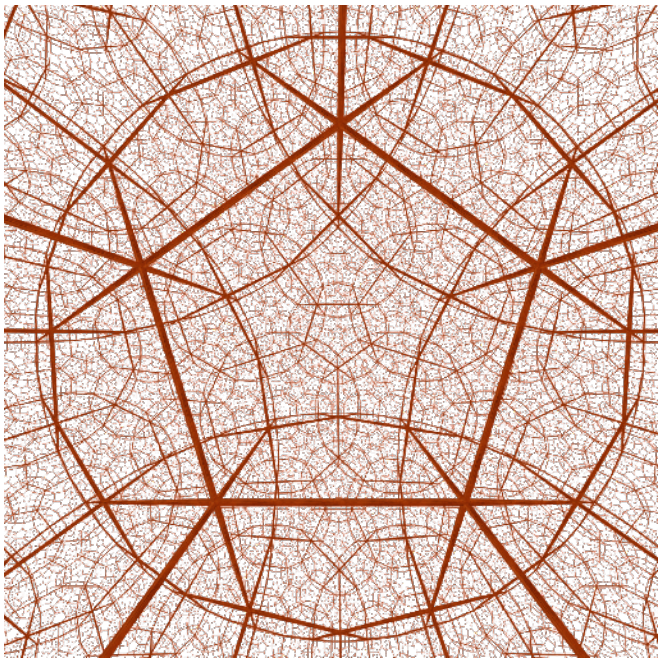
Master 2 — 2020-21 — Grenoble
Groups and Geometries



The geometry of many interesting objects is quite often more involved (and interesting) than the euclidean geometry.

For instance, the so-called closed surfaces can be equipped with a uniformized geometry: spherical, flat or locally hyperbolic, the later being applicable for all such closed surfaces except the sphere, and the torus.

Another example: quite often, the complement of knot, or a link, in $\mathbf{S}^3$ can be equipped with a complete metric that is locally hyperbolic, locally modeled on the real hyperbolic space of dimension 3.

Those geometries can be studied in the point of view of the Erlangen program of Felix Klein, by the algebra of a Lie group like SL(2, **R**), or SL(2, **C**), or SO(n, 1) or SU(n, 1).
But the importance of groups in these descriptions does not stop there. From a model space, like the hyperbolic space $\mathbf{H}^3$ to the

manifold, one has a covering map, and the action of the discrete fundamental group by deck transformations. This group inherits some geometric and algebraic (eg cohomological) properties, and this entanglement makes a work-ground for its study, and makes it suitable for studying the corresponding manifold. The picture of the previous page shows a regular tessellation of the hyperbolic space $\mathbf{H}^3$ by hyperbolic dodecahedra, tessellation encoded by the action of a discrete group of isometries, fundamental group of some hyperbolic quotient object.

In this interaction between groups and geometry, we will enter in a lively topic, at the level of contemporary research.

Program.

September 2020. Preliminary Course, 9h. (2 weeks)
Fundamental groups, covering, presentations of groups.

September-December 2020. Choice of **2 courses** among three. (33h and 18h of exercises):

- —Hyperbolic spaces : Geometry and Discrete Groups.
  by Anne Parreau and Pierre Will
- —Algorithmic Topology and Groups,
  by Francis Lazarus (2/3), and François Dahmani (1/3)
- —Representation theory and homological algebra,
  by Claire Amiot and Estanislao Herscovich

January-April 2021. Choice of **1 course** among 2. (24h)

- —Effective methods for arithmetic groups,
  by Martin Deraux
- —Hyperbolicities in discrete groups,
  by François Dahmani

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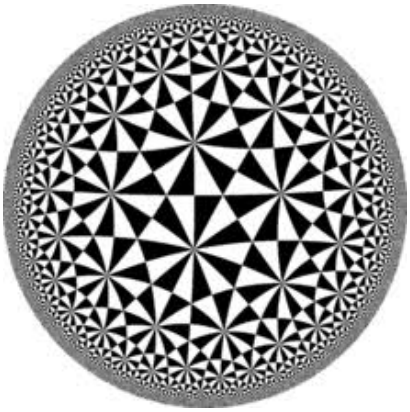
1 — **Hyperbolic Spaces : Geometry and Discrete Groups** (Anne Parreau and Pierre Will, 33h, 18h of tutoring, 12 ECTS)



Hyperbolic spaces play a central role in many domains of geometry and are today a well established domain of study. They can be studied as model spaces for their geometry, and for their discrete groups of isometries, and quotients. The purpose of the course is to give a description of these spaces, and construct examples of lattices. We will start by the fundamental example of Fuchsian groups, and the real hyperbolic plane. Tools and methods will be illustrated here. Then we will turn to higher dimensional real hyperbolic spaces, and also complex hyperbolic spaces. If time permits, we will discuss non-hyperbolic higher rank spaces

Content:
—Fuchsian groups, and the hyperbolic plane.
—Real hyperbolic spaces, of higher dimension.
—Complex Hyperbolic spaces, and/or higher rank symmetric spaces.

Possible internship topics:
— Complex Hyperbolic spaces.
— Higher rank symmetric spaces.

Bibliography

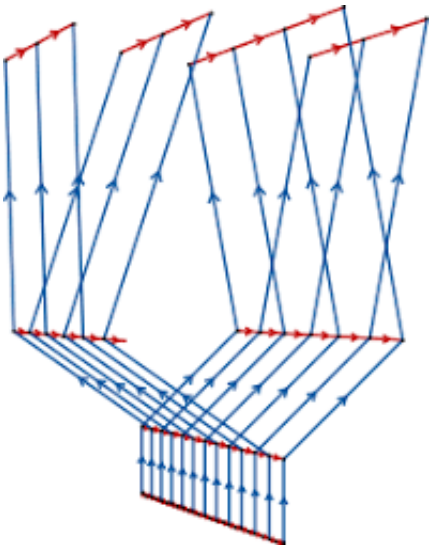
A. Beardon, The geometry of discrete groups. Corrected reprint of the 1983 original. Graduate Texts in Mathematics, 91. *Springer-Verlag, New York,* 1995. xii+337 pp. ISBN: 0-387-90788-2

F. Bonahon, Low-Dimensional Geometry: From Euclidean Surfaces to Hyperbolic Knots, Student Mathematical Library Vol. 49; 2009; 384 pp. ISBN: 978-0-8218-4816-6

W. Thurston : Three dimensional geometry and topology, vol. 1, Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. x+311 pp. ISBN: 0-691-08304-5

Prerequisite: basic notions of topology, algebra, and differential calculus.

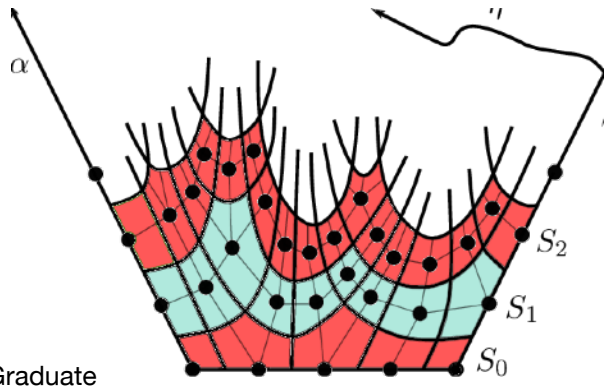
2 — **Algorithmic Topology and Groups**, Francis Lazarus, and François Dahmani (33h, 18h of tutoring, 12 ECTS)



Describing objects by discrete structures has many advantages, but makes all the more clear the limits of our understanding. For instance, one can compute easily presentations of fundamental groups of simplicial complexes, but in general, determining whether this group is trivial is impossible (undecidable). This course proposes an algorithmic approach to treat, in favorable cases, some fundamental problems in topology, often interpreted in terms of problems in fundamental groups. These problems are the homotopy between curves, the existence of homeomorphisms between spaces, and the triviality of a knot. The case of surface groups will be important to us.

Content:
— Triangulations
— Word and conjugacy problem for surface groups
— Normal surfaces in a triangulated 3-manifold
— Geometry of the word problem: Snowflake groups, right-angled Artin groups.

Possible internships:
— Right-angled Coxeter and Artin Groups, and special CAT(0) cube complexes
— Conjugacy problem in GL(n,Z) or MCG(Sigma)



Picture credit: Anne Thomas

Bibliography

J. Stillwell. Classical topology and combinatorial group theory. Graduate Texts in Mathematics, 72. *Springer-Verlag, New York,* 1993. xii+334 pp. ISBN: 0-387-97970-0

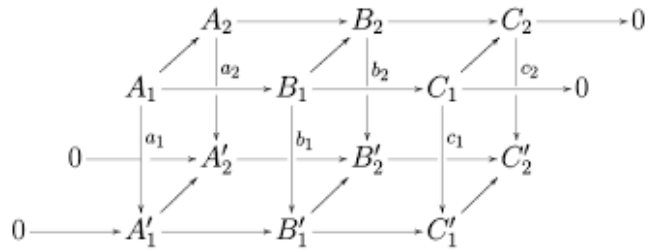
J. Gross, T. Tucker. Topological Graph Theory. Dover Publications, Inc., Mineola, NY, 2001. xvi+361 pp. ISBN: 0-486-41741-7

B. Mohar, C. Thomassen. Graphs on surfaces. Johns Hopkins Studies in the Mathematical Sciences. 2001. xii+291 pp. ISBN: 0-8018-6689-8

N. Brady, T. Riley, H. Short, The geometry of the word problem for finitely generated groups, Advanced Courses in Mathematics. CRM Barcelona. *Birkhäuser Verlag, Basel,* 2007. x+206 pp. ISBN: 978-3-7643-7949-0

http://www.gipsa-lab.fr/~francis.lazarus/Enseignement/compuTopo.html

3 — Representation Theory and Homological Algebra (Claire Amiot, Estanislao Herscovich, 33h et 18h of tutoring, 12 ECTS)



Roughly speaking, representation theory is the study of symmetries of linear spaces. Leaving aside its beauty, it has many applications in combinatorics, geometry, number theory, probability, and even quantum theory, to mention a few. The first part of the course will deal with basic notions of representation theory and category theory. The second part will cover the basics of homological algebra, which is, grosso modo, a pervasive set of tools to build invariants allowing to distinguish and study quite complicated objects in algebra, topology and geometry. From these invariants stem the main importance of homological algebra: its possibility to produce tools that allow to prove statements outside the scope of homological algebra. Some major examples of this phenomenon are the proof of the central theorem in algebraic geometry stating that the localisation of a regular local ring at a prime ideal is also regular, or that every regular local ring is an UFD.

Contents:
Part I. Representation theory
— Module theory. — Categories, functors, and natural transformations
— Simple, free, projective and injective modules. — Artin-Wedderburn theorem
— Maschke's theorem. — Krull-Schmidt theorem

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0 & \to & \mathrm{Hom}(H_p,G) & \xrightarrow{q^*} & \mathrm{Hom}(Z_p,G) & \xrightarrow{j^*} & \mathrm{Hom}(B_p,G) & & \\
 & & & & f^* \downarrow\uparrow i^* & & \downarrow \partial^* & & \\
 & & \mathrm{Hom}(C_{p-1},G) & \xrightarrow{\delta} & \mathrm{Hom}(C_p,G) & \xrightarrow{\delta} & \mathrm{Hom}(C_{p+1},G) & & \\
 & & \downarrow i^* & & \uparrow \partial^* & & & & \\
 & & \mathrm{Hom}(Z_{p-1},G) & \xrightarrow{j^*} & \mathrm{Hom}(B_{p-1},G) & \to & \mathrm{Ext}(H_p,G) & \to & 0 \\
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Part II. Homological algebra
— Complexes and homology — Snake lemma — Projective and injective resolutions — Derived functors — Tor and Ext, Künneth formula — Group (co)homology— Eckmann-Shapiro lemma

Possible internships:
—Cohomology of Hopf algebras
—Koszul property for quiver algebras;
—Representation theory of a skew-group alegbra.

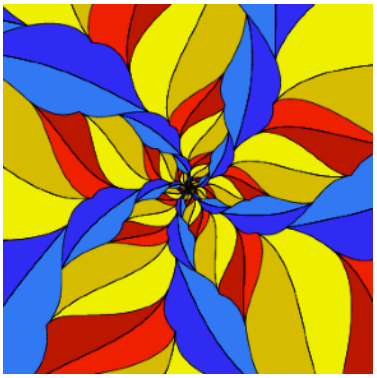
Bibliography

Assem, Ibrahim. Algèbres et modules: cours et exercices. Presses Université Ottawa, 1997. 330 pp.
Weibel, Charles A. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp

Prerequisite:
- A first course on abstract algebra (covering group theory and ring theory).
- A course on basic topology.

Advanced courses.

1 — Effective Methods for Arithmetic Groups (Martin Deraux, 24h, 6 ECTS)



Arithmetic groups are essentially groups that can be written as the set of all matrices with integer entries in a suitable linear group. They are objects of fundamental interest in many areas of mathematics, including geometry, topology and number theory. The aim of this course is to develop tools to find a fundamental domain for the action of a given arithmetic group of isometries of a (real or complex) hyperbolic space. This will allow us to write explicit finite presentations, to solve the word problem, and to get a list of conjugacy classes of finite subgroups, for instance.

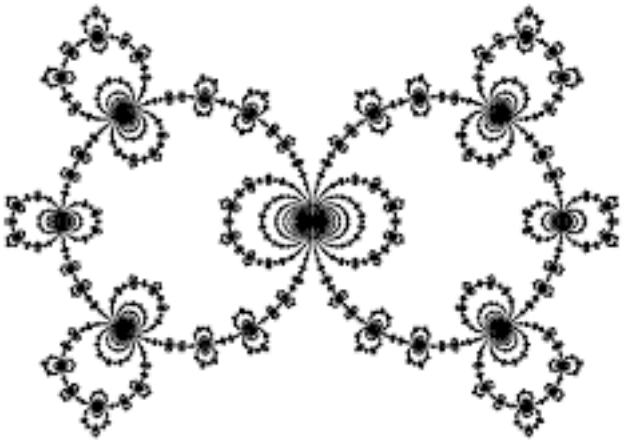
Content:
— Arithmetic groups of isometries of real and complex hyperbolic spaces.
— Fundamental domains, and their identifications.
— Torsion, and word problem.
— Selberg's lemma and effective aspects
— Reidemeister-Schreier method.

Bibliography:
C. McLachlan, A. Reid, The Arithmetic of Hyperbolic 3-Manifolds, GTM, volume 219, Springer.
D. Witte-Morris, Introduction to Arithmetic Groups, https://arxiv.org/abs/math/0106063

Prerequesites:

First semester courses (at least « Hyperbolic Spaces Geometry and Discrete Groups »).

Picture credit: R. Schwartz



2 — Hyperbolicities in Discrete Groups (François Dahmani, 24h, 6 ECTS)

Max Dehn used hyperbolicity of the complex disc to study fundamental groups of surfaces, but the full strength of this point of view was only explained in 1986 by Misha Gromov, who developed the theory of hyperbolic groups. Recently partial hyperbolicity properties have been discovered and used in various settings. This course will visit them, and many examples.

Content:
— Hyperbolic groups,
— Relatively hyperbolic groups,
— The Dehn filling theorem,
— Acylindrical hyperbolicity,
— Applications.

Bibliography:
M. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften, 319. *Springer-Verlag, Berlin,* 1999. xxii+643 pp. ISBN: 3-540-64324-9

Picture credit : K. Ruane, C. Hruska,

Prerequisite: first semester course including preferably « Algorithmic topology and groups »

Prerequisite for the program.

The program of M2-MF is a second year of the Master of Mathematics and applications, for which the students are assumed to have studied the curriculum of the M1-General-Mathematics. In the context of this program, algebra, geometry, topology are predominant.

Scientific and pedagogical team

Claire Amiot	https://www-fourier.ujf-grenoble.fr/~amiot/indexfr.html
François Dahmani	https://www-fourier.ujf-grenoble.fr/~dahmani/
Martin Deraux	https://www-fourier.ujf-grenoble.fr/~deraux/
Estanislao Herscovich	https://www-fourier.ujf-grenoble.fr/~eherscov/
Francis Lazarus	http://www.gipsa-lab.grenoble-inp.fr/~francis.lazarus/
Anne Parreau	https://www-fourier.ujf-grenoble.fr/~parreau/
Pierre Will	https://www-fourier.ujf-grenoble.fr/~will/