**On lower bounds of the density of packings of equal spheres of $\mathbb{R}^n$**

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**Abstract**

We study lower bounds of the packing density of non-overlapping equal spheres in $\mathbb{R}^n, n \geqslant 2,$ as a function of the maximal circumradius of its Voronoi cells. Our viewpoint is that of Delone sets which allows to investigate the gap between the upper bounds of Rogers or Kabatjanskiĭ-Levenštein and the Minkowski-Hlawka type lower bounds for the density of lattice-packings. As a consequence we provide explicit asymptotic lower bounds of the covering radii (holes) of the Barnes-Wall, Craig and Mordell-Weil lattices, respectively $BW_n, \mathbb{A}_n^{(r)}$ and $MW_n$, and of the Delone constants of the BCH packings, when $n$ goes to infinity.

**Résumé**

Nous étudions des bornes inférieures de la densité d'un système d'empilement de sphères égales dans $\mathbb{R}^n, n \geqslant 2,$ qui ne se recouvrent pas en fonction du rayon maximal d'inscription de ses cellules de Voronoi. Notre point-de-vue est celui des ensembles de Delaunay qui permet de se donner les moyens de l'étude de l'écart laissé entre les bornes supérieures de Rogers ou de Kabatjanskiĭ-Levenštein et les bornes inférieures de type Minkowski-Hlawka pour la densité d'empilement de réseau. Comme conséquence nous donnons des bornes inférieures explicites du rayon de couverture (trou) des réseaux de Barnes-Wall, Craig et Mordell-Weil, respectivement $BW_n, \mathbb{A}_n^{(r)}$ et $MW_n$, et les constantes de Delaunay d'empilements BCH, lorsque $n$ tend vers l'infini.

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## 1. Introduction

The fundamental problem of knowing the maximal density of packings of equal spheres in $\mathbb{R}^n$ has received a lot of attention ([Hand], Oesterlé [O], Rogers [R], Gruber and Lekkerkerker [GL], Martinet [Ma], Conway and Sloane [CS], Cassels [Ca], Zong [Z]). Similar problems are encountered in coding theory, data transmission, combinatorial geometry and cryptology [Hof]. We will tackle it by the notion of Delone set for which we will give explicit lower bounds of the the density of a Delone set as a function of $n$ and its so-called Delone constant $R$ expressing the maximal size of its holes.

Blichfeldt, Rogers, Levenštein, Sidel'nikov, Kabatjanskiǐ- Levenštein [GL] [Hand] [CS] have given upper bounds while lower bounds were given by Minkowski, Davenport-Rogers, Ball [Ba], etc, in the lattice-packing case (see section 2). In between the situation is considered as fairly vague. The present contribution brings a light to the range between both types of bounds.

For this we will use the language of uniformly discrete sets and Delone sets instead of that of systems of spheres. Let us recall it. A discrete subset $\Lambda$ of $\mathbb{R}^n$ is said to be uniformly discrete of constant $r$ if there exists a real number $r > 0$ such that $x, y \in \Lambda, x \neq y$ implies $\|x-y\| \geqslant r$. Uniformly discrete sets of constant $1$ will be called $\mathscr{UD}$-sets and the set of $\mathscr{UD}$-sets will be denoted by $\mathscr{UD}$ (without mentioning the dimension $n$ of the ambiant space). There is a one-to-one correspondence between the set $SS$ of systems of equal spheres of radius $1/2$ and the set $\mathscr{UD}$ : $\Lambda = (a_i)_{i\in\mathbb{N}} \in \mathscr{UD}$ is the set of sphere centres of $\mathscr{B}(\Lambda) = \{a_i + B(0,1/2) \mid i \in \mathbb{N}\} \in SS$ where $B(z,t)$ denotes generically the closed ball centred at $z \in \mathbb{R}^n$ of radius $t > 0$. We will take $1/2$ in the sequel for the common radius of spheres to be packed and will consider $\mathscr{UD}$-sets instead of systems of equal spheres of radius $1/2$.

Assume $\Lambda \in \mathscr{UD}$ of minimal interpoint distance equal to one. The density of the system of spheres $\mathscr{B}(\Lambda)$ is defined by

$$\delta(\mathscr{B}(\Lambda)) := \limsup_{R\to+\infty} \left[ \mathrm{vol}\left( \left( \bigcup_{i\in\mathbb{N}} (a_i + B) \right) \bigcap B(0,R) \right) /\mathrm{vol}(B(0,R)) \right]$$

Let us denote by $\mathscr{L}$ the space of ($n$-dimensional) lattices of $\mathbb{R}^n$. We will denote by

$$\delta := \sup_{\Lambda\in\mathscr{UD}} \delta(\mathscr{B}(\Lambda)), \ \text{resp.} \ \delta_L := \sup_{\Lambda\in\mathscr{L}\cap\mathscr{UD}} \delta(\mathscr{B}(\Lambda))$$

the supremum of densities of sphere packings whose centres form a $\mathscr{UD}$-set, resp. a lattice which is a $\mathscr{UD}$-set, of minimal interpoint distance one, and will call it the *packing density*, resp. the *lattice-packing density*.

A $\mathscr{UD}$-set $\Lambda$ is said to be a Delone set of constant $R > 0$ if, for all $z \in \mathbb{R}^n$, there exists an element $\lambda \in \Lambda$ such that $\|z-\lambda\| \leqslant R$ (relative denseness property with constant $R$). $R$ will be called the *Delone constant* of $\Lambda$ when it is taken minimal for this property of relative denseness. Let $X_R \subset \mathscr{UD}$ be the subset of Delone sets of constant $R > 0$ of $\mathbb{R}^n$. Elements of

$$\bigcup_{R>0} X_R$$

will be called Delone sets (of $\mathbb{R}^n$). A Delone set is never empty: there exists $R_c = R_c(n) > 0$ such that its constant $R$ is greater than $R_c$. We will call it the *covering constant* (which only depends upon $n$). It is the smallest value $R$ such that $X_R$ is not empty.

In section 3 we will explicitely give expressions of lower bounds of $\delta(\mathscr{B}(\Lambda))$, therefore of $\delta$, as a function of $n$ and the Delone constant $R$ attached to $\Lambda$ making a link with the problem of the most economical covering of a ball of radius $R$ by balls of radius $1/2$ [VG1]. This constant $R$ is the maximal circumradius of the Voronoi cells in the Voronoi decomposition of space by $\Lambda$; if $\Lambda$ is a lattice, it means as a function of the covering radius, if $\Lambda$ is a non periodic $\mathscr{UD}$-set, it means as a function of the "maximal size of the holes in $\Lambda$". In section 4 we will considerably improve the results of section 3 following another viewpoint. Namely, we will prove

THEOREM 1.1. — *Assume $n \geqslant 2$. If $\Lambda$ is a Delone set of $\mathbb{R}^n$ of constant $R$, then*

$$(2R)^{-n} \leqslant \delta(\mathscr{B}(\Lambda)) \leqslant \delta \qquad \textit{for all} \quad R_c \leqslant R. \tag{1}$$

Let us denote $\mu_n(R) := (2R)^{-n}$. The $(2R)^{-n}$ dependence of the expression of $\mu_n(R)$ with $n$ is very important and constitutes a key result. It allows to study the minimal asymptotic values of the covering constant $R_c(n)$ when $n$ tends to infinity. Namely, we will prove

THEOREM 1.2. — *For all $\epsilon > 0$ there exists $n(\epsilon)$ such that, for all $n > n(\epsilon)$, we have*

$$X_R = \emptyset \qquad \textit{for all} \quad R < 2^{-0.401} - \epsilon.$$

REMARK.— Theorem 1.2 asserts the existence of an infinite collection of *middle-sized* Voronoi cells in any densest or saturated packing of equal spheres of $\mathbb{R}^n$ of radius $1/2$ of circumradii greater than

$$2^{-0.401} + o(1) = 0.757333... + o(1).$$

The small values of $R$ between the bound $\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}$ and $1$ are discussed in the section 3.

In section 5, as an application of theorem 1.1, we will obtain explicit lower bounds as a function of $n$ of the covering radii (holes) of known lattices, namely Barnes-Wall $BW_n$, Craig $\mathbb{A}_n^{(r)}$, Mordell-Weil $MW_n$, and of the Delone constants of BCH packings.

In section 6 we will show the pertinency of the lower bound $\mu_n(R)$ by comparing it to known classical asymptotic bounds. From the theorem 1.1 the conjecture stating that the packing density $\delta$ is $2^{-0.5990n}$ (after [KL]), when $n$ is sufficiently large, is now equivalent to the existence of a Delone set of $\mathbb{R}^n$ of minimal interpoint distance equal to one and of Delone constant less than $2^{-0.401} + o(1)$.

## 2. Context

The upper bounds of $\delta$, as a function of $n$, are recalled in Table 1, the best one being the one of Kabatjanskiĭ and Levenštein.

Table 1

| | |
|---|---|
| $\frac{n}{e}\,2^{-n/2}$ | Rogers [R3], 1958 |
| $2^{(-0.5096+o(1))n}$ | Sidel'nikov [S], 1973 |
| $2^{(-0.5237+o(1))n}$ | Levenštein [Lv], 1979 |
| $2^{(-0.5990+o(1))n}$ | Kabatjanskiĭ and Levenštein [KL], 1978 |

As for lower bounds of the density of densest packings of equal spheres, up to our knowledge, the basic result is concerned with lattices and no result exists for general densest aperiodic packings of equal spheres: the conjecture of Minkowski (1905) proved by Hlawka [Ca] [GL] states

$$\frac{\zeta(n)}{2^{n-1}} \leqslant \delta_L \tag{1}$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ denotes the Riemann $\zeta$-function. This lower bound is non-effective in the sense that its proof does not allow explicit constructions of very dense lattices. This lower bound was improved by Davenport and Rogers [DR]

$$(\ln\sqrt{2} + o(1))\; n\,2^{-n}\,, \qquad\qquad n \text{ sufficiently large} \tag{2}$$

and by other authors (Ball [Ba] recently obtained better: $2(n-1)\zeta(n)2^{-n}$ ) ; and still exhibits the $2^{-n}$ dependence. For details, see [Hand], chapter VI in Cassels [Ca], Sloane in [CS] chapter 9, Gruber and Lekkerkerker [GL], or Zong [Z].

Bounds for the lattice-packing density are linked to holes. If a lattice $\Lambda$ of $\mathbb{R}^n$ is a Delone set of constant $R$ where $R$ is taken minimal for the property of relative denseness, then classically the quantity $R = R(\Lambda)$ is called the *covering radius* of $\Lambda$:

$$R = \sup_{x\in\mathbb{R}^n}\ \inf_{\lambda\in\Lambda}\ \|x-\lambda\|$$

Given a $\mathscr{UD}$-set $\Lambda := \{\lambda_i\}$, to each element $\lambda_i \in \Lambda$ is associated its local cell $C(\lambda_i,\Lambda)$, also denoted by $C(\lambda_i,\mathscr{B}(\Lambda))$, defined by the closed subset (not necessarily bounded), called Voronoi cell at $\lambda_i$

$$C(\lambda_i,\Lambda) \ := \ \{x\in\mathbb{R}^n \mid \|x-\lambda_i\| \leqslant \|x-\lambda_j\| \ \text{for all} \ \ j \neq i\}.$$

As soon as $\Lambda$ is a Delone set of constant $R > 0, R < +\infty$, all the Voronoi cells at its points are bounded closed convex polyhedra. In this case, for all $\lambda_i \in \Lambda$, we have

$$C(\lambda_i,\Lambda) \ := \ \{x\in\mathbb{R}^n \mid \|x-\lambda_i\| \leqslant \|x-\lambda_j\| \ \text{for all} \ \ j\neq i \ \text{with} \ \|\lambda_j-\lambda_i\| < 2R\}.$$

By definition the circumradius of the Voronoi cell at $\lambda_i$ is

$$\rho_i \ := \ \max_{v}\|\lambda_i - v\|$$

where the supremum (reached) is taken over all the vertices $v$ of the Voronoi cell $C(\lambda_i,\Lambda)$ at $\lambda_i$.

In the case of a lattice $L$ the covering radius $R(L)$ is the circumradius of the Voronoi cell of the lattice $\Lambda$ at the origin. Any vertex of this Voronoi cell at a distance $R(\Lambda)$ from $\Lambda$ is called a (spherical) *hole* of $\Lambda$ (or of $\Lambda + B(0,1/2)$). Sometimes these holes are called deep holes or deepest holes while the other vertices of the Voronoi cell at a distance less than $R(\Lambda)$ from $\Lambda$ are simply called holes or shallow holes [CS].

All the vertices of the Voronoi cell of a lattice at the origin may be simultaneously (deepest) holes when this Voronoi cell is highly symmetrical [VG].

The determination of (say) *the minimal hole constant*

$$R_L := \min_{L \in \mathcal{UD} \cap \mathcal{L}} R(L)$$

over all lattices $L$ of $\mathbb{R}^n$ which are $\mathcal{UD}$-sets is an important problem already mentioned by Fejes-Toth [Ft]. It corresponds to the smallest possible holes in lattice-packings $L + B$. Our knowledge about it is comparatively limited and the lattices for which the covering radius is equal to the minimal hole constant are unknown as soon as $n$ is large enough. In Table 2 we summarize some values and known upper bounds of $R_L = R_L(n)$.

Table 2 : Minimal hole constant $R_L(n)$ for lattice-packings of spheres of radius $1/2$ in $\mathbb{R}^n$.

| | | |
|---|---|---|
| $n = 3$ | Böröczky [Bo1] | $= \sqrt{5}/(2\sqrt{3}) \simeq 0.645497...$ |
| $n = 4$ | Horvath [Ho] | $= (\sqrt{3}-1)3^{1/4}/\sqrt{2} \simeq 0.68125...$ |
| $n = 5$ | Horvath [Ho] | $= \sqrt{9+\sqrt{13}}/(2\sqrt{6}) \simeq 0.72473...$ |
| $n \geqslant 2$ | Rogers [R4] | $< 1.5$ |
| $n \geqslant 2$ | Henk [He] | $\leqslant \sqrt{21}/4 \simeq 1.1456...$ |
| $n >> 1$ | Butler [Bu] | $\leqslant n^{(\log_2 \ln n + c)/n} = 1 + o(1)$ ( $c$ is a constant) |

The following theorem is fundamental but non-constructive.

THEOREM 2.1. — ( Butler )

$$R_L(n) \leqslant 1 + o(1) \qquad \textit{when } n \textit{ is sufficiently large.}$$

($Q_0$) Does there exist $n_0$ such that the inequality $R_L(n) \geqslant 1$ holds for all $n \geqslant n_0$ ?

If the answer to this fundamental question is yes, then the theorem of Butler [Bu] would imply that $R_L(n) = 1 + o(1)$. Then this result would be a very important step towards a proof of the conjecture stating that the strict inequality

$$\delta > \delta_L$$

holds for $n$ large enough. It is conjectured that the answer is yes [CS]. Consequently, the search of lower bounds of $R_L(n)$ is crucial.

The easy lower bound $\sqrt{2}/2 + o(1)$ for $R_L(n)$ was given by Blichfeldt (Butler [Bu] p 722), when $n$ is large enough.

Because of the difficulty of computing explicitely the Voronoi cells of a lattice from the lattice itself when $n$ is large, the information between a lattice and its population of holes is limited (see Chapter 22 by Norton in [CS]). However, let us mention the Leech lattice $\Lambda_{24}$ for which the theorem of Conway, Parker and Sloane (in [CS] Chapter 23) gives an extremely small value of the covering radius (here $\Lambda_{24}$ is normalized such that its minimal interpoint distance is exactly 1):

$$R(\Lambda_{24}) = \sqrt{2}/2.$$

## 3. Saturated Delone sets and lower bounds of the packing density

We will say that a $\mathscr{UD}$-set $\Lambda$ is *saturated*, or *maximal*, if it is impossible to add a sphere to $\mathscr{B}(\Lambda)$ without destroying the fact that it is a packing of spheres, i.e. without creating an overlap of spheres. The set $SS$ of systems of spheres of radius $1/2$, is partially ordered by the relation $\prec$ defined by

$$\Lambda_1, \Lambda_2 \in \mathscr{UD}, \quad \mathscr{B}(\Lambda_1) \prec \mathscr{B}(\Lambda_2) \iff \Lambda_1 \subset \Lambda_2.$$

By Zorn's lemma, maximal sphere packings exist. The saturation operation of a sphere packing consists in adding spheres to obtain a maximal sphere packing. It is fairly arbitrary and may be finite or infinite. Note that it is not because a sphere packing is maximal (saturated) that its density is equal to $\delta$.

Let us call $X^{(s)}$ the subset of saturated Delone sets of $\mathbb{R}^n$. By saturating a Delone set of constant $R > 0$ we will always obtain a Delone set of constant less than 1, but not a Delone set of constant $= R_c$ in general. Let $R^{(s)}$ be the supremum of the values of $R$ such that a saturated Delone set is a Delone set of constant $R$. In other words

$$\bigcup_{R \geqslant R^{(s)}} \{\Lambda \in X_R \mid \Lambda \notin X_{R'} \text{ for all } R' < R\}$$

contains no saturated Delone set. It is easy to check that

$$1/2 < R_c < R^{(s)} = 1 \tag{1}$$

Hence

$$X^{(s)} \subset \bigcup_{R_c \leqslant R < R^{(s)} = 1} X_R \subset X_1$$

If the supremum $\delta$ of the density is reached, then obviously it will be reached on $X^{(s)}$.

Remark.— If $\Lambda = \{\lambda_i\}$ is a Delone set of constant $R$, saturated or not, where $R$ is assumed minimal for the property of relative denseness, then

$$R = \max_i \rho_i$$

where $\rho_i$ is the circumradius of the Voronoi cell centred at $\lambda_i$.

The following inequality of Blichfeldt is very instructive to compute a lower bound of the minimal constant $R$ of a saturated Delone set from its generic Voronoi cell.

LEMMA 3.1. — *If* $\Lambda$ *is a Delone set of* $\mathbb{R}^n$ *of constant* $R$, $n \geqslant 1$, *then*

$$\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}} \leqslant R$$

*Proof.* — It is an application of lemma 1 in Rogers [R] p 79 (or Blichfeldt [Bl]) since the distance from the centre of a Voronoi cell to any point of its $(n-i)$-dimensional plane, in the Voronoi decomposition of space by $\Lambda$, is at least

$$\frac{1}{2}\sqrt{\frac{2i}{i+1}} \qquad \text{for all } 1 \leqslant i \leqslant n$$

Taking $i = n$ in the above inequality gives the result. Note that in the constructions of Rogers, packings of the unit ball $B(0,1)$ are considered whereas we are packing the ball $B = B(0,1/2)$; this justifies the factor $1/2$ in front of the expression. ☐

As a corollary, the minimal Delone constant $R_c = R_c(n)$, relative to $\mathbb{R}^n$, $n \geqslant 1$, satisfies

$$\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}} \leqslant R_c(n), \text{ that is } \quad R_c(n) \geqslant \frac{\sqrt{2}}{2}(1+O(1/n)) \quad \text{for } n \text{ sufficiently large.}$$

We will call $\sqrt{2}/2$ the Blichfeldt bound. As we have recalled it, it is reached with the Leech lattice for $n = 24$.

If $n = 1$, we see that $X_{R_c} = X_{1/2}$ is not empty since it contains $\mathbb{Z}$. If $n = 2$, the set $X_{R_c} = X_{\frac{1}{\sqrt{3}}}$ is not empty since it contains the lattice generated by the points of coordinates $(1,0)$ and $(1/2, \sqrt{3}/2)$ in the plane (extreme lattice) in an orthonormal basis. What happens for $n \geqslant 3$ ? The set $X_{\frac{\sqrt{2}}{2}\sqrt{\frac{n}{n+1}}}$ is certainly empty since the minimal Voronoi cell is not tiling periodically the ambiant space as soon as $n \geqslant 3$ [R] (Hales [H] for $n = 3$).

($Q_1$) For which values of $n$ and $R$ is $X_R$ not empty ?

This question is partially answered by the theorem 1.2. This question is fairly old. In the spirit of the works of Delone, Ryshkov [Ry] already in 1975 named Delone sets $(r,R)$-systems and already asked several questions about the minimal/maximal density of $(r,R)$-systems.

In order to state the following theorem and to fix the notations, we need to recall the notion of *covering density* of $\mathbb{R}^n$. Following Rogers [R] we will say that a system $\mathscr{K}$ of translates

$$B + a_i \qquad\qquad (i = 1,2,\ldots)$$

of the ball $B$ by a sequence of points $a_i \in \mathbb{R}^n$ forms a covering if each point of $\mathbb{R}^n$ lies in at least one of the set of the system. If the collection $\{a_i\}$ is a lattice, we will say that $\mathscr{K}$ forms a *lattice covering*. We will assume as usual that the set $\{a_i\}$ constitutes a uniformly discrete set of constant $> 0$. Then we can associate a density $\rho(\mathscr{K})$ with the system $\mathscr{K}$: it is the limiting ratio of the sum of the Lebesgue measures of those sets of the system $\mathscr{K}$, which lie in a large ball, to the Lebesgue measure of the ball, as it becomes infinitely large. Equivalently it will be the mean of the multiplicity function associated with the system $\mathscr{K}$: recall that the multiplicity function is equal to $m$ at the point $x \in \mathbb{R}^n$ if $x$ belongs to $m$ balls $B + a_i$ of the system $\mathscr{K}$.

Let us denote by $\vartheta$, resp. $\vartheta_L$, the density of the most economical (minimal) covering, resp. the density of the most economical (minimal) lattice covering, of $\mathbb{R}^n$ by $B$:

$$\vartheta \ := \inf_{\mathscr{K},\{a_i\}} \rho(\mathscr{K}) \quad , \text{ resp. } \vartheta_L \ := \inf_{\mathscr{K},\{a_i\}\in\mathscr{L}} \rho(\mathscr{K}) \tag{2}$$

where the infimum is taken over all coverings, resp. all lattice coverings, of space, built from uniformly discrete sets. These quantities are respectively called the *covering density* and the *lattice covering density* of $\mathbb{R}^n$.

It is clear that we always have

$$\delta_L \leqslant \delta \leqslant 1 \leqslant \vartheta \leqslant \vartheta_L$$

Many attempts were done to give upper bounds of $\vartheta_L$ or $\vartheta$ ( Kerschner [K] for $n = 2$; see [R], [GL], [Z], [CS] for general $n$). We will consider the strict upper bound $n \ln n + n \ln(\ln n) + 5n$ of $\vartheta$ (for all $n \geqslant 3$ and even for $n = 2$ ([R] p 47)) given by Rogers in [R1].

THEOREM 3.2. — *A Delone set $\Lambda$ of $\mathbb{R}^n$ has a strictly positive density. More precisely, there exists a real number $\nu_{R,n} > 1$ depending only upon $R$ and $n$ such that*

$$\nu_{R,n}^{-1} \leqslant \delta(\mathscr{B}(\Lambda)) \,, \qquad \Lambda \in X_R \tag{3}$$

*If $n \geqslant 2$, with $\vartheta_n = n \ln n + n \ln(\ln n) + 5n$, we have*

$$1 < \nu_{R,n} \leqslant \begin{cases} e\vartheta_n (2R)^n & \text{if } R \geqslant n/2, \\ n\vartheta_n (2R)^n & \text{if } \frac{n}{2\ln n} \leqslant R < \frac{n}{2}. \end{cases}$$

*In addition, if $n \geqslant 9$ we have*

$$1 < \nu_{R,n} \leqslant \frac{4e(2R)^n n\sqrt{n}}{\ln n - 2}\left(n \ln n + n \ln(\ln n) + n \ln(2R) + \frac{1}{2}\ln(144n)\right) \tag{4}$$

*for all $R_c \leqslant R < \frac{n}{2\ln n}$.*

*Proof.* — The real number $\nu_{R,n}$ will appear as the minimal number of balls of radius $1/2$ which can cover a ball of radius $R$ in $\mathbb{R}^n$ (therefore it will be an integer). The quantity $\vartheta_n$ can be replaced by any upper bound of the covering density of $\mathbb{R}^n$ to obtain a better bound of $\nu_{R,n}$.

Let us take $R \geqslant R_c$. Let $x_1, x_2, \ldots, x_\nu$ be points of $\mathbb{R}^n$ such that

$$\bigcup_{j=1}^{\nu} \left(x_j + B(0,1/2)\right) \ \supset \ B(0,R) \tag{5}$$

and assume $\Lambda \in X_R$. Let us show that

$$\mathbb{R}^n \subset \bigcup_{i=1}^{\nu} (x_i + B(0,1/2) + \Lambda)$$

Indeed, since $\Lambda$ is a Delone set of constant $R$, for all $z \in \mathbb{R}^n$, there exists $\lambda \in \Lambda$ such that $z - \lambda \in B(0,R)$. Hence, $z$ belongs to $\Lambda + B(0,R)$. By eq. (5) there exists an integer $k \in \{1,2,\ldots,\nu\}$ such that

$$z \in \ \Lambda + x_i + B(0,1/2)$$

from which we deduce the claim.

This covering of $\mathbb{R}^n$ leads to the inequality

$$1 \leqslant \sum_{i=1}^{\nu} \chi_{\mathscr{B}(x_i+\Lambda)}(x) \qquad\qquad x \in \mathbb{R}^n$$

where $\chi_{\mathscr{B}(x_i+\Lambda)}$ is the characteristic function of the set $\mathscr{B}(x_i+\Lambda)$. We deduce, for all $t > 0$,

$$1 \leqslant \sum_{i=1}^{\nu} \frac{1}{\mathrm{vol}(tB)} \int_{tB} \chi_{\mathscr{B}(x_i+\Lambda)}(x)\,dx$$

hence

$$1 \leqslant \sum_{i=1}^{\nu} \limsup_{t\to+\infty} \left( \frac{1}{\mathrm{vol}(tB)} \int_{tB} \chi_{\mathscr{B}(x_i+\Lambda)}(x)\,dx \right)$$

Finally, since the density of $\Lambda$ is invariant by any translation of $\Lambda$ (Rogers [R], theorem 1.7), we obtain

$$1 \leqslant \nu\delta(\mathscr{B}(\Lambda))$$

It is clear that there exists a minimal integer $\nu = \nu_{R,n}$ which depends upon $R$ and $n$ which corresponds to the most economical way of covering $B(0,R)$ by balls of radius $1/2$. This minimal integer is exactly the minimal number of balls $B(0,1)$ which can cover the ball $B(0,2R)$. This integer is not known explicitely except for $n=1$ (obvious) and $n=2$ when $R<1$ (Kerschner [K]).

In order to obtain an explicit lower bound for the density of any $\Lambda \in X_R$ we have to provide an explicit upper bound of $\nu_{R,n}$ as a function of $R$ and $n$. But this is exactly the object of theorem 3.2 in Rogers ([R], p 43) and of the content of the proof of theorem 3 in Rogers ([R1], pp 163-164), replacing $R$ by $2R$ in the corresponding formulae. Therefore, we deduce the claims from these two sources. $\square$

In the above theorem we have left aside the search of the best upper estimate of the integer $\nu_{R,n}$. This search amounts to replace, if possible, $\vartheta_n$ by the best upper bound of the covering density of $\mathbb{R}^n$ when $R \geqslant n/(2\ln n)$ and $n \geqslant 2$: in the case $R < n/(2\ln n)$, $n \geqslant 9$, it seems that the upper bound of $\nu_{R,n}$ obtained by Rogers in [R] was not improved by more recent contributions. For instance, Schramm [Sc] admits in the remark p 184 in [Sc] that Rogers did better than him in [R1]. See also Bourgain and Lindenstrauss [BL] and Raigorodskii [Ra]. Let us observe that the computation of the maximal number of caps, uniformly discretely distributed on a sphere, is central in [KL] and closely related to the problem of the determination of the integers $\nu_{R,n}$.

Remark 1.— When $R \geqslant R_c$ is fixed, the number $\nu_{R,n}$ tends to infinity when $n$ goes to infinity. Therefore the sets $\{\nu_{R,n} \mid 3 \leqslant n\}$ are infinite. Similarly, when $n$ is fixed (large enough), the number $\nu_{R,n}$ tends to infinity as $R \to +\infty$.

($Q_2$) What are the sets of integers $\{\nu_{R,n} \mid R_c(n) \leqslant R, 3 \leqslant n\} \subset \mathbb{N}$?

Remark 2.— The theorem 3.2 cannot be improved in the case of lattices which are Delone sets of constant $R$. Indeed, the problem of finding the most economical way to cover the ball

$B(0,R)$ by balls of radius $1/2$ (eq.(5)), which is the key ingredient to the above theorem, is independent of the fact that a Delone set is periodical or not. This property of covering economically the ball $B(0,R)$ was used by Golay and Paige, actually not with balls but with cubes but the idea is the same (see Leech [Le] §2.3 p 670), to obtain codes at the origin of the very dense so-called Leech lattice in dimension 24.

It is usual to compare upper bounds and lower bounds of the packing density $\delta$ as a function of $n$ ([R], [GL], §19 and §38, p 390-391, [CS], chap. 1 and 9, [Z], chap. 3). Rogers had considered a regular simplex of side 2 of $\mathbb{R}^n$ whose vertices are the centres of spheres of radius 1 and denoted by $\sigma_n$ the ratio of the volume of the part of the simplex covered by the spheres to the volume of the whole simplex. We have, using the Schläfli's function $F_n(\alpha)$ (see Coxeter [Co] or Leech [Le]),

$$\sigma_n = 2^{-3n/2}(n+1)^{1/2}(n!)^2 F_n((\operatorname{arcsec} n)/2)\frac{\pi^{n/2}}{\Gamma(1+n/2)}$$

A good asymptotic approximation of $\sigma_n$ when $n$ is large is given by

$$\sigma_n \sim \frac{n}{e}2^{-n/2}$$

(Daniels' asymptotic formula, [R3]). From the theorem 7.1 in Rogers [R] $\sigma_n$ is an upper bound of the packing density $\delta$. Leech has computed the function $F_n$ for small values of $n$ and discussed this bound in terms of kissing numbers of spheres [Co] [Le] [CS] [GL] [Bo]. Recall that the maximal kissing number is bounded above by the Coxeter-Böröczky [Bo] upper bound

$$\frac{\sqrt{\pi n^3}}{e\sqrt{2}}\,2^{n/2}$$

or better by the asymptotic estimate of Kabatjanskiĭ-Levenštein [KL]

$$2^{0.401n+o(n)}$$

and below by the (nonconstructive) lower bound of Wyner [Wy]

$$2^{0.207581\ldots n+o(n)}$$

when $n$ goes to infinity. The upper bound $\sigma_n$ is known to be better than the (old) upper bound of the lattice-packing density given previously by Blichfeldt [Bl] which is $cn/(\sqrt{2})^n$ this one exhibiting also a $2^{-n/2}$-dependence with $n$.

Here the viewpoint is not that of explicit constructions. Working with packings of spheres arising from Delone sets for which we only control the constant $R$ seems more convenient to estimate the highest possible densities than considering precise constructions in which the local environments around spheres are rendered optimally compact, with spheres touching each other at the best, etc, with the difficulty to calculate accurately the kissing numbers from these models. Indeed, it seems natural to consider that very dense packings can be obtained by 'almost-touching' spheres everywhere. This means that $R$ is as small as possible, allowing many possibilities of local clustering of spheres around a central one nevertheless, between the Coxeter-Böröczky/ Kabatjanskiĭ-Levenštein bounds and the bound of Wyner for the kissing number. By 'as small as possible' we mean slightly above $R_c(n)$. But the exact expression of $R_c(n)$ is unknown. We have only an asymptotic lower bound for it, as given by the theorem 1.2. These considerations are concerned with values of the density close to $\delta$ but not with the exact description of all the arrangements of spheres in a densest sphere packing which can be extremely diversified (see Hales's works [H] [H1] on Hales-Ferguson theorem, i.e. Kepler's conjecture, for $n = 3$).

## 4. Proofs of the theorems 1.1 and 1.2

1. *Proof of the theorem 1.1.*— Let $R_c \leqslant R$ and $T > R$ be a real number. If $\Lambda$ is a Delone set of constant $R$ of $\mathbb{R}^n$, then $(B(0,R)+\Lambda) \cap B(0,T)$ covers the ball $B(0, T-R)$. Hence, the number of elements of $\Lambda \cap B(0,T)$ is at least $((T-R)/R)^n$. On the other hand, since all the balls of radius $1/2$ centred at the elements of $\Lambda \cap B(0,T)$ lie within $B(0, T+1/2)$, the proportion of space they occupy in $B(0, T+1/2)$ is at least

$$\left(\frac{T-R}{R}\right)^n \frac{\mathrm{vol}(B(0,1/2))}{\mathrm{vol}(B(0,T+1/2))} = \left(\frac{T-R}{2R(T+1/2)}\right)^n.$$

When $T$ tends to infinity the above quantity tends to $(2R)^{-n}$ which is a lower bound of the density $\delta(\mathscr{B}(\Lambda))$.

2. *Proof of the theorem 1.2.*— Let $\sigma_{KL}(n) = 2^{-0.599n}$ be the upper bound of Kabatjanskiĭ-Levenštein of the packing density $\delta$. By the theorem 1.1 we deduce that, with $R_c \leqslant R \leqslant 1$,

$$\mu_n(R) \leqslant \delta \leqslant 2^{-0.599n}$$

Raising this equation to the power $1/n$ immediately gives $2R \geqslant 2^{0.599} + o(1)$ that is $R \geqslant 2^{-0.401} + o(1)$.

## 5. Asymptotic behaviour of holes in sequences of lattices and packings

The expression of the bound $\mu_n(R)$ will be used to compute a lower bound of the Delone constant of a Delone set, or a lower bound of the covering radius of a given lattice $L \in \mathscr{U}\mathscr{D} \cap \mathscr{L}$, when its density and its minimal interpoint distance are known.

In the case of a lattice $L$, the minimal interpoint distance of $L$ is the square root of the norm $N(L)$ of the lattice (Martinet [Ma]). We will consider the normalized lattice $\frac{1}{\sqrt{N(L)}} L$ instead of the lattice $L$ to apply the preceding considerations with packings of spheres of common radius $1/2$. The situation is similar for a Delone set which will be normalized by its minimal interpoint distance.

We will denote by $\mathrm{dens}(L)$ the density of the system of spheres $L + B(0, \sqrt{N(L)}/2)$ if $L$ is a lattice and by $\mathrm{dens}(\Lambda)$ the density of the system of spheres $\Lambda + B(0, n(\Lambda)/2)$ if $\Lambda$ is a Delone set of minimal interpoint distance $n(\Lambda)$. We have (theorem 1.7 in Rogers [R])

$$\mathrm{dens}(L) = \delta(\mathscr{B}(L/\sqrt{N(L)})) \quad \text{and} \quad \mathrm{dens}(\Lambda) = \delta(\mathscr{B}(\Lambda/n(\Lambda))).$$

In the following the constructions of the mentioned lattices and packings will not be recalled. The expressions of their density and minimal interpoint distance (or norm) are taken from the cited sources. Interested readers will refer to the references given in each subsection below for each case.

Proposition 5.1. — *Let $n \geqslant 3$. If $\Lambda \in \mathcal{UD}$ is a Delone set of minimal interpoint distance $n(\Lambda)$ and of constant $R$, resp. a lattice $L \in \mathcal{UD} \cap \mathcal{L}$ of $\mathbb{R}^n$, of norm $N(L)$ and of density $dens(\Lambda)$, resp. $dens(L)$, its constant $R$, resp. its covering radius $R(L)$, satisfies:*

$$n(\Lambda)\,\tilde{t_\Lambda} \;\leqslant\; R\;, \quad \textit{resp.}\; \sqrt{N(L)}\,\tilde{t_L} \;\leqslant\; R(L)$$

*where $\tilde{t_\Lambda}$, resp. $\tilde{t_L}$, is the unique solution of the equation*

$$\mu_n(t) \;=\; dens(\Lambda)\;, \quad \textit{resp.}\; \mu_n(t) \;=\; dens(L).$$

*Proof.* — Let us prove it for a lattice of norm $1$ then of norm $N(L) \neq 1$. The proof is similar for an arbitrary Delone set.

Assume that $L$ is a lattice of norm $1$. It is easy to check that for all $n \geqslant 3$ the function $t \to \mu_n(t)$ defined on $(\sqrt{2}/2, +\infty)$ is strictly decreasing. It tends to zero when $t$ tends to infinity. The equation $\mu_n(t) = C$ has only one solution for all constant $C$ in the range $(0, \max_{t\in(\sqrt{2}/2,+\infty)} \mu_n(t))$. Therefore with $C = \delta(\mathcal{B}(L))$ there exists a unique value of $t$, say $t_L$, such that $\mu_n(t) = \delta(\mathcal{B}(L))$. Now, if we assume that $L$ is a Delone set of constant $R(L)$ strictly smaller than $t_L$ we would have $\delta(\mathcal{B}(L)) \geqslant \mu_n(R(L)) > \mu_n(t_L)$. This is impossible. We deduce the claim.

Now let us assume that $L$ is a lattice of norm $N(L) \neq 1$. By considering $L/\sqrt{N(L)}$, the preceding case shows that

$$t_{L/\sqrt{N(L)}} \;\leqslant\; R(L/\sqrt{N(L)})$$

Now $R(L/\sqrt{N(L)}) = R(L)/\sqrt{N(L)}$. As for $t_{L/\sqrt{N(L)}}$ it is obtained as unique solution of $\mu_n(t) = \delta(\mathcal{B}(L/\sqrt{N(L)}))$. But the density $\delta(\mathcal{B}(L/\sqrt{N(L)}))$ is exactly the density of the system of spheres $L + B(0, \sqrt{N(L)}/2)$ which is the density $\mathrm{dens}(L)$ of $L$. Therefore $\tilde{t_L} = t_{L/\sqrt{N(L)}}$. This proves the claim. $\square$

In the following the notations will be used:

$$t_L \;:=\; \sqrt{N(L)}\,\tilde{t_L} \quad \text{and} \quad t_\Lambda \;:=\; n(\Lambda)\,\tilde{t_\Lambda}$$

for $L$ and $\Lambda$ as in proposition 5.1.

Remark .— Let us test the lower bounds $\mu_{24}(R)$ on the covering radius of the Leech lattice $\Lambda_{24}$ in $\mathbb{R}^{24}$ for which the density $\delta(\Lambda_{24}) = \pi^{12}/479001600 = 0.001930...$ and the covering radius $R(\Lambda_{24}) = \sqrt{2}/2$ are both known [CS]. Here $N(\Lambda_{24}) = 1$. We obtain $t_{\Lambda_{24}} = 0.6487...$ This low value informs us interestingly and indirectly about the proportion of deepest holes of the lattice, which is fairly elevated.

Indeed, in a general way for a lattice $L$, we expect that the two values $t_L$ and $R(L)$ are close when the proportion of vertices of the Voronoi cell (of the lattice $L$ at the origin) which are very deep holes (including deepest holes) is weak (note that the number of vertices of this Voronoi cell which are deepest holes is at least one by definition) and that the majority of the other vertices are very shallow holes. If the proportion of deepest holes of this Voronoi cell is large and that the other vertices of this Voronoi cell are also “almost deepest” holes, the closeness of the values $t_L$ and $R(L)$ will be less good. In the case of the Leech lattice $\Lambda_{24}$, the deep holes are numerous and all of them are classified (Theorem 2 in [CS] chapter 23): there are 23 inequivalent deep holes under the congruences of $\Lambda_{24}$ and they are in one-to-one correspondance with the 23 Niemeier lattices. This indicates why both values $R(\Lambda_{24})$ and $t_{\Lambda_{24}}$ are not so close one to the other.

Let us now apply proposition 5.1 to some known sequences of lattices and packings, as given by Conway and Sloane ([CS], Chapters 5 and 8) and Martinet ([Ma] chapter V).

### 5.1. Barnes-Wall lattices

The density of the Barnes-Wall lattice $BW_n$ (Leech [Le], [CS] p 234 or p 151) in $\mathbb{R}^n$, $n = 2^m$, $m \geqslant 2$, is equal to

$$2^{-5n/4} n^{n/4} \pi^{n/2} / \Gamma(1 + n/2).$$

The norm $N(BW_n)$ is (Leech [Le] p 678) equal to $n$.

Corollary 5.2. — *Let $n = 2^m$ with $m \geqslant 2$. The covering radius $R(BW_n) \geqslant t_{BW_n}$ of the Barnes-Wall lattice $BW_n$ is such that the size of its (deepest) hole tends to infinity as (and better than)*

$$t_{BW_n} := \frac{2^{-1/4}}{\sqrt{\pi e}} n^{3/4} \, (1 + o(1))$$

*when $n$ goes to infinity.*

*Proof.* — Raising to the power $1/n$ the equation

$$2^{-5n/4} n^{n/4} \pi^{n/2} / \Gamma(1 + n/2) \;=\; \delta(\mathscr{B}(BW_n/\sqrt{n})) \;=\; \mu_n(t)$$

and allowing $n$ to tend to infinity leads easily to the claimed asymptotic expression of $t_{BW_n/\sqrt{n}}$ as a function of $n$. The multiplication of $t_{BW_n/\sqrt{n}} = t_{\tilde{BW_n}}$ by the minimal interpoint distance $\sqrt{n}$ gives the claimed lower bound $t_{BW_n}$ of the covering radius $R(BW_n)$ of $BW_n$. □

### 5.2. BCH packings

In this subsection, we will refer to [CS] p 155. Let $n = 2^m, m \geqslant 4$. The packings of equal spheres we are considering are obtained using extended BCH codes in construction C of length $n$. They are not lattices. There are two packings (a and b) which use two different codes of the Hamming distances. Let us denote the second one by $P_{nb}$. Its density $\mathrm{dens}(P_{nb})$ satisfies

$$\log_2 \mathrm{dens}(P_{nb}) \;\simeq\; -\frac{1}{2} n \log_2 \log_2 n \qquad \text{as } n \to +\infty$$

and its minimal interpoint distance is ([CS] p 150) $n(P_{nb}) = \sqrt{\gamma}\, 2^a$ with $\gamma = 2$ and $a = [(m-1)/2]$. From this behaviour we deduce

Corollary 5.3. — *Let* $n = 2^m$ *with* $m \geqslant 4$. *The Delone constant* $R(P_{nb}) \geqslant t_{P_{nb}}$ *of the BCH packing* $P_{nb}$ *tends to infinity as (and better than)*

$$t_{P_{nb}} = 2^{-\frac{1}{2}+[(-1+\log_2 n)/2]}\sqrt{\log_2 n}\,(1+o(1)) \;\simeq\; \frac{1}{\sqrt{2}}\log_2 n\,(1+o(1))$$

*when* $n$ *goes to infinity.*

The proof of this corollary can be made with the same arguments as in the proof of corollary 5.2.

### 5.3. Craig lattices

These lattices are known to be among the densest ones (Martinet [Ma] p 136-143, Conway and Sloane [CS] p 222-224).

The density $\mathrm{dens}(\mathbb{A}_n^{(r)})$ of the Craig lattice $\mathbb{A}_n^{(r)}$, $n \geqslant 1$, $r \geqslant 1$, in $\mathbb{R}^n$ is at least

$$\frac{(r/2)^{n/2}}{(n+1)^{r-1/2}}\frac{\pi^{n/2}}{\Gamma(1+n/2)}$$

with equality if the norm of the lattice is $2r$.

The norm of Craig lattices is not known in general and lower bounds of $N(\mathbb{A}_n^{(r)})$ were obtained by Craig [Ma] [BB] [Cr]. The determination of $N(\mathbb{A}_n^{(r)})$ is equivalent to the so-called Tarry-Escott problem in combinatorics and does not seem to be solved yet. However, for some values of $n$ and $r$ this norm is known.

Theorem 5.4. — *(i) (Craig) If* $n+1$ *is a prime number* $p$ *and* $r < n/2$, *then* $N(\mathbb{A}_n^{(r)}) \geqslant 2r$.

*(ii) (Bachoc and Batut) If* $n+1$ *is a prime number* $p$ *with* $r$ *a strict divisor of* $n = p-1$, *then* $N(\mathbb{A}_n^{(r)}) = 2r$.

Bachoc and Batut [BB] made the following more general conjecture from an exhaustive investigation of many Craig lattices.

Conjecture 1. — *(Bachoc - Batut) For all* $n$ *such that* $n+1$ *is a prime number* $p$ *and for all* $1 \leqslant r \leqslant (p+1)/4$, *we have*

$$N(\mathbb{A}_n^{(r)}) \;=\; 2r.$$

This conjecture was partially proved by Elkies, cited in Gross [Gr], in the case $p \equiv 3 \bmod 4$ and $r = (p+1)/4$. Elkies's result arises from the general theory of Mordell-Weil lattices developped by Elkies and Shioda concerning the groups of rational points of elliptic curves over function fields [Sh].

Using the assertion *(ii)* in the theorem 5.4 we obtain the following corollary.

COROLLARY 5.5. — *Assume $n$ such that $n+1$ is a prime number $p$ and $r$ a strict divisor of $n$. Then, the covering radius $R(\mathbb{A}_n^{(r)}) \geqslant t_{\mathbb{A}_n^{(r)}}$ of the Craig lattice $\mathbb{A}_n^{(r)}$ is such that the size of its (deepest) hole tends to infinity as (and better than)*

$$t_{\mathbb{A}_n^{(r)}} := \frac{1}{\sqrt{2\pi e}}\sqrt{n}(1+o(1))$$

*when $n$ goes to infinity.*

Let us remark that $t_{\mathbb{A}_n^{(r)}}$ is independant of $r$ when $n$ is large enough.

If the conjecture 1 is true, then the assumption of the above corollary 5.5 can be replaced by the following assumption

"For all $n$ such that $n+1$ is a prime $p$ and for all $1 \leqslant r \leqslant (p+1)/4$"

and this replacement would lead to the same result.

As shown by the corollaries 5.2 and 5.5 the deep holes of the Barnes-Wall and Craig lattices $BW_n$ and $\mathbb{A}_n^{(r)}$ have a size which goes to infinity with $n$ ($r$ fixed). In order to allow comparison between them and with the theorem of Butler (theorem 2.1), we have to consider the normalized lattices

$$\frac{1}{\sqrt{n}}BW_n \quad \text{and} \quad \frac{1}{\sqrt{2r}}\mathbb{A}_n^{(r)}$$

assuming that $n$ is such that $n+1$ is a prime number and that $1 \leqslant r \leqslant (n+2)/4$ (conjecture 1). In the first case, the covering radius tends to infinity with $n$ leaving no hope to obtain very dense packings of spheres from the lattices $BW_n$ when $n$ is large enough. In the second case, since

$$t_{\mathbb{A}_n^{(r)}/\sqrt{2r}} \;=\; \frac{1}{2\sqrt{\pi e}}\sqrt{\frac{n}{r}}$$

we see that $t_{\mathbb{A}_n^{(r)}/\sqrt{2r}} > 1$ if $r < \frac{1}{4\pi e}n$. Let us recall, from the theorem 2.1, that the existence of very dense lattices (of minimal interpoint distance one) of covering radius as close as $1$ is expected. Therefore we can expect to find very dense Craig lattices satisfying this condition when $r = r(n)$ is a suitable function of $n$ and large enough, namely:

$$r(n) > \frac{1}{4\pi e}n$$

for which the lower bound $t_{\mathbb{A}_n^{(r)}/\sqrt{2r}}$ of $R(\mathbb{A}_n^{(r)}/\sqrt{2r})$ is less than unity. Since we are under the assumptions of conjecture 1, the possible range for $r(n)$ for such an opportunity becomes

$$\frac{1}{4\pi e}n \;<\; r(n) \;\leqslant\; \frac{n+2}{4}.$$

On the other hand, the density $\text{dens}(\mathbb{A}_n^{(r)})$ reaches its maximum when $r$ is the integer the closest to $\frac{n}{2\ln(n+1)}$ (obtained by cancelling the derivative of $\text{dens}(\mathbb{A}_n^{(r)})$ with respect to $r$, with $n$ fixed).

($Q_3$) Does there exist normalized Craig lattices $\mathbb{A}_n^{(r)}/\sqrt{N(\mathbb{A}_n^{(r)})}$ (for general $n$ and $r$) which exhibit a Delone constant (covering radius) smaller than $1$ ?

### 5.4. Mordell-Weil lattices

We will refer here to the class of Mordell-Weil lattices given by the following theorem of Shioda ([Sh1] Theorem 1.1).

THEOREM 5.6. — ( Shioda ) *Let* $p$ *be a prime number such that* $p+1 \equiv 0\ (mod\, 6)$ *and* $k$ *any field containing* $\mathbb{F}_{p^2}$. *The Mordell-Weil lattice* $E(K)$ *of the elliptic curve* $E$

$$y^2 = x^3 + 1 + u^{p+1} \tag{1}$$

*defined over the rational function field* $K$, *where* $K = k(u)$, *is a positive-definite even integral lattice with the following invariants:*

$$\begin{array}{rcl} \text{rank} & = & 2p-2 \\ \det & = & p^{\frac{p-5}{3}} \\ N(E(K)) & = & \frac{p+1}{3} \\ \text{centre density } \Delta & = & \frac{\left(\frac{p+1}{12}\right)^{p-1}}{p^{(p-5)/6}} \\ \text{kissing number} & \geqslant & 6p(p-1) \end{array}$$

Recall that the centre density $\Delta$ is the quotient of the density of the lattice by the volume $\pi^{n/2}/\Gamma(1+n/2)$ of the unit ball of $\mathbb{R}^n$.

Such a lattice in $\mathbb{R}^{2p-2}$, denoted by $MW_n$ with $n = 2p-2$, has a minimal interpoint distance equal to $\sqrt{(p+1)/3}$ and a density

$$\text{dens}(MW_n) = \Delta \frac{\pi^{p-1}}{\Gamma(p)}$$

We deduce that

$$\begin{aligned} \tilde{t}_{MW_n} &\simeq \frac{1}{2} \frac{\sqrt{\pi}}{(\Gamma(p))^{1/(2p-2)}} \frac{\left(\frac{p+1}{12}\right)^{1/2}}{p^{(p-5)/(12(p-1))}} \\ &\simeq \frac{\sqrt{\pi\, e}}{4\sqrt{3}}\, p^{-1/12} \simeq 2^{-2+1/12} \frac{\sqrt{\pi\, e}}{\sqrt{3}}\, n^{-1/12} \end{aligned}$$

This value goes to zero while

$$t_{MW_n} \simeq 2^{1/12} \frac{\sqrt{\pi\, e}}{12\sqrt{2}}\, n^{5/12}$$

goes to infinity when $p$ (or $n$) tends to infinity. This result indicates that the deep holes of the normalized Mordell-Weil lattice $MW_n/\sqrt{N(MW_n)}$ are in fact very shallow, and may be probably bounded above independently of $n$. This leads to ask the following question.

($Q_4$) Does there exist normalized Mordell-Weil lattices $MW_n/\sqrt{N(MW_n)}$ which exhibit a Delone constant (covering radius) smaller than 1 ?

## 6. Comments and conjecture

The lower bound $\mu_n(R)$ of $\delta$ is particularly interesting for Delone sets of constant $R$ of $\mathbb{R}^n$ which are saturated, that is when $R < 1$. Indeed, it exhibits a dependence with $n$ which is in

$$(2R)^{-n} = 2^{-n(1+\log_2 R)}.$$

Now taking $R = R^{(s)} = 1$ (maximal value of the constant $R$ for saturated Delone sets) gives a $2^{-n}$ dependence typical of the Minkowski-Hlawka type lower bounds of $\delta_L$ while taking $R = \sqrt{2}/2$ (the Blichfeldt bound, lemma 3.1) provides a $2^{-n/2}$ dependence typical of the Rogers bound $\sigma_n$. Inbetween all values of $R$ are formally possible but the range is limited (theorem 1.2).

The theorem 1.1 actually gives a partial answer to old expectations when $R$ is close to $R_c(n)$. Indeed, recall the words of Gruber, Lekkerkerker and Rogers.

First, in [GL] p 391, *the best known upper and lower bounds for* $\delta$ *differ by a factor which is approximately* $2^{n/2}$. *This means that the problem of closest packing of spheres is still far from its solution (except for low values of* n*).*

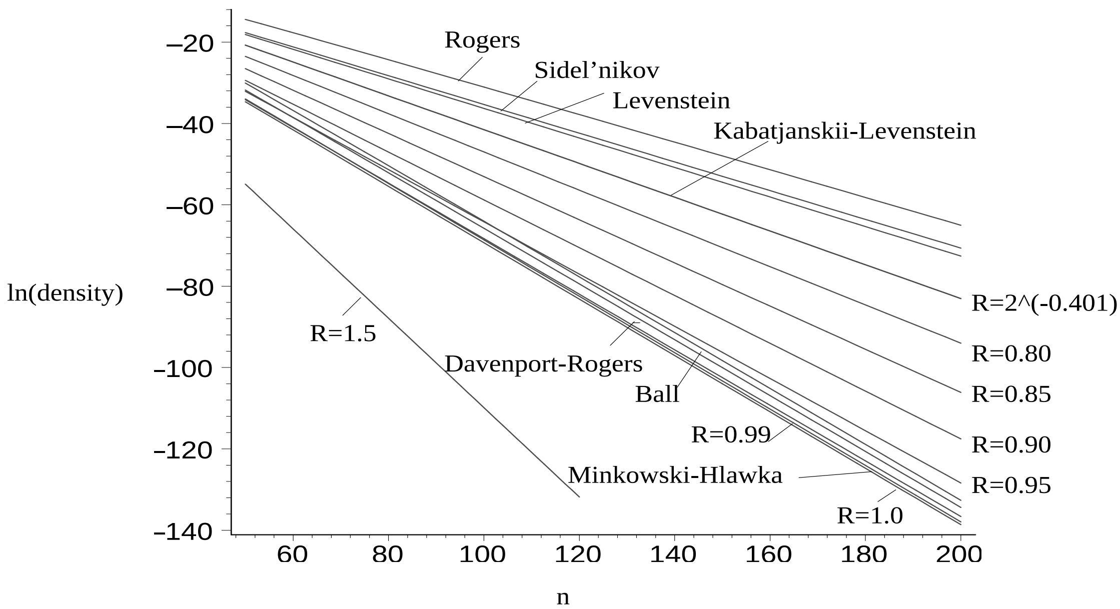


Figure 1 : Upper bounds of the packing density $\delta$ and lower bounds of the lattice-packing density $\delta_L$. The $R$-dependent lower bounds $\mu_n(R)$ are plotted for $R = 2^{-0.401}$, 0.8, 0.85, 0.90, 0.95, 0.99, 1.5 as a function of the dimension $n$.

If we cite [R] p 9, we were still up till the present results in the situation where "*There remains a wide gap between the results of the Minkowski-Hlawka type, ..., and the results of Blichfeldt type, ...*".

In Figure 1 are plotted the $R$-dependent bound $\mu_n(R)$ for several values of $R$, the upper bounds of Rogers, Sidel'nikov, Levenštein, Kabatjanskiĭ-Levenštein, the lower bounds of Davenport-Rogers, Ball and of Minkowski-Hlawka, as a function of the dimension $n$. All values between these two types of bounds can be reached by $\mu_n(R)$ when $R$ is suitably chosen below 1.

The curve $n \to \mu_n(R)$ for $R = 1$ is slightly below the Minkowski-Hlawka bound. When $R$ is greater than 1, the curves $n \to \mu_n(R)$ are entirely below the Minkowski-Hlawka bound. On the contrary, when $R < 1$ is close to unity, the curve $\mu_n(R)$ lies below the Minkowski-Hlawka bound up till a certain value of $n$ and then dominates it, as expected asymptotically. When $2^{-0.401} < R < 1$ lies far enough from 1 the entire curve $n \to \mu_n(R)$ lies strictly between the two types of bounds (Kabatjanskiĭ-Levenštein and Minkowki-Hlawka).

| Type | Name | $\log_2 \Delta$ |
|---|---|---|
| constructions | Barnes-Wall $BW_{65536}$ | 180224 |
| | $B_{65536}$ | 290998 |
| | $\eta(\overline{\Lambda_{32}})$ | 295120 |
| | Craig $\mathbb{A}_{65536}^{(2954)}$ | 297740 |
| (existence) lower bounds | Minkowski-Hlawka | 324603 |
| of $\delta_L$ | Davenport-Rogers | 324616 |
| | Ball | 324620 |
| $\mu_{65536}(R)$ | $R = 1.5$ | 286266 |
| | $R = 1.0$ | 324602 |
| lower bounds | $R = 0.99$ | 325553 |
| from theorem 1.1 | $R = 0.95$ | 329452 |
| | $R = 0.90$ | 334564 |
| | $R = 0.85$ | 339968 |
| | $R = 0.80$ | 345700 |
| | $R = 2^{-0.401}$ | 350882 |
| upper bounds | Kabatjanskiĭ-Levenštein | 350882 |
| | Levenštein | 355818 |
| of $\delta$ | Sidel'nikov | 356742 |
| | Rogers | 357385 |

Table 3 : Table 1.4 of [CS] chap. 1 to which we have added the lower bounds $\mu_{65536}(R)$ for different values of $R$ (the values of the centre density $\log_2 \Delta$ are recomputed from the original references).

The theorem 1.2 does not say anything about the frequency and the density of such middle-sized Voronoi cells of circumradius $R$ approximately equal to $2^{-0.401}$ in a general saturated Delone set of $\mathbb{R}^n$ of constant $R$ when $n$ is sufficiently large, in particular in the densest ones.

To allow comparison with known results in literature and to follow Conway and Sloane [CS] we have taken $n$ fairly large, namely $n = 65536$. To appreciate the pertinency of the formula given by the theorem 1.1 we have reproduced in Table 3 the Table 1.4 of [CS] chap.

1 and added therein the values of the centre density $\Delta$ deduced from $\mu_{65536}(R)$ for $R = 2^{-0.401}, 0.8, 0.85, 0.90, 0.95, 0.99, 1.0, 1.5$. The value of (the logarithm in base 2 of) the centre density $\Delta$ computed from $\mu_{65536}(R)$ now sticks to the Kabatjanskiĭ-Levenštein's bound when $R$ is at its asymptotic maximum $R = 2^{-0.401}$. Is this value reached by the Delone constant of a Delone set?

When $n$ is large enough, the sensitivity of $\mu_n(R)$ to the Delone constant $R$ can be perceived by the following comparison (see Table 3): the centre density 324602 relative to the bound $\mu_{65536}(1)$ is slightly below the lower bound 324603 of Minkowski-Hlawka, as expected, whereas the centre density 325553 relative to $\mu_{65536}(0.99)$ is slightly above the best lower bound 324620 of Ball. This gives credit to the conjecture that lattices do not exhibit a covering radius less than 1 when $n$ is sufficiently large.

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