Covering a ball with smaller equal balls in $\mathbb{R}^n$

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Abstract

We give an explicit upper bound of the minimal number of balls of radius $1/2$ which form a covering of a ball of radius $T > 1/2$ in $\mathbb{R}^n$, $n \geqslant 2$.

1. Introduction

Let $T > 1/2$ and $\nu_{T,n}$ the minimal number of (closed) balls of radius $1/2$ which can cover a (closed) ball of radius T in $\mathbb{R}^n$, $n \geqslant 2$. In [R1] (pp 163-164 and theorem 2) Rogers has obtained the following result.

THEOREM 1.1. — *(i) If $n \geqslant 3$, with $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$, we have*

$$1 < \nu_{T,n} \leqslant \begin{cases} e\vartheta_n (2T)^n & \text{if } T \geqslant n/2, \\ n\vartheta_n (2T)^n & \text{if } \frac{n}{2\ln n} \leqslant T < \frac{n}{2}. \end{cases} \quad (1)$$

(ii) If $n \geqslant 9$ we have

$$1 < \nu_{T,n} \leqslant \frac{4e(2T)^n n\sqrt{n}}{\ln n - 2}\left(n \ln n + n \ln(\ln n) + n \ln(2T) + \frac{1}{2}\ln(144n)\right) \quad (2)$$

for all $1/2 < T < \frac{n}{2\ln n}$.

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The assertion (i) can easily be extended to the case $n = 2$ by invoking Rogers [R] (p 47) so that the strict upper bound $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$ of the covering density of equal balls in $\mathbb{R}^n$ is still a valid one in this case. Thus the inequalities (1) are still true for $n = 2$. In the case $n = 2$, see also Kershner [K]. On the other hand, the result (ii) does not seem to have been improved since then, see for instance [Hand], Fejes-Toth [Ft], Schramm [Sc], Raigorodski [Ra] or Bourgain et al [BL]. This problem is linked to the existence of explicit lower bounds of the packing constant of equal spheres in $\mathbb{R}^n$ [MVG] and to various problems [MR] [IM] [FF] [Ma].

In this contribution we give an improvment of the upper bound of $\nu_{T,n}$ given by the assertion (ii), i.e. when the radius T is less than $n/(2 \ln n)$. Namely, we will prove

THEOREM 1.2. — *Let $n \geqslant 2$. The following inequalities hold:*

(i) $n < \nu_{T,n} \leqslant$

$$\frac{7^{4(\ln 7)/7}}{4}\sqrt{\frac{\pi}{2}}\,\frac{n\sqrt{n}\left[(n-1)\ln(2Tn)+(n-1)\ln(\ln n)+\frac{1}{2}\ln n+\ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n}-2}\right)\right]}{T\,(1-\frac{2}{\ln n})(1-\frac{2}{\sqrt{\pi\, n}})(\ln n)^2}\,(2T)^n$$

$$\textit{if}\ \ 1 < T < \frac{n}{2\ln n} \qquad (3)$$

(ii) $n < \nu_{T,n} \leqslant$

$$\sqrt{\frac{\pi}{2}}\,\frac{\sqrt{n}\left[(n-1)\ln(2Tn)+(n-1)\ln(\ln n)+\frac{1}{2}\ln n+\ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n}-2}\right)\right]}{T\,(1-\frac{2}{\ln n})(1-\frac{2}{\sqrt{\pi\, n}})}\,(2T)^n$$

$$\textit{if}\ \ 1/2 < T \leqslant 1 \qquad (4)$$

The following question seems fundamental: what are the integers $\nu_{T,n}$ when $1/2 < T$, $2 \leqslant n$ and the corresponding configurations of balls of radius $1/2$ when they form the most economical covering of the closed ball $B(0, T)$ of radius T centred at the origin ?

2. Proof of the theorem 1.2

The idea of the proof is simple: (i) when T is small enough, it amounts to show that the sphere $S(0, T)$ can be covered by a collection of N balls of radius $1/2$ suitably placed at equidistance from the origin, and that this covering to which we add the central ball $B(0, 1/2)$ actually covers the ball $B(0, T)$ itself; in the subsection 2, an upper bound of the minimal value of N is calculated from the results given by the lemmas of subsection 1; (ii) when T is larger, we proceed recursively using (i) to give an upper bound of N. The configuration of balls of radius $1/2$ covering $B(0, T)$ is then ordered by layers, the last layer of balls of radius $1/2$ being at an optimal distance from the origin so as to cover the sphere $S(0, T)$.

1. *Caps and sectors.*—Let $T > 1/2$ and $n \geqslant 2$ in the following. If the closed ball $B(0, T)$ is covered by N smaller balls of radius $1/2$, the smaller balls will intersect the sphere $S(0, T)$,

for a certain proportion of them. The intersection of a closed ball of radius $1/2$ and the sphere $S(0,T)$, if it is not empty, is called a (spherical) cap. For fixing the notations let us define properly what is a cap and the sector it generates in $B(0,T)$.

Let $h \geqslant 0$ and u be a unit vector of $\mathbb{R}^n$. Let us denote by $H_{h,u}$ the affine hyperplane $\{z + hu \mid z \in \mathbb{R}^n, z \cdot u = 0\}$ of $\mathbb{R}^n$. Assume that $H_{h,u}$ intersects the ball $B(0,T)$, i.e. $h \leqslant T$. We will denote

$$C_{T,h,u} := \{z \in S(0,T) \mid \frac{z \cdot u}{\|z\|} \geqslant \frac{h}{T}\}$$

The $n-2$-dimensional sphere $H_{h,u} \cap C_{T,h,u}$ admits $x = \sqrt{T^2 - h^2}$ for radius. The correspondence between $x \in [0,T]$ and $h \in [0,T]$ is one-to-one. We will say that $C_{T,h,u}$ is the cap of chord $2x$ and of centre Tu. If a subset Y of $S(0,T)$ is such that there exists $h \geqslant 0$ and u a unit vector of $\mathbb{R}^n$ such that $Y = C_{T,h,u}$, then we will say that Y is a cap of chord $2x$ of $S(0,T)$.

Every cap $C_{T,h,u}$ of chord $2x$ of $S(0,T)$ generates a sector in $B(0,T)$. We will denote it by

$$\mathscr{S}(T,h,u) \ := \ \{z \in B(0,T) \mid \frac{z \cdot u}{\|z\|} \geqslant \frac{h}{T}\}$$

We will denote by $V_{(T,x)}$ (indexing with x instead of h) the volume of a sector generated by a cap of chord $2x$ in $S(0,T)$ with $x \leqslant T$. Let $\omega_n := \pi^{n/2}/\Gamma(1+n/2)$ so that the (n-dimensional) volume of a ball of radius T in $\mathbb{R}^n$ is $\omega_n T^n$.

Lemma 2.1. — *We have*

$$\frac{\omega_{n-1}}{\omega_n} \geqslant \frac{1}{\sqrt{2\pi}}\sqrt{n}(1 - \frac{2}{\sqrt{\pi n}}) \tag{1}$$

Proof. — The following inequalities are classical ([Va] p 171):

$$\frac{\omega_{n-1}}{\omega_n} \geqslant \begin{cases} \frac{1}{\sqrt{2\pi}}\sqrt{n}\left(1 + \left(\frac{n}{2e}\right)^{n/2}\frac{1}{(n/2)!}\right)^{-1} & \text{if } n \text{ is even} \\ \frac{1}{\sqrt{2\pi}}\sqrt{n+1}\left(1 - \left(\frac{n+1}{2e}\right)^{(n+1)/2}\frac{1}{((n+1)/2)!}\right) & \text{if } n \text{ is odd} \end{cases} \tag{2}$$

By Stirling's formula we deduce the result. □

Lemma 2.2. — *Let $0 < x < T$. Let n be odd and put $y = (n-1)/2$. The volume $V_{(T,x)}$ of a sector in $B(0,T) \subset \mathbb{R}^n$ generated by a cap of chord $2x$ in $S(0,T)$ is equal to*

$$\omega_{n-1}x^{n-1}\left[\frac{\sqrt{T^2-x^2}}{n} + \frac{2(T-\sqrt{T^2-x^2})}{n+1}\sum_{j=0}^{y}\frac{y!(y+1)!}{(y+1+j)!\,(y-j)!}\left(\frac{T-\sqrt{T^2-x^2}}{T+\sqrt{T^2-x^2}}\right)^j\right] \tag{3}$$

It satisfies the relations

$$(i) \quad V_{(T,x)} \ = \ x^n V_{(T/x,1)} \tag{4}$$

$$(ii) \quad \frac{T}{nx} \ \leqslant \ \frac{2n(T/x) + (1-n)\sqrt{(T/x)^2-1}}{n(n+1)} \ \leqslant \ \frac{1}{\omega_{n-1}}\,V_{(T/x,1)} \tag{5}$$

Proof. — Let us show (3). The first term $\omega_{n-1}x^{n-1}\frac{\sqrt{T^2-x^2}}{n}$ is the volume of the truncated cone $\{z \in \mathscr{S}(T,h,u) \mid z\cdot u \leqslant h\}$ with $h = \sqrt{T^2-x^2}$. The second term in (3) is the volume of $\{z \in \mathscr{S}(T,h,u) \mid z\cdot u \geqslant h\}$: any point of $C_{T,\sqrt{T^2-x^2},u}$ which is at distance t from $H_{\sqrt{T^2-x^2},u}$ is at distance $\left(x^2-t^2-2t\sqrt{T^2-x^2}\right)^{1/2}$ from the line $\mathbb{R}u$. Hence, this volume equals to

$$\int_0^{T-\sqrt{T^2-x^2}} \omega_{n-1}\left[x^2-t^2-2t\sqrt{T^2-x^2}\right]^{(n-1)/2} dt$$

It is obtained by integration by parts, y times, of the integral

$$\omega_{n-1}\int_0^{\alpha}(\alpha-t)^y(t-\beta)^y dt \tag{6}$$

with $\alpha = T-\sqrt{T^2-x^2}$ and $\beta = -T-\sqrt{T^2-x^2}$.

The relation (4) is obvious. Let us show (5). We deduce it from the fact that the summation in (3) has positive terms and is greater than its first term which is 1. □

LEMMA 2.3. — *Assume $n \geqslant 2$ even and $0 < x < 1$. The volume $V_{(T,x)}$ of a sector in $B(0,T) \subset \mathbb{R}^n$ generated by a cap of chord $2x$ in $S(0,T)$ satisfies the relations:*

$$(i)\quad V_{(T,x)} = x^n V_{(T/x,1)} \tag{7}$$

$$(ii)\quad \frac{T}{nx} \leqslant \frac{2n(T/x)+(2-n)\sqrt{(T/x)^2-1}}{n(n+2)} \leqslant \frac{1}{\omega_{n-1}}V_{(T/x,1)} \tag{8}$$

Proof. — The equality (7) is obvious. In order to prove (8), let us observe that the function $t \to (\alpha-t)(t-\beta)$ defined on the interval $[0,\alpha]$ is valued in the interval $[0,1]$ since it lies below the horizontal line of y-coordinate $-\alpha\beta = x^2 < 1$. We deduce the following inequalites

$$(\alpha-t)^{\frac{n+1}{2}}(t-\beta)^{\frac{n+1}{2}} \leqslant (\alpha-t)^{\frac{n}{2}}(t-\beta)^{\frac{n}{2}} \leqslant (\alpha-t)^{\frac{n-1}{2}}(t-\beta)^{\frac{n-1}{2}}$$

for all $t \in [0,\alpha]$. From (6) in the proof of lemma 2.2 we deduce a lower bound of the volume of the convex hull of $C_{T,\sqrt{T^2-x^2},u}$ for n even using the preceding n odd case of lemma 2.2: changing n to $n+1$ now odd in the computation of the lower bound of the summation in (3). Let us note that the computation of the volume of $\{z \in \mathscr{S}(T,h,u) \mid z\cdot u \leqslant h\}$ with $h = \sqrt{T^2-x^2}$ still gives $\omega_{n-1}x^{n-1}\frac{\sqrt{T^2-x^2}}{n}$ for n even so that the first term of $V_{(T,x)}$ remains the same as in the n odd case. We deduce the inequality (8). □

LEMMA 2.4. — *Let $0 < x \leqslant 1/2$. Let D be a point of the cap $C_{T,\sqrt{T^2-1/4},u} \subset S(0,T) \subset \mathbb{R}^n$ at a distance x from the line $\mathbb{R}u$. Let B denote the unique point which lies in the intersection of $C_{T,\sqrt{T^2-1/4},u} \cap H_{\sqrt{T^2-1/4},u}$ with the plane $(0,D,Tu)$ with the property that it is the closest to D. If η denotes the distance between D and the line OB, we have the following relation between x, T and η:*

$$x = \frac{1}{2}\sqrt{1-\left(\frac{\eta}{T}\right)^2} - \frac{\eta}{2}\sqrt{4-\frac{1}{T^2}} \quad \textit{equivalently}\ \eta = \frac{1}{2}\sqrt{1-\left(\frac{x}{T}\right)^2} - \frac{x}{2}\sqrt{4-\frac{1}{T^2}} \tag{9}$$

Proof. — Let ψ be the angle between the lines OB and OD, ψ' the angle between the lines OD and $\mathbb{R}u$, so that $\sin(\psi) = \eta/T$ and $\sin(\psi') = x/T$. Since $\sin(\psi+\psi') = 1/(2T)$ we obtain

$$1 \;=\; 2\,x\sqrt{1-(\eta/T)^2} + 2\,\eta\sqrt{1-(x/T)^2}.$$

This expression is symmetrical in x and η. It is now easy, from it,to deduce the expression of x as a function of η, as stated by the eq.(9). □

LEMMA 2.5. — *Let us assume that a collection of N balls $(B(c_j, 1/2))_{j=1,2,\cdots,N}$ of $\mathbb{R}^n$ is such that (i) for all $j = 1,2,\cdots,N$, $B(c_j,1/2)\cap S(0,T)$ is a cap of chord 1 in $S(0,T)$ and (ii) these N caps form a covering of $S(0,T)$. Then (i) if $T > \sqrt{2}/2$, the union*

$$\bigcup_{j=1}^{N} B(c_j,1/2) \quad \textit{covers the annulus} \quad \{z\in\mathbb{R}^n \mid T-\frac{1}{2T}\leqslant \|z\| \leqslant T\}$$

of the ball $B(0,T)$; (ii) if $1/2 < T \leqslant \sqrt{2}/2$ this union covers $B(0,T)$.

Proof. — Any such ball $B(c_j,1/2)$ covers the part of the sector

$$\{z\in \mathscr{S}(T,\sqrt{T^2-1/4}, Oc_j/\|Oc_j\|) \mid \alpha T \leqslant \|z\|\}$$

with α to be determined. To compute α, let us consider two adjacent balls, say $B(c_1,1/2)$ and $B(c_2,1/2)$, such that the intersection of the respective caps $B(c_1,1/2)\cap S(0,T)$ and $B(c_2,1/2)\cap S(0,T)$ is reduced to one point. Then, on the line $\mathrm{O}\frac{c_1+c_2}{2}$, it is easy to check that all the points z such that $T-1/2T \leqslant \|z\| \leqslant T$ are covered. This gives $\alpha = 1-1/2T^2$. Now, since the caps $B(c_j,1/2)\cap S(0,T)$ form a covering of $S(0,T)$, the balls $B(c_j,1/2)$ form a covering of the annulus $\{z\in\mathbb{R}^n \mid \alpha T\leqslant\|z\|\leqslant T\}$. The last assertion is obvious. □

Let us consider $N(\geqslant 1)$ distinct points $M_1, M_2,\cdots,M_N$ of $S(0,T)\subset\mathbb{R}^n$. We will consider that they are the respective centres of caps of chord $2x$ of $S(0,T)$. We will denote by $\theta_{(T,x)}(M_1,M_2,\cdots,M_N)$ the proportion of $S(0,T)$ occupied by these caps. In other terms, with $u_i := \frac{OM_i}{\|OM_i\|}$ for all $i=1,2,\cdots,N$, we have

$$\theta_{(T,x)}(M_1,M_2,\cdots,M_N) := \frac{\mathrm{Vol}_{n-1}(\bigcup_{i=1}^{N} C_{T,\sqrt{T^2-x^2},u_i})}{\mathrm{Vol}_{n-1}(S(0,T))}$$

LEMMA 2.6. — *Let $N\geqslant 1$, $x\in(0,1/2]$. The mean $E\theta(N,T,x)$ of $\theta_{(T,x)}(M_1,M_2,\cdots,M_N)$ over all possibilites of collections of N distinct points $(M_1,M_2,\cdots,M_N)$ of $S(0,T)$ is equal to*

$$E\theta(N,T,x) = 1-\left(1-\frac{V_{(T,x)}}{\omega_n T^n}\right)^N$$

Proof. — Let $M_1,M_2,\cdots,M_N$ be N points of $S(0,T)$. We define

$$p_i \;=\; \frac{\mathrm{Vol}_{n-1}(C_{T,\sqrt{T^2-x^2},u_i})}{\mathrm{Vol}_{n-1}(S(0,T))} \qquad , \; i=1,2,\cdots,N,$$

the probability that a point $M\in S(0,T)$ belongs to the cap of chord $2x$ of centre M_i. It is the probability, hence independent of i, that M_i belongs to the cap of chord $2x$ of centre M.

We have $p_i = \frac{V_{(T,x)}}{\omega_n T^n}$. Therefore, the probability that M belongs to none of the caps of chord $2x$ of centre M_i for all $i = 1, 2, \cdots, N$ is, by the independence of the points, the product of the probabilities that none of the M_i's belongs to the cap of chord $2x$ of centre M, that is the product

$$\left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right)^N$$

This value is independent of the collection of points $\{M_i\}$. We deduce the mean $E\theta(N, T, x)$ by complementarity. □

2. *Proof of the theorem 1.2.—*

Proposition 2.7. — *Let* $0 < x < 1/2$. *With* $\eta(x) = \frac{1}{2}\sqrt{1 - \left(\frac{x}{T}\right)^2} - \frac{x}{2}\sqrt{4 - \frac{1}{T^2}}$, *if*

$$N \geqslant \frac{\omega_n T^n}{V_{(T,x)}} \ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right) \tag{10}$$

then there exists a collection of N *distinct caps of centres* $M_1, M_2, \cdots, M_N$ *of chord* 1 *of* $S(0, T) \subset \mathbb{R}^n$ *satisfying*

$$\ln\left(\frac{1}{1 - \theta_{(T,x)}(M_1, M_2, \cdots, M_N)}\right) > N\frac{V_{(T,x)}}{\omega_n T^n} \tag{11}$$

which covers $S(0, T)$.

Proof. — Given $x \in (0, 1/2]$ there exists at least one collection of caps $\{C_{T,\sqrt{T^2-x^2},u_i} \mid i = 1, 2, \cdots, N\}$ of centres $M_1, M_2, \cdots, M_N$, where the unit vectors $u_i := OM_i/\|OM_i\|$ are all distinct, such that the relation (11) is true since, after lemma 2.6, the mean $E\theta(N, T, x)$ is equal to $1 - \left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right)^N$ and that

$$\ln\left(\frac{1}{1 - E\theta(N, T, x)}\right) = -N\ln\left(1 - \frac{V_{(T,x)}}{\omega_n T^n}\right) > N\frac{V_{(T,x)}}{\omega_n T^n}. \tag{12}$$

Let us note that the points $M_1, M_2, \cdots, M_n$ depend upon x. Keeping fixed the centres $M_1, M_2, \cdots, M_N$ and putting around them caps of chord 1 instead of $2x$, we obtain a new collection of caps. Let us show that this new collection of caps of chords 1 of $S(0, T)$ forms a covering. We will assume that it does not and will show the contradiction.

Then there exists a point $M \in S(0, T)$ such that

$$M \notin \bigcup_{i=1}^{N} C_{T,\sqrt{T^2-1/4},u_i}$$

Let us write $u := OM/\|OM\|$ for the unit vector on the line OM. At worse, M lies close to the boundary of the domain $\bigcup_{i=1}^{N} C_{T,\sqrt{T^2-1/4},u_i}$, hence close to the boundary of one of the caps $C_{T,\sqrt{T^2-1/4},u_i}$ of chord 1. We can now apply lemma 2.4 as if M were on this boundary: $\eta = \eta(x)$ is strictly positive since $x < 1/2$ by the eq.(9). Therefore the cap $C_{T,\sqrt{T^2-\eta(x)^2},u}$ is not trivial and is disjoint from the union

$$\bigcup_{i=1}^{N} C_{T,\sqrt{T^2-x^2},u_i}$$

This means that

$$1 - \theta_{(T,x)}(M_1, M_2, \cdots, M_N) \;>\; \theta_{(T,\eta(x))}(M) > 0$$

Therefore

$$\ln\left(\frac{1}{1 - \theta_{(T,x)}(M_1, M_2, \cdots, M_N)}\right) \;<\; \ln\left(\frac{1}{\theta_{(T,\eta(x))}(M)}\right)$$

From the eq.(12) we deduce the relation

$$N\,\frac{V_{(T,x)}}{\omega_n T^n} \;<\; \ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right)$$

Hence the contradiction. □

By lemma 2.1 and the eq.(4), (5), (7), (8), we deduce

$$\frac{\omega_n T^n}{V_{(T,x)}}\,\ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right) = \frac{\omega_n}{\omega_{n-1}}\frac{n\,T^n}{x^n}\frac{\omega_{n-1}}{n\,V_{(T/x,1)}}\,\ln\left(\frac{\omega_n}{\omega_{n-1}}\frac{n\,T^n}{(\eta(x))^n}\frac{\omega_{n-1}}{n\,V_{(T/(\eta(x)),1)}}\right)$$

$$\leqslant \sqrt{\frac{\pi}{2}}\frac{\sqrt{n}\,(2T)^n\;(1-4\eta(x))^{-n/2}}{T\left(1-\frac{2}{\sqrt{\pi n}}\right)}\left[-(n-1)\ln(\eta(x)) + (n-1)\ln T + \ln\left(\frac{\sqrt{2}\pi\,n}{\sqrt{\pi n}-2}\right)\right] \tag{13}$$

In the proposition 2.7, we can take any x, hence any η, in the open interval $(0, 1/2)$ such that the condition (11) is satisfied. We will chose η and $x = x(\eta)$ as functions of n only with η tending monotonically to zero when n goes to infinity, hence x tending to $1/2$. This will give a minimal integer

$$\lfloor\frac{\omega_n T^n}{V_{(T,x)}}\,\ln\left(\frac{\omega_n T^n}{V_{(T,\eta(x))}}\right)\rfloor + 1$$

for obtaining the covering property of $S(0,T)$ as a function of n and T only.

The problem consists now in finding, in the set of strictly positive monotone decreasing functions $f(x)$ defined on $(1/4,+\infty)$ such that $\lim_{x\to+\infty} f(x) = 0$, one function for which $-(1-4\,f(x))^{-x/2}\ln(f(x))$ goes the slowest to $+\infty$ when x tends to $+\infty$. We will not solve this problem here. We will simply take $f(x) = 1/(2xu(x))$ with $u(x)$ an increasing monotone continuous function such that $\lim_{x\to+\infty} u(x) = +\infty$, in particular $u(x) = \ln x$. By reporting this function in the eq.(13) we take $\eta = 1/(2n\ln n)$, $n \geqslant 3$. This gives an expression of x as a function of n from the eq.(9). This function represents a fairly good compromise.

The second member of the inequality (10) appears as a configurational entropy which has to be exceeded for the existence of (at least one) a certain configuration of equal caps of chord 1 for covering $S(0,T)$. But the condition (11) is non-constructive.

We will now explicit the second member of the inequality (13) with $\eta = 1/(2n\ln n)$. Thus, for all $n \geqslant 2$, since $(1-2/(n\ln n))^{-n/2} < (1-2/(\ln n))^{-1}$, we obtain

$$\sqrt{\frac{\pi}{2}}\,\frac{\sqrt{n}}{T}\left(\frac{(2T)^n}{(1-\frac{2}{\ln n})(1-\frac{2}{\sqrt{\pi\,n}})}\right)\left[(n-1)\ln(2T\,n\ln n) + \frac{1}{2}\ln n + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n}-2}\right)\right] \tag{14}$$

for $1/2 < T \leqslant 1$.

By lemma 2.5, if $1/2 < T \leqslant 1$, then, in order to cover the ball $B(0,T)$ by balls of radius $1/2$, it suffices to put a ball of radius $1/2$ centred at the origin (not necessary if $1/2 < T \leqslant \sqrt{2}/2$) and to put a collection of N balls (with N chosen minimal) given by the proposition 2.7 around such that their intersections with $S(0,T)$ are caps of chord 1 which cover $S(0,T)$. This total number of balls, $N+1$, is certainly exceeded by (14). This proves the assertion (i) in the theorem 1.2.

Let us prove the assertion (ii) in the theorem 1.2. If $T > 1$, we proceed inductively using the lemma 2.5. We will cover $B(0,T)$ as follows. We put a ball of radius $1/2$ centred at the origin. Then we put balls of radius $1/2$ in such a way that their intersections with the spheres $S(0,T_m)$ are caps of chord 1 which cover $S(0,T_m)$, where the decreasing sequence $\{T_m\}$ is defined by $T_0 = T, T_1 = T_0 - \frac{1}{2T_0}, \cdots, T_m = T_{m-1} - \frac{1}{2T_{m-1}}, \cdots$ with $m \in \{0,1,\cdots,m_0\}$ and m_0 defined by the condition that $T_{m_0} \leqslant 1$ and $T_{m_0-1} > 1$. Since, for all integer $m \in \{0,1,\cdots,m_0\}$, we have

$$T - \frac{m}{2T} \geqslant T_m$$

the total number of balls of radius $1/2$ disposed in such a configuration required for covering $B(0,T)$ is certainly less than

$$\sum_{m=0}^{m_0} \left(2\left(T - \frac{m}{2T}\right)\right)^n \frac{\sqrt{\pi n}\left(1 - \frac{2}{\ln n}\right)^{-1}}{T\sqrt{2}\left(1 - \frac{2}{\sqrt{\pi n}}\right)} \left[(n-1)\ln(2(T - \frac{m}{2T})n\ln n) + \frac{\ln n}{2} + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n} - 2}\right)\right]$$

$$\leqslant \frac{\sqrt{\pi n}\left(1 - \frac{2}{\ln n}\right)^{-1}}{T\sqrt{2}\left(1 - \frac{2}{\sqrt{\pi n}}\right)} \left[(n-1)\ln(2Tn\ln n) + \frac{\ln n}{2} + \ln\left(\frac{\pi\sqrt{2n}}{\sqrt{\pi n} - 2}\right)\right] \sum_{m=0}^{m_0} (2(T - \frac{m}{2T}))^n$$

But

$$\sum_{m=0}^{m_0} (2(T - \frac{m}{2T}))^n \leqslant (2T)^n \sum_{m=0}^{m_0} e^{-\frac{nm}{T^2}} \leqslant (2T)^n \sum_{m=0}^{+\infty} e^{-\frac{nm}{T^2}} = \frac{(2T)^n}{1 - e^{-n/T^2}}$$

Since $T < n/(2\ln n)$, we have

$$\frac{e^{n/T^2}}{e^{n/T^2} - 1} < \frac{e^{4(\ln n)^2/n}}{e^{4(\ln n)^2/n} - 1} < \frac{n}{4(\ln n)^2} e^{4(\ln n)^2/n}$$

The function $t \to (\ln t)^2/t$ reaches its maximum on $[2,+\infty)$ at $t = e^2$. Hence, for all integer $n \geqslant 2$, we have $(\ln n)^2/n \leqslant (\ln 7)^2/7$. We deduce that

$$\sum_{m=0}^{m_0} (2(T - \frac{m}{2T}))^n \leqslant \frac{e^{4(\ln 7)^2/7}}{4} \frac{n\,(2T)^n}{(\ln n)^2}$$

with a constant $e^{4(\ln 7)^2/7}/4 = 2.176...$ This gives the assertion (ii).

As for the strict lower bound n in the eq.(3) and (4), it obviously comes from the dimension of the ambiant space: n balls being placed along the n coordinates axis of any basis of $\mathbb{R}^n$ never cover $B(0,T)$ when $T > 1/2$.

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