On spectral decomposition of regular operator algebras

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Résumé. Soit $\mathcal{A}$ une sous algèbre de Banach commutative, régulière (pas forcément semi-simple) de l'algèbre des opérateurs d'un espace de Banach X. Pour tout $x \in X$ et $a \in \mathcal{A}$, on introduit leur spectre local $Sp(x)$ et $Sp(a)$ dans l'espace M des idéaux maximaux de $\mathcal{A}$ et pour tout $\Lambda \subset M$ le sous-espace spectral $X(\Lambda)$ des x ayant un spectre dans Λ. Le résultat principal est de montrer que dans ces conditions l'algèbre $\mathcal{A}$ est décomposable i.e. pour tout recouvrement d'ouverts $U_1, ..., U_n$ de M, l'espace X est décomposable en une somme de sous-espaces spectraux $X(\overline{U_k}), 1 \leq k \leq n$, i.e. $X = X(\overline{U_1}) + ... + X(\overline{U_n})$.

Abstract. Let $\mathcal{A}$ be a commutative algebra of operators on a Banach space. For each $x \in X$ and $a \in \mathcal{A}$, we introduce their local spectra $Sp(x)$ and $Sp(a)$ as subsets of the maximal ideal space M of $\mathcal{A}$, and for each subset $\Lambda \subset M$ we define the spectral subspace $X(\Lambda)$ consisting of x with $Sp(x) \subset \Lambda$. The main result of the paper states that if $\mathcal{A}$ is a regular, not necessarily semisimple commutative Banach algebra of operators on a Banach space X, then $\mathcal{A}$ is decomposable in the sense that for every open cover $U_1, ..., U_n$ of M, the space X is decomposed into a sum of closed spectral subspaces $X(\overline{U_k}), 1 \leq k \leq n$, i.e. $X = X(\overline{U_1}) + ... + X(\overline{U_n})$.

1. Introduction. The theory of non-quasianalytic representations of locally compact abelian groups was constructed in [LMF]. In this theory, Banach algebra methods (developed, in particular, by Domar [D])) have played an important role. It is related to the general theory of decomposable operators [CF] (see also [DS], [EL], [V], [L]) , which has received an extensive development in the last thirty years and is still an

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active area of research. The literature on this subject is too vast, so we refer the reader to the cited monographs for details.

In recent years there has been a considerable interest in non-quasianalytic groups and semigroups of operators on Banach spaces (see e.g. [BY], [Vu], [H], [HNR],[HN], [DV].). This has inspired us to reexamine the original papers of Lyubich and Matsaev [LM] and Lyubich, Matsaev and Feldman [LMF]. In particular, we have paid attention to the role that the semisimplicity condition of the underlying operator algebras plays in guaranteeing the corresponding decomposability properties. It turns out that the semisimplicity property does not hold, in general, for operator algebras generated by non-quasianalytic (or even by isometric) representations of locally compact Abelian groups (see [MV], [FV]). On the other hand, results of Colojoara-Foias ([CF, Chapter 6, Section 2]) and of many other authors on decomposability of multiplication operators or multipliers on Banach algebras (cf. [Fl], [Fr], [Ne], [ELN]) all depend heavily on the semisimplicity condition (aside from the regularity, which is the natural condition for decomposability).

During our work, we have found a general algebraic framework for constructing spectral subspaces for regular, not necessarily semisimple, algebras of operators on a Banach space X. Thus, our approach gives a unified treatement of the cases considered in the above mentioned papers [LMF], [CF] and, moreover, describes some new situations when the semisimplicity condition does hot hold. More precisely, given a commutative regular subalgebra $\mathcal{A}$ (with unit) of the algebra $L(X)$ of all bounded linear operators on a Banach space X, we introduce, for each element $x \in X$ and $a \in \mathcal{A}$, their local spectra $Sp(x), Sp(a)$ as *generalized Beurling spectra*, i.e. as hulls of closed ideals generated by x (respectively, by a). The closure M_0 of the union of $Sp(x)$, for all x, is called the *support* of the algebra of operators $\mathcal{A}$. For each closed subset F of the support M_0 (or the Gelfand space M) of $\mathcal{A}$, we introduce the spectral subspace $X(F)$ which consists of all elements x such that $Sp(x) \subset F$ and prove that $X(F)$ is a closed hyperinvariant subspace. Using the regularity of the algebra $\mathcal{A}$, we show that for every open cover $U_i, i = 1, 2, ..., n$ of the support M_0 (or of M), the space X is decomposed into a sum of the corresponding spectral subspaces $X(\overline{U_i})$. Although our approach provides a (joint) spectral decomposition property for families of operators in the algebra $\mathcal{A}$, as well as for individual operators $T \in \mathcal{A}$, which is to some extent analogous to the well known spectral decomposition properties for operators and commuting families of operators that the reader can find in the numerous articles cited above, it should be emphasized that the decomposition property introduced in this paper is, in general, different from the well known decomposition properties considered in the traditional theory of decomposable operators. For instance, if T is a bounded linear operator on X and $x \in X, x \neq 0$, then the local spectrum $\sigma_T(x)$ is a non-empty subset of $\mathbb{C}$, while $Sp(x)$ is a subset of the Gelfand space M and in general it can happen that $Sp(x) = \emptyset$ for $x \neq 0$. As a consequence, $X(\emptyset)$ can be a non-zero subspace, in our definition, but in the context of decomposable operators we must have $X(\emptyset) = \{0\}$. Despite these differences, it is hoped that our results shed some new light to the decomposability theory for regular, non-semisimple commutative Banach algebra of operators.

We will use standard notions and basic results from the theory of commutative Banach algebras as presented, for instance, in [G], [N]. Given a commutative Banach algebra $\mathcal{A}$, we denote the spectrum or the Gelfand space of A by M and we denote its radical (i.e. the set of topological niltopent elements) by $\mathcal{R}$. For each $a \in \mathcal{A}$, we denote by $\hat{a} : M \to \mathbb{C}$ its Genfand transform which is a continuous function on M. The kernel of the homomorphism $a \mapsto \hat{a}$ is in fact the radical $\mathcal{R}$. Recall that the algebra $\mathcal{A}$ is called regular if for every closed subset $F \subset M$ and every $\gamma_0 \in M \setminus F$ there exists an element $a \in \mathcal{A}$ such that $\hat{a}|F = 0, \hat{a}(\gamma_0) = 1$. If $\mathcal{A}$ is regular, $F \subset M$ is closed and $\gamma_0 \notin F$, then there exist an open set $V \supset F$ and $a \in \mathcal{A}$ such that $\hat{a}|V = 0, \hat{a}(\gamma_0) = 1$.

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2. A preliminary result. Let $\mathcal{A}$ be a regular commutative Banach algebra, $\mathcal{R} \subset \mathcal{A}$ its radical. Let M be the Gelfand space of $\mathcal{A}$, and $F \subset M$ a closed subset. Define

$$I_0(F) := \{a \in \mathcal{A} : \hat{a}|U = 0 \text{ for some neighborhood } U \supset F\}.$$

It is clear that $I_0(F)$ is an ideal in $\mathcal{A}$. For an ideal I, let $h(I)$ denotes its hull, i.e. the set of all maximal ideals containing I. In other words,

$$h(I) := \{\gamma \in M : \hat{a}(\gamma) = 0 \text{ for all } a \in I\}.$$

If F is a hull, that is a closed subset in the Gelfand space M, then its kernel $k(F)$ is by definition the intersection of all maximal ideals in F, i.e.

$$k(F) := \{a \in \mathcal{A} : \hat{a}(\gamma) = 0 \text{ for all } \ \gamma \in F\}.$$

Clearly, $k(F)$ is a closed ideal, $k(h(I)) \supset I$ for every ideal I, and $k(F) = \cap_{\gamma \in F} I(\{\gamma\})$.

The following is a modified version of a well-known result for commutative Banach algebras (see [N, p.225, Theorem 4]).

Theorem 1. For any closed subset $F \subset M$, we have $h(I_0(F)) = F$ and $I_0(F)$ is the smallest ideal among those ideals $I \subset \mathcal{A}$ such that $I \supset \mathcal{R}, \ h(I) = F$.

Proof. It is clear that $I_0(F) \supset \mathcal{R}$ and $h(I_0(F) \supset F$. On the other hand, if $\gamma \notin F$ then there exist an open set $U \supset F$ and $a \in A$ such that $\hat{a}|U = 0$ and $\hat{a}(\gamma) = 1$. Thus,

$a \in I_0(F)$ and $\hat{a}(\gamma) = 1$, so that $\gamma \notin h(I_0(F))$. Therefore, $h(I_0(F)) = F$. Now let I be an ideal such that $h(I) = F$ and $I \supset \mathcal{R}$. We show that $I \supset I_0(F)$. Let $y \in I_0(F)$. Then there exists a neighborhood U of F such that $\hat{y}|U = 0$. Let $F_1 = M \setminus U$. Then $F \cap F_1 = \emptyset$, and there exists $z \in A$ such that $z \in I$ and $\hat{z}|F_1 = 1$ (see e.g. [N, p.223]). Therefore, $\hat{y}(\gamma) - \hat{y}(\gamma)\hat{z}(\gamma)$ for all $\gamma \in M$, which implies that $y = yz \in \mathcal{R}$. Hence, $y \in (yz + \mathcal{R}) \subset I + \mathcal{R} \subset I$, i.e. $I_0(F) \subset I$.

Remark that if $F = \emptyset$, then $I_0(F) = I_0(\emptyset)$ consists of elements whose Gelfand transforms have compact support. If $F = M$, then $I_0(F) = I_0(M) = \mathcal{R}$, the radical. Note also that a set F is called *a set of spectral synthesis* or *a S-set* if $\overline{I_0(F)} = I$ for every closed ideal I such that $h(I) = F$.

3. The local spectrum .

Now let X be a Banach space, $L(X)$ the Banach algebra of all bounded linear operators on X, and let $\mathcal{A} \subset L(X)$ be a commutative Banach subalgebra. We also assume that $\mathcal{A}$ contains the identity operator I. This is not a restriction since we can always consider the algebra $\widetilde{\mathcal{A}}$ generated by $\mathcal{A}$ and the identity operator I.

Let $\mathcal{R}$ denote the radical of $\mathcal{A}$, and put

$$X_0 := \overline{span}\{Nx : x \in X, N \in \mathcal{R}\}. \tag{1}$$

It is clear that X_0 is a closed invariant subspace for every operator in $\mathcal{A}$. Note that if $X_0 = \{0\}$ if and only if $\mathcal{A}$ is semisimple.

For each element $x \in X$ let

$$I_x := \{a \in \mathcal{A} : ax \in X_0\}.$$

Then I_x is a closed ideal of $\mathcal{A}, I_x \supset \mathcal{R}$, hence

$$\mathcal{R}_0 := \cap\{I_x : x \in X\}$$

also is a closed ideal and $\mathcal{R}_0 \supset \mathcal{R}$.

For each element $a \in \mathcal{A}$, let

$$I_a := \{b \in \mathcal{A} : ab \in \mathcal{R}_0\}.$$

Then I_a is a closed ideal of $\mathcal{A}$ such that $I_a \supset \mathcal{R}_0$.

We define the *local generalized Beurling spectrum, or simply the spectrum,* $Sp(x)$ of the element x (with respect to $\mathcal{A}$) by

$$Sp(x) := h(I_x) = \{\gamma \in M : \hat{a}(\gamma) = 0 \text{ for all } a \in I_x\},$$

and, similarly, *the local generalized Beurling spectrum, or simply the spectrum,* $Sp(a)$, of the element a by

$$Sp(a) := h(I_a) = \{\gamma \in M : \hat{b}(\gamma) = 0 \text{ for all } b \in I_a\}.$$

For a subset $\Lambda \subset M$, let

$$X(\Lambda) := \{x : Sp(x) \subset \Lambda\},$$

and

$$A(\Lambda) := \{a \in \mathcal{A} : Sp(a) \subset \Lambda\}.$$

Further, let

$$M_0 := \overline{\cup\{Sp(x) : x \in X\}},\ M_1 := \overline{\cup\{Sp(a) : a \in A\}}.$$

The algebra $\mathcal{A}$ is called *decomposable*, if

(i) For every closed subset $F \subset M_0$, the corresponding subspace $X(F)$ is closed;

(ii) If $U_i, i = 1, 2, ..., n$ is an open cover of M, then

$$X = X(\overline{U_1}) + X(\overline{U_2}) + ... + X(\overline{U_n}).$$

The main result of this paper states that any regular commutative subalgebra $\mathcal{A}$ of $L(X)$ is decomposable (Theorem 8).

As we will see, $Sp(x) \neq \emptyset$ if and only if $x \notin X_0$ (and similarly $Sp(a) \neq \emptyset$ if and only if $a \notin \mathcal{R}_0$). Thus, if $X_0 \neq X$, then there are elements with non-empty spectrum, a fact which is a crucial condition for a spectral decomposition. In general, it may happen that $X_0 = X$, which implies that every element x has an empty spectrum, or $M_0 = \emptyset$, so that our analysis will give no information about decomposability. On the other hand, if $X_0 = \{0\}$, then $\mathcal{A}$ is semisimple and decomposability is possible, provide that the algebra $\mathcal{A}$ is regular. This case has been considered in [CF, chapter 6], [Fr], [Fl] (cf. [ELN], [Ne]). Our results even in this semisimple case, although very natural, do not seem to have been stated in the literature. When X_0 is a nontrivial subspace, i.e. $X_0 \notin \{\{0\}, X\}$, which is a case of non-semisimple algebras, our approach gives a decomposition of X into corresponding spectral subspaces.

We next give some illustrating examples.

Examples.

1. Let $\mathcal{A} \subset L(X)$ be a semisimple regular commutative subalgebra containing the identity operator I. Then, as noted before, $X_0 = \{0\}$. From Proposition 2-v it follows that $\mathcal{R}_0 = \mathcal{R} = \{0\}$. In this case, $Sp(x)$ and $Sp(a)$ are the standard Beurling spectra.

2. Let $X = \mathbb{C}^3$, and

$$\mathcal{A} = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Then

$$\mathcal{R} = \left\{ \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : b \in \mathbb{C} \right\}.$$

and

$$X_0 = \{\mathbf{x} = (x_1, 0, 0) : x_1 \in \mathbb{C}\}.$$

It is easy to see that, for a vector $\mathbf{x} := (x_1, x_2, x_3)$, we have $I_{\mathbf{x}} = \mathcal{A}$ if $x_2 = 0$ and $I_{\mathbf{x}} = \mathcal{R}$ if $x_2 \neq 0$. Hence, $\mathcal{R}_0 = \mathcal{R}$.

3. Let Y_1, Y_2 be Banach spaces, $N : Y_1 \to Y_1$ be a quasinilpotent operator such that $\overline{NY_1} = Y_1$, and $\mathcal{B}$ be the Banach algebra generated by N and I_{Y_1}. Put $X = Y_1 \oplus Y_2$ and

$$\mathcal{A} = \left\{ \begin{bmatrix} A & 0 \\ 0 & \lambda I_{Y_2} \end{bmatrix} : A \in \mathcal{B}, \lambda \in \mathbb{C} \right\},$$

Then $X_0 = Y_1$, and for each $\mathbf{x} = (y_1, y_2)$, we have $I_{\mathbf{x}} = \mathcal{B}$ if $y_2 \neq 0$ and $I_{\mathbf{x}} = \mathcal{A}$ if $y_2 = 0$. Therefore, $\mathcal{R}_0 = \mathcal{B} \neq \mathcal{R}$.

4. The spectral decomposition of regular operator algebras.

Now assume that $\mathcal{A} \subset L(X)$ is a commutative regular Banach subalgebra which contains the identity operator I.

Proposition 2. Let $a, b \in \mathcal{A}, x \in X$. Then
(i) $Sp(x) = \emptyset$ if and only if $x \in X_0$;
(ii) $Sp(a) = \emptyset$ if and only if $a \in \mathcal{R}_0$;
(iii) If $C \in \{\mathcal{A}\}'$, then $Sp(Cx) \subset Sp(x)$;
(iv) $Sp(ax) \subset Sp(a) \cap Sp(x)$;
(v) $Sp(ab) \subset Sp(a) \cap Sp(b)$;
(vi) If $U \subset M$ is an open subset and $\hat{a}|U = 0$, then $Sp(a) \subset U^c$;
(vii) If $Y \subset X$ is a dense subset, then $Sp(a) = \overline{\cup\{Sp(ax) : x \in Y\}}$;
(viii) If $\mathcal{B} \subset \mathcal{A}$ is a dense subset, then $Sp(x) = \overline{\cup\{Sp(ax) : a \in \mathcal{B}\}}$;

Proof. (i) If $Sp(x) = \emptyset$, then $h(I_x) = \emptyset$. By Theorem 1, $I_x \supset I_0(\emptyset) = \mathcal{A}$. Hence $I_x = \mathcal{A}$, or $x \in X_0$. Conversely, if $x \in X_0$, then $I_x = \mathcal{A}$, hence $Sp(x) = \emptyset$.

(ii) If $Sp(a) = \emptyset$, then $h(I_a) = \emptyset$, hence $I_a \supset I_0(\emptyset) = \mathcal{A}$, i.e. $I_a = \mathcal{A}$, or $a \in \mathcal{R}_0$. Conversely, if $a \in \mathcal{R}_0$, then $I_a = \mathcal{A}$, hence $Sp(a) = \emptyset$.

(iii) is obvious since $I_x \subset I_{Cx}$.

(iv) If $b \in I_a$, then $ba \in \mathcal{R}_0$, so that $ba \in I_x$, or $bax \in X_0$ for all x, hence $b \in I_{ax}$. Therefore, $Sp(ax) \subset Sp(a)$. By (iii), $Sp(ax) \subset Sp(x)$, hence $Sp(ax) \subset Sp(a) \cap Sp(x)$.

(v) If $c \in I_a$, then $ac \in \mathcal{R}_0$ or $ac \in I_x$ for all $x \in X$. Hence $abc \in I_x$ for all $x \in X$ or $abc \in \mathcal{R}_0$, so that $c \in I_{ab}$. Thus, $I_a \subset I_{ab}$, which implies $Sp(ab) \subset Sp(a)$, so that $Sp(ab) \subset (Sp(a) \cap Sp(b))$.

(vi) Let $b \in I_0(U^c)$. Then there is a neighborhood V of U^c such that $\hat{b}|V = 0$. Therefore, $\hat{a}\hat{b} = 0$, i.e. $ab \in \mathcal{R}$, hence $b \in I_a$. Thus, $I_0(U^c) \subset I_a$, and an application of Theorem 1 gives $Sp(a) = h(I_a) \subset h(I_0(U^c)) = U^c$.

(vii) The inclusion

$$\overline{\cup_{x \in Y} Sp(ax)} \subset Sp(a)$$

follows from (iv). To show the converse inclusion, we assume the contrary, i.e. there exists $\gamma_0 \in Sp(a)$ such that $\gamma_0 \notin F := \overline{\cup_{x \in Y} Sp(ax)}$. Then there exist a neighborhood U of F and element $b \in \mathcal{A}$ such that $\hat{b}|U = 0$ and $\hat{b}(\gamma_0) = 1$. It follows from (vi) that $Sp(b) \subset U^c$, hence

$$Sp(abx) \subset U^c \cap Sp(ax) = \emptyset, \text{ for all } x \in Y,$$

so that $abx \in X_0$ for each $x \in Y$. Since Y is dense, this implies $abx \in X_0$ for each $x \in X$, hence $ab \in I_x$ for all x, or $ab \in \mathcal{R}_0$, so that $b \in I_a$, which in turn implies $\hat{b}(\gamma_0) = 0$, a contradiction.

(viii) The inclusion $E := \overline{\cup\{Sp(ax) : a \in \mathcal{B}\}} \subset Sp(x)$ follows from (iv).

To show the converse inclusion, we assume the contrary, i.e. there exists $\gamma_0 \in Sp(x)$ such that $\gamma_0 \notin E$. Then there exist $b \in \mathcal{A}$ and a neighborhood U of E such that $\hat{b}|U = 0$ and $\hat{b}(\gamma_0) = 1$. By (vi), $Sp(b) \subset U^c$ and

$$Sp(abx) \subset U^c \cap Sp(ax) = \emptyset, \text{ for all } a \in \mathcal{B},$$

hence $abx \in X_0$, for all a. Therefore, $bx \in X_0$, or $b \in I_x$, so that $\hat{b}(\gamma_0) = 0$, a contradiction.

Proposition 3. The following statements hold:

(i) $M_0 = M_1$.

(ii) $M_0 = \cap\{F : \ F \text{ is closed and } \ X(F) = X\}$.

(iii) $M_0 = \cap\{F : \ F \text{ is closed and } \ A(F) = A\}$.

(iv) $M = M_0$ if and only if $\mathcal{R} = \mathcal{R}_0$;

(v) If $X_0 = \{0\}$, then $M_0 = M$ and hence $\mathcal{R}_0 = \mathcal{R}$.

Proof. (i) Let $\gamma \in M_1$. Then for every open set U such that $\gamma \in U$, there exists $a \in \mathcal{A}$ such that $U \cap Sp(a) \neq \emptyset$. By Proposition 2-vii, there exists an element $x \in X$ such that $U \cap Sp(ax) \neq \emptyset$. This implies $\gamma \in M_0$, so that $M_1 \subset M_0$. The proof of the inclusion $M_0 \subset M_1$ is analogous.

(ii) Since $X(M_0) = X$, we have $M_0 \supset (\cap\{F : \ F\text{is closed and } \ X(F) = X\})$. The converse inclusion follows from the fact that if F is a closed subset such that $X(F) = X$, then $Sp(x) \subset F$ for all $x \in X$, so that $M_0 \subset F$.

(iii) The proof is analogous to (ii).

(iv) Assume $\mathcal{R} = \mathcal{R}_0$ and $\gamma_0 \in M \setminus M_0$. Then there is a neighborhood U of M_0 and an element $a \in \mathcal{A}$ such that $\hat{a}|U = 0, \hat{a}(\gamma_0) = 1$. By Proposition 2-vi, $Sp(a) \subset U^c$, hence $Sp(a) = \emptyset$ so that $a \in \mathcal{R}_0 = \mathcal{R}$, a contradiction to $\hat{a}(\gamma_0) = 1$.

Conversely, assume that $\mathcal{R} \neq \mathcal{R}_0$. Then there is $a \in \cap\{I_x : x \in X\}$, and $a \notin \mathcal{R}$. Therefore, $\hat{a}(\gamma) = 0$ for all $\gamma \in M_0$, but $\hat{a}$ is not identically zero. It implies $M \neq M_0$.

(v) Assume that there exists $\gamma_0 \in M \setminus M_0$. Then, since $\mathcal{A}$ is regular, there exist an open set $V \supset M_0$ and $a \in \mathcal{A}$ such that $\hat{a}|V = 0$, $\hat{a}(\gamma_0) = 1$. It follows, by Proposition 2-iv, that $Sp(a) \subset V^c$, hence $Sp(ax) \subset V^c \cap M_0 = \emptyset$, so that $ax \in X_0 = \{0\}$ for all $x \in X$. Thus, $a = 0$, a contradiction.

The set $M_0 = M_1$ is called *the support* of the operator subalgebra $\mathcal{A}$ of $L(X)$.

Proposition 4. The following statements hold:

(i) For any $x, y \in X$ we have $Sp(x + y) \subset Sp(x) \cup Sp(y)$;

(ii) $X(\Lambda)$ is a linear hyperinvariant subspace, $X(\emptyset) = X_0, X(M) = X(M_0) = X$ and $X(\Lambda) \supset X_0$ for every $\Lambda \subset M$;

(iii) If $\Lambda_1 \subset \Lambda_2$, then $X(\Lambda_1) \subset X(\Lambda_2)$. If Λ_α is a family of subsets of M, then $X(\cap_\alpha \Lambda_\alpha) = \cap_\alpha X(\Lambda_\alpha)$;

(iv) Let Λ be a closed subset. Then $x \in X(\Lambda)$ if and only if $ax \in X_0$ for every $a \in I_0(\Lambda)$;

(v) If Λ is a closed subset, then $X(\Lambda)$ is a closed hyperinvariant subspace;

Proof. (i) Let $\gamma_0 \notin \Lambda := Sp(x) \cup Sp(y)$. Choose a neighborhood V of Λ and an element $b \in \mathcal{A}$ such that $\hat{b}|V = 0, \hat{b}(\gamma_0) = 1$.Then $Sp(bx) = Sp(by) = \emptyset$, hence $bx \in X_0, by \in X_0$, so that $b(x+y) \in X_0$. Hence $b \in I_{x+y}$, and therefore $\gamma_0 \notin Sp(x+y)$.

(ii) This follows from (i) and Proposition 2-(i),(iii);

(iii) is obvious;

(iv) If $x \in X(\Lambda)$, then $Sp(x) \subset \Lambda$, i.e. $h(I_x) \subset \Lambda$. By Theorem 1, $I_x \supset I_0(\Lambda)$, or $ax \in X_0$ for all $a \in I_0(\Lambda)$. Conversely, if $ax \in X_0$ for all $a \in I_0(\Lambda)$, then $I_0(\Lambda) \subset I_x$, hence $Sp(x) = h(I_x) \subset h(I_0(\Lambda)) = \Lambda$.

(v) Follows immediately from (iv).

Proposition 5. The following statements hold:

(i) $Sp(a + b) \subset Sp(a) \cup Sp(b)$;

(ii) $A(\Lambda)$ is an ideal of $\mathcal{A}$; $A(\emptyset) = \mathcal{R}_0$, $A(M) = A(M_1) = \mathcal{A}$, and $A(\Lambda) \supset \mathcal{R}_0$ for all Λ;

(iii) If $\Lambda_1 \subset \Lambda_2$, then $A(\Lambda_1) \subset A(\Lambda_2)$. If Λ_α is a family of subsets of M, then $A(\cap\Lambda_\alpha) = \cap A(\Lambda_\alpha)$;

(iv) If Λ is a closed subset, then $a \in A(\Lambda)$ if and only if $ab \in \mathcal{R}_0$ for every $b \in I_0(\Lambda)$;

(v) If Λ is a closed subset, then $A(\Lambda)$ is a closed ideal;

Proof. (i) We show that if $\gamma_0 \notin F := Sp(a) + Sp(b)$, then $\gamma_0 \notin Sp(a+b)$. Let U be a neighborhood of F and c be an element in $\mathcal{A}$ such that $\hat{c}|U = 0, \hat{c}(\gamma_0) = 1$. Then $Sp(ca) = Sp(cb) = \emptyset$, hence $ca \in \mathcal{R}_0, cb \in \mathcal{R}_0$, or $c(a+b) \in \mathcal{R}_0$. Thus, $c \in I_{a+b}$. Since $\hat{c}(\gamma_0) = 1$, we have $\gamma_0 \notin Sp(a+b)$.

(ii) follows from (i) and Proposition 2(v).

(iii) is obvious.

(iv) Let $Sp(a) \subset \Lambda$, i.e. $h(I_a) \subset \Lambda$. By Theorem 1, $I_a \supset I_0(h(I_a)) \supset I_0(\Lambda)$, or $ba \in \mathcal{R}_0$ for all $b \in I_0(\Lambda)$. Conversely, let $ba \in \mathcal{R}_0$ for all $b \in I_0(\Lambda)$. Then $I_0(\Lambda) \subset I_a$. Hence $Sp(a) = h(I_a) \subset h(I_0(\Lambda)) = \Lambda$.

(v) follows immediately from (iv).

Proposition 6. If $a \in \mathcal{R}_0$, then $\hat{a}|M_0 = 0$. Conversely, if there is a neighborhood U of M_0 such that $\hat{a}|U = 0$, then $a \in \mathcal{R}_0$. In other words, $I_0(M_0) \subset \mathcal{R}_0 \subset I(M_0)$ or $k(\mathcal{R}_1) = M_0$.

Proof. Assume that $a \in \mathcal{R}_0 := \cap\{I_x : x \in X\}$. Then $\hat{a}(\gamma) = 0$ for all $\gamma \in h(I_x)$, i.e. $\hat{a}|Sp(x) = 0$, for all $x \in X$, which implies $\hat{a}|M_0 = 0$. Conversely, if $\hat{a}|U = 0$ for some open neighborhood U of M_0, then $a \in I_0(h(I_x)) \subset I_x$ for all x, hence $a \in \mathcal{R}_0$.

Theorem 7. Assume that $\mathcal{A} \subset L(X)$ is a commutative regular subalgebra containing the identity operator. Then the algebra of multiplication operators in $\mathcal{A}$ is decomposbale. More precisely, if $U_1, U_2, ..., U_n$ form an open cover of $\mathcal{M}_0$, then $\mathcal{A} = A(\overline{U_1}) + A(\overline{U_2}) + ... + A(\overline{U_n})$.

Proof. Let $V_1 = U_1 \cup (M \setminus M_0), V_k = U_k, 2 \leq k \leq n$. Then $(V_k)_{k=1}^n$ is an open cover of M. Since the algebra $\mathcal{A}$ is regular, there exist $b_1, b_2, ..., b_n \in \mathcal{A}$ such that $\hat{b}_k|V_k^c = 0 (1 \leq k \leq n)$, and $\sum_{k=1}^n \hat{b}_k = 1$. If V is a closed subset such that $V \supset V_k$, then $\hat{b}_k|V^c = 0$, hence by Proposition 2-iv we have $Sp(b_k) \subset V$. Therefore, $Sp(b_k) \subset \overline{V}_k, k = 1, ..., n$.

Let $a \in \mathcal{A}$. Put $a_i = ab_i, i = 1, 2, ..., n$ and $a_{n+1} = a - \sum_{k=1}^n ab_k$. Then $Sp(a_k) \subset Sp(b_k) \subset \overline{V_k}$, or $a_k \in A(\overline{V_k})$ for all $k = 1, ..., n$. Since $\hat{a}_{n+1} = \hat{a} - \sum_{k=1}^n \hat{a}\hat{b}_k = 0$, it follows $\hat{a}_{n+1} \in \mathcal{R} \subset A(\overline{V_k})$ for all k. It is easy to see that $A(\overline{V_k}) = A(\overline{U_k})$ for all k. Thus, $a = \sum_{k=1}^n a_k + a_{n+1} =$ is the required decomposition (we can include a_{n+1} into any of the ideals $A(\overline{U_k}), 1 \leq k \leq n$).

Theorem 8. Assume that $\mathcal{A} \subset L(X)$ is a commutative regular subalgebra containing the identity operator. Then $\mathcal{A}$ is decomposbale. More precisely, if $U_1, U_2, ..., U_n$ is an open cover of $\mathcal{M}_0$. Then $X = X(\overline{U_1}) + X(\overline{U_2}) + ... + X(\overline{U_n})$.

Proof. Define $V_k, k = 1, ..., n$, and choose $b_1, b_2, ..., b_n \in \mathcal{A}$ as in the proof of Proposition 7, i.e. $\hat{b}_k|V_k^c = 0 (k = 1, ..., n)$, and $\sum_{k=1}^n \hat{b}_k = 1$. For each $a \in \mathcal{A}$, let

$a_k := ab_k, k = 1, ..., n, a_{n+1} = a - (a_1 + ... + a_n)$.

Fix $x \in X$. Put $x_k = b_k x$ for $k = 1, 2, ..., n$ and $x_{n+1} = x - \sum_{k=1}^n b_k x$. Then $Sp(x_k) \subset \overline{V_k}$, so that $x_k \in X(\overline{V_k}) = X(\overline{U_k}), 1 \leq k \leq n$.

We have $Sp(ax_{n+1}) = Sp((a - \sum_{k=1}^n ab_k)x) \subset Sp(a_{n+1}) = \emptyset$, hence $Sp(x_{n+1}) = \emptyset$ or $x_{n+1} \in X_0 \subset X(\overline{U_k})$ for all $k = 1, 2, ...$, and the required decomposition is obtained.

We recall the following well known result that in combination with Theorems 7 an 8 will give an alternative way of defining spectral decompositions.

Proposition 9. Let M be a locally compact Hausdorff space and let K be a compact subset of M. For every open cover $\{G_i\}_{i=1}^n$ of K there is an open cover $\{H_i\}_{i=1}^n$ of K such that the sets H_i are relatively compact and $\overline{H_i} \subset G_i, i = 1, ..., n$.

For the proof, see [EL], p. 37.

From Theorems 7-8 and Proposition 9 we immediately obtain the following decomposition property.

Theorem 10. Assume that $U_1, ..., U_n$ is an open cover of M_0. Then

$$X = X(U_1) + X(U_2) + ... + X(U_n),$$
$$\mathcal{A} = A(U_1) + A(U_2) + ... + A(U_n).$$

Theorem 8 gives, in particular, the existence of nontrivial hyperinvariant subspaces if the support of the algebra $\mathcal{A}$ contains more than one element.

Corollary 11. If M_0 contains more than one point, then

(i) $\mathcal{A}$ has a non-trivial hyperinvariant subspace.

(ii) The algebra of multiplication operators on $\mathcal{A}$ has non-trivial hyperinvariant subspace.

5. The algebra of multiplication operators on a regular commutative Banach algebra.

Assume that B is a commutative regular Banach algebra with unit. Denote by R its radical. Let $X := B$ and $\mathcal{A}$ consist of multiplication operators on X, i.e.

$$\mathcal{A} := \{T_a : B \to B, a \in B, \ T_a b := ab \text{ for all } b \in B\}.$$

Let X_0 be the corresponding subspace of $X(= B)$, $\mathcal{R}$ and $\mathcal{R}_0$ be the corresponding ideals in $\mathcal{A}$. It is easy to see that $T_a \in \mathcal{R}$ if and only if $a \in R$, and that $X_0 = R$, $\mathcal{R}_0 = \mathcal{R}$. By Proposition 2, we have $M_0 = M$. We can naturally identify $\mathcal{A}$ with B via the isomorphism $a \mapsto T_a$.

Applying Theorem 8 to this case, we obtain the following result.

Theorem 12. The algebra of multiplication operators on a regular commutative Banach algebra is decomposable.

The following proposition gives an equivalent description of the spectrum $Sp(x)$ for every $x \in X(= B)$ in terms of the support of the function $\hat{x}$. From this proposition it follows, via Theorem 2.4 of [CF, p.200] (see also [Ne], Theorem 1.4), that for the case of semisimple regular algebra B and a single multiplication operator T_a on B, our spectral subspace $X(F)$ coincides with the spectral subspace in the sense of Foias. Recall that if $x \in B$, then the support of its Gelfand transform $\hat{x}$ is defined by $supp(\hat{x}) = \overline{\{\gamma \in M(B) : \hat{x}(\gamma) \neq 0\}}$. Thus, $\gamma_0 \in supp(\hat{x})$ if and only if for every open neighborhood U of γ_0, there exists $\gamma \in U$ such that $\hat{x}(\gamma) \neq 0$.

Proposition 13. For each element $x \in B$, we have $Sp(x) = supp(\hat{x})$.

Proof. Assume that $\gamma_0 \notin Sp(x)$. Then there exists $a \in I_x$ such that $\hat{a}(\gamma_0) \neq 0$. Since $\hat{a}$ is continuous, there exists an open neighborhood U of γ_0 such that $\hat{a}(\gamma) \neq 0$ for all $\gamma \in U$. Since $a \in I_a$, we have $T_a x \in R$ or $ax \in R$, hence $\hat{a}\hat{x} = 0$. This implies that $\hat{x}|U = 0$ so that $\gamma_0 \notin supp(\hat{x})$.

Conversely, assume that $\gamma_0 \notin supp(\hat{x})$. Then there is an open neighborhood U of γ_0 such that $\hat{x}(\gamma) = 0$ for all $\gamma \in U$. Let V be an open neighborhood of γ_0 such that $\overline{V} \subset U$ and, since B is regular, let a be an element of B such that $\hat{a}(\gamma_0) = 1$, $\hat{a}|V^c = 0$. Then $\widehat{ax} = \hat{a}\hat{x} = 0$, hence $ax \in R$, or $a \in I_x$. Since $\hat{a}(\gamma_0) \neq 0$, we have $\gamma_0 \notin Sp(x)$.

Remark that under the assumption that B is semisimple, and for the standard spectral subspaces, the decomposability of multiplication operators on B in the standard sense has been shown in [CF]. Proposition 13 shows that our results include this result of [CF] (for the case of semisimple algebras. Finally, let us note that in [Fl] it is shown that if a bounded linear operator A on a Banach space X generates a semisimple regular algebra, which satisfies one more additional assumption, then there exists a nontrivial invariant subspace. Our result shows that the the operator A is even decomposable without the additional assumption and the semisimplicity condition, and hence possesses a nontrivial invariant subspace (of course, if the spectrum contains more than one point).

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