$.

*(iii)* $x_{j+1}\notin\mathfrak p$ *for all* $\mathfrak p\in\bigcup_{i=1}^d \operatorname{Att}(0:_{H^i_{\mathfrak m}(M)}(x_1,\dots,x_j)R))\setminus\{\mathfrak m\}$ *and all* $j=0,\dots,k-1$.

*(iv)* $x_{j+1}\notin\mathfrak p$ *for all* $\mathfrak p\in\bigcup_{i=1}^d \operatorname{Att}(0:_{H^i_{\mathfrak m}(M)}(x_1^{n_1},\dots,x_j^{n_j})R)\setminus\{\mathfrak m\}$, *all positive integers* $n_1,\dots,n_k$ *and all* $j=0,\dots,k-1$.

*(v)* $(x_1,\dots,x_k)$ *is an f-sequence of* $\widehat{R}-$*module* $D^i(M)$ *for all* $i=1,\dots,d$, *where* $D^i(M)$ *is the Matlis dual of* $H^i_{\mathfrak m}(M)$ *for all* $i=1,\dots,d$.

*Proof.* (i)⇒(ii) follows by Lemma 3.4. (ii)⇒ (iv) follows by Lemma 3.3. (iv)⇒(iii) is trivial. (iii)⇔(v) follows by the formula (3) in the proof of Lemma 3.4. So we need only to prove (iii)$\Rightarrow$ (i). In fact, with the same method used in the proof of Lemma 3.4, for all $j = 1, \dots, k$, we can prove the equality (4) for the case $n_1 = \dots = n_j = 1$. Therefore $(x_1, \dots, x_k)$ is a strictly f-sequence of $M$. $\square$

## 4. Proof of Theorem 1.2

To prove Theorem 1.2, we introduce here the notion of permutable strictly f-sequence.

**Definition 4.1.** A sequence $\underline{x} = (x_1, \dots, x_k)$ of elements in $\mathfrak{m}$ is called *permutable strictly f-sequence* if it is strictly f-sequence of $M$ in any order.

**Lemma 4.2.** *Let $k$ be an positive integer. Then there always exists a permutable strictly f-sequence $(x_1, \dots, x_k)$ of $M$.*

*Proof.* We prove by induction on $k$. Let $k = 1$. Set $C_1 = \bigcup_{i=1}^d \mathrm{Att}(H^i_{\mathfrak{m}}(M)) \setminus \{\mathfrak{m}\}$. Let $x_1 \in \mathfrak{m}$ such that $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_1$. It is clear that $x_1$ is a permutable strictly f-sequence of $M$. Let $k > 1$ and assume that there exists a permutable strictly f-sequence $(x_1, \dots, x_{k-1})$ of $M$. Set

$$C_k = \bigcup_{I \subseteq \{1,\dots,k-1\}} \bigcup_{i=1}^{d} \mathrm{Att}(0 :_{H^i_{\mathfrak{m}}(M)} \sum_{j \in I} x_j R) \ \setminus \{\mathfrak{m}\}.$$

Let $x_k \in \mathfrak{m}$ such that $x_k \notin \mathfrak{p}$ for all $\mathfrak{p} \in C_k$. We will prove that $(x_1, \dots, x_k)$ is a permutable strictly f-sequence of $M$. Let $(y_1, \dots, y_k)$ be a permutation of $(x_1, \dots, x_k)$. Denote by $D^i(M)$, $i = 1, \dots, d$, the Matlis dual of $H^i_{\mathfrak{m}}(M)$. Then by Corollary 3.5, the lemma is proved if we can show that $(y_1, \dots, y_k)$ is an f-sequence of $\widehat{R}-$module $D^i(M)$ for all $i = 1, \dots, d$. Assume that $(y_1, \dots, y_k)$ is not an f-sequence of $D^i(M)$ for some $i$. Then there exists an integer $n, 1 \leq n \leq k)$ such that $y_n \in \widehat{\mathfrak{p}}$ for some prime ideal $\widehat{\mathfrak{p}} \in \mathrm{Ass}_{\widehat{R}}(D^i(M)/(y_1, \dots, y_{n-1})D^i(M)) \setminus \{\widehat{\mathfrak{m}}\}$. Then

$$\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}} \in \mathrm{Ass}(D^i(M)/(y_1, \dots, y_{n-1})D^i(M))_{\widehat{\mathfrak{p}}}.$$

On the other hand, we get by induction hypothesis and by the choice of $x_k$ that $x_k = y_j$ for some $j < n$. Therefore $(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n, x_k)$ is a strictly f-sequence of $M$ contained in $\widehat{\mathfrak{p}}$ and hence $(\overline{y}_1, \dots, \overline{y}_{j-1}, \overline{y}_{j+1}, \dots, \overline{y}_n, \overline{x}_k)$ is a regular sequence of $D^i(M))_{\widehat{\mathfrak{p}}}$. Since any permutaion of a regular sequence is again a regular sequence, $\overline{y}_n$ is a regular element of $D^i(M)/(y_1, \dots, y_{n-1})D^i(M)_{\widehat{\mathfrak{p}}}$. It follows that

$$\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}} \notin \mathrm{Ass}(D^i(M)/(y_1, \dots, y_{n-1})D^i(M))_{\widehat{\mathfrak{p}}}.$$

This gives a contradiction and the lemma is proved. $\square$

For each artinian $R-$module $A$ we recall now some notions from [SH]: The *stability index* of $A$, denoted by $s(A)$, is the least positive integer $s$ such that $\mathfrak{m}^s A = \mathfrak{m}^n A$ for all $n \geq s$; we denote by $Rl(A)$ the length of $A/\mathfrak{m}^{s(A)}A$ and it is called *residual length* of $A$. Note that if $x \in \mathfrak{m}$ is an element such that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Att}\, A \setminus \{\mathfrak{m}\}$ then $\ell(A/x^n A) = Rl(A)$ for all $n \geq s(A)$.

**Proof of Theorem 1.3.** Let $(x_1, \dots, x_d)$ be a permutable strictly f-sequence of $M$. We will prove that the length of the generalized fraction $1/(x_1^{n_1}, \dots, x_d^{n_d}, 1)$ is not a polynomial for $\underline{n}$ large enough. Set $M_{\underline{n},0} = M$; $M_{\underline{n},k} = M/(x_1^{n_1}, \dots, x_k^{n_k})M$, $k =$

$1,\dots,d$. Since $(x_1^{n_1},\dots,x_d^{n_d})$ is also an f-sequence of $M$, it is derived from [CM, Lemma 2.4] and [SH, Proposition 2.2] the following exact sequences

$$\begin{aligned}
&0 \longrightarrow H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1})/x_k^{n_k}H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1}) \\
(5)\qquad &\longrightarrow U(M_{\underline{n},k})_{d-k+1}^{-d+k-1}M_{\underline{n},k} \overset{\Psi_{d-k+2}}{\longrightarrow} U(M_{\underline{n},k-1})_{d-k+2}^{-d+k-2}M_{\underline{n},k-1}
\end{aligned}$$

for all $k=1,\dots,d-1$, where $\Psi_{d-k+2}$ is defined by

$$\Psi_{d-k+2}(\overline{m}/(u_{k+1},\dots,u_d,1)) = m/(x_k^{n_k},u_{k+1},\dots,u_d,1),$$

for all $m\in M_{\underline{n},k-1}$ and for all $(u_{k+1},\dots,u_d,1)\in U(M_{\underline{n},k})_{d-k+1}$ (here we denote by $\overline{m}$ the image of $m$ in $M_{\underline{n},k}$). Set

$$s_{\underline{n},k-1} = s(H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1})),\quad k=1,\dots,d,$$

the stability index of $H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1})$. Since $(x_1^{n_1},\dots,x_d^{n_d})$ is again a strictly f-sequence of $M$ by Corollary 3.5, $\ell(H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1})/x_k^{n_k}H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1}))<\infty$ for all $k=1,\dots,d-1$. It follows that

$$\operatorname{Ker}(\Psi_{d-k+2}) = H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1})/\mathfrak{m}^{s_{\underline{n},k-1}}H_{\mathfrak{m}}^{d-k}(M_{\underline{n},k-1}),$$

for $n_k\ge s_{\underline{n},k-1}$. Hence $\operatorname{Ker}(\Psi_{d-k+2})$ is independent of $n_k$ for all $n_k\ge s_{\underline{n},k-1}$. Note that $\operatorname{Ker}(\Psi_{d-k+2})$ is of finite length, therefore there exist finitely many elements $f_1,\dots,f_v$ which generate $\operatorname{Ker}(\Psi_{d-k+2})$ for all $n_k\ge s_{\underline{n},k-1}$. On the other hand, it follows from [SZ2] that

$$U(M_{\underline{n},k})_{d-k+1}^{-d+k-1}M_{\underline{n},k} = \bigcup_{n_{k+1},\dots,n_d\ge 0} M_{\underline{n},k}(1/(x_{k+1}^{n_{k+1}},\dots,x_d^{n_d},1)).$$

Moreover, it is clear that if $n_i\ge m_i,\ i=k+1,\dots,d$, then

$$M_{\underline{n},k}(1/(x_{k+1}^{m_{k+1}},\dots,x_d^{m_d},1)) \subseteq M_{\underline{n},k}(1/(x_{k+1}^{n_{k+1}},\dots,x_d^{n_d},1)).$$

Therefore, given $n_k\ge s_{\underline{n},k-1}$, there exists some integer $r(M_{\underline{n},k})$ (depends on $M_{\underline{n},k}$) such that

$$\operatorname{Ker}(\Psi_{d-k+2}) = (f_1,\dots,f_v)R \subseteq M_{\underline{n},k}(1/(x_{k+1}^{n_{k+1}},\dots,x_d^{n_d},1))$$

for all $n_{k+1},\dots,n_d\ge r(M_{\underline{n},k})$. So, the exact sequences (5) imply the exact sequences

$$0\to \operatorname{Ker}(\Psi_{d-k+2}) \to M_{\underline{n},k}(1/(x_{k+1}^{n_{k+1}},\dots,x_d^{n_d},1)) \overset{\Psi_{d-k+2}}{\longrightarrow} M_{\underline{n},k-1}(1/(x_k^{n_k},\dots,x_d^{n_d},1))\to 0,$$

for all $k = 1, \dots, d-1$, for all $n_k \geq s_{\underline{n},k-1}$, and all $n_{k+1}, \dots, n_d \geq r(M_{\underline{n},k})$. Therefore we have

$$\begin{aligned} q_{\underline{x};M}(\underline{n}) &= q_{(x_2,\dots,x_d);M_{\underline{n},1}}(n_2,\dots,n_d) - Rl(H^{d-1}_{\mathfrak{m}}(M)) \\ &= q_{(x_3,\dots,x_d);M_{\underline{n},2}}(n_3,\dots,n_d) - \sum_{i=1}^{2} Rl(H^{d-i}_{\mathfrak{m}}(M_{\underline{n},i-1})) \\ &= \dots \\ &= q_{x_d;M_{\underline{n},d-1}}(n_d) - \sum_{i=1}^{d-1} Rl(H^{d-i}_{\mathfrak{m}}(M_{\underline{n},i-1})) \\ &= n_d e(x_d; M_{\underline{n},d-1}) - \sum_{i=1}^{d-1} Rl(H^{d-i}_{\mathfrak{m}}(M_{\underline{n},i-1})) \end{aligned}$$

for all $n_1 \geq s_{\underline{n},0}$, $n_j \geq \max\{s_{\underline{n},j-1}, r(M_{\underline{n},1}), \dots, r(M_{\underline{n},j-1})\}, j = 2, \dots, d-1$, and $n_d \geq r(M_{\underline{n},d-1})$. Since $(x_1^{n_1}, \dots, x_d^{n_d})$ is an f-sequence of $M$, it follows that

$$n_d \; e(x_d, M_{\underline{n},d-1}) = n_1 n_2 \dots n_d \; e(\underline{x}; M).$$

Therefore

$$J_{\underline{x};M}(\underline{n}) = \sum_{i=1}^{d-1} Rl(H^{d-i}_{\mathfrak{m}}(M_{\underline{n},i-1})) \tag{6}$$

for all $n_1 \geq s_{\underline{n},0}$, $n_j \geq \max\{s_{\underline{n},j-1}, r(M_{\underline{n},1}), \dots, r(M_{\underline{n},j-1})\}, j = 2, \dots, d-1$, and $n_d \geq r(M_{\underline{n},d-1})$. Now, assume that $q_{\underline{x};M}(\underline{n})$ is a polynomial for $n_1, \dots, n_d$ large enough. Then there exists a polynomial $f(X_1, \dots, X_d)$ of degree $pf(M)$ such that

$$J_{\underline{x};M}(n_1, \dots, n_d) = f(n_1, \dots, n_d) \tag{7}$$

for $n_1, \dots, n_d$ large enough. It follows by (6) and (7) that for all $n_1 \geq s_{\underline{n},0}$, $n_j \geq \max\{s_{\underline{n},j-1}, r(M_{\underline{n},1}), \dots, r(M_{\underline{n},j-1})\}, j = 2, \dots, d-1$, and $n_d \geq r(M_{\underline{n},d-1})$, the polynomial $f(n_1, \dots, n_d)$ does not depend on $n_d$. Therefore the variable $X_d$ can not appear in any term of $f(X_1, \dots, X_d)$. For any $m \in M$ and any permutation $\pi$ of $\{1, \dots, d\}$, we have by [SH, Corollary 2.5] that

$$m/(x_1^{n_1}, \dots, x_d^{n_d}, 1) = (\mathrm{sign}(\pi)) m/(x_{\pi(1)}^{n_{\pi(1)}}, \dots, x_{\pi(d)}^{n_{\pi(d)}}, 1).$$

Therefore we can repeat the above process for the strictly f-sequence $(x_2, x_3, \dots, x_d, x_1)$ and we get that the variable $X_1$ can not appear in any term of $f(X_1, \dots, X_d)$. Continue the above process for the strictly f-sequences $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d, x_j)$ for $j = 2, \dots, d-1$, none of the variables $X_2, \dots, X_{d-1}$ can appear in any term of $f(X_1, \dots, X_d)$. Therefore $f(X_1, \dots, X_d)$ must be a constant, a contradiction because the degree of $f(X_1, \dots, X_d)$ is $pf(M) > 0$. □

**Proof of Corollary 1.3.** (i)⇒(ii). Suppose that $M$ is pseudo generalized Cohen-Macaulay, i.e $pf(M) \leq 0$. Let $\underline{x}$ be a system of parameters of $M$. Since $J_{\underline{x};M}(\underline{n})$ is a increasing function and bounded above by a constant, $J_{\underline{x};M}(\underline{n})$ must be a constant for $n_1, \dots, n_d$ large enough. Hence $q_{\underline{x};M}(\underline{n})$ is a polynomial for $\underline{n}$ large enough.

(ii)⇒(iii). Trivial.

(iii)⇒ (i). Suppose that $M$ is not pseudo generalized Cohen-Macaulay. Then $pf(M) > 0$. It follows by Theorem 1.2 that there exists a strictly f-sequence $\underline{x}$ of $M$ such that $q_{\underline{x};M}(\underline{n})$ is not polynomial. This gives a contradiction. □

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