

3-FOLD SYMMETRIC PRODUCTS OF CURVES AS HYPERBOLIC HYPERSURFACES IN \mathbb{P}^4

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ABSTRACT. We construct new examples of Kobayashi hyperbolic hypersurfaces in \mathbb{P}^4 . They are generic projections of the triple symmetric product $V = C(3)$ of a generic genus $g \geq 6$ curve C , smoothly embedded in \mathbb{P}^7 .

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INTRODUCTION

Given a generic smooth projective curve C of genus g over \mathbb{C} , we consider the threefold symmetric product $T := C(3)$ of C . It is known that T is Kobayashi hyperbolic if and only if $g \geq 5$ [ShZa1, Cor. 3.3]. Recall that by Brody's Theorem [Br], a projective algebraic variety X is Kobayashi hyperbolic (or simply *hyperbolic*) if there is no non-constant holomorphic map $f : \mathbb{C} \rightarrow X$.

We suppose that T is embedded in \mathbb{P}^m for some m . We notice that, since the irregularity of T is $q(T) := h^1(T, \mathcal{O}_T) = g$, a well known theorem of Barth-Larsen [BaLa, Thm. 1] ensures that $m \geq 6$ as far as $g \geq 1$. Actually, if one believes to Hartshorne conjecture (see [LaVe]), then $m \geq 7$.

We consider the projection $\pi : \mathbb{P}^m \dashrightarrow \mathbb{P}^4$ from a general subspace $\Pi \subseteq \mathbb{P}^m$ of codimension 5, and we let T' be the image of T under π , so that T' is a hypersurface in \mathbb{P}^4 birational to T . Our main result in this paper is the following theorem:

Theorem 0.1. *If a genus g curve C with $g \geq 7$ is neither hyperelliptic nor trigonal and the jacobian $J(C)$ is simple, then a generic projection T' to \mathbb{P}^4 of the threefold symmetric product $T = C(3)$ of C , arbitrarily embedded into \mathbb{P}^m as a threefold of degree d , is a hyperbolic hypersurface in \mathbb{P}^4 of degree d . The same conclusion holds for a certain special embedding $T = C(3) \hookrightarrow \mathbb{P}^9$ in the case where C is a general plane quintic ($g = 6$, $d = 125$).*

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It is known [Br, Za] that hyperbolicity for projective hypersurfaces is an open property. Thus small deformations of T' are again hyperbolic. Consequently, the above theorem enables us to conclude that a very general hypersurface of degree $d = \deg T$ in \mathbb{P}^4 is *algebraically hyperbolic*, that is it does not contain neither rational and elliptic curves, nor abelian surfaces (see e.g., [Cl1, Cl2, Pa1, Pa2, Vo, Ra] and the references therein for related results and optimal degree bounds).

Theorem 0.1 extends to \mathbb{P}^4 a similar construction from [ShZa1] of hyperbolic surfaces in \mathbb{P}^3 birational to symmetric squares of curves. Concerning other explicit constructions of Kobayashi hyperbolic projective hypersurfaces, see e.g., [ShZa2, ShZa3] and the literature therein.

The proof is divided as follows. In §1 we reduce the proof of hyperbolicity of T' to that of algebraic hyperbolicity of the double surface S' of T' . In §2 we show that the only rational or elliptic curves in S' could be irreducible components of the triple curve Γ' of T' . This is based on a technical result, of independent interest, on deformations of hyperelliptic and bielliptic curves inside abelian varieties which generalizes an earlier one due to Pirola [Pi]. In §3 we exclude the possibility that Γ' contains rational or elliptic components, thus completing the proof that S' is algebraically hyperbolic. We believe that a study of the triple curves here deserves to be done on its own right. Finally, in §4 we discuss the problem of finding, for a given curve, projective embeddings of its symmetric products.

The lowest degree of a hyperbolic threefold T' in \mathbb{P}^4 that we can find with our methods is 125, attained by the threefold symmetric product T of a general plane quintic, naturally embedded in \mathbb{P}^9 . Examples of lower degree can be found in [ShZa2]. Our approach here is mainly geometrical and exploits only minimum of analysis. Essentially, the only analytical fact we need is the Bloch Conjecture (see e.g., §9 of [De]). Notice that the proofs in [ShZa2] depend heavily on the value distribution theory.

1. FIRST PART OF THE PROOF: REDUCTION TO THE HYPERBOLICITY OF THE DOUBLE SURFACE

In this section we go on keeping the conventions and the notation introduced above. Namely C is a general curve of genus $g \geq 5$ i.e., C is neither hyperelliptic nor trigonal and $J(C)$ is simple. We frequently use below the following elementary observation.

Lemma 1.1. *For a curve C as above, the threefold symmetric product $T = C(3)$ does not contain curves of geometric genus $< g$.*

Proof. Since C is neither hyperelliptic nor trigonal the Abel-Jacobi map $\alpha : T = C(3) \rightarrow J(C)$ is injective. Moreover, as $J(C)$ is supposed to be simple the image $\alpha(E) \subseteq J(C)$ of a curve $E \subseteq T$ cannot be of geometric genus $< g$. \square

In particular, C as above does not admit a $2 : 1$ or a $3 : 1$ map onto a curve of smaller genus.

1.2. We suppose that $T = C(3)$ is embedded in \mathbb{P}^m , and we let T' be the general projection of T to \mathbb{P}^4 . We notice that T' is singular. We will not describe in full details the singularities of T' here. However, if one takes into account the description of the singularities of the generic projection to \mathbb{P}^3 of a non-degenerate, smooth surface in \mathbb{P}^m , $m \geq 5$ (see [Mo], p.60), one can see that:

- (i) $\text{Sing}(T')$ is an irreducible surface S' whose general point is a double point of T' with tangent cone formed by two distinct hyperplanes;
- (ii) there is a curve $\Gamma' \subseteq S'$ (possibly reducible or empty) such that a general point of any irreducible component of Γ' is a triple point of T' with tangent cone formed by three independent hyperplanes. A general point of any component of Γ' is a triple point for S' ,

with tangent cone formed by three independent planes meeting along the tangent line to Γ' at that point;

(iii) there is a curve $\Delta' \subseteq S'$ (possibly reducible or empty) such that a general point of any irreducible component of Δ' is a double point of T' with tangent cone formed by a hyperplane counted twice. A general point of any component of Δ' is smooth for S' ;

(iv) worse singularities for T' and S' occur at isolated points.

Remark 1.3. If H is a general hyperplane in \mathbb{P}^m containing the centre of projection Π then the restriction $\pi|_H$ projects generically the hyperplane section $T \cap H$ to \mathbb{P}^3 . If $H' := \pi(H) \subseteq \mathbb{P}^4$ then $\text{Sing}(\pi(T \cap H)) = S' \cap H'$ is the double curve of the surface $\pi(T \cap H) = T' \cap H'$. Furthermore, $\Gamma \cap H'$ consists of the triple points of $T' \cap H'$, whereas $\Delta' \cap H'$ consists of its pinch points. This yields (i) – (iv).

1.4. We denote by S the pull-back of S' via π . Again we will not describe in full details S and the induced map $\pi : S \rightarrow S'$. We notice that $\pi : S \rightarrow S'$ is finite of degree 2, and S , as well as S' , is singular. Indeed, if Γ is the pull-back of Γ' via π , the surface S is singular along Γ having, at a general point of any irreducible component of Γ , a double point with tangent cone formed by two distinct planes meeting along the tangent line to Γ at that point. We consider the following diagram:

$$\begin{array}{ccccccccc} \tilde{S} & \rightarrow & \hat{S} & \rightarrow & S & \hookrightarrow & T & \hookrightarrow & \mathbb{P}^m \\ \downarrow & & \downarrow & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ \tilde{S}' & \rightarrow & \hat{S}' & \rightarrow & S' & \hookrightarrow & T' & \hookrightarrow & \mathbb{P}^4 \end{array}$$

where $\hat{S} \rightarrow S$ [resp. $\hat{S}' \rightarrow S'$] is a normalization morphism, $\tilde{S} \rightarrow \hat{S}$ [resp. $\tilde{S}' \rightarrow \hat{S}'$] is a minimal desingularization, and the vertical arrows $\tilde{S} \rightarrow \tilde{S}'$ and $\hat{S} \rightarrow \hat{S}'$ are induced by π . Notice that $\pi : T \rightarrow T'$ is also a normalization morphism.

The following lemma reduces the proof of Theorem 0.1 to establishing hyperbolicity of S' rather than that of T' .

Lemma 1.5. *In the above setting, T' is hyperbolic if and only if S' is hyperbolic.*

Proof. If T' is hyperbolic, of course so is $S' \subseteq T'$. Conversely, suppose S' is hyperbolic whereas T' is not. Let $f : \mathbb{C} \rightarrow T'$ be a non-constant holomorphic map. Then $f(\mathbb{C})$ is not contained in S' , and therefore f can be lifted to a non-constant holomorphic map $\tilde{f} : \mathbb{C} \rightarrow T$. But since T is hyperbolic (see Corollary 3.3 of [ShZa1]), we arrive at a contradiction. \square

Next we reduce the proof of hyperbolicity of S' to that of *algebraic hyperbolicity* of S' . The latter means that S' does not contain neither rational nor elliptic curves¹ or, equivalently, that every morphism $f : E \rightarrow S'$, where E is an elliptic curve, must be constant. This is based on the following lemma.

Lemma 1.6. *The irregularity of \tilde{S}' is $q(\tilde{S}') \geq g \geq 5$.*

Proof. Let x be a general point of S' , and let $\pi^{-1}(x) = \{x_1, x_2\}$ with $x_1 \neq x_2$. Fix an Abel-Jacobi map $\alpha : T \rightarrow J(C)$, and consider the point $y_x := \alpha(x_1) + \alpha(x_2)$. Thus we have a rational map $S' \ni x \mapsto y_x \in J(C)$, which determines a rational map, whence a morphism, $\sigma : \tilde{S}' \rightarrow J(C)$. Since we are assuming that $J(C)$ is simple then either $q(\tilde{S}') \geq g$ or $q(\tilde{S}') \geq g$ and σ is constant. We denote this constant, that depends on the choice of the centre Π of projection, by c_Π . Thus we have a rational map $\mathbb{G}(n-5, n) \ni \Pi \mapsto c_\Pi \in J(C)$ which in turn is constant i.e., $c := c_\Pi$ does not depend on the center Π of projection, whence we may assume $c = 0$. Let now x_1, x_2 be two general points of T . By choosing Π to be a general \mathbb{P}^{m-5}

¹In principle, we have to add 'and is not dominated by an abelian surface', but, for our purposes, we do not really need this more restrictive definition.

meeting the line $\langle x_1, x_2 \rangle$, we conclude that $\alpha(x_1) + \alpha(x_2) = 0$, which is clearly impossible. Therefore $q(\tilde{S}') \geq g$. \square

Now we perform our first reduction step:

Proposition 1.7. *In the above setting, T' is hyperbolic if and only if S' is algebraically hyperbolic.*

Proof. By Lemma 1.5, T' is hyperbolic if and only if so is S' . If S' is hyperbolic then it is algebraically hyperbolic. Conversely, suppose S' is algebraically hyperbolic and assume it is not hyperbolic. Then there is a non-constant holomorphic map $f : \mathbb{C} \rightarrow S'$. If f were *algebraically degenerate* i.e., if the Zariski closure Z of $f(\mathbb{C})$ were not the whole S' , then we would get a contradiction. Indeed in that case Z would be a curve of genus 0 or 1, contrary to the assumption that S' is algebraically hyperbolic. Thus f is not algebraically degenerate, and therefore we can lift f to a non-constant map $\tilde{f} : \mathbb{C} \rightarrow \tilde{S}'$. Since by Lemma 1.6 $q(\tilde{S}') \geq 5$, by Bloch's Theorem \tilde{f} is algebraically degenerate (see e.g., §9 of [De]). This implies that f itself is algebraically degenerate, a contradiction. \square

2. SECOND PART OF THE PROOF: DEFORMATIONS OF HYPERELLIPTIC AND BIELLIPTIC CURVES IN ABELIAN VARIETIES

We go on with the same hypotheses and notation as above. In this section we make another step towards the proof of 0.1 by proving

Proposition 2.1. *For any rational or elliptic curve E and for any non-constant morphism $f : E \rightarrow S'$, the image $f(E)$ must be contained in the triple curve Γ' of S' .*

Our proof relies on a proposition on deformations of hyperelliptic and bielliptic curves on an abelian variety partially due to Pirola [Pi]. We will actually prove an extension of Pirola's result. As a matter of fact, it would not be difficult to prove an even more general version concerning deformations of a curve inside an abelian variety A which is a double cover of another curve², but we will not dwell on this here. A first version of this result is contained in unpublished notes in collaboration between G. van der Geer and the first author.

2.2. In order to state our result, we need to introduce some notation. We let Y be a projective variety, X be a smooth, irreducible, projective curve of genus $\gamma \geq 2$ and $f : X \rightarrow Y$ be a morphism birational onto its image. We denote by β the homology class of $f(X)$, and we consider the *Kontsevich space* $\mathcal{M}_\gamma(Y, \beta)$ (see §0.4 of [FuPa]), so that the isomorphism class of $f : X \rightarrow Y$ corresponds to a point of $\mathcal{M}_\gamma(Y, \beta)$. One has an obvious forgetful map $\phi : \mathcal{M}_\gamma(Y, \beta) \rightarrow \mathcal{M}_\gamma$ to the moduli space \mathcal{M}_γ of curves of genus γ . Suppose that the isomorphism class of X in \mathcal{M}_γ lies in some irreducible subvariety \mathcal{H} of \mathcal{M}_γ . We denote by $\mathcal{H}(Y, \beta)$ the scheme $\phi^*(\mathcal{H})$; again the class of $f : X \rightarrow Y$ corresponds to a point in $\mathcal{H}(Y, \beta)$. We let $\delta_f(\mathcal{H}(Y, \beta))$ be the dimension of $\mathcal{H}(Y, \beta)$ at the point corresponding to the class of $f : X \rightarrow Y$, that is the dimension of the deformation space of f within $\mathcal{H}(Y, \beta)$.

We let \mathcal{H}_γ be the hyperelliptic locus and \mathcal{E}_γ be the bielliptic locus in \mathcal{M}_γ . For curves in abelian varieties, we have the following rigidity result.

Proposition 2.3. *Let X be a smooth, irreducible, projective curve of genus γ , let A be an abelian variety, and let $f : X \rightarrow A$ be a morphism which is birational onto its image, whose homology class in A is β . One has:*

- (i) *if X is hyperelliptic then $\delta_f(\mathcal{H}_\gamma(A, \beta)) = \dim A$;*
- (ii) *if A contains no elliptic curve and X is bielliptic then $\delta_f(\mathcal{E}_\gamma(A, \beta)) \leq \dim A + 1$ ³.*

²That is, which possesses a non-trivial holomorphic involution.

³This bound is sharp; see Example 1 in [Pi, Sect. 1].

Proof. We notice that A acts in a natural way on $\mathcal{H}_\gamma(A, \beta)$. Indeed $a \in A$ acts on the class of $f : X \rightarrow A$ by sending it to the class of $\tau_a \circ f : X \rightarrow A$, where $\tau_a : A \rightarrow A$ is the translation by a . Since this action is faithful, we see that $\delta_f(\mathcal{H}_\gamma(A, \beta)) \geq \dim A$. In order to prove (i) one has to prove that, up to the above action of A , there is no non-trivial deformation of $f : X \rightarrow A$ to a family of maps of hyperelliptic curves to A .

Let $\mathcal{X} \rightarrow B$ be a deformation of X over a disc B , such that its general fiber is still hyperelliptic, and let $F : \mathcal{X} \rightarrow A$ be a deformation of f . We may assume that $\mathcal{X} \rightarrow B$ has a section which is a Weierstrass point of the g_2^1 on each curve of the family. Using this we can change F by translations in A in such a way that the image of all these Weierstrass points is 0 in A .

For a general point $b \in B$, we let X_b be the corresponding fiber of $\mathcal{X} \rightarrow B$, and we let $x_1 + x_2$ be a divisor of the g_2^1 on X_b . As g_2^1 is parametrized by \mathbb{P}^1 and A does not contain rational curves, the point $F(x_1) + F(x_2) \in A$ does not depend on $x_1 + x_2 \in g_2^1$, whence it must be 0 . Consequently, the Weierstrass points of the g_2^1 are sent by F to points of order 2 of A .

Let N_f be the normal sheaf to the map f , defined as the cokernel of the differential $df : \Theta_X \rightarrow f^*\Theta_A$. Recall (see [ArCo1], p. 344 or [ArCo2], §§5,6) that there is an exact sequence

$$0 \rightarrow \mathcal{T} \rightarrow N_f \rightarrow N'_f \rightarrow 0,$$

where \mathcal{T} is a torsion sheaf supported by the ramification divisor R_f of f (actually, $\mathcal{T} \cong \mathcal{O}_{R_f}$), and N'_f is locally free of rank $\dim A - 1$. Let $s \in H^0(X, N_f)$ be the section corresponding to the deformation F of f [Ho]. This gives us a section $s' \in H^0(X, N'_f)$ which vanishes at the $2\gamma + 2$ Weierstrass points of the g_2^1 on X . Indeed, as we have seen above, their images are fixed under the deformation F of f . Since

$$c_1(N'_f) \leq c_1(N_f) = c_1(\omega_X) = 2\gamma - 2$$

we see that s' is identically zero. Hence s vanishes at the general point of X , and so it must be identically zero too. This proves (i).

The proof of (ii) is quite similar. Indeed, we have to show that, up to the action of A , there is at most a one-parameter family of deformations of $f : X \rightarrow A$ to a family of maps of bielliptic curves to A .

Let $\mathcal{X} \rightarrow B$ be a deformation of X over a disc B such that its general fiber is bielliptic, and let $F : \mathcal{X} \rightarrow A$ be a non-trivial deformation of f . Arguing as in case (i), we may assume that $\mathcal{X} \rightarrow B$ has a section which is a branch point of the bielliptic involution on each curve of the family. Therefore we can change F by composing it with appropriate translations in A in such a way that the image of all these branch points is 0 in A .

For a general point $b \in B$, we let X_b be the corresponding fiber of $\mathcal{X} \rightarrow B$, and we let $x_1 + x_2$ be a divisor of the bielliptic involution on X_b . Since A does not contain elliptic curves, the point $F(x_1) + F(x_2) \in A$ does not depend on $x_1 + x_2$ in the bielliptic involution, whence (by virtue of the above convention) it must be 0 . Henceforth the $2\gamma - 2$ branch points of the bielliptic involution are sent by F to points of order 2 of A .

Let $s \in H^0(X, N_f)$ be a non-zero section corresponding to the deformation F of f [Ho]. Then the associate section $s' \in H^0(X, N'_f)$ vanishes at $2\gamma - 2$ distinct points on X , which, in principle, gives no contradiction. We claim however that any two such non-zero sections s_1 and s_2 define the same line subbundle of N_f . Indeed, N_f is a quotient of the trivial bundle $f^*\Theta_A$, whence is spanned. If $\dim A = d$ then we can find sections $s_3, \dots, s_{d-1} \in H^0(X, N_f)$ such that, at a general point $x \in X$, the vectors $s_3(x), \dots, s_{d-1}(x) \in N_{f,x} \cong \mathbb{C}^{d-1}$ are linearly independent and, moreover, the linear subspaces $\text{span}(s_1(x), s_2(x))$ and $\text{span}(s_3(x), \dots, s_{d-1}(x))$ of $N_{f,x}$ are transversal.

From the exact sequence

$$0 \rightarrow \Theta_X \xrightarrow{df} f^*\Theta_A \rightarrow N_f \rightarrow 0$$

one obtains (see e.g., [Ha], Ch. II, Exercises 5.16(d) and 6.11-6.12):

$$\bigwedge^{d-1} N_f = \det N_f \cong (\det \Theta_X)^{-1} = \omega_X.$$

On the other hand, the holomorphic section $\sigma := s_1 \wedge s_2 \wedge s_3 \wedge \dots \wedge s_{d-1}$ of $\bigwedge^{d-1} N_f \cong \omega_X$ vanishes to order 2 at $2\gamma - 2$ points, whence it must be identically zero. This implies that at a general point $x \in X$, the vectors $s_1(x), s_2(x)$ are linearly dependent, proving our claim.

Now we can conclude that there is no pair s_1, s_2 of linearly independent sections as above, thus proving the assertion. Indeed for a general point $x \in X$ there exists a non-zero linear combination s'' of s_1, s_2 which vanishes at x . As s'' also vanishes at the $2\gamma - 2$ points where s_1 and s_2 vanish, s'' must be identically zero and so, s_1, s_2 must be linearly dependent, as claimed. \square

As a consequence, we have the following

Corollary 2.4. *We let X be a smooth, irreducible, projective curve of genus γ , A be an abelian variety and Y be a closed, irreducible subvariety of A . If $f : X \rightarrow Y$ is a morphism birational onto its image, whose homology class is β , then the following hold.*

(i) *If X is hyperelliptic then $\delta_f(\mathcal{H}_\gamma(Y, \beta)) \leq \dim Y$.*

(ii) *If A contains no elliptic curve and X is bielliptic then $\delta_f(\mathcal{E}_\gamma(Y, \beta)) \leq \dim Y + 1$.*

Proof. Let us prove (i); the proof of (ii) is similar and we leave it to the reader. By Proposition 2.3, for any point $a \in A$ the maximal deformation family of f such that the image of X under the corresponding maps contains a , consists of just a 1-dimensional family of translations of $f : X \rightarrow A$. Therefore for any point $a \in Y$, there is at most a 1-dimensional family of deformations of $f : X \rightarrow Y$ in $\mathcal{H}_\gamma(Y, \beta)$ such that the image of X under the corresponding maps passes through a . This immediately implies the assertion. \square

We are now in a position to prove 2.1.

Proof of Proposition 2.1. First we exclude the existence of a morphism $f : \mathbb{P}^1 \rightarrow S'$ birational onto its image Z' , where $Z' \not\subseteq \Gamma'$. We suppose that such a morphism does exist for a generic projection $\pi : \mathbb{P}^m \dashrightarrow \mathbb{P}^4$, and we consider the curve $Z := \pi^{-1}(Z')$ and its normalization $f : X \rightarrow Z \subseteq T$. As $Z' \not\subseteq \Gamma'$, the smooth curve X admits a 2-to-1 morphism $\pi \circ f$ to Z' , whence it has an induced 2-to-1 morphism to \mathbb{P}^1 . Since by our assumption C is neither hyperelliptic nor trigonal, the 3-fold $T = C(3)$ contains no rational curve. Henceforth the curve X is irreducible and hyperelliptic. We show below that $\delta_f(\mathcal{H}_\gamma(T, \beta)) \geq 5$, which contradicts part (i) of Corollary 2.4. Indeed, since $T = C(3)$ contains no rational curve, any Abel-Jacobi map $\alpha : T = C(3) \rightarrow J(C)$ is injective, and so in Corollary 2.4 we can take $A = J(C)$ and $Y = \alpha(T)$.

We may assume that $T \subseteq \mathbb{P}^7$. We chose a general plane $\Pi \subseteq \mathbb{P}^7$ for the centre of projection to \mathbb{P}^4 . We consider a hyperelliptic curve X and a map $f : X \rightarrow T$ as above. For a general divisor $D = x_1 + x_2$ in the g_2^1 on X , we consider the line $\ell_D = \langle f(x_1), f(x_2) \rangle \subseteq \mathbb{P}^7$. The Zariski closure of the union of all lines ℓ_D with $D \in g_2^1 \simeq \mathbb{P}^1$ is a surface scroll Σ , and the plane Π intersects any fiber of the induced ruling $\Sigma \dashrightarrow \mathbb{P}^1$. Notice that Σ cannot be a plane. Indeed otherwise $Z = f(X) \subseteq \Sigma$ would be a conic, which is impossible, because T does not contain rational curves. Thus the maximal dimension of a family \mathfrak{S} of planes intersecting any fiber of the ruling of Σ is 10 attained in the case where Σ is a cone and the planes in question are passing through its vertex. When Π runs over \mathfrak{S} , and only for those planes Π , Z' and

$f : X \rightarrow Z$ do not vary. Due to our assumption above, this clearly yields

$$\delta_f(\mathcal{H}_\gamma(Y, \beta)) \geq \dim \mathbb{G}(2, 7) - 10 = 5,$$

proving (i).

Repeating word-by-word the above arguments and making use of Corollary 2.4(ii) one can exclude the existence, for a generic projection $\pi : \mathbb{P}^m \dashrightarrow \mathbb{P}^4$, of a morphism $f : E \rightarrow S'$ from an elliptic curve E birational onto its image $Z' \not\subseteq \Gamma'$. This gives (ii). We leave the details to the reader. \square

3. THIRD PART OF THE PROOF: HYPERBOLICITY OF THE TRIPLE CURVE

Again we keep the same conventions as above. That is, we still assume C to be a neither hyperelliptic nor trigonal curve of genus $g \geq 5$ with a simple jacobian $J(C)$.

In this section we conclude the proof of Theorem 0.1. Its first claim follows from Propositions 1.7 and 2.1 by virtue of the following result.

Proposition 3.1. *In the above setting, if in addition $g \geq 7$, then no irreducible component of the triple curve Γ' of T' is rational or elliptic.*

Let us introduce some notation and make several useful comments. For the time being (until Lemma 3.19(ii) below) it will be sufficient to suppose $g \geq 5$.

3.2. Assuming that $T \subseteq \mathbb{P}^7$, we can factor the generic projection $\pi : \mathbb{P}^7 \dashrightarrow \mathbb{P}^4$ into a projection $\bar{\pi} : \mathbb{P}^7 \dashrightarrow \mathbb{P}^6$ from a general point, which we fix once and forever, and a projection $\pi_L : \mathbb{P}^6 \dashrightarrow \mathbb{P}^4$ from a general line $L \subseteq \mathbb{P}^6$, which we let vary. Thus $T' := T'_L$ and $\Gamma' := \Gamma'_L$ depend on L . We observe that, when we project to \mathbb{P}^6 , the image \bar{T} of T acquires at worst finitely many double points. In particular, Lemma 1.1 equally applies to \bar{T} .

3.3. We consider the incidence relation

$$\begin{array}{ccc} \mathcal{J} & \subseteq & T(3) \times \mathbb{G}(1, 6) \\ & p_1 \swarrow & \searrow p_2 \\ & T(3) & \mathbb{G}(1, 6) \end{array}$$

where \mathcal{J} is the Zariski closure of the set of pairs $(x_1 + x_2 + x_3, L)$ such that $L \cap \bar{T} = \emptyset$ and the points $y_i := \bar{\pi}(x_i) \in \bar{T}$ ($i = 1, 2, 3$) are distinct, whereas $\pi_L(y_1) = \pi_L(y_2) = \pi_L(y_3) \in T'$ is a point of Γ'_L .

3.4. For a general point $\xi := x_1 + x_2 + x_3 \in T(3)$, we let $\Lambda_\xi := \langle y_1, y_2, y_3 \rangle$ be the trisecant plane to \bar{T} in \mathbb{P}^6 through the points y_1, y_2, y_3 . From the General Position Theorem [ACGH, p. 109] (see also [ChCo, Cor. 1.3]) it follows that $\Lambda_\xi \cap \bar{T} = \{y_1, y_2, y_3\}$, and so the map $\xi \mapsto \Lambda_\xi$ is generically one-to-one.

Furthermore, $\pi_L(y_1) = \pi_L(y_2) = \pi_L(y_3)$ if and only if $L \subseteq \Lambda_\xi$. Thus the fiber $p_1^{-1}(\xi)$ of the first projection $p_1 : \mathcal{J} \rightarrow T(3)$ can be naturally identified with the dual projective plane $\Lambda_\xi^* \simeq \mathbb{P}^2$. Since $T(3)$ and the general fibers Λ_ξ^* of p_1 are irreducible, there is only one irreducible component \mathcal{J}_0 of \mathcal{J} which dominates $T(3)$ via the first projection. One has $\dim \mathcal{J}_0 = \dim T(3) + \dim \Lambda_\xi^* = 11$. Moreover the following holds.

Lemma 3.5. *The map $p_2|_{\mathcal{J}_0} : \mathcal{J}_0 \rightarrow \mathbb{G}(1, 6)$ is surjective. Therefore also the map $p_2|_{\mathcal{J}} : \mathcal{J} \rightarrow \mathbb{G}(1, 6)$ is surjective, and its fiber over a general point $L \in \mathbb{G}(1, 6)$ is birational to the triple curve Γ'_L . In particular, $\Gamma'_L \neq \emptyset$.*

Proof. To prove the first assertion, we must show that a general line L in \mathbb{P}^6 is contained in a 3-secant plane Λ to \bar{T} . To this point, we consider a general hyperplane H in \mathbb{P}^6 containing L . Clearly, the hyperplane section $H \cap \bar{T}$ is a smooth linearly non-degenerate surface in $H \simeq \mathbb{P}^5$. By a result of Chiantini and Coppens [ChCo, Sect. 2], if L were not contained in a 3-secant plane to $H \cap \bar{T}$ then $H \cap \bar{T}$ would be either a cone or a rational normal surface of degree 4

in $H \simeq \mathbb{P}^5$. Anyhow, it would be covered by a family of rational curves, which is excluded by Lemma 1.1. This proves the first assertion; the second one follows easily. \square

3.6. By Lemma 3.5, for any irreducible component \mathcal{J}' of \mathcal{J} that dominates $\mathbb{G}(1, 6)$ via the second projection p_2 , the fiber $p_2^{-1}(L) \cap \mathcal{J}'$ over a general point $L \in \mathbb{G}(1, 6)$ is a curve birational to the union of some irreducible components of the triple curve Γ'_L . Thus we have $\dim \mathcal{J}' = \dim \mathbb{G}(1, 6) + \dim \Gamma'_L = 11$. Furthermore, \mathcal{J}' being irreducible, the monodromy of the family $p_2|_{\mathcal{J}'} : \mathcal{J}' \rightarrow \mathbb{G}(1, 6)$ acts transitively on the set of irreducible components of the fiber $p_2^{-1}(L) \cap \mathcal{J}'$. Hence all these components have the same geometric genus, which we denote by $\delta(\mathcal{J}')$.

3.7. To prove Proposition 3.1 we must show that $\delta(\mathcal{J}') \geq 2$ for any irreducible component \mathcal{J}' of \mathcal{J} that dominates $\mathbb{G}(1, 6)$ via the second projection. Arguing by contradiction, we suppose in the sequel that $\delta(\mathcal{J}') \leq 1$. We begin by considering a component \mathcal{J}' of \mathcal{J} different from \mathcal{J}_0 , assuming it does exist. Thus \mathcal{J}' does not dominate $T(3)$ via the first projection p_1 (see 3.4).

3.8. Clearly, for a general point $(\xi = x_1 + x_2 + x_3, L)$ of such a component \mathcal{J}' , the points $y_i := \bar{\pi}(x_i) \in \bar{T}$ ($i = 1, 2, 3$) are collinear (cf. 3.4), and so belong to a 3-secant line l_ξ to \bar{T} that meets L . Thus given a general point $\xi \in p_1(\mathcal{J}') \subseteq T(3)$, the fiber $p_1^{-1}(\xi) \subseteq \mathcal{J}'$ over ξ is contained in the set of all pairs (ξ, L) such that the line L meets l_ξ . We denote by $G(l_\xi)$ the set of all lines L in $\mathbb{G}(1, 6)$ with $L \cap l_\xi \neq \emptyset$. We have the following lemma.

Lemma 3.9. *The fiber of $p_1|_{\mathcal{J}'}$ over a general point ξ in $p_1(\mathcal{J}')$ is:*

$$p_1^{-1}(\xi) \cap \mathcal{J}' = \{\xi\} \times G(l_\xi).$$

Thus $\dim p_1(\mathcal{J}') = 11 - \dim G(l_\xi) = 5$.

Proof. A general point $(\xi, L) \in \mathcal{J}'$ does not belong to any other irreducible component of \mathcal{J} , and the point $\pi_L(y_1) = \pi_L(y_2) = \pi_L(y_3) \in \Gamma'_L$ is smooth (see 3.6). For any line $L' \in G(l_\xi)$ we still have $\pi_{L'}(y_1) = \pi_{L'}(y_2) = \pi_{L'}(y_3) \in \Gamma'_{L'}$ and so, clearly, (ξ, L') varies in \mathcal{J}' when L' varies in $G(l_\xi)$. Notice finally that $G(l_\xi)$ is an irreducible variety which is fibered over $l_\xi \simeq \mathbb{P}^1$ with fibers isomorphic to \mathbb{P}^5 . This proves our assertions. \square

3.10. We let again $\xi \in p_1(\mathcal{J}')$ be a general point with $\xi = x_1 + x_2 + x_3$. As we have seen in 3.8, the points $y_i := \bar{\pi}(x_i) \in \bar{T}$ ($i = 1, 2, 3$) belong to a 3-secant line l_ξ to \bar{T} . We let $\mathcal{V}(\mathcal{J}')$ be the closure in $\mathbb{G}(1, 6)$ of the set of all these lines l_ξ as ξ varies in $p_1(\mathcal{J}')$. As \bar{T} does not contain lines (see Lemma 1.1) the map $p_1(\mathcal{J}') \rightarrow \mathcal{V}(\mathcal{J}')$, $\xi \mapsto l_\xi$, is birational and has finite fibers. Thus by Lemma 3.9, $\mathcal{V}(\mathcal{J}')$ is an irreducible variety of dimension 5.

Let us also consider the variety $V(\mathcal{J}') := \cup_{l \in \mathcal{V}(\mathcal{J}')} l$ in \mathbb{P}^6 .

Lemma 3.11. *$V(\mathcal{J}')$ is an irreducible hypersurface in \mathbb{P}^6 .*

Proof. There is a natural incidence relation

$$\begin{array}{ccc} \mathcal{I} = \mathcal{I}(\mathcal{J}') & \subseteq & \mathcal{V}(\mathcal{J}') \times \mathbb{P}^6 \\ & \searrow q_1 & \swarrow q_2 \\ & \mathcal{V}(\mathcal{J}') & \mathbb{P}^6 \end{array}$$

which projects onto $\mathcal{V}(\mathcal{J}')$ with irreducible general fibers l_ξ . Therefore, \mathcal{I} and also $V(\mathcal{J}') = q_2(\mathcal{I})$ are irreducible. Let us show that $V(\mathcal{J}')$ meets a general line L in \mathbb{P}^6 , whence $\dim V(\mathcal{J}') \geq 5$. Indeed, as $L \in p_2(\mathcal{J}') = \mathbb{G}(1, 6)$ (see 3.6-3.7) there exists a point $(\xi = x_1 + x_2 + x_3, L) \in \mathcal{J}'$, and so the secant line $l_\xi \in \mathcal{V}(\mathcal{J}')$ meets L .

To show that $\dim V(\mathcal{J}') \leq 5$ we exploit the theory of foci [ChCi]. We consider the second projection $q_2 : \mathcal{V}(\mathcal{J}') \times \mathbb{P}^6 \rightarrow \mathbb{P}^6$ and the induced homomorphism dq_2 of the tangent bundles. The kernel $T(q_2) := \ker(dq_2)$ is a rank 5 subbundle of $T(\mathcal{V}(\mathcal{J}') \times \mathbb{P}^6)$ whose restriction to every fiber $\{\tau\} \times \mathbb{P}^6$ of q_1 is the normal bundle of this fiber and is clearly trivial. We pick a

general point $\tau := l_\xi \in \mathcal{V}(\mathcal{J}')$, and we let $\mathcal{I}_\tau := q_1^{-1}(\tau) \cap \mathcal{I} \simeq l_\xi \simeq \mathbb{P}^1$ be the fiber of the first projection $q_1|_{\mathcal{I}}$ over τ . Then the restriction of the vector bundle $T(q_2)$ to \mathcal{I}_τ identifies with the trivial bundle $(\mathcal{O}_{\mathbb{P}^1})^5$ (see [ChCi], (1.3)).

The image under q_2 of the fiber \mathcal{I}_τ is the 3-secant line l_ξ of \bar{T} , with the normal bundle $N_{l_\xi/\mathbb{P}^6} = N_{\mathbb{P}^1/\mathbb{P}^6} \simeq (\mathcal{O}_{\mathbb{P}^1}(1))^5$. The homomorphism of normal bundles induced by the projection $q_2 : (\mathcal{J}' \times \mathbb{P}^6, \mathcal{I}_\tau) \rightarrow (\mathbb{P}^6, l_\xi)$ restricts to $T(q_2)|_{\mathcal{I}_\tau}$ giving the so called *characteristic map*

$$\lambda : (\mathcal{O}_{\mathbb{P}^1})^5 \rightarrow (\mathcal{O}_{\mathbb{P}^1}(1))^5$$

of the family of lines $\mathcal{V}(\mathcal{J}')$ at $\tau = l_\xi$. Fixing coordinates we can write λ via a 5×5 matrix Φ_λ of linear binary forms on $l_\xi \simeq \mathbb{P}^1$, called the *focal matrix* of $\mathcal{V}(\mathcal{J}')$ at τ . We let $F_\lambda := \det \Phi_\lambda$; this is a binary form of degree 5.

We assume that $V(\mathcal{J}') = \mathbb{P}^6$. Then $q_2 : \mathcal{I} \rightarrow \mathbb{P}^6$ is surjective and generically finite. Thus the focal matrix Φ_λ is non-degenerate at a general point of $l_\xi \simeq \mathbb{P}^1$, whence F_λ is not identically zero.

The equation $F_\lambda = 0$ defines the so called *focal points* of the family $\mathcal{V}(\mathcal{J}')$ on l_ξ ([ChCi], (1.5)). This family can also have *fundamental points* and *cuspidal points*. Cuspidal points sitting on the image of a general fiber \mathcal{I}_t correspond to its singular points ([ChCi], (1.6)). Since in our setting $\mathcal{I}_t \simeq l_\xi$ is smooth, there is no such point on l_ξ . Hence by Proposition 1.7 in [ChCi], the focal points of the family $\mathcal{V}(\mathcal{J}')$ on l_ξ coincide with its fundamental points, that is with the fixed points of 1-parameter families of deformations of the line l_ξ within our family $\mathcal{V}(\mathcal{J}')$. These are exactly the points where the rank of the focal matrix Φ_λ drops.

If $\xi = x_1 + x_2 + x_3$ then the subfamily of $\mathcal{V}(\mathcal{J}')$ of lines through the point $y_i := \bar{\pi}(x_i) \in l_\xi$ ($i = 1, 2, 3$) is at least 2-dimensional. Indeed, letting $\nu : T^3 \rightarrow T(3)$ be the natural map, we consider the preimage $P_1 := \nu^{-1}(p_1(\mathcal{J}')) \subseteq T^3$. By Lemma 3.9 it has dimension 5, whence the fiber of the first projection $\text{pr}_1 : P_1 \rightarrow T$ over the point x_i is at least 2-dimensional.

It follows that $y_i \in l_\xi$ ($i = 1, 2, 3$) is a fundamental point for the family $\mathcal{V}(\mathcal{J}')$, where the rank of Φ_λ drops at least by 2 (cf. the proof of Proposition 1.7 in [ChCi]). We observe that the matrix Φ_λ is left-right equivalent to a non-degenerate 5×5 diagonal matrix of binary forms $\text{diag}(d_1(u : v), \dots, d_5(u : v))$, where $d_i|d_{i+1}$ and d_{i+1}/d_i are the *invariant polynomials* of Φ_λ (see e.g., [Ga], Thm. 3 in §VI.3 or [Co], Thm. 3 in Appendix to Ch. 6). If y is a point of $l_\xi \simeq \mathbb{P}^1$ where the rank of Φ_λ drops at least by 2 then at least 2 of the d_i vanish at y , whence $F_\lambda = \prod_{i=1}^5 d_i$ has a multiple root at y . In particular, this is so for $y = y_i$ ($i = 1, 2, 3$). Hence, as $\deg F_\lambda = 5$, this polynomial must be identically zero, a contradiction. This proves the lemma. \square

3.12. From the previous analysis we deduce the following. We let L be a general line in \mathbb{P}^6 through a general point $x \in V(\mathcal{J}')$. By Lemma 3.11 the general fibers of the surjection $q_1 : \mathcal{I} \rightarrow V(\mathcal{J}')$ are 1-dimensional. Hence there is a 1-dimensional family of lines of $\mathcal{V}(\mathcal{J}')$ through x . This family is parametrized by some of the irreducible components of the fiber $p_2^{-1}(L) \cap \mathcal{J}'$ over L , that is by some components of the triple curve Γ'_L (see 3.6). Thus our family of all 3-secant lines to \bar{T} passing through x describes several cones with vertex x , say, $\Lambda_{x,i}$ ($1 \leq i \leq \nu$), over the corresponding irreducible components of Γ'_L . These are rational or elliptic as we suppose $\delta(\mathcal{J}') \leq 1$ (see 3.7).

3.13. We fix in the sequel an Abel-Jacobi map $\alpha : T = C(3) \rightarrow J(C)$, and we let $\alpha_3 : T(3) \rightarrow J(C)$ be the extension of α to the symmetric product $T(3)$ via

$$\xi = x_1 + x_2 + x_3 \mapsto \alpha_3(\xi) := \alpha(x_1) + \alpha(x_2) + \alpha(x_3).$$

Lemma 3.14. *The image $W(\mathcal{J}') := \alpha_3(p_1(\mathcal{J}')) \subseteq J(C)$ has dimension $\mu \leq 1$.*

Proof. We use the following notation. For a general point $t \in W(\mathcal{J}')$ we let $\mathcal{V}_t \subset \mathbb{G}(1, 6)$ be the family of trisecant lines l_ξ to \bar{T} corresponding to the points ξ in the fiber of $\alpha_3 :$

$p_1(\mathcal{J}') \rightarrow W(\mathcal{J}')$ over t . Thus $\dim \mathcal{V}_t = 5 - \mu$. For an irreducible component \mathcal{V}'_t of \mathcal{V}_t , we let $V'_t := \cup_{l \in \mathcal{V}'_t} l$, so that V'_t is swept out by the $(5 - \mu)$ -dimensional family of lines \mathcal{V}'_t . We note that $\mathcal{V}'_t \subseteq \mathcal{V}(\mathcal{J}')$ and $V'_t \subseteq V(\mathcal{J}')$.

If $l \in \mathcal{V}'_t$ is a line through a general point $x \in V'_t$ then l is contained in one of the cones $\Lambda_{x,i}$ ($1 \leq i \leq \nu$) with vertex x (see 3.12), which we denote by $\Lambda_x(l)$. As $\Lambda_x(l)$ is a cone over a rational or elliptic curve (see 3.12), it is swept out by lines l' with $l' \in \mathcal{V}'_t$. There is henceforth a 1-dimensional family of lines of \mathcal{V}'_t through a general point of V'_t , and so $\dim V'_t = \dim \mathcal{V}'_t = 5 - \mu$. Moreover, as V'_t contains the cone $\Lambda_x(l)$ we have $5 - \mu \geq 2$.

If $5 - \mu = 2$ then $V'_t = \Lambda_x(l)$ is a surface in \mathbb{P}^6 with a 2-dimensional family of lines \mathcal{V}'_t , whence is a plane (see Thm. 1(1) in [Ro]). This plane $\Lambda_x(l)$ cuts out on \bar{T} a cubic curve, which contradicts Lemma 1.1. Therefore $\mu \leq 2$.

If $\mu = 2$ then V'_t is a threefold covered by a 3-dimensional family of lines \mathcal{V}'_t . Hence, by a theorem of Severi-Segre (see Thm. 1(2) in [Ro]), V'_t is either a \mathbb{P}^3 , an irreducible quadric, or a scroll in planes⁴.

If V'_t is a \mathbb{P}^3 described by a 3-dimensional family of trisecant lines to \bar{T} then V'_t cuts \bar{T} along a cubic surface. Thus \bar{T} must contain a rational curve, which contradicts Lemma 1.1.

If V'_t is a quadric then the tangent space to V'_t at x cuts out on V'_t a quadric cone $\Lambda_x(l) \subseteq V'_t$. We consider the curve $\bar{\Gamma} := \Lambda_x(l) \cap \bar{T}$. As $\bar{\Gamma}$ is met by the lines of $\Lambda_x(l)$ in three points, $\bar{\Gamma}$ is either a plane cubic or a space curve of degree 6. Anyhow, the geometric genus of $\bar{\Gamma}$ is at most 4, which again contradicts Lemma 1.1.

Similarly, if V_t is a scroll in planes then $\Lambda_x(l)$ is a plane which cuts out on \bar{T} a cubic curve. This yields to a contradiction as above.

Thus $\mu \leq 1$, as required. \square

3.15. We turn further to considering the component $\mathcal{J}' = \mathcal{J}_0$ of \mathcal{J} (see 3.4 and 3.7). In Lemma 3.18 below we establish an analog of Lemma 3.14 for \mathcal{J}_0 . We begin again by some preliminary observations.

For a general point $\xi = x_1 + x_2 + x_3 \in T(3)$, the dual $\Lambda_\xi^* \simeq \mathbb{P}^2$ of the 3-secant plane $\Lambda_\xi := \langle y_1, y_2, y_3 \rangle \simeq \mathbb{P}^2$ to \bar{T} is naturally embedded in the grassmanian $\mathbb{G}(1, 6)$ (cf. 3.4). We let

$$\mathcal{J}_\xi := p_2^{-1}(\Lambda_\xi^*) \cap \mathcal{J}_0 \quad \text{and} \quad \Omega_\xi := p_1(\mathcal{J}_\xi) \subset T(3).$$

If $(\zeta, L) \in \mathcal{J}_0$ is a general point with $\zeta = z_1 + z_2 + z_3 \in T(3)$ then $(\zeta, L) \in \mathcal{J}_\xi$ if and only if $L = \Lambda_\xi \cap \Lambda_\zeta$, that is iff $\pi_L \circ \bar{\pi} : T \rightarrow T' = T'_L$ sends both x_i and z_i ($i = 1, 2, 3$) to the triple curve Γ'_L (see 3.2). Thus ξ and ζ determine the line $L = L_{\xi, \zeta}$ in a unique way. Therefore the morphism $p_1 : \mathcal{J}_\xi \rightarrow \Omega_\xi$ is birational, and so $\dim \Omega_\xi = \dim \mathcal{J}_\xi = 3$. Furthermore, a general point $\zeta \in T(3)$ belongs to Ω_ξ if and only if the planes Λ_ξ and Λ_ζ meet along a line, that is along $L = L_{\xi, \zeta}$.

Lemma 3.16. (i) For any pair of general points $\zeta, \zeta' \in \Omega_\xi$, the planes Λ_ζ and $\Lambda_{\zeta'}$ meet at a point⁵.

(ii) For any 3 general points $\zeta_1, \zeta_2, \zeta_3 \in \Omega_\xi$, ξ is an isolated point of the intersection $\bigcap_{i=1}^3 \Omega_{\zeta_i}$.

Proof. (i) Assuming on the contrary that the planes Λ_ζ and $\Lambda_{\zeta'}$ meet along a line $L_{\zeta, \zeta'}$, we let $\Pi_{\xi, \zeta} \simeq \mathbb{P}^3$ be the 3-subspace in \mathbb{P}^6 spanned by the planes Λ_ξ and Λ_ζ . Then, clearly, for a general point $\zeta' \in \Omega_\xi$ the lines $L_{\xi, \zeta'}$ and $L_{\zeta, \zeta'}$ are distinct and are contained in $\Pi_{\xi, \zeta}$, whence also $\Lambda_{\zeta'} \subseteq \Pi_{\xi, \zeta}$. This yields a rational map

$$\Omega_\xi \ni \zeta' \longmapsto \Lambda_{\zeta'} \in \Pi_{\xi, \zeta}^*$$

⁴that is, there is a unique \mathbb{P}^2 through a general point of V'_t .

⁵which is the intersection point of the lines $L_{\xi, \zeta} := \Lambda_\zeta \cap \Lambda_\xi$ and $L_{\xi, \zeta'} := \Lambda_{\zeta'} \cap \Lambda_\xi$.

which is generically one-to-one (see 3.4). Thus a general $\Lambda \in \Pi_{\xi, \zeta}^*$ is a trisecant plane of \bar{T} . It follows that $\Pi_{\xi, \zeta}$ meets \bar{T} along a cubic curve. This contradicts Lemma 1.1. Now (i) is proven.

(ii) The planes Λ_{ζ_i} ($i = 1, 2, 3$) meet Λ_ξ along the 3 lines L_{ξ, ζ_i} in general position. We let

$$p_{i,j} := L_{\xi, \zeta_i} \cap L_{\xi, \zeta_j} = \Lambda_{\zeta_i} \cap \Lambda_{\zeta_j} \subseteq \Lambda_\xi \quad (1 \leq i < j \leq 3).$$

Let us pick a point $\zeta \in \bigcap_{i=1}^3 \Omega_{\zeta_i}$ close enough to ξ . Then Λ_ζ intersects every plane Λ_{ζ_i} along a line $L_{\zeta, \zeta_i} \subseteq \Lambda_\zeta$ (see 3.15), and by (i) we have

$$p_{i,j} \in L_{\zeta, \zeta_i} \cap L_{\zeta, \zeta_j} \subseteq \Lambda_\zeta \cap \Lambda_{\zeta_i} \cap \Lambda_{\zeta_j} \quad (1 \leq i < j \leq 3).$$

Hence Λ_ζ is the plane through the 3 non-collinear points $p_{1,2}, p_{1,3}, p_{2,3}$. This holds also for Λ_ξ , therefore $\Lambda_\zeta = \Lambda_\xi$. Thus $\Lambda_\zeta \cap \bar{T} = \Lambda_\xi \cap \bar{T}$, and so $\zeta = \xi$ (see 3.4), as required. \square

3.17. Recall (see 3.6) that, for a general line $L \in \mathbb{G}(1, 6)$, all irreducible components of the fiber $p_2^{-1}(L) \cap \mathcal{J}_0$ have the same geometric genus $\delta(\mathcal{J}_0)$. To prove Proposition 3.1 we must show that $\delta(\mathcal{J}_0) \geq 2$. Arguing by contradiction, we assume as in 3.7 that $\delta(\mathcal{J}_0) \leq 1$.

For a general point $\xi \in T(3)$, we denote by $\Omega_\xi(\xi)$ the irreducible component of Ω_ξ which contains ξ . Thus $\Omega_\xi(\xi)$ is birational to the irreducible component $\mathcal{J}_\xi(\xi)$ of \mathcal{J}_ξ which contains the fiber $p_1^{-1}(\xi) \cap \mathcal{J}_\xi = \{\xi\} \times \Lambda_\xi^*$ (cf. 3.4 and 3.15). Clearly, $\mathcal{J}_\xi(\xi)$ is saturated by those irreducible components of the p_2 -fibers over the points $L \in \Lambda_\xi^*$ which meet $p_1^{-1}(\xi)$. As ξ is general, there is just one such component $\mathcal{J}_\xi(\xi)$ of \mathcal{J}_ξ , and it has dimension 3 (see 3.15). The following analog of Lemma 3.14 holds.

Lemma 3.18. *If $\delta(\mathcal{J}_0) \leq 1$ then, for a general point $\xi \in T(3)$, the Abel-Jacobi map $\alpha_3 : T(3) \rightarrow J(C)$ is constant on $\Omega_\xi(\xi)$.*

Proof. Letting $\xi = x_1 + x_2 + x_3$ we pick a general point $\zeta = z_1 + z_2 + z_3$ in $\Omega_\xi(\xi)$. We claim that $\alpha(z_1) + \alpha(z_2) + \alpha(z_3) = \alpha(x_1) + \alpha(x_2) + \alpha(x_3)$. Indeed, we let $L \subset \Lambda_\xi$ be a general line and $\Gamma_{(\xi, L)}$ be the irreducible component of the fiber $p_2^{-1}(L) \cap \mathcal{J}_0$ containing (ξ, L) . It is birational to an irreducible component of the triple curve Γ'_L , which has geometric genus $\delta(\mathcal{J}_0) \leq 1$ (see 3.17). As we suppose that the jacobian $J(C)$ is simple, the map α_3 must be constant along the curve $p_1(\Gamma_{(\xi, L)})$ through ξ . Henceforth, for any point $\zeta = z_1 + z_2 + z_3 \in p_1(\Gamma_{(\xi, L)})$, we have $\alpha(z_1) + \alpha(z_2) + \alpha(z_3) = \alpha(x_1) + \alpha(x_2) + \alpha(x_3)$. On the other hand, if $L \subseteq \Lambda_\xi$ is a general line and $(\zeta, L) \in \Gamma_{(\xi, L)}$ is a general point, then ζ is a general point in $\Omega_\xi(\xi)$, which proves our claim. \square

We need in the sequel the following

Lemma 3.19. (i) *For any linear series g_9^r on a curve C of genus $g \geq 5$ one has $r \leq 4$.*

(ii) *Furthermore, if C is neither hyperelliptic nor trigonal and $g \geq 7$ then $r \leq 3$.*

Proof. (i) By the Riemann-Roch Theorem and the Serre duality, for a divisor $D \in g_9^r$ we have:

$$r = h^0(D) - 1 = 9 - g + h^0(K_C - D).$$

Thus if D is non-special then $r = 9 - g \leq 4$ as soon as $g \geq 5$. If D is special then by Clifford's Theorem, $r \leq d/2 = 4.5$, so again $r \leq 4$. This proves (i).

(ii) Suppose now that C has a g_9^4 and $g \geq 6$, so that the series is special. If the g_9^4 has a base point then C is hyperelliptic by Clifford's theorem, which is excluded by our assumptions. Assume the series is base point free. Then it defines a birational morphism $C \rightarrow C'$ onto a non-degenerate curve C' of degree 9 in \mathbb{P}^4 . The Castelnuovo genus bound [ACGH, Ch. III, p. 116] tells us that $g \leq 7$. But if $g = 7$ then the residual series $|K - D|$ of the g_9^4 with respect to the canonical series $K := K_C$ is a g_3^1 , which is also excluded. \square

We are ready now to prove 3.1.

Proof of Proposition 3.1. Assume that $\delta(\mathcal{J}') \leq 1$ for an irreducible component \mathcal{J}' of \mathcal{J} which dominates $\mathbb{G}(1, 6)$ via the projection p_2 . There is a natural finite surjective map $\nu_3 : T(3) \rightarrow C(9)$. By abuse of notation, we still denote by $\alpha : C(9) \rightarrow J(C)$ an Abel-Jacobi map for $C(9)$ chosen in such a way that for $\xi = x_1 + x_2 + x_3 \in T(3)$ one has $\alpha(\nu_3(\xi)) = \alpha_3(\xi)$ (see 3.13).

We consider first the case $\mathcal{J}' = \mathcal{J}_0$. By virtue of Lemma 3.18 and of Abel's Theorem we conclude that for general points $\xi \in T(3)$ and $\zeta \in \Omega_\xi(\xi)$, all divisors $D_\zeta := \nu_3(\zeta) \in C(9)$ are contained in the same complete linear system g_9^r with $r \geq 3$, which we denote by \mathcal{D}_ξ (so $\mathcal{D}_\xi \simeq \mathbb{P}^r$). And also $D_{\zeta'} \in \mathcal{D}_\xi$ for a general point $\zeta' \in \Omega_\zeta(\zeta)$.

We claim that, actually, one must have $r \geq 5$, which contradicts Lemma 3.19(i).

We let $\mathcal{D}'_\xi := \nu_3^{-1}(\mathcal{D}_\xi) \subseteq T(3)$. As we have seen above, for a general point $\zeta \in \Omega_\xi(\xi)$ one has $\Omega_\zeta(\zeta) \subseteq \mathcal{D}'_\xi$. We observe that the map $\nu_3 : T(3) \rightarrow C(9)$ is unramified outside the diagonal. Since $\xi \in T(3)$ is a general point, there is no ramification at ξ , and therefore ξ is a smooth point of \mathcal{D}'_ξ . By Lemma 3.16(ii), for any 3 general points $\zeta_i \in \Omega_\xi(\xi)$ ($i = 1, 2, 3$), the 3-folds $\Omega_{\zeta_i}(\zeta_i) \subseteq \mathcal{D}'_\xi$ meet at ξ , and ξ is an isolated point of the intersection $\bigcap_{i=1}^3 \Omega_{\zeta_i}(\zeta_i)$. Clearly, this would be impossible if $r = \dim \mathcal{D}_\xi = \dim \mathcal{D}'_\xi$ were at most 4. This proves the assertion in the case $\mathcal{J}' = \mathcal{J}_0$.

We turn next to the case $\mathcal{J}' \neq \mathcal{J}_0$. By Lemma 3.14, α_3 sends the 5-dimensional subvariety $p_1(\mathcal{J}')$ of $T(3)$ either to a point or to a curve. In virtue of Lemma 3.19(i) the former is impossible, whereas in the latter case C admits a linear system g_9^4 . Since we are assuming that $g \geq 7$ and C is neither hyperelliptic nor trigonal, this contradicts Lemma 3.19(ii). Now the proof is completed. \square

3.20. This concludes the proof of the first part of Theorem 0.1. Let us point out that the only place in the proof where one really needs the hypothesis $g \geq 7$ (rather than $g \geq 5$) is just at the end of the argument, in excluding the existence of a component $\mathcal{J}' \neq \mathcal{J}_0$ with $\delta(\mathcal{J}') \leq 1$.

We use this observation in the proof of Proposition 3.22 below, which provides the second part of Theorem 0.1.

3.21. For a smooth plane curve C of degree d , we let $\mathcal{L} = \mathcal{O}_C(1)$. In 4.2 below we associate to \mathcal{L} a line bundle $\mathcal{L}(3)^s$ on $C(3)$. According to Lemma 4.3(ii) and Proposition 4.4, $\mathcal{L}(3)^s$ embeds $C(3)$ into \mathbb{P}^9 as a smooth threefold T of degree d^3 . This map $\phi_{\mathcal{L}(3)^s} : C(3) \rightarrow T \subseteq \mathbb{P}^9$ can be described more explicitly as follows.

We let $\gamma : \mathbb{P}^2 \rightarrow (\mathbb{P}^2)^*$ be a natural isomorphism and, for a point $x \in \mathbb{P}^2$, we let $\gamma_x = \gamma(x)$. We identify \mathbb{P}^9 with the linear system $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)))$ of cubic curves in \mathbb{P}^2 , and we let $\phi : \mathbb{P}^2(3) \rightarrow \mathbb{P}^9$,

$$x_1 + x_2 + x_3 \mapsto \phi(x_1 + x_2 + x_3) := \gamma_{x_1} \cdot \gamma_{x_2} \cdot \gamma_{x_3}.$$

Since the natural map $\text{Sym}^3 H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \rightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ is an isomorphism, the restriction of ϕ to $C(3) \subseteq \mathbb{P}^2(3)$ coincides with $\phi_{\mathcal{L}(3)^s}$, whereas its restriction to the small diagonal $\Delta = \mathbb{P}^2 \subseteq \mathbb{P}^2(3)$ yields the triple Veronese embedding of \mathbb{P}^2 in \mathbb{P}^9 .

If $d \geq 6$ then all hypotheses of the first part of Theorem 0.1 are met, and therefore the generic projection T' of T in \mathbb{P}^4 is hyperbolic. The second part of Theorem 0.1 deals with the case $d = 5$, and is provided by the following

Proposition 3.22. *If C is a general plane quintic and $\mathcal{L} = \mathcal{O}_C(1)$ then the line bundle $\mathcal{L}(3)^s$ embeds $C(3)$ in \mathbb{P}^9 as a smooth threefold T of degree 125 whose generic projection T' to \mathbb{P}^4 is hyperbolic.*

The proof is based on the following two lemmas.

Lemma 3.23. *We let C be a smooth non-hyperelliptic curve of genus 6. If there exists a component $\mathcal{J}' \neq \mathcal{J}_0$ of \mathcal{J} with $p_2(\mathcal{J}') = \mathbb{G}(1, 6)$ and $\delta(\mathcal{J}') \leq 1$ then for any linear system g_9^4*

on C , any divisor $D \in g_9^4$ and any decomposition $D = x_1 + x_2 + x_3$ with $x_i \in C(3)$, the points $y_i = \bar{\pi}(x_i) \in \bar{T}$ ($i = 1, 2, 3$) are collinear.

Proof. We note that the special linear systems g_9^4 on C form just a 1-dimensional family $W_9^4(C) = \{|K - q|\} \quad (q \in C)$. As C is not hyperelliptic, every such series g_9^4 is base point free and defines a birational embedding of C onto a non-degenerate degree 9 curve $C' \subseteq \mathbb{P}^4$ (see the proof of Lemma 3.19). Moreover, the monodromy of this series acts on a general divisor $D \in g_9^4$ giving the full symmetric group (see [ACGH], Lemma on p. 111). That is, the ramified covering map $\varrho : C^9 \rightarrow C(9)$ is irreducible over every $g_9^4 = |K - q| \subseteq C(9)$. The map ϱ admits a natural factorization :

$$C^9 \xrightarrow{\nu_1} T^3 \xrightarrow{\nu_2} T(3) \xrightarrow{\phi_3} C(9).$$

Let us suppose that a component $\mathcal{J}' \neq \mathcal{J}_0$ with $p_2(\mathcal{J}') = \mathbb{G}(1, 6)$ and $\delta(\mathcal{J}') \leq 1$ does exist. We keep below the notation as in the proof of Lemma 3.14. By this Lemma, the image $\alpha_3(p_1(\mathcal{J}')) \subseteq J(C)$ is a curve $W(\mathcal{J}') = W_9^4(C) \simeq C$. For a general point $q \in C = W_9^4(C)$, we consider the family $\mathcal{V}_q \subset \mathbb{G}(1, 6)$ of trisecant lines l_ξ to \bar{T} corresponding to the points ξ in the fiber $P_q := \alpha_3^{-1}(q) \cap p_1(\mathcal{J}')$ of $\alpha_3 : p_1(\mathcal{J}') \rightarrow W_9^4(C) \subseteq J(C)$ over q . It has dimension 4, and every irreducible component \mathcal{V}'_q of \mathcal{V}_q and every component P'_q of the fiber P_q are also of dimension 4. Therefore $\nu_3(P'_q) = |K - q| \subseteq C(9)$.

Since the monodromy action on $|K - q|$ is irreducible, the pull-back $\tilde{P}_q := \varrho^{-1}(|K - q|) \subseteq C^9$ is also irreducible. Hence P'_q coincides with the image of \tilde{P}_q in $T(3)$. The latter shows, by the way, that $P_q = P'_q$ is actually irreducible. As the symmetric group S_9 acts on \tilde{P}_q , it follows that for any divisor $D \in g_9^4 = |K - q|$ and any decomposition $D = x_1 + x_2 + x_3$ with $x_i \in C(3)$, the points $y_i = \bar{\pi}(x_i) \in \bar{T}$ ($i = 1, 2, 3$) belong to a trisecant line l_ξ , where $\xi = x_1 + x_2 + x_3 \in P_q = \nu_2 \circ \nu_1(\tilde{P}_q)$, as stated. \square

Lemma 3.24. *If C is a general plane quintic then, for a general g_9^4 on C , a general divisor $D \in g_9^4$ and a decomposition $D = x_1 + x_2 + x_3$ with $x_i \in C(3)$, the points $\phi(x_i) \in T \subseteq \mathbb{P}^9$ ($i = 1, 2, 3$) are not collinear, where $\phi : C(3) \rightarrow \mathbb{P}^9$ is as in 3.21.*

Proof. Clearly, it is sufficient to exhibit just one such smooth plane quintic and just one divisor $D \in g_9^4$ on C as above. To construct a quintic with the required property, we take a smooth conic C_0 , and we fix four points q and p_i ($i = 1, 2, 3$) on C_0 . By Bertini's Theorem there exists a smooth quintic C cutting on C_0 the divisor $3(p_1 + p_2 + p_3) + q$. Conversely, the divisor cut out by C_0 on C is $3(p_1 + p_2 + p_3) + q \in |K_C| = g_{10}^5$. Then the divisor $D := 3(p_1 + p_2 + p_3)$ on C varies in the linear system $|K_C - q| = g_9^4$ cut out by conics through q .

We let $x_i := 3p_i \in C(3)$, $i = 1, 2, 3$. The corresponding points in \mathbb{P}^9 lie on the Veronese image V_3 of $\mathbb{P}^2 = \Delta$ via cubics (see 3.21). Since V_3 does not contain lines and is the intersection of quadrics through V_3 , it does not admit trisecant lines neither. Hence the points $\phi(x_i) \in V_3 \cap T \subseteq \mathbb{P}^9$ ($i = 1, 2, 3$) cannot be collinear. This finishes our proof. \square

Now we can prove 3.22.

Proof of Proposition 3.22. It is well known that a general plane quintic C has gonality 4 and the jacobian $J(C)$ is simple (e.g., see [ACGH], p. 218, [Bal] and [CiGe], Cor. (1.2)). Thereby, similarly as in the proof of Proposition 3.1 it follows that $\delta(\mathcal{J}_0) \geq 2$. Thus it suffices to exclude the existence of a component $\mathcal{J}' \neq \mathcal{J}_0$ of \mathcal{J} with $p_2(\mathcal{J}') = \mathbb{G}(1, 6)$ and $\delta(\mathcal{J}') \leq 1$. But for such a component the conclusion of Lemma 3.23 must hold, which contradicts Lemma 3.24. Indeed, since the points $\phi(x_i) \in T \subseteq \mathbb{P}^9$ ($i = 1, 2, 3$) are not collinear, for a generic projection $\bar{\pi} : \mathbb{P}^9 \supseteq T \rightarrow \bar{T} \subseteq \mathbb{P}^6$ the corresponding points $y_i = \bar{\pi} \circ \phi(x_i) \in \bar{T}$ ($i = 1, 2, 3$) are not collinear either. \square

We note that the hyperbolic hypersurfaces of degree 125 which we have found are of the lowest degree we can construct with these methods (cf. Lemma 4.9 below). See [ShZa2] for a better result.

4. ON PROJECTIVE EMBEDDINGS OF SYMMETRIC PRODUCTS

In this section we address the problem of finding, for a genus g curve C , embeddings of the symmetric product $C(n)$ ($n \geq 2$) into a projective space.

4.1. Let us start by recalling some basic facts about divisor classes on $C(n)$. Given $x \in C$, we let $\Xi_n(x)$ be the subset of $C(n)$ consisting of all divisors $D \in C(n)$ such that $x \leq D$. Clearly, $\Xi_n(x)$ is a reduced divisor on $C(n)$ isomorphic to $C(n-1)$. We let ξ be the class of $\Xi_n(x)$ in the Neron-Severi group $NS(C(n))$. If $B = \sum_i a_i x_i$ is a divisor on C , we denote by $\Xi_n(B)$ the divisor $\sum_i a_i \Xi_n(x_i)$ on $C(n)$. This provides a homomorphism $\Xi_n : \text{Pic}(C) \rightarrow \text{Pic}(C(n))$, whose image in $NS(C(n))$ lies in $\mathbb{Z}\langle \xi \rangle$. Similarly, to a line bundle \mathcal{L} on C we associate a line bundle $\Xi_n(\mathcal{L})$ on $C(n)$ with $\Xi_n(\mathcal{L}) \equiv (\deg \mathcal{L})\xi$ ⁶, called the *symmetrization* of \mathcal{L} (cf. [Ma]). It is defined only up to isomorphism.

The divisor Δ in $C(n)$ given by the diagonal is divisible by 2, because it is the branching divisor of the natural map $C^n \rightarrow C(n)$. We let δ be the divisor class of $\frac{\Delta}{2}$ in $NS(C(n))$.

A third basic class $\theta \in NS(C(n))$ is the class of the pull-back of a theta divisor on $J(C)$ via an Abel-Jacobi map $\alpha : C(n) \rightarrow J(C)$. The three classes ξ, δ, θ are related by

$$\delta = (n + g - 1)\xi - \theta$$

(see e.g., [ACGH], Prop. (5.1), p. 358 or [Ko, L. 7]).

The intersection form in the submodule $\mathbb{Z}\langle \xi, \theta \rangle$ of $NS(C(n))$ is given by

$$(4.1) \quad \xi^i \cdot \theta^{n-i} = \frac{g!}{(g-n+i)!}, \quad i = 0, \dots, n,$$

as dictated by Poincaré's formula [ACGH, p. 25] (see [Ko, L. 1]).

We notice that $NS(C(n)) = \mathbb{Z}\langle \xi, \theta \rangle$ for every $n \in \mathbb{N}$ if and only if $\text{End}(J(C)) \cong \mathbb{Z}$.

4.2. We consider again the natural map $C^n \rightarrow C(n)$, and we let $p_i : C^n \rightarrow C$ be the projection to the i -th factor. If \mathcal{L} is a line bundle on C , we denote by \mathcal{L}^n the line bundle $\otimes_{i=1}^n p_i^*(\mathcal{L})$ on C^n . By Künneth's formula one has $H^0(C^n, \mathcal{L}^n) \cong H^0(C, \mathcal{L})^{\otimes n}$. The symmetric group S_n acts on $H^0(C^n, \mathcal{L}^n)$, and two of the related irreducible representations are $\text{Sym}^n H^0(C, \mathcal{L})$ and $\bigwedge^n H^0(C, \mathcal{L})$. If σ is a non-zero section in one of these subspaces then the divisor $\sigma^*(0)$ is stable under the natural S_n -action on C^n , and so it is the pull-back of a divisor on $C(n)$. Actually (cf. [Gö]) there are two line bundles on $C(n)$, which we denote by $\mathcal{L}(n)^s$, respectively, $\mathcal{L}(n)^a$, such that

$$(4.2) \quad H^0(C(n), \mathcal{L}(n)^s) \cong \text{Sym}^n H^0(C, \mathcal{L}), \quad H^0(C(n), \mathcal{L}(n)^a) \cong \bigwedge^n H^0(C, \mathcal{L}).$$

We need the following facts concerning the line bundle $\mathcal{L}(n)^s$.

Lemma 4.3. *One has:*

- (i) $\mathcal{L}(n)^s = \Xi_n(\mathcal{L})$;
- (ii) $(\mathcal{L}(n)^s)^n = (\deg \mathcal{L})^n$.

Proof. If s_1, \dots, s_n are n sections of \mathcal{L} , we denote their symmetric product by $s_1 \odot \dots \odot s_n$. Given a divisor $D = x_1 + \dots + x_n \in C(n)$, one has

$$(4.3) \quad s_1 \odot \dots \odot s_n(D) = \sum_{(i_1, \dots, i_n)} s_{i_1}(x_1) \dots s_{i_n}(x_n),$$

⁶As usual, \equiv stands for numerical equivalence.

where the sum runs over all permutations (i_1, \dots, i_n) of the set $\{1, \dots, n\}$. Thus if $A = s^*(0) = x_1 + \dots + x_k \in \text{Div}_k(C)$, where $s \in H^0(C, \mathcal{L})$, then

$$s^{\odot n}(y_1 + \dots + y_n) = n!s(y_1) \dots s(y_n) = 0 \Leftrightarrow y_i = x_j$$

for some $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$. Hence $(s^{\odot n})^*(0) = \Xi_n(A) \in \text{Div}(C(n))$. This clearly implies (i). Part (ii) follows from (i) and the fact that $\xi^n = 1$ (see (4.1)). \square

The following result extends Lemma 3.10 of [ShZa1].

Proposition 4.4. \mathcal{L} is very ample on C if and only if $\mathcal{L}(n)^s$ is very ample on $C(n)$.

Proof. Let us show that, if the line bundle $\mathcal{L}(n)^s$ on $C(n)$ is very ample, then so is \mathcal{L} as well. This trivially holds if $n = 1$. Then we proceed by induction, and we restrict $\mathcal{L}(n)^s$ to $\Xi_n(x) \simeq C(n-1)$. Since, obviously, $\mathcal{L}(n)^s$ restricts to $\mathcal{L}(n-1)^s$, we conclude the induction.

To prove the converse, we assume that \mathcal{L} is very ample on C . Let us show first that $\mathcal{L}(n)^s$ separates points of $C(n)$. We let

$$D = p_0x_0 + p_1x_1 + \dots + p_kx_k \quad \text{and} \quad D' = p'_0x_0 + p'_1x_1 + \dots + p'_kx_k$$

be two distinct effective divisors of degree n on C regarded as points of $C(n)$, where x_1, \dots, x_k are distinct points of C and p_i, p'_i are non-negative integers with $0 \leq p_0 < p'_0$. By the very ampleness of \mathcal{L} , we can find two sections $s_0, s_1 \in H^0(C, \mathcal{L})$ such that $s_0(x_0) \neq 0$ and $s_1(x_0) = 0$, whereas $s_1(x_i) \neq 0$ for all $i = 1, \dots, k$. We let $\sigma := s_0^{\odot p_0} \odot s_1^{\odot(n-p_0)} \in H^0(C(n), \mathcal{L}(n)^s)$. Then by (4.3) we obtain:

$$\sigma(D) = k!s_0^{p_0}(x_0)s_1^{p_1}(x_1) \dots s_1^{p_k}(x_k) \neq 0$$

whereas $\sigma(D') = 0$, as $p'_0 > p_0$ and $s_1(x_0) = 0$. Thus σ separates D and D' .

Next we show that $\mathcal{L}(n)^s$ separates tangent vectors. Let $D = x_1 + \dots + x_n$ be a point of $C(n)$ and $X \in T_D(C(n))$ be a non-zero tangent vector. We need to find a section σ of $\mathcal{L}(n)^s$ such that $\sigma(D) = 0$ whereas $X(\sigma) \neq 0$. If D is formed by n distinct points, the argument is similar to those developed in [ShZa1], *loc.cit.*, and we will not repeat it now. To the other extreme, suppose that $D = nx$ sits on the small diagonal. We will prove the statement in this case only, since the intermediate cases, where only some of the points of D come together, can be treated in a similar way.

Let t be a local coordinate on C in a small disc W around x . Then near (x, \dots, x) , C^n is isomorphic to the polydisc W^n with coordinates, say, (t_1, \dots, t_n) . Whereas $C(n)$ about nx looks like $V := W(n)$, which is still a polydisk with coordinates (z_1, \dots, z_n) , where z_i is the i -th symmetric function on (t_1, \dots, t_n) .

Take now sections $s_i \in H^0(C, \mathcal{L})$ ($i = 1, \dots, n$) which have the following expressions in W :

$$s_1 = t + o(t), \quad s_i = 1 + a_it + o(t), \quad i = 2, \dots, n.$$

Since \mathcal{L} is very ample we can find sections as above with arbitrary values of a_i , $i = 2, \dots, n$. Our objective is to show that one can find a section of the form $\sigma = s_1 \odot \dots \odot s_n$ such that in $V = W(n)$ it looks like

$$(4.4) \quad \sigma = b_1z_1 + \dots + b_nz_n + (\text{higher order terms in } z_1, \dots, z_n)$$

with arbitrary b_i , $i = 1, \dots, n$. This will prove the assertion.

First of all, if we let $a_i = 0$ ($i = 2, \dots, n$) then

$$\sigma = (n-1)!z_1 + (\text{higher order terms}).$$

Thus referring to (4.4) above, we can find a section corresponding to the n -tuple $(b_1, \dots, b_n) = (1, 0, \dots, 0)$. Let us show by induction that for every n -tuple $(0, \dots, 0, 1_j, 0, \dots, 0)$, where $j = 2, \dots, n$, there exist corresponding sections. By induction it suffices to prove that for any

$j = 2, \dots, n$ we can find a section corresponding to an n -tuple of the form $(b_1, \dots, b_j, 0, \dots, 0)$ with $b_j \neq 0$. For this it suffices to take $a_i = 1$, $i = 2, \dots, j$, $a_i = 0$, $i = j + 1, \dots, n$, since then

$$\sigma = b_1 z_1 + \dots + b_{j-1} z_{j-1} + j!(n-j)! z_j + (\text{higher order terms})$$

with some $b_1, \dots, b_{j-1} \in \mathbb{Z}$, which we do not care to compute. This ends our proof. \square

4.5. In the remainder of this section we turn to line bundles of the form $\mathcal{L}(n)^a$. The following analog of Lemma 4.3 holds.

Lemma 4.6. (i) *The class of $\mathcal{L}(n)^a$ in $NS(C(n))$ is*

$$(\deg \mathcal{L} - g - n + 1)\xi + \theta = (\deg \mathcal{L})\xi - \delta.$$

(ii) *Consequently,*

$$(\mathcal{L}(n)^a)^n = \sum_{i=0}^n \binom{n}{i} \frac{g!}{(g-n+i)!} (\deg \mathcal{L} - g - n + 1)^i.$$

Proof. We give a proof only in the case where the line bundle $\mathcal{L}(n)^a$ has enough sections (cf. Remark 4.7 below for an alternative approach). By virtue of (4.2) we may regard the sections of $\mathcal{L}(n)^a$ as those of $\wedge^n H^0(C, \mathcal{L})$. Suppose that $\sigma = s_1 \wedge \dots \wedge s_n \in \wedge^n H^0(C, \mathcal{L})$ is such that the divisor $\sigma^*(0)$ on $C(n)$ is reduced, where s_1, \dots, s_n are linearly independent sections of \mathcal{L} . Notice that the class of $\sigma^*(0)$ in the group $NS(C(n))$ does not depend on the choice of σ . Given $D = x_1 + \dots + x_n \in C(n) \setminus \Delta$, one has

$$\sigma(D) = s_1 \wedge \dots \wedge s_n(D) = \det(s_i(x_j))_{i,j=1,\dots,n} = 0$$

if and only if there is a non-trivial linear combination s of s_1, \dots, s_n such that $s(x_i) = 0$ for all $i = 1, \dots, n$, that is $D \leq s^*(0)$. It is not difficult to extend this remark to the case $D \in \Delta$. That is, also in this case D belongs to $\sigma^*(0)$ if and only if $D \leq s^*(0)$ for a certain non-zero section $s \in V := \text{span}\{s_1, \dots, s_n\}$. We leave the details to the reader. Thus $\sigma^*(0) = \Gamma_n(V)$, where $\Gamma_n(V)$ denotes the set of all $D \in C(n)$ subordinated to the members of the linear system $|V|$. Now (i) follows by Lemma (3.2) in [ACGH, p. 342], whereas (ii) follows from (i) by the Poincaré Formula (4.1). \square

Remark 4.7. Let $p : C \times C(n-1) \rightarrow C$ be the first projection and $\alpha : C \times C(n-1) \rightarrow C(n)$ be the map sending $(x, \sum_{i=1}^{n-1} x_i)$ to $x + \sum_{i=1}^{n-1} x_i$. For a line bundle \mathcal{L} over C , its *Mattuck symmetrization* [Ma, Sect. 1] is defined by the push-pull formula :

$$\mathcal{E}_{n,\mathcal{L}} := \alpha_* p^* \mathcal{L}.$$

This is a rank n vector bundle over $C(n)$. There is a canonical isomorphism [Ma, Corollary of Prop. 2] :

$$H^0(C(n), \mathcal{E}_{n,\mathcal{L}}) \cong H^0(C, \mathcal{L}).$$

Indeed, for $s \in H^0(C, \mathcal{L})$, p^*s is constant along the fibers of α . Also, for any open subset $U \subseteq C(n)$ one has [Ma, Prop. 1] :

$$(4.5) \quad H^0(U, \mathcal{E}_{n,\mathcal{L}}) \cong [H^0(\varrho^{-1}(U), \bigoplus_{i=1}^n p_i^* \mathcal{L})]^{S_n},$$

where as above $\varrho : C^n \rightarrow C(n)$ stands for the orbit map of the action of the symmetric group S_n on C^n .

Similarly, if $\Sigma_n \subseteq C \times C(n)$ is the natural incidence relation then the *Göttsche symmetrization* of \mathcal{L} is defined as follows [Gö] :

$$\mathcal{E}'_{n,\mathcal{L}} := (p_2)_* (p_1)^* \mathcal{L},$$

where $p_1 : \Sigma_n \rightarrow C$ and $p_2 : \Sigma_n \rightarrow C(n)$ are the canonical projections⁷. The map

$$\Sigma_n \ni \left(x, x + \sum_{i=1}^{n-1} x_i\right) \longmapsto \left(x, \sum_{i=1}^{n-1} x_i\right) \in C \times C(n-1)$$

yields an isomorphism $\Sigma_n \simeq C \times C(n-1)$ which agrees with the above projections. This gives rise to a natural identification

$$\mathcal{E}_{n,\mathcal{L}} = \mathcal{E}'_{n,\mathcal{L}}.$$

By [Gö, Thm. A.1] one has:

$$(4.6) \quad \mathcal{L}(n)^a \cong \det \mathcal{E}_{n,\mathcal{L}} \cong \mathcal{L}(n)^s \otimes \mathcal{O}_{C(n)}(-\delta),$$

which by virtue of 4.3(i) provides another proof of 4.6(i). Moreover (4.5) and (4.6) yield the second part of (4.2).

In particular, if $\mathcal{L} = \omega_C = \Omega_C^1$ then [Ma, Sect. 1, (2)] $\mathcal{E}_{n,\mathcal{L}} \cong \Omega_{C(n)}^1$, whence by (4.6) we have

$$(4.7) \quad \omega_{C(n)} \cong \omega_C(n)^a.$$

4.8. We notice that a direct analog of Proposition 4.4 does not hold in general for the bundles $\mathcal{L}(n)^a$. For instance, if C is a general plane quartic then $\mathcal{L} := \omega_C = \mathcal{O}_C(1)$ is very ample, although $\mathcal{L}(2)^a \cong \omega_{C(2)}$ is ample but not very ample (see [Ko] or [ShZa1, (4.16.b')]).

Suppose however that $\mathcal{L} \rightarrow C$ is n -very ample i.e., any subscheme of length $n+1$ imposes independent conditions to the sections in $H^0(C, \mathcal{L})$ (see [BeSo]). Then

$$\phi_{\mathcal{L}} : C \rightarrow \mathbb{P}(H^0(C, \mathcal{L})^*)$$

is an embedding, and there is no $(n+1)$ -secant \mathbb{P}^{n-1} to C in this embedding. The map

$$\phi_{\mathcal{L}(n)^a} : C(n) \rightarrow \mathbb{P}\left(\bigwedge^n H^0(C, \mathcal{L})^*\right)$$

can be described as follows. Given a divisor $D \in C(n)$, we regard it as a subscheme of $C \subseteq \mathbb{P}(H^0(C, \mathcal{L})^*)$. Then D spans a \mathbb{P}^{n-1} which we denote by $\langle D \rangle$, and we let

$$\phi_{\mathcal{L}(n)^a}(D) := \langle D \rangle \in \mathbb{G}(n-1, \mathbb{P}(H^0(C, \mathcal{L})^*)) \subseteq \mathbb{P}\left(\bigwedge^n H^0(C, \mathcal{L})^*\right).$$

As no $\langle D \rangle \simeq \mathbb{P}^{n-1}$ is $(n+1)$ -secant to C , the map $\phi_{\mathcal{L}(n)^a}$ is injective. Actually by [CaGö], $\phi_{\mathcal{L}(n)^a}$ is an embedding i.e., $\mathcal{L}(n)^a$ is very ample, if and only if \mathcal{L} is n -very ample⁸.

Anyhow, the minimal degree of projective embeddings of $C(3)$ given by line bundles of the form $\mathcal{L}(3)^a$ is higher than those provided by line bundles of the form $\mathcal{L}(3)^s$. Indeed, we have the following lemma.

Lemma 4.9. *If a curve C is neither hyperelliptic nor trigonal then for any 3-ample line bundle \mathcal{L} on C one has $(\mathcal{L}(3)^a)^3 > 125$.*

Proof. If $\deg \mathcal{L} = d$ then according to Lemma 4.6(ii) we have

$$(4.8) \quad \begin{aligned} (\mathcal{L}(3)^a)^3 &= g(g-1)(g-2) + 3g(g-1)(d-g-2) \\ &\quad + 3g(d-g-2)^2 + (d-g-2)^3. \end{aligned}$$

In view of (4.8) and the classification of space curves [Ha, IV.6.4.2], it is easily seen that $(\mathcal{L}(3)^a)^3 > 125$ unless, maybe, for one of the following pairs :

$$(4.9) \quad (g, d) = (5, 7), (6, 5), (6, 7), (6, 8), (7, 8) \quad \text{or} \quad (8, 8).$$

⁷Actually, [Gö] is dealt with projective surfaces rather than with curves.

⁸Notice that $\phi_{\mathcal{L}(n)^a}$ and the map φ_{n-1} considered in [CaGö] are dual to each other.

By the result of Catanese and Göttsche cited in 4.8, the bundle $(\mathcal{L}(3)^a)^3$ is very ample and so, there is no 4-secant plane to the image $\varphi_{|\mathcal{L}|}(C)$ of C under the embedding defined by the linear system $|\mathcal{L}|$. In particular, we must have $\dim |\mathcal{L}| \geq 4$.

On the other hand, $d - g \leq 2$ in all cases of (4.9). Therefore the linear system $|\mathcal{L}|$ is special and different from the canonical one. Since C is neither hyperelliptic nor trigonal, Clifford's Theorem gives the strict inequality $\dim |\mathcal{L}| < d/2$, which yields $d \geq 9$. This excludes the six remaining possibilities as in (4.9). \square

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