Involution and commutator length for complex hyperbolic isometries

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Abstract

We study decompositions of complex hyperbolic isometries as products of involutions. We show that PU(2,1) has involution length 4 and commutator length 1, and that for all $n \geqslant 3$ PU(n,1) has involution length at most 8.

1 Introduction

Riemannian symmetric spaces are characterized by the existence of special isometries called *central involutions*: for each point p of such a space X, there exists an involution $I_p \in \mathrm{Isom}(X)$ such that p is an isolated fixed point of I_p and $d_pI_p = -\mathrm{Id} \in \mathrm{GL}(T_pX)$. The group of displacements of a Riemannian symmetric space X is the subgroup of the isometry group $\mathrm{Isom}(X)$ which is generated by pairwise products of central involutions. It is a classical fact that for connected symmetric spaces, it coincides with the identity component $\mathrm{Isom}^0(X)$ (see for example Proposition IV-1.4 of [L]). This means that every isometry in the identity component is a product of a finite (even) number of central involutions.

It is then a natural question to ask, given a symmetric space X, what the central involution length of $\mathrm{Isom}^0(X)$ is, i.e. the smallest $n \in \mathbb{N}$ (if any) such that any element of $\mathrm{Isom}^0(X)$ is a product of at most n central involutions. One can also relax the question to more general involutions, which is also of geometric interest as it allows for example to consider reflections, which have fixed-point loci of maximal (rather than minimal) dimension.

Basmajian and Maskit investigated in [BM] the involution length of $\mathrm{Isom}(X)$ when X is a symmetric space of constant (sectional) curvature, i.e. one of the model spaces S^n, E^n or H^n. They found that, allowing orientation-reversing involutions the involution length is always 2, whereas if one restricts to orientation-preserving involutions (i.e. involutions in the identity component $\mathrm{Isom}^0(X)$) it is 2 or 3, depending explicitly on the space and the congruence class of n mod. 4. They deduce from these facts that every element of $\mathrm{Isom}^0(X)$ is a commutator, i.e. the commutator length of $\mathrm{Isom}^0(X)$ is 1. This follows from the remark that every square of a triple product of involutions is a commutator. Indeed, for any triple of involutions (I_1, I_2, I_3), we have

$$(I_1I_2I_3)^2 = [I_1I_2, I_3I_2]. \tag{1}$$

In this paper we study the analogous question in $\mathrm{Isom}(X)$ when X is complex hyperbolic space $\mathrm{H}^n_{\mathbb{C}}$, the model complex symmetric space of constant negative holomorphic sectional curvature. Here $\mathrm{Isom}(X)$ has 2 connected components, one consisting of all holomorphic isometries (the identity component, isomorphic to $\mathrm{PU}(n,1)$) and the other consisting of all antiholomorphic isometries. It is well known that any element of $\mathrm{PU}(n,1)$ is a product of 2 antiholomorphic involutions (usually called *real reflections*); this was originally observed by Falbel and Zocca in [FZ] when $n = 2$ then for all values of n by Choi in [C] (see also [GT], and [N] for the elliptic case, corresponding to $\mathrm{U}(n)$). However, only special elements of $\mathrm{PU}(n,1)$ are products of two holomorphic involutions (see Lemma 4 in the case of PU(2,1)). The involution length of $\mathrm{PU}(n,1)$ is thus at least 3 (for $n \geqslant 2$). Our main result is the following:

Theorem 1 *The involution length of* $\mathrm{PU}(2,1)$ *is 4.*

(We also show the analogous statements where "involution" is replaced by "central involution", or by "complex reflection of order 2".) More specifically, we show that all loxodromic and parabolic isometries in $\mathrm{PU}(2,1)$

are triple products of involutions, whereas some elliptic conjugacy classes are not. More precisely, we give in Proposition 10 and Corollary 4 a precise description of those regular elliptic elements that are not products of three involutions. This description is made in terms of the *angle pair* of an elliptic isometry. Elliptic isometries preserve two orthogonal complex lines on which they act by rotation; the angle pair is the pair formed by these two rotation angles (see Section 3.3.3). The angle pair determines the conjugacy class of an elliptic element. It should be noted that there is a slight subtlety here. Loxodromic conjugacy classes in PU(2,1) are determined unambiguously by the trace of any lift to SU(2,1). This is not the case for elliptic isometries: for a given value of the trace, there are generically three possible angle pairs, which correspond to the various possible relative positions of eigenvectors and the light-cone in $\mathbb{C}^{2,1}$ (see Section 3.3.3). In particular, one can show that any complex number can be realized as the trace of a triple product of involutions though not all isometries are products of three involutions.

Our method is based on the use of the product map on the product of two semisimple conjugacy classes (see e.g. [FW2, P]). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two conjugacy classes. The product map on $\mathcal{C}_1 \times \mathcal{C}_2$ is defined by

$$\begin{aligned} \tilde{\mu} \quad : \quad & \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{G} \\ & (A, B) \longmapsto [AB], \end{aligned} \tag{2}$$

where $\mathcal{G}$ is the space of conjugacy classes of PU(2,1) (see Section 3.3.4) and $[\cdot]$ denotes the conjugacy class of an element. We review the main properties of this map in Section 4. The image by $\tilde{\mu}$ of reducible pairs (A, B) form the so-called *reducible walls* that divide $\mathcal{G}$ into *chambers*. The crucial fact is that when $\mathcal{C}_1$ and $\mathcal{C}_2$ are semisimple classes, these chambers are either full or empty, i.e. $\operatorname{Im} \tilde{\mu}$ is a union of chamber closures (see Sections 4.2, 4.3). In our case we consider this map when $\mathcal{C}_1$ is the conjugacy class of a product of two involutions and $\mathcal{C}_2$ is the conjugacy class of an involution. Applying this method we are able to determine which elliptic and loxodromic conjugacy classes are triple products of involutions. We have to deal with parabolic conjugacy classes separately as they aren't semisimple and cannot be separated from conjugacy classes of complex reflections. To prove that the involution length of PU(2,1) is 4, we show that the map $\tilde{\mu}$ becomes surjective when both $\mathcal{C}_1$ and $\mathcal{C}_2$ are conjugacy classes of products of two involutions.

We also obtain as a byproduct of these results that $\mathrm{PU}(2,1)$ has commutator length 1 (Theorem 5), but slightly more indirectly than in [BM]. Indeed, we show that even though not every element of $\mathrm{PU}(2,1)$ is a triple product of involutions, it is the square of a triple product of involutions and conclude using (1).

In higher dimensions, i.e. in $\mathrm{PU}(n,1)$ with $n \geqslant 3$, the involution length will be at least 3, for the same reason as above (pairwise products of involutions have special properties). However the finer methods that we use in this work to improve the lower bound to 4 (so, prove that not every element is a product of 3 involutions), and provide an upper bound of 4 (so, prove that every element is a product of 4 involutions) do not extend easily to higher dimensions, as they rely on a detailed understanding the chamber structure in the space of elliptic conjugacy classes in $\mathrm{PU}(2,1)$ (see Section 4 for more details), which gets significantly more complicated in higher dimensions. Djokovic and Malzan proved in [DM1] that the length of $\mathrm{SU}(n)$ with respect to complex reflections of order 2 is $2n-1$, and in [DM2] that the corresponding length in $\mathrm{SU}(p,q)$ (with $p, q \geqslant 1$) is $p+q+2$ or $p+q+3$ (depending on the parity of $p+q$). By combining our results for $n = 2$ with results of [GT] (namely their bound on the involution length of $\mathrm{SU}(n)$) we obtain the following result (Theorem 2):

Theorem 2 *For all $n \geqslant 2$, the involution length of PU(n,1) is at most 8.*

The paper is organized as follows. In Section 2, we present some classical facts on products of isometries in the Poincaré disk for later reference. Section 3 is devoted to the description of conjugacy classes in PU(2,1). In Section 4, we introduce the product map and describe the general strategy to determine its image. We then apply this strategy in Sections 5 and 6, to determine which loxodromic and regular elliptic isometries are products of three involutions. We deal with parabolic conjugacy classes in Section 7. Finally, in Section 8, we apply these results to study the involution length and commutator length.

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2 Some classical hyperbolic geometry

Proposition 1 *(1) Every element of PSL(2,$\mathbb{R}$) is a product of two reflections.*
(2) Every antiholomorphic isometry of the Poincaré disk is a product of three reflections.
(3) Every element of PSL(2,$\mathbb{R}$) is a product of at most three half-turns.

Proof. The first part of Proposition 1 is classical (see for instance Sections 7.32 to 7.35) of [Bear]). The second part follows, as any antiholomorphic isometry of the Poincaré disk is the product of an element of PSL(2,$\mathbb{R}$) and a reflection (e.g. $z \longmapsto -\overline{z}$). For the third part, we proceed by case by case analysis.
(a) Any hyperbolic element h is a product of two half turns with fixed points a distance $\frac{\ell}{2}$ apart on its invariant axis, where ℓ is the translation length of h.
(b) To see that elliptic elements are products of three half-turns, consider a triangle $T = (p_1, p_2, p_3)$ in the Poincaré disk, with internal angles $\theta_i \in [0, \pi)$, $i = 1, 2, 3$. Let I_k be the half-turn about the midpoint of the edge $[p_{k+1}, p_{k+2}]$ of T, where indices are taken modulo 3 (see Figure 1). Then $I_1I_2I_3$ is elliptic (it fixes p_2), and it is a simple exercice in plane hyperbolic geometry to see that its rotation angle is $\theta = \theta_1 + \theta_2 + \theta_3 \in (0, \pi)$. Changing $I_1I_2I_3$ to its inverse $I_3I_2I_1$, we see that any non-zero rotation angle in $(-\pi, \pi)$ can be obtained this way. Elliptic elements with angle π are obtained in the case where $I_1 = I_2 = I_3$.
(c) For parabolic elements, consider an ideal triangle $T = (p_1, p_2, p_3)$ in the Poincaré disk, and let I_k be the half-turn fixing the orthogonal projection of p_k onto the opposite edge (see Figure 2). The product $I_1I_2I_3$ fixes p_2, and is parabolic. This can be seen for instance by considering the orbit of a horosphere based at p_2 (one can also argue that the group by $\langle I_1, I_2, I_3\rangle$ is conjugate to an index 3 subgroup of the modular group PSL(2,$\mathbb{Z}$), and that $I_1I_2I_3$ corresponds to the cube of the parabolic element $z \longmapsto z + 1$ under this conjugation). $\square$

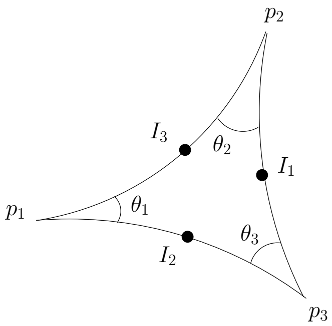

Figure 1: An elliptic triple product of half-turns

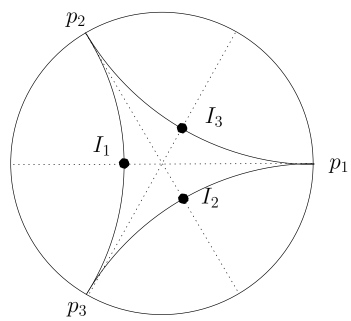

Figure 2: A parabolic triple product of half-turns

For later use, we describe the possible conjugacy classes for the product of two isometries of the Poincaré disk lying in certain prescribed conjugacy classes.

Proposition 2 *(1) Let $\mathcal{C}$ be a hyperbolic conjugacy class in PSL(2,$\mathbb{R}$). (a) The product of an element $h \in \mathcal{C}$ and a half-turn can belong to any nontrivial conjugacy class. In particular, it can be elliptic with arbitrary rotation angle. (b) The product of an element $h \in \mathcal{C}$ and a reflection is a glide reflection with arbitrary translation length.*
(2) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two elliptic conjugacy classes in PSL(2,$\mathbb{R}$), corresponding to rotation angles θ_1 and θ_2 (with $\theta_i \in [0, 2\pi)$). If $E_1 \in \mathcal{C}_1$ and $E_2 \in \mathcal{C}_2$ are such that E_1E_2 is elliptic, then the rotation angle of E_1E_2 can take any value in $[\theta_1 + \theta_2, 2\pi)$ (resp. $(2\pi, \theta_1 + \theta_2]$) if $\theta_1 + \theta_2 < 2\pi$ (resp. $\theta_1 + \theta_2 > 2\pi$).
(3) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two hyperbolic conjugacy classes in PSL(2,$\mathbb{R}$), corresponding to translation lengths ℓ_1 and ℓ_2. Then every elliptic isometry is a product h_1h_2 with $h_1 \in \mathcal{C}_1$ and $h_2 \in \mathcal{C}_2$.

Proof. (1a) Let $h \in \mathcal{C}$ (with translation length denoted ℓ), and let ι be a half-turn. Let γ_1 be the geodesic orthogonal to the axis of h through the fixed point of ι, and σ_1 the reflection about γ_1 (see Figure 4). Let σ_2 be the unique reflection such that $h = \sigma_1\sigma_2$; it fixes pointwise a geodesic γ_2 which is at distance $\frac{\ell}{2}$ from γ_1. The half-turn ι is the product of σ_1 and the reflection σ_3 about the geodesic γ_3 orthogonal to γ_1 through p. The product ιh is equal to $\sigma_2\sigma_3$. As in Figure 4, we see that when p moves away from the axis of h, the relative position of γ_2 and γ_3 varies continuously from orthogonal (when p is on the axis of h) to disjoint with arbitrarily

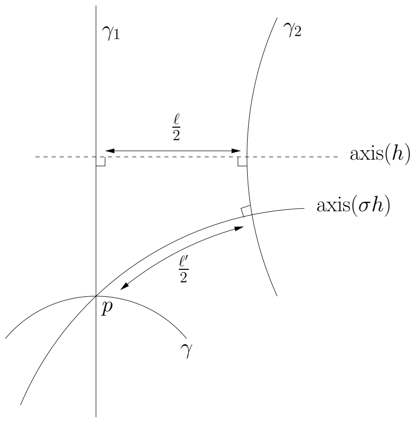

Figure 3: Product of a hyperbolic isometry and a reflection

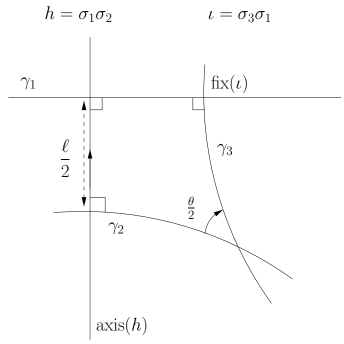

Figure 4: Product of a hyperbolic isometry and a half-turn (elliptic case)

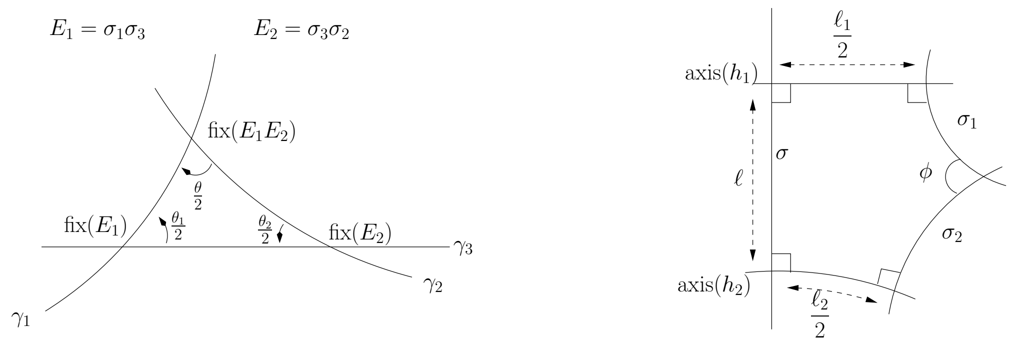

Figure 5: Elliptic product of two elliptic (left) or hyperbolic (right) isometries in given conjugacy classes

large distance (when p goes to infinity along γ_1). We thus obtain elliptic classes with any rotation angle $\theta \in [0, \pi[$ (when γ_2 and γ_3 intersect), a parabolic class (when γ_2 and γ_3 are asymptotic), and any hyperbolic class (when γ_2 and γ_3 are ultra-parallel). The other elliptic classes are obtained by applying the reflection about the axis of h, which reverses orientation.

(1b) Let $h \in \mathcal{C}$, with translation length denoted ℓ. Write $h = \sigma_1\sigma_2$, where σ_1 and σ_2 are reflections about geodesics γ_1 and γ_2 orthogonal to the axis of h, and a distance $\frac{\ell}{2}$ apart. Now consider a geodesic γ, orthogonal to γ_1 and σ the reflection about it. The product $\sigma\sigma_1$ is the half-turn about the point $p = \gamma \cap \gamma_1$. Therefore σh is the product of a reflection and a half-turn, which is a glide reflection (as p is not fixed by σ_2). As γ moves away from the axis of h, the translation length ℓ' of σh can take any positive value (see Figure 3).

(2) Let γ_3 be the geodesic connecting the fixed points of E_1 and E_2, and σ_3 the associated reflection. Decompose the two elliptics as products $E_1 = \sigma_1\sigma_3$ and $E_2 = \sigma_3\sigma_2$, where σ_1 and σ_2 are reflections about geodesics through the fixed points of E_1 and E_2. The geodesics γ_1 and γ_2 intersect γ_3 with angles $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$, as indicated on Figure 5. The product $E_1E_2 = \sigma_1\sigma_2$ is elliptic if and only if γ_1 and γ_2 intersect inside the disk. The result follows by studying the possible angles of the triangle bounded by γ_1, γ_2 and γ_3 when the distance between the fixed points of E_1 and E_2 varies.

(3) The argument is about the same as for the previous item. Consider the right hand side of Figure 5. The two hyperbolic isometries are $h_1 = \sigma_1\sigma$ and $h_2 = \sigma\sigma_2$. When the distance ℓ varies from 0 to ∞ the product h_1h_2 varies from identity (when $\ell = 0$) to hyperbolic with arbitrarily large translation length. In particular, the angle ϕ can take any value between 0 and π. $\square$

3 Complex hyperbolic space and its isometries

3.1 Basic definitions

The standard reference for complex hyperbolic geometry is [G1]. For the reader's convenience we include a brief summary of key definitions and facts. Our main result concerns the case of dimension $n = 2$, but the general setup is identical for higher dimensions so we state it for all $n \geqslant 1$.

Distance function: Consider $\mathbb{C}^{n,1}$, the vector space $\mathbb{C}^{n+1}$ endowed with a Hermitian form $\langle \cdot\,, \cdot \rangle$ of signature $(n, 1)$. Let $V^- = \{Z \in \mathbb{C}^{n,1} | \langle Z, Z\rangle < 0\}$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}\mathrm{P}^n$ denote projectivization. Define $\mathrm{H}^n_{\mathbb{C}}$ to be $\pi(V^-) \subset \mathbb{C}\mathrm{P}^n$, endowed with the distance d (Bergman metric) given by:

$$\cosh^2\Big(\frac{d(\pi(X), \pi(Y))}{2}\Big) = \frac{|\langle X, Y\rangle|^2}{\langle X, X\rangle\langle Y, Y\rangle}. \tag{3}$$

Different choices of Hermitian forms of signature $(n, 1)$ give rise to different models of $\mathrm{H}^n_{\mathbb{C}}$. The two most standard choices are the following. First, when the Hermitian form is given by $\langle Z, Z\rangle = |z_1|^2+\cdots+|z_n|^2-|z_{n+1}|^2$, the image of V^- under projectivization is the unit ball of $\mathbb{C}^n$, seen in the affine chart $\{z_{n+1} = 1\}$ of $\mathbb{C}P^n$. This model is referred to as the *ball model* of $\mathrm{H}^n_{\mathbb{C}}$. Secondly, when $\langle Z, Z\rangle = 2\mathrm{Re}(z_1\overline{z_{n+1}}) + |z_2|^2 + \cdots + |z_n|^2$, we obtain the so-called *Siegel model* of $\mathrm{H}^n_{\mathbb{C}}$, which generalizes the Poincaré upper half-plane.

Isometries: From (3) it is clear that $\mathrm{PU}(n, 1)$ acts by isometries on $\mathrm{H}^n_{\mathbb{C}}$, where $\mathrm{U}(n, 1)$ denotes the subgroup of $\mathrm{GL}(n + 1, \mathbb{C})$ preserving $\langle\cdot, \cdot\rangle$, and $\mathrm{PU}(n, 1)$ its image in $\mathrm{PGL}(n + 1, \mathbb{C})$. In fact, $\mathrm{PU}(n,1)$ is the group of holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$, and the full group of isometries is $\mathrm{PU}(n, 1)\ltimes\mathbb{Z}/2$, where the $\mathbb{Z}/2$ factor corresponds to a real reflection (see below). Holomorphic isometries of $\mathrm{H}^n_{\mathbb{C}}$ can be of three types, depending on the number and location of their fixed points. Namely, $g \in \mathrm{PU}(n, 1)$ is :

- *elliptic* if it has a fixed point in $\mathrm{H}^n_{\mathbb{C}}$
- *parabolic* if it has (no fixed point in $\mathrm{H}^n_{\mathbb{C}}$ and) exactly one fixed point in $\partial\mathrm{H}^n_{\mathbb{C}}$
- *loxodromic*: if it has (no fixed point in $\mathrm{H}^n_{\mathbb{C}}$ and) exactly two fixed points in $\partial\mathrm{H}^n_{\mathbb{C}}$

Totally geodesic subspaces: A *complex k-plane* is a projective k-dimensional subspace of $\mathbb{C}P^n$ intersecting $\pi(V^-)$ non-trivially (so, it is an isometrically embedded copy of $\mathrm{H}^k_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$). Complex 1-planes are usually called *complex lines.* If $L = \pi(\tilde{L})$ is a complex $(n - 1)$-plane, any $v \in \mathbb{C}^{n+1} - \{0\}$ orthogonal to $\tilde{L}$ is called a *polar vector* of L. Such a vector satisfies $\langle v, v\rangle > 0$, and we will usually normalize v so that $\langle v, v\rangle = 1$.

A *real k-plane* is the projective image of a totally real $(k+1)$-subspace W of $\mathbb{C}^{n,1}$, i. e. a $(k+1)$-dimensional real vector subspace such that $\langle v, w\rangle \in \mathbb{R}$ for all $v, w \in W$. We will usually call real 2-planes simply real planes, or $\mathbb{R}$-planes. Every real n-plane in $\mathrm{H}^n_{\mathbb{C}}$ is the fixed-point set of an antiholomorphic isometry of order 2 called a *real reflection* or $\mathbb{R}$-reflection. The prototype of such an isometry is the map given in affine coordinates by $(z_1, ..., z_n) \mapsto (\overline{z_1}, ..., \overline{z_n})$ (this is an isometry provided that the Hermitian form has real coefficients).

In $\mathrm{H}^2_{\mathbb{C}}$, the relative position of complex lines can be determined using using the following Lemma.

Lemma 1 *Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be distinct vectors in $\mathbb{C}^{2,1}$ such that $\langle \mathbf{n}_k, \mathbf{n}_k\rangle \neq 0$. When $\mathbf{n}_k$ has negative type we denote by n_k its projection to $\mathrm{H}^2_{\mathbb{C}}$, when it has positive type, we denote by L_k its polar complex line. Consider*

$$\kappa = \frac{|\langle \mathbf{n}_1, \mathbf{n}_2\rangle|^2}{\langle \mathbf{n}_1, \mathbf{n}_1\rangle\langle \mathbf{n}_2, \mathbf{n}_2\rangle}. \tag{4}$$

1. *If $\mathbf{n}_1$ and $\mathbf{n}_2$ both have negative type, then $\kappa > 1$ and $\kappa = \cosh^2(d/2)$ where $d = d(n_1, n_2)$.*
2. *If $\mathbf{n}_1$ and $\mathbf{n}_2$ have opposite types, say $\mathbf{n}_1$ has positive type and $\mathbf{n}_2$ negative type, then $\kappa \leqslant 0$ and $\kappa = -\sinh^2(d/2)$, where $d = d(L_1, n_2)$. In particular $\kappa = 0$ if and only if n_2 belongs to L_1.*
3. *If $\mathbf{n}_1$ and $\mathbf{n}_2$ both have positive type, then:*

(a) L_1 and L_2 are ultraparallel $\iff \kappa > 1$; in that case $\kappa = \cosh^2(d/2)$, where $d = d(L_1, L_2)$,

(b) L_1 and L_2 intersect inside $\mathrm{H}^2_{\mathbb{C}} \iff 0 \leqslant \kappa < 1$; in that case $\kappa = \cos^2(\theta)$, where θ is the angle between L_1 and L_2,

(c) L_1 and L_2 are asymptotic if and only if $\kappa = 1$.

Proof. The first item comes from the distance between two points in $\mathrm{H}^2_{\mathbb{C}}$, which is given by (3). The third one is a reformulation of Section 3.3.2 of [G1]. To prove the second one, we note that if $\mathbf{n}_1$ is polar to L_1, then the orthogonal projection of n_2 on L_1 is given by the vector

$$\mathbf{v} = \mathbf{n}_2 - \frac{\langle \mathbf{n}_2, \mathbf{n}_1 \rangle}{\langle \mathbf{n}_2, \mathbf{n}_2 \rangle} \mathbf{n}_1.$$

The distance between n_1 and L_2 is then obtained by applying (3) to $\mathbf{v}$ and $\mathbf{n}_2$, and this gives the result. □

Remark 1 Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be two positive type vectors polar to two complex lines L_1 and L_2. The two vectors are linearly independant if and only if L_1 and L_2 are disinct. When this is the case there exists a (unique up to scalar multiples) vector $\mathbf{n}$ which is orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. This vector can be taken to be $\mathbf{n} = \mathbf{n}_1 \boxtimes \mathbf{n}_2$ (where $\boxtimes$ denotes the Hermitian cross product, as defined in Section 2.2.7. of [G1]). It is sometimes useful to note that if L_1 and L_2 are distinct, they are asymptotic if and only if the family $(\mathbf{n}_1, \mathbf{n}, \mathbf{n}_2)$ is linearly dependent. This can be seen easily, for instance by considering the Gram matrix of this family for $\langle \cdot, \cdot \rangle$.

3.2 The 2-dimensional Siegel model

We provide a few details about the 2-dimensional Siegel model, as we will use it a lot when working with parabolic isometries. It is the one associated to the Hermitian form given by

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{5}$$

The complex hyperbolic plane corresponds to the domain given by $|z_2|^2 - 2\mathrm{Re}(z_1) < 0$ for $(z_1, z_2) \in \mathbb{C}^2$ (seen as the affine chart $\{z_3 = 1\}$ of $\mathbb{C}P^2$). Any point in $\mathrm{H}^2_{\mathbb{C}}$ lifts to a unique vector in $\mathbb{C}^3$ of the form

$$\mathbf{m}_{z,t,u} = \begin{bmatrix} -|z|^2 - u + it \\ z\sqrt{2} \\ 1 \end{bmatrix} \text{ where } z \in \mathbb{C}, t \in \mathbb{R} \text{ and } u > 0. \tag{6}$$

The triple (z, t, u) is called *horospherical coordinates* for m. In these coordinates, the boundary of $\mathrm{H}^2_{\mathbb{C}}$ is formed by those points for which $u = 0$, that is the projections of the vectors $\mathbf{m}_{z,t,0}$, together with the point at infinity, which is the projection of $\mathbf{q}_\infty = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. In turn, the boundary of $\partial \mathrm{H}^2_{\mathbb{C}}$ is the one point compactification of the 3-dimensional Heisenberg group $\mathbb{C} \times \mathbb{R}$, with group law

$$[z, t] \cdot [w, s] = [z + w, t + s + 2\Im(z\overline{w})]. \tag{7}$$

We will see below that left Heisenberg multiplication corresponds to the action of a unipotent parabolic isometry fixing the point at infinity (see Section 3.3.2). These parabolics preserve each level set of u (which are in fact the horospheres centred at q_∞). We often call $[z, t]$ the Heisenberg coordinates of the point in the boundary of $\mathrm{H}^2_{\mathbb{C}}$ given by $\mathbf{m}_{z,t,0}$.

Note that $\langle \mathbf{m}_{z,t,u}, \mathbf{m}_{z,t,u} \rangle = -2u$; in particular, if $u < 0$, then the vector $\mathbf{m}_{z,t,u}$ is polar to a complex line. In fact, a complex line is either polar to a vector $\mathbf{m}_{z,t,u}$ for some $u < 0$ (if it does not contain q_∞), or polar to a vector of the form $\begin{bmatrix} -z\sqrt{2} & 1 & 0 \end{bmatrix}$ (if it does). The latter vector is polar to the complex line connecting q_∞ to the boundary point with Heisenberg coordinates $[z, 0]$.

3.3 Conjugacy classes in PU(2,1)

We denote by $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{E}$ the spaces of loxodromic, parabolic and elliptic conjugacy classes in PU(2,1). We will say an eigenvalue of a transformation $A \in \mathrm{SU}(2,1)$ has *positive type* (resp. *null type*, resp. *negative type*) if it corresponds to a positive (resp. null, resp. negative) type eigenvector.

3.3.1 Loxodromic classes.

In the Siegel model, any loxodromic isometry is conjugate to one given by the diagonal matrix

$$L_\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \frac{\overline{\lambda}}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\overline{\lambda}} \end{bmatrix}, \tag{8}$$

for some $|\lambda| > 1$ (the attracting eigenvalue of L_λ). The parameter λ is uniquely defined up to multiplication by a cube root of 1 (this corresponds to the three lifts to SU(2,1) of an element in PU(2,1)). Writing $\lambda = re^{-i\theta/3}$ with $r > 1$, we see that $\mathcal{L}$ is homeomorphic to the cylinder $S^1 \times \mathbb{R}^+$, where S^1 is the interval $[0, 2\pi]$ with endpoints identified. The parameter θ is the *rotation angle* of L_λ; the translation length of L_λ is given by $\ell = 2\ln|\lambda|$. Note that the unit modulus eigenvalue of a loxodromic element does not determine its conjugacy class, but it determines its rotation angle, in particular the vertical line of the cylinder $\mathcal{L}$ to which it belongs.

We will call *hyperbolic* any loxodromic isometry with angle $\theta = 0$ (that is, conjugate to L_r for some $r \in (1, +\infty)$). Similarly, we will call *half-turn loxodromics* those with rotation angle $\theta = \pi$ (conjugate to L_{-r}, with $r \in (1, +\infty)$). The axis of a loxodromic isometry L is contained in an S^1-family $(P_\alpha)_{\alpha\in[0,\pi)}$ of real planes on which L acts by rotation: $P_\alpha \longmapsto P_{\alpha+\theta}$, where θ is the rotation angle of L. In particular, hyperbolic (resp half-turn loxodromic) isometries preserve each real plane containing their axis and act on it as a hyperbolic isometry (resp. glide reflection). We will denote by $\mathcal{H}$ the space of hyperbolic conjugacy classes. Vertical lines in $\mathcal{L}$ are those with fixed value of θ. Hyperbolic and half-turn loxodromic classes form the vertical lines $\theta = 0$ and $\theta = \pi$. Using (8), it is easy to see that a loxodromic map is hyperbolic (resp. half-turn loxodromic) if and only if it has a lift to SU(2,1) with real trace larger than 3 (resp. less than -1).

3.3.2 Parabolic classes

Parabolic isometries are those whose lifts to SU(2,1) are non-diagonalizable. A parabolic isometry is called *unipotent* if it has a unipotent lift to SU(2,1); otherwise, it is called *screw-parabolic* (or *ellipto-parabolic*, see e.g. [CG] or [G1]). A unipotent parabolic isometry is called either *2-step* or *3-step*, according to whether the minimal polynomial of its unipotent lift is $(X-1)^2$ or $(X-1)^3$ (see section 3.4 of [CG]). In the first case (also called *vertical Heisenberg translation* the unipotent lift is conjugate to one of the following matrices:

$$\begin{bmatrix} 1 & 0 & \pm i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

In the second case, (also called *horizontal Heisenberg translation*), the unipotent lift is conjugate to

$$\begin{bmatrix} 1 & -\sqrt{2} & -1 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

The terms horizontal and vertical Heisenberg translation refer to the fact that the boundary of complex hyperbolic space can be identified with the Heisenberg group, and unipotent parabolics acts on the boundary as left Heisenberg translations. We refer the reader to Chapter 4 of [G1], or to Section 2.3 of [W3]. Screw-parabolic isometries have a lift conjugate to a matrix of the form

$$\begin{bmatrix} 1 & 0 & it \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } \theta \in [0, 2\pi). \tag{11}$$

Note the latter matrix does not have determinant 1. The parameter θ is called the *rotation angle* of the screw-parabolic, and t its translation length. Screw-parabolic isometries preserve a complex line, on which they act as a usual parabolic isometry of the Poincaré disk, and they rotate through an angle θ around this line. In particular, we will call *half-turn parabolic maps* those screw parabolic maps with rotation angle π. Parabolic isometries having a lift to SU(2,1) with real trace are either unipotent or half-turn parabolic. Screw parabolics and 2-step unipotent parabolic have a stable complex line (in (9) and (11), it is the one polar to the second vector of the canonical basis of $\mathbb{C}^3$). On the other hand, they preserve no real plane. Likewise, 3-step unipotent parabolics preserve a real plane (in the case of (10), it is the projection of $\mathbb{R}^3 \subset \mathbb{C}^3$ to $\mathrm{H}^2_{\mathbb{C}}$), but no complex line.

As explained in Section 3.2, the boundary of $\mathrm{H}^2_{\mathbb{C}}$ can be identified to the one point compactification of the Heisenberg group. All unipotent isometries can be written under the form

$$T_{[z,t]} = \begin{bmatrix} 1 & -\overline{z}\sqrt{2} & -|z|^2 + it \\ 0 & 1 & z\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \text{ where } z \in \mathbb{C} \text{ and } t \in \mathbb{R}. \tag{12}$$

It is a direct verification to see that these matrices respect the group multiplication law given in (7):

$$T_{[z,t]} \cdot T_{[w,s]} = T_{[z,t]\cdot[w,s]} \tag{13}$$

For that reason, unipotent parabolics are often called *Heisenberg translations*. In particular, the representatives of the unipotent conjugacy classes given in (9) and (10) are $T_{[0,\pm 1]}$ and $T_{[1,0]}$.

3.3.3 Elliptic classes

An elliptic isometry g is called *regular* if any of its matrix representatives $A \in \mathrm{U}(n,1)$ has distinct eigenvalues. The eigenvalues of a matrix $A \in \mathrm{U}(n,1)$ representing an elliptic isometry g have modulus one. Exactly one of these eigenvalues has eigenvectors in V^- (projecting to a fixed point of g in $\mathrm{H}^n_{\mathbb{C}}$), and such an eigenvalue will be called *of negative type*. Regular elliptic isometries have an isolated fixed point in $\mathrm{H}^n_{\mathbb{C}}$. A non regular elliptic isometry is called *special*. Among the special elliptic isometries are the following two types (which exhaust all special elliptic types when $n = 2$):

1. A *complex reflection* is an elliptic isometry $g \in \mathrm{PU}(n,1)$ whose fixed-point set is a complex $(n-1)$-plane. In other words, any lift of such an isometry to U(n,1) has a negative type eigenvalue of multiplicity n.
2. A *complex reflection in a point* is an elliptic isometry whose lifts have a simple eigenvalue of negative type and another eigenvalue of multiplicity n. In other words, such an isometry is conjugate to $\lambda\mathrm{Id} \in \mathrm{U}(n)$ (for some $\lambda \in \mathrm{U}(1)$) , where $\mathrm{U}(n)$ is the stabilizer of the origin in the ball model. Complex reflections in a point of order 2 are also called *central involutions*; these are the symmetries that give $\mathrm{H}^n_{\mathbb{C}}$ the structure of a symmetric space.

In the ball model of $\mathrm{H}^2_{\mathbb{C}}$, any lift of an elliptic isometry g is conjugate to a diagonal matrix of the form:

$$\begin{bmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & e^{i\gamma} \end{bmatrix}, \text{ where } \alpha, \beta, \gamma \in [0, 2\pi). \tag{14}$$

Here the negative type eigenvalue is $e^{i\gamma}$. The two positive eigendirections correspond to a pair of (orthogonal) stable complex lines in $\mathrm{H}^2_{\mathbb{C}}$, and the negative one to a fixed point inside $\mathrm{H}^2_{\mathbb{C}}$. Projectively, the isometry g acts on its stable lines as rotations, through angles $\theta_1 = \alpha - \gamma$ and $\theta_2 = \beta - \gamma$ respectively. The conjugacy class of an elliptic isometry is determined by this (unordered) pair of angles. In particular, the eigenvalue spectrum of a lift to SU(2,1) of an elliptic isometry does not determine it conjugacy class there are generically three possible angle pairs for a given triple of eigenvalues. Conversely, an elliptic conjugacy class with angle pair $\{\theta_1, \theta_2\}$ is represented by the following matrix in SU(2,1):

$$E_{\theta_1,\theta_2} = \begin{bmatrix} e^{i\frac{2\theta_1-\theta_2}{3}} & 0 & 0 \\ 0 & e^{i\frac{2\theta_2-\theta_1}{3}} & 0 \\ 0 & 0 & e^{-i\frac{\theta_1+\theta_2}{3}} \end{bmatrix}. \tag{15}$$

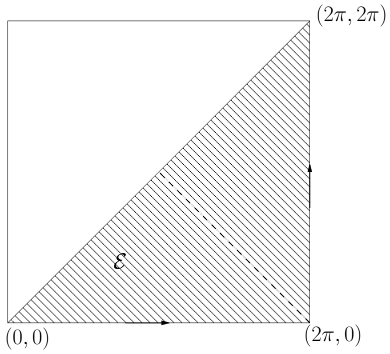

Figure 6: The space of elliptic conjugacy classes. Arrows on the edges of the triangles indicate identifications. The dashed segment represent angle pairs of real elliptics.

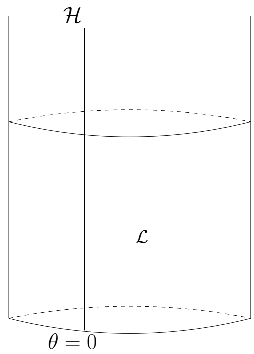

Figure 7: The space of loxodromic conjugacy classes

We denote by $\mathcal{E}$ the space of elliptic conjugacy classes in PU(2,1). From the above discussion, we may identify $\mathcal{E}$ with the quotient of $S^1 \times S^1$ under the relation $\{\theta_1, \theta_2\} \simeq \{\theta_2, \theta_1\}$, in other words with:

$$\Delta/\sim, \text{ where } \Delta = \{(\theta_1, \theta_2), 0 \leqslant \theta_2 \leqslant \theta_1 \leqslant 2\pi\}, \tag{16}$$

with identifications $(0, \theta) \sim (\theta, 2\pi)$ for all θ; see Figure 6). An elliptic isometry is said to be *real elliptic with angle* θ if its angle pair is of the form $\{2\pi - \theta, \theta\}$ with $\theta \in [0, \pi]$. One of the lifts to SU(2,1) of such an isometry has eigenvalues $\{e^{i\theta}, e^{-i\theta}, 1\}$ (with 1 of negative type), and trace$1 + 2\cos\theta \in \mathbb{R}$. The two conditions of having a lift with real trace and negative type eigenvalue equal to 1 characterize real elliptics among elliptics. Moreover, that lift is conjugate to an element of $\mathrm{SO}(2,1) \subset \mathrm{SU}(2,1)$; in particular, real elliptics preserve a real plane, on which they act by rotation through angle θ.

Remark 2 There are two conjugacy classes of involutions in PU(2,1):

1. *Central involutions* (or complex relections in a point of order 2) are the isometries conjugate to $(z_1, z_2) \longmapsto (-z_1, -z_2)$ in the ball model. They have angle pair $\{\pi, \pi\}$, i.e. are real elliptics with angle π. Central involutions have an isolated fixed point in $\mathrm{H}^2_{\mathbb{C}}$, and preserve every complex line through that fixed point, acting on it as a half-turn.

2. *Complex symmetries* (or complex reflections of order 2) are the isometries conjugate to $(z_1, z_2) \longmapsto (z_1, -z_2)$ in the ball model. They have angle pair $\{\pi, 0\}$. Complex symmetries fix pointwise a unique complex line in $\mathrm{H}^2_{\mathbb{C}}$, called their *mirror*. They preserve every complex line orthogonal to the mirror, acting on it as a half-turn.

Both types of involutions can be lifted to SU(2,1) as follows. Let n be a point in $\mathbb{C}P^2 \setminus \partial\mathrm{H}^2_{\mathbb{C}}$. Let $\mathbf{n}$ be a lift of n such that $\langle \mathbf{n}, \mathbf{n} \rangle = 2\varepsilon$, with $\varepsilon \in \{-1, 1\}$. If $\varepsilon = -1$ (resp. 1), n is a point of $\mathrm{H}^2_{\mathbb{C}}$ (resp. is polar to a complex line in $\mathrm{H}^2_{\mathbb{C}}$). Consider the linear involution of $\mathbb{C}^{2,1}$ defined by

$$I_n(Z) = -Z + \varepsilon\langle Z, \mathbf{n}\rangle\mathbf{n}, \text{ for } Z \in \mathbb{C}^{2,1}. \tag{17}$$

I_n acts on $\mathrm{H}^2_{\mathbb{C}}$ as the central involution fixing the point $n \in \mathrm{H}^2_{\mathbb{C}}$ when $\varepsilon = -1$, and the complex symmetry across $\mathbf{n}^{\perp}$ when $\varepsilon = 1$. Note that given an involution in PU(2,1), its lift of the form (17) is the unique lift which is also an involution. We will often identify a holomorphic involution with this lift.

3.3.4 The space of conjugacy classes

We will be interested in the space $\mathcal{G}$ of conjugacy classes of the group $G = \mathrm{PU}(2,1)$ (see section 3.1 for basic definitions). As a topological space (with the quotient topology), this space is not Hausdorff; more specifically,

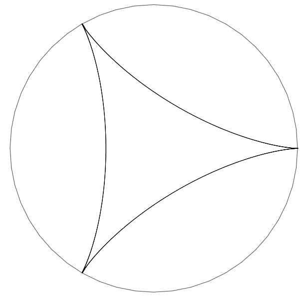

Figure 8: The null-locus of the polynomial f inscribed in the circle of radius 3 centered at the origin.

the conjugacy class of complex reflections with a given (nonzero) rotation angle has the same neighborhoods as the screw-parabolic class with the same angle, and likewise, the identity and the 3 unipotent classes all share the same neighborhoods. For most of our purposes it will be sufficient to consider the set $\mathcal{G}^{reg}$ of *regular semisimple* classes, i.e. those classes of elements whose lifts are semisimple with distinct eigenvalues (so, loxodromic or regular elliptic). However, it will also be useful to consider as in [FW2] the maximal Hausdorff quotient $c(\mathcal{G})$ of the full space of conjugacy classes in G.

Concretely, $c(\mathcal{G})$ consists of the open dense set $\mathcal{G}^{reg}$, together with the set $\mathcal{B}$ of equivalence classes of complex reflections and screw-parabolics, as well as the identity and unipotents, which are identified in the quotient; we will call such classes *boundary classes*. We will denote by $\mathcal{L}$ (respectively $\mathcal{E}$, $\mathcal{E}^{reg}$) the subsets of $\mathcal{G}$ consisting of loxodromic (resp. elliptic, resp. regular elliptic) elements of G; the conjugacy class of an element $A \in G$ will be denoted $[A]$. The global topology of $\mathcal{G}$ can be described as follows: $\mathcal{E}$ is closed (in fact, compact), $\mathcal{L}$ and $\mathring{\mathcal{E}} = \mathcal{E} \setminus \mathcal{B}$ are open and disjoint, and $\mathcal{E} \cap \overline{\mathcal{L}} = \mathcal{B}$. Note that $\mathcal{L}$ and $\mathring{\mathcal{E}}$ have natural smooth structures (which were used in [FW1], [FW2] and [P]), whereas boundary classes are singular points of $\mathcal{G}$, as they have arbitrarily small neighborhoods homeomorphic to 3 half-disks glued along a common diameter (2 of them in $\mathcal{E}$, 1 in $\mathcal{L}$).

As in the classical case of the Poincaré disk, the isometry type of an isometry is closely related to the trace of a lift to SU(2,1). The following Proposition can be found in Chapter 7 of [G1]; see Figure 8.

Proposition 3 (Goldman) *Let f denote the function defined for $z \in \mathbb{C}$ by:*

$$f(z) = |z|^4 - 8Re(z^3) + 18|z|^2 - 27. \tag{18}$$

Then, for any isometry $g \in \mathrm{PU}(2,1)$ with lift $A \in SU(2,1)$:

- *g is regular elliptic* $\iff$ $f(\mathrm{tr}(A)) < 0$.
- *g is loxodromic* $\iff$ $f(\mathrm{tr}(A)) > 0$.
- *g is special elliptic or screw-parabolic* $\iff$ $f(\mathrm{tr}(A)) = 0$ *and* $\mathrm{tr}(A) \notin 3C_3$.
- *g is unipotent or the identity* $\iff$ $\mathrm{tr}(A) \in 3C_3$.

Combining the latter proposition with Section 3.3.1 gives the following:

Remark 3 An element A in SU(2,1) represents a hyperbolic isometry if and only if $\mathrm{tr}(A) = \omega x$, where $x \in (3, +\infty)$ and ω is a cube root of unity. An element A in SU(2,1) represents a half-turn loxodromic isometry if and only if $\mathrm{tr}(A) = -\omega x$, where $x \in (-\infty, -1)$ and ω is a cube root of unity.

3.4 Double products of involutions

There is one conjugacy class of antiholomorphic involutions in $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$: *real reflections*, that fix pointwise an embedded copy $\mathrm{H}^n_{\mathbb{R}} \subset \mathrm{H}^n_{\mathbb{C}}$. The standard example is the map $Z \longmapsto \overline{Z}$ in the unit ball of dimension n. It is well known that any holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$ is a product of two real reflections (this is due to Falbel and Zocca [FZ] in dimension two, and to Choi [C] in higher dimensions). In contrast, only very few elements of PU(2,1) are products of two holomorphic involutions.

Proposition 4 *Let $g \neq Id$ be a holomorphic isometry of $\mathrm{H}^2_{\mathbb{C}}$. Then,*

1. *g is a product of two central involutions if and only if it is hyperbolic.*
2. *g is a product of a complex symmetry and a central involution if and only if it is half-turn loxodromic or a complex symmetry.*
3. *g is a product of two complex symmetries if and only if it is hyperbolic, 3-step unipotent or real elliptic.*

Proof. Let I_1 and I_2 be two involutions, lifted as in (17) to the linear maps $I_k(Z) = -Z + \varepsilon_k\langle Z, \mathbf{n}_k\rangle\mathbf{n}_k$ with $k = 1, 2$ and ε_k in $\{-1, 1\}$. As in Lemma 1, we denote by n_k the projection to $\mathrm{H}^2_{\mathbb{C}}$ of $\mathbf{n}_k$ when it is negative, and by L_k its polar complex line if it is positive. Then:

$$I_1I_2(Z) = Z - \varepsilon_1\langle Z, \mathbf{n}_1\rangle\mathbf{n}_1 - \varepsilon_2\langle Z, \mathbf{n}_2\rangle\mathbf{n}_2 + \varepsilon_1\varepsilon_2\langle Z, \mathbf{n}_2\rangle\langle\mathbf{n}_2, \mathbf{n}_1\rangle\mathbf{n}_1. \tag{19}$$

Let $\mathbf{n}$ be a vector orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$; as the latter are linearly independant, by Remark 1, $(\mathbf{n}_1, \mathbf{n}, \mathbf{n}_2)$ is a basis for $\mathbb{C}^3$ except if $\mathbf{n}_1$ and $\mathbf{n}_2$ are both positive and represent asymptotic lines. Assuming this is not the case, we can compute the trace of I_1I_2 in this basis using (19). This gives:

$$\mathrm{tr}(I_1I_2) = -1 + \varepsilon_1\varepsilon_2|\langle\mathbf{n}_1, \mathbf{n}_2\rangle|^2 = -1 + 4\kappa, \tag{20}$$

where κ was defined by (4). This expression remains valid when the two lines are asymptotic (in which case the above triple of vectors is no longer a basis). Hence the product of two involutions in SU(2,1) always has real trace (up to multiplication by a cube root of 1). From Sections 3.3.1 to 3.3.3, such a product can only be hyperbolic, half-turn loxodromic, unipotent, half-turn parabolic or real elliptic.

There are three different cases, depending on the respective types of $\mathbf{n}_1$ and $\mathbf{n}_2$. The results are obtained directly from Lemma 1.

1. If $\varepsilon_1 = \varepsilon_2 = -1$, then I_1 and I_2 are central involutions and κ can take any value in $(1, +\infty)$. We obtain all hyperbolic classes this way.
2. If $\varepsilon_1 = -\varepsilon_2 = 1$, then I_1 is a complex symmetry, and I_2 is a central involution. In this case κ can take any value in $(-\infty, 0]$. For negative values of κ, we obtain all possible half-turn loxodromic isometries. If $\kappa = 0$, then n_2 belongs to the mirror of I_1, and I_1I_2 is the complex symmetry about the line orthogonal to L_1 through n_2.
3. If $\varepsilon_1 = \varepsilon_2 = 1$ then the cases where $\kappa > 1$ give all possible hyperbolic classes. In case $0 \leqslant \kappa < 1$, the vector $\mathbf{n}$ orthogonal to $\mathbf{n}_1$ and $\mathbf{n}_2$ has negative type, and is an eigenvector with eigenvalue 1 of I_1I_2 (this follows directly from (19)). Therefore, the eigenvalue spectrum of I_1I_2 is $\{1, e^{i\alpha}, e^{-i\alpha}\}$, where $\kappa = \cos^2\alpha/2$ (see Lemma 1). In particular, I_1I_2 is real elliptic with rotation angle α. Finally, if $\kappa = 1$ the two complex lines L_1 and L_2 are asymptotic, and the product I_1I_2 is parabolic. To verify that I_1I_2 is 2-step unipotent, pick a vector $\mathbf{n}$ such that $\langle\mathbf{n}, \mathbf{n}_2\rangle = 0$ and $\langle\mathbf{n}, \mathbf{n}_1\rangle \neq 0$ ($\mathbf{n}$ corresponds to a point in L_2 but not in L_1). The triple $(\mathbf{n}_1, \mathbf{n}, \mathbf{n}_2)$ is a basis of $\mathbb{C}^3$, and the matrix of I_1I_2 in this basis is equal to

$$M = \begin{bmatrix} 3 & -\langle\mathbf{n}, \mathbf{n}_1\rangle & \langle\mathbf{n}_2, \mathbf{n}_1\rangle \\ 0 & 1 & 0 \\ -\langle\mathbf{n}_1, \mathbf{n}_2\rangle & 0 & -1 \end{bmatrix}$$

A straightforward verification using $|\langle\mathbf{n}_1, \mathbf{n}_2\rangle|^2 = 4$ shows that $(M - id)^2$ has rank one. □

From Proposition 4, we see that generic holomorphic isometries are not products of two holomorphic involutions, in other words that PU(2,1) has involution length at least 3. In the next sections, we are going to determine which elements of PU(2,1) are products of three holomorphic involutions. To that end, we will use the following remarks.

Remark 4 To any involution I, we associate a sign : $+1$ if it is a complex symmetry, or -1 if it is a central involution (this is the sign of $\langle \mathbf{n}, \mathbf{n} \rangle$ for $\mathbf{n}$ as in (17)). To any triple of involutions (I_1, I_2, I_3) is thus associated a triple of signs $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. We will often shorten this notation by omitting the 1 and only keeping the signs, e.g. $(+,+,-)$ will stand for $(1,1,-1)$. We will say that an isometry $A \in \mathrm{PU}(2,1)$ is a triple product of type $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ if it is a product $I_1 I_2 I_3$ where I_k has sign ε_k.

1. Fix a triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{-1,+1\}^3$. If A is a triple product of type $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, then it is also a triple product of type $(\varepsilon_{\sigma(1)}, \varepsilon_{\sigma(2)}, \varepsilon_{\sigma(3)})$ for any permutation $\sigma \in \mathfrak{S}_3$. To verify this, is suffices to note that conjugating by I_1 amouts to applying the 3-cycle $(1,2,3)$ and that two neighboring signs ε and ε' can always be exchanged. For example if I_1 is a complex symmetry and I_2 a central involution, then $I_1 I_2 I_1$ is conjugate to I_2 and is thus a central involution, denoted I_2'. We then have $I_1 I_2 = I_2' I_1$ and thus $I_1 I_2 I_3 = I_2' I_1 I_3$. Thus it is enough to study the four triples $(+,+,+)$, $(+,+,-)$, $(+,-,-)$ and $(-,-,-)$.

2. Proposition 4 shows in particular that any product of two central involutions is also a product of two complex symmetries. This implies the following two facts.

   (a) To prove that an isometry $A \in \mathrm{PU}(2,1)$ is a triple product of any type, its suffices to prove that it is a triple product of type $(+,-,-)$ and $(-,-,-)$. In other words, it suffices to prove that A is the product of an hyperbolic element and an involution of either type.

   (b) If an isometry $A \in \mathrm{PU}(2,1)$ cannot be written as a $(+,+,+)$ triple product of type nor as a $(+,+,-)$ triple product, then it is not a product of three holomorphic involutions.

4 Conjugacy classes and products of isometries

To analyze products of three holomorphic involutions $I_1 I_2 I_3$, we will view them as products of two isometries, one of which is a product of two involutions. As we have seen in Section 3.4, being a product of two holomorphic involutions gives restrictions on the conjugacy class. We are therefore led to study the following product map.

4.1 The product map

We consider as in [FW2] and [P] the following question: given two conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ in $G = \mathrm{PU}(2,1)$, what are the possible conjugacy classes for the product AB as A varies in $\mathcal{C}_1$ and B varies in $\mathcal{C}_2$? More specifically, given two semisimple conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$, the problem is to determine the image of the map:

$$\begin{array}{rccc} \tilde{\mu}: & \mathcal{C}_1 \times \mathcal{C}_2 & \longrightarrow & \mathcal{G} \\ & (A,B) & \longmapsto & [AB] \end{array}, \tag{21}$$

where $\mathcal{G}$ is the set of conjugacy classes in PU(2,1) and $[\cdot]$ denotes the conjugacy class of an element. When studying this question, *reducible pairs* play a crucial role.

Definition 1 *We say that a subgroup $\Gamma < \mathrm{PU}(2,1)$ is* reducible *if it fixes a point in $\mathbb{C}P^2$ (so, either all elements of Γ have a common fixed point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, or they all preserve a common complex line), and* irreducible *otherwise. Likewise we will say that a pair $(A,B) \in \mathrm{PU}(2,1)^2$ is reducible (resp. irreducible) if it generates a reducible (resp. irreducible) group.*

The strategy used in [FW2] and [P] consists of the following four parts:

1. Prove that $\mathrm{Im}\,\tilde{\mu}$ is closed;

2. Prove that images of irreducible pairs are interior points of $\mathrm{Im}\,\tilde{\mu}$;

3. Determine the set $W_{\mathrm{red}} = \{[AB] \,|\, (A,B) \in \mathcal{C}_1 \times \mathcal{C}_2 \text{ reducible}\}$ of *reducible walls*;

4. Determine which *chambers*, i.e. connected components of $\mathcal{G} \setminus W_{\mathrm{red}}$, are in the image - by parts 1 and 2, $\mathrm{Im}\,\tilde{\mu}$ is a union of chamber closures.

Parts 1 and 2 follow respectively from sections 4.2 and 4.3 below. They imply the following crucial fact (see Section 2.5 of [P]), which justifies part 4.

Theorem 3 *Any chamber of* $\mathcal{G} \setminus W_{\text{red}}$ *is either full or empty.*

Note that parts 1 and 2 are obtained once and for all. In contrast, 3 and 4 require a case by case analysis.

Remark 5

1. In the cases we consider we will observe that the intersection of the reducible walls with $\mathcal{E}$ consists of a finite collection of linear segments that have slope -1, 2 or 1/2. We refer to [P] for a general proof of this fact. In particular, this shows that the diagonal segment $\{(\theta,\theta), \theta \in [0, 2\pi)\}$ cannot contain any reducible walls.
2. A useful consequence of Theorem 3 is the following fact. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ be three conjugacy classes, with $\mathcal{C}_1, \mathcal{C}_2$ semisimple. Assume that there exist two pairs (A_1, B_1) and (A_2, B_2) such that $A_iB_i \in \mathcal{C}_3$ for $i = 1, 2$, with (A_1, B_1) reducible and (A_2, B_2) irreducible. As (A_1, B_1) is reducible $\mathcal{C}_3$ corresponds to a point on a reducible wall. As (A_2, B_2) is irreducible, $\mathcal{C}_3$ is interior to the image of the product map. This implies that all chambers having the point $\mathcal{C}_3$ in their closure are full.

We start with two general observations about reducible and irreducible pairs. Recall that a *special elliptic* isometry in $\mathrm{PU}(2,1)$ is one whose lifts have repeated eigenvalues; geometrically this means that its angle pair has the form $\{\theta, 0\}$ (in which case it is a complex reflection about a line) or $\{\theta, \theta\}$ (in which case it is a complex reflection in a point).

Lemma 2 *Let* $A, B \in \mathrm{PU}(2,1)$*. If* A *and* B *(resp.* A *and* AB*) are both special elliptic then the group* $\langle A, B\rangle$ *is reducible.*

Proof. Complex reflections about lines preserve all complex lines perpendicular to their mirror, and complex reflections about points preserve all complex lines containing their isolated fixed point. In all cases, either A and B have a fixed point in common in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ or they preserve a common complex line. □

Lemma 2 is very useful in determining which special elliptic elements are attained as products. It was used in the following form in the proof of Proposition 4.1 of [P]:

Corollary 1 *If one of* $\mathcal{C}_1$ *or* $\mathcal{C}_2$ *is a conjugacy class of special elliptic elements, then any chamber of* $\mathrm{Im}\,\tilde{\mu} \cap \mathcal{E}$ *containing an open interval of the diagonal in its closure is empty.*

Proof. Asssume we have such a chamber C that is full. Then if A is special elliptic and the angle pair of AB lie on the diagonal, AB is special elliptic, and by Lemma 2 the pair (A, AB) is reducible. As the pair (A, AB) generates the group $\langle A, B\rangle$, this implies that (A, B) is also reducible. This means that the diagonal interval lying in the closure of C is (part of) a reducible wall. This contradicts Remark 5. □

The second observation is that any pair can be deformed to an irreducible pair, unless prohibited by Lemma 2:

Proposition 5 *Let* $\mathcal{C}_1, \mathcal{C}_2$ *be 2 semisimple conjugacy classes in* $\mathrm{PU}(2,1) \setminus \{\mathrm{Id}\}$*, at least one of which is regular semisimple. Then irreducible pairs form an open dense subset of* $\mathcal{C}_1 \times \mathcal{C}_2$*.*

Proof. Recall that a pair $(A, B) \in \mathrm{PU}(2,1)^2$ is reducible if (any lifts of) A, B have a common eigenvector in $\mathbb{C}^3$. Therefore reducible pairs form a closed subset of $\mathrm{PU}(2,1)^2$, and irreducible pairs an open subset.

Now if say $\mathcal{C}_1$ is regular semisimple and $\mathcal{C}_2$ semisimple (and not the identity), then eigenspaces in $\mathbb{C}^3$ of lifts to $\mathrm{U}(2,1)$ of elements of $\mathcal{C}_1$ have (complex) dimension 1, and likewise eigenspaces in $\mathbb{C}^3$ of lifts to $\mathrm{U}(2,1)$ of elements of $\mathcal{C}_2$ have (complex) dimension at most 2. Then, if $(A_0, B_0) \in \mathcal{C}_1 \times \mathcal{C}_2$ is reducible, let $v \in \mathbb{C}^3$ be a common eigenvector of (lifts of) A_0, B_0 and V_A (resp. V_B) the eigenspace of A_0 (resp. B_0) containing v. Then in a neighborhood of (A_0, B_0), any pair (A, B) with $V_A \cap V_B = \{0\}$ is irreducible, and since V_A has dimension 1 and V_B dimension at most 2 there exist such pairs arbitrarily close to (A_0, B_0). □

Corollary 2 *If $\mathcal{C}_1, \mathcal{C}_2$ are 2 semisimple conjugacy classes in* $\mathrm{PU}(2,1) \setminus \{\mathrm{Id}\}$*, at least one of which is regular semisimple, then every reducible wall bounds at least one full chamber.*

Proof. By Proposition 5, any reducible pair, corresponding to a point on a reducible wall W can be deformed into an irreducible pair. This either gives a point in one of the chambers bounding W, which is therefore full, or a point on the reducible wall which is then also the image of an irreducible pair. As in Remark 5 (2), both chambers bounded by that wall are then full. □

4.2 The product map is closed

Let G be the identity component of the isometry group of a Riemannian symmetric space with negative sectional curvature. The *translation length* $|g|$ of an isometry $g \in \mathrm{Isom}(X)$ is defined as $|g| = \mathrm{Inf}\{d(x, gx) : x \in X\}$. An isometry g is called *semisimple* if the infimum is attained, i. e. if there exists $x \in X$ such that $|g| = d(x, gx)$. In the case of hyperbolic spaces, semisimple isometries are the non-parabolic ones (in other words, an isometry is semisimple if its matrix representatives are semisimple).

Theorem 4 below is the key point in this section. This compactness result, which as stated is Proposition 2 of [FW2] and is essentially Theorem 3.9 of [Be], is sometimes called the Bestvina-Paulin compactness theorem,. It is obtained by taking Gromov-Hausdorff limits to get an action on an $\mathbb{R}$-tree.

Theorem 4 *Let X be a negatively curved Riemannian symmetric space, $G = \mathrm{Isom}^0(X)$, and (g_i), (h_i) two sequences of semisimple elements of G with uniformly bounded translation length. Then either:*
(1) there exists $f_i \in G$ such that $f_i g_i f_i^{-1}$ and $f_i h_i f_i^{-1}$ converge in G (after passing to a subsequence), or
(2) the sequence of translation lengths $|g_i h_i|$ is unbounded.

We will use the following consequence of this result (Theorem 2 of [FW2]). Recall from the end of section 3.3 that we denote $\mathcal{G}$ the space of conjugacy classes of G, and $c(\mathcal{G})$ the maximal Hausdorff quotient of $\mathcal{G}$.

Corollary 3 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two semisimple conjugacy classes in G, and consider the diagonal action of G on $\mathcal{C}_1 \times \mathcal{C}_2$ by conjugation. Then:*
(a) the product map $\mu : (A, B) \longrightarrow AB$ descends to a map $\bar{\mu} : \mathcal{C}_1 \times \mathcal{C}_2 / G \longrightarrow c(\mathcal{G})$ that is proper.
(b) The image of $\bar{\mu}$ is closed in $c(\mathcal{G})$.

Proof. (a) If K is a compact subset of $c(\mathcal{G})$ and $(g_i, h_i) \in G \times G$ is (a choice of representatives of) a sequence in $\bar{\mu}^{-1}(K)$, the sequence of translation lengths $|g_i h_i|$ is bounded, therefore by Theorem 4 $\bar{\mu}^{-1}(K)$ is compact.
(b) If (c_i) is a sequence in $\mathrm{Im}\,\bar{\mu}$ converging to $c \in c(\mathcal{G})$, let as above $(g_i, h_i) \in G \times G$ be a choice of representatives of preimages of c_i. Then the sequence of translation lengths $|g_i h_i|$ is bounded, therefore by Theorem 4 (after conjugating) (g_i) and (h_i) converge in G, say to g, h respectively. Then by continuity of $\bar{\mu}$, $c = \bar{\mu}(g, h)$ is in $\mathrm{Im}\,\bar{\mu}$, which is therefore closed. □

4.3 The product map is open

Proposition 6 *Let $\mathcal{C}_1, \mathcal{C}_2$ be 2 semisimple conjugacy classes in $G = \mathrm{PU}(2,1)$, and (A, B) be an irreducible pair in $\mathcal{C}_1 \times \mathcal{C}_2$. Then the differential of $\tilde{\mu}$ at (A, B) is surjective and thus $\tilde{\mu}$ is locally surjective at that point.*

The key point in the proof is the following lemma (see Lemma 2.4 of [P], the proof of Prop. 4.2 of [FW1] or the final section of [G2] in a different context). Denoting $\mu : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow G$ the product map, and $\mathfrak{z}(A, B)$ the Lie algebra of the centralizer of the group generated by A and B:

Lemma 3 *The differential at a pair (A, B) of the product map μ satisfies $Im(d_{(A,B)}\mu) = \mathfrak{z}(A, B)^{\perp}.AB$, where the orthogonal is taken with respect to the Killing form of G.*

Now if (A, B) is irreducible, then $\mathfrak{z}(A, B) = \{0\}$, so μ is a submersion at such a point (the Killing form is non-degenerate). The proposition follows since the projection $\pi : G \longrightarrow \mathcal{G}$ is open, as a quotient map.

4.4 Reducible walls in the elliptic-elliptic case

In this section we review the case where the two classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are elliptic as well as their product. This situation has been analyzed in detail in [P], but we recall it briefly for self-containedness. Assume that the two classes correspond to angle pairs (θ_1, θ_2) and (θ_3, θ_4) with $0 \leqslant \theta_1 \leqslant \theta_2 < 2\pi$ and $0 \leqslant \theta_3 \leqslant \theta_4 < 2\pi$. The possible reducible configurations for a pair (A, B) in $\mathcal{C}_1 \times \mathcal{C}_2$ fall into three types, which correspond to points and segments in the affine chart Δ (see Section 3.3.3).

Totally reducible pairs. This is when A and B commute. In that case, A and B have a common fixed point in $\mathrm{H}^2_{\mathbb{C}}$ and the same stable complex lines. The corresponding angle pairs are thus given by the two points $\{\theta_1 + \theta_3, \theta_2 + \theta_4\}$ and $\{\theta_1 + \theta_4, \theta_2 + \theta_3\}$ (the precise order of the coordinates depends on the values of the θ_i).

Spherical reducible pairs. This is when A and B have a common fixed point in $\mathrm{H}^2_{\mathbb{C}}$. In that case, A and B can be lifted to U(2,1) as a pair

$$A = \begin{bmatrix} \tilde{A} & \\ & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} \tilde{B} & \\ & 1 \end{bmatrix},$$

where $\tilde{A}$ and $\tilde{B}$ are matrices in U(2) with respective spectra $\{e^{i\theta_1}, e^{i\theta_2}\}$ and $\{e^{i\theta_3}, e^{i\theta_4}\}$. The problem is thus reduced to the similar one in U(2). The set of angle pairs of spherical reducible pairs is then the segment of slope -1 connecting the two totally reducible vertices. This segment may appear as disconnected in Δ (see [P]).

Hyperbolic reducible pairs. This is when A and B preserve a common complex line L in $\mathrm{H}^2_{\mathbb{C}}$. If A and B are regular, they each preserve two complex lines. When the product AB is elliptic, we will denote by θ_C its rotation angle in the line L and by θ_N its rotation angle in the normal direction. There are four families of hyperbolic reducible configurations that correspond to the possible choices of rotation angles of A and B in the common stable complex line. The possible values of the angle pairs for hyperbolic reducible configurations are those lying on the projection to $\mathcal{E}$ of one of the four segments C_{ij} in Δ, where $i \in \{1, 2\}$, $j \in \{3, 4\}$, and C_{ij} is the affine segment defined by the conditions

$$\theta_C = 2\theta_N + (\theta_i + \theta_j) - 2(\theta_k + \theta_l), \text{ with } \begin{cases} \theta_i + \theta_j < \theta_C < 2\pi \text{ if } \theta_i + \theta_j < 2\pi \\ \\ 2\pi < \theta_C < \theta_i + \theta_j \text{ if } \theta_i + \theta_j > 2\pi \end{cases}, \tag{22}$$

where we use the convention that $\{k, l\}$ and $\{i, j\}$ are disjoint. The wall C_{ij} corresponds to the case where A and B rotate through angles θ_i and θ_j respectively in the complex line L. For example, the segment C_{14} corresponds to the case when A rotates through θ_1 and B through θ_4. Then A and B can be conjugated in U(2,1) so that:

$$A = \begin{bmatrix} e^{i\theta_2} & \\ & \tilde{A} \end{bmatrix}, B = \begin{bmatrix} e^{i\theta_3} & \\ & \tilde{B} \end{bmatrix}, \text{and } AB = \begin{bmatrix} e^{i(\theta_2+\theta_3)} & \\ & \tilde{C} \end{bmatrix}$$

where $\tilde{A}$ and $\tilde{B}$ are matrices in U(1,1) with respective spectra $\{e^{i\theta_1}, 1\}$ and $\{e^{i\theta_4}, 1\}$. The eigenvalues of $\tilde{C}$ are $e^{i\alpha}$ (positive type) and $e^{i\beta}$ (negative type), for some α and β in $[0, 2\pi)$. The angle pair of AB is given by

$$\theta_C = \alpha - \beta \text{ and } \theta_N = \theta_2 + \theta_3 - \beta.$$

On the other hand, the relation $\det(AB) = \det(A)\det(B)$ gives the relation $\alpha + \beta = \theta_1 + \theta_4 \mod 2\pi$. The precise range given in (22) is obtained by applying Proposition 2.

5 Loxodromic triple products

In this section, we apply the strategy described in Section 4.1 to prove that any loxodromic isometry is a product of three involutions of any kind. We start by giving a description of the reducible walls in the space $\mathcal{L}$ of loxodromic classes, in angle-length coordinates $(\theta, \ell) \in S^1 \times \mathbb{R}^+$ (see Section 3.3.1).

Proposition 7 *Let $\mathcal{C}_1, \mathcal{C}_2$ be 2 semisimple conjugacy classes in $G = \mathrm{PU}(2,1)$. We denote by W_{red} the corresponding set of reducible walls.*

(a) If $\mathcal{C}_1, \mathcal{C}_2$ are both loxodromic (or $\mathcal{C}_1$ loxodromic and $\mathcal{C}_2$ a complex reflection in a point), then $W_{red} \cap \mathcal{L}$ consists of the single wall $\{\theta_1 + \theta_2\} \times \mathbb{R}^+$, where θ_1 and θ_2 are the rotation angles of $\mathcal{C}_1$ and $\mathcal{C}_2$.

(b) If $\mathcal{C}_1$ is loxodromic and $\mathcal{C}_2$ is regular elliptic (or a complex reflection), let θ_1 be the rotation angle of $\mathcal{C}_1$ and $\{\alpha_2, \beta_2\}$ the angle pair of $\mathcal{C}_2$. Then $W_{red} \cap \mathcal{L}$ consists of the two walls

$$\left\{\theta_1 + \alpha_2 - \frac{\beta_2}{2}\right\} \times \mathbb{R}^+ \text{ and } \left\{\theta_1 + \beta_2 - \frac{\alpha_2}{2}\right\} \times \mathbb{R}^+. \tag{23}$$

(c) If $\mathcal{C}_1, \mathcal{C}_2$ are both regular elliptic, then $W_{red} \cap \mathcal{L}$ consists of three or four walls of the form $\{\alpha + \beta\} \times \mathbb{R}^+$, where α (resp. β) is one of the two rotation angles of $\mathcal{C}_1$ (resp. $\mathcal{C}_2$).

Part (a) is actually contained in [FW2] but we include a more detailed proof. Note that if A, B are both special elliptic then the group $\langle A, B\rangle$ is always reducible by Lemma 2.

Proof. Let $(A, B) \in \mathcal{C}_1 \times \mathcal{C}_2$ be a reducible pair of semisimple isometries, with loxodromic product. In particular, A and B have a common eigenvector in $\mathbb{C}^3$. If A and B both admit $\mathbf{e}$ as an eigenvector, with respective eigenvalues u_A and u_B, then the product AB has $\mathbf{e}$ as an eigenvector with eigenvalue $u_A u_B$. As AB is loxodromic the value of $u_A u_B$ determines the conjugacy class of AB if it has non unit modulus, and it determines the vertical line of $\mathcal{L}$ to which $[AB]$ belongs if it has unit modulus (see Section 3.3.1). The result will therefore depend on the respective type and number of eigenvectors of A and B.

(a) If A and B are both loxodromic with attracting eigenvalues u_A and u_B then from the general form (8) for loxodromics, their product has either $\overline{u_A u_B}/u_A u_B$ as a positive type eigenvalue, or one of $u_A u_B$, $1/\overline{u_A u_B}$ $u_A/\overline{u_B}$ and $u_B/\overline{u_A}$ as a null type eigenvalue. All these complex numbers determine the same vertical line in $\mathcal{L}$. This gives the result when A and B are both loxodromic. If, say, B is a complex reflection in a point, then A and B can only have a positive type common eigenvector. As all positive eigenvectors for B have the same eigenvalue the same conclusion holds as for pairs of loxodromics.

(b) If A is loxodromic and B is regular elliptic or a complex reflection, then only a positive type vector can be a common eigenvector. If B has angle pair $\{\alpha_2, \beta_2\}$, we see using (8) and (15) that the conjugacy class of AB belongs to one of the two vertical lines in $\mathcal{L}$ given by (23).

(c) If A and B are both regular elliptic and AB is loxodromic, then again only a positive type vector can be a common eigenvector for A and B. As A and B each have two distinct (unit modulus) eigenvalues of positive type, this leaves three or four possibilities. Indeed, denoting by (α_1, α_2) and (β_1, β_2) the respective (pairs of) arguments of the positive eigenvalues of A and B, the possible arguments of the positive type eigenvalue of AB belong to

$$\{\alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1, \alpha_2 + \beta_2\}.$$

Since A and B are regular elliptic, $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$ so at least three of these four angles are distinct.

To finish proving Proposition 7, we need to see that all points one the vertical lines described above are indeed attained by the product map. Once the rotation angle θ of AB is fixed, the parameter r (in the notation of Section 3.3.1) describes the translation length of the corresponding loxodromic element. We now show that the cases where A and B preserve a common complex line suffice to cover the whole vertical line. Indeed, in that case the translation length of AB is attained at any point of its axis, which is contained in the complex line preserved by A and B, a copy of the Poincaré disk. Now if A and B have fixed conjugacy classes in PSL(2,$\mathbb{R}$), their product AB can be hyperbolic with any translation length. This can either be seen as a simple exercice in plane hyperbolic geometry in the spirit of Propositions 2 and 2 or as a consequence of Theorem 3, applied to the case where X is the Poincaré disk. □

Proposition 7 (a) shows that when $\mathcal{C}_1$ and $\mathcal{C}_2$ are loxodromic, the reducible walls do not disconnect $\mathcal{L}$. As shown by Falbel and Wentworth in [FW2], this implies the following proposition, which is Theorem 1 of [FW2].

Proposition 8 ([FW2]) *Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be three loxodromic conjugacy classes in* $\mathrm{PU}(2,1)$*. Then there exists* $(A, B, C) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ *such that* $ABC = \mathrm{Id}$.

Proof. Applying the first item of Proposition 7, we see that if $(A, B) \in \mathcal{C}_1 \times \mathcal{C}_2$ is a reducible pair, the conjugacy class of AB can take any value in a fixed vertical line $\mathbb{R} \times \{\theta\}$. As this vertical line does not disconnect $\mathcal{L}$, Theorem 3 and Corollary 2 imply that $\mathrm{Im}\,\tilde{\mu}$ contains $\mathcal{L}$. □

Remark 6 The first item of Proposition 7 implies that the conclusion of Proposition 8 still holds if one of the three conjugacy classes is a complex reflection about a point. However, if at least two of A, B and C are complex reflections about points, then the group they generate is reducible, by Lemma 2.

Proposition 9 *Any loxodromic element in PU(2,1) is a product of three holomorphic involutions of any kind.*

We now prove Proposition 9 using the ideas of Section 4. In section 7 we will give an alternate proof which provides explicit configurations of involutions.

Proof. By Remark 4, we only need to prove that a loxodromic isometry is a triple product of types $(-,-,-)$ and $(+,-,-)$. By Proposition 4 it suffices to show that any loxodromic map can written both

1. as a product of a hyperbolic map and a central involution, and

2. as a product of a hyperbolic map and a complex symmetry.

To prove the first item, it suffices to apply Proposition 8 and Remark 6 in the case where A is hyperbolic and B is a central involution. We see that AB can belong to any loxodromic class.

For the second item, in the notation of Proposition 7, we have $\theta_1 = 0$, $\alpha_2 = \pi$ and $\beta_2 = 0$. The two reducible walls are thus $\{\theta = \pi\}$ and $\{\theta = -\pi/2\}$. To prove the result, we apply the strategy suggested by the second item of Remark 5, namely we show that any half-turn loxodromic (that is with rotation angle $\theta = \pi$) can also be obtained as a product hI, where h is hyperbolic, I a complex symmetry, and the pair (h, I) is irreducible. Consider the real plane $P = \mathrm{H}^2_{\mathbb{R}} \subset \mathrm{H}^2_{\mathbb{C}}$, and h a hyperbolic map preserving P (that is, a hyperbolic element of $\mathrm{SO}(2,1) \subset \mathrm{SU}(2,1)$). Consider a geodesic γ in P, with endpoints distinct from those of the axis of h, and L the complex line containing it. The complex reflection I about L preserves P and acts on it as the reflection about the geodesic γ. The product hI preserves P and $(hI)_{|P}$ is the product of a hyperbolic map and a reflection, thus either a glide reflection or a reflection. This means that hI is either a complex symmetry or a half-turn loxodromic. Applying Proposition 2, we see that hI can have any translation length, thus lie in any half-turn loxodromic conjugacy class. Now, the pair (h, I) is irreducible because h and I have no common fixed point in $\mathbb{C}P^2$. This proves that the two chambers bounded by $\{\pi\} \times \mathbb{R}^+$ are full. Hence, $\tilde{\mu}$ is onto $\mathcal{L}$ in that case. □

6 Regular elliptic triple products

Our goal in this section is to show that not all elliptic isometries are triple products of involutions, and to determine precisely which regular elliptic conjugacy classes cannot be written as triple products of involutions. By Remark 4, it suffices to determine those classes that can be written neither as a product of type $(+,+,-)$ nor of type $(+,+,+)$. To do so, we will study the products of an involution of any type with the product of two complex symmetries. By Proposition 4, this means we have to study pairs (I, A) where I is an involution of any type and A is either hyperbolic, 2-step unipotent or real elliptic (see Section 3.3.3 for definition). As hyperbolic and real elliptic isometries are semisimple we will apply the strategy described in Section 4 to determine those elliptic classes that can be written as such products. We will see that these classes correspond to angle pairs lying in a union of polygons in the triangle Δ. Now, 2-step unipotent isometries can be seen both as limits of sequences of hyperbolic isometries and of sequences of real elliptics. This can be seen by considering a sequence of pairs of complex lines (L_n, L'_n) with respective complex symmetries I_n and I'_n. If L_n and L'_n are ultraparallel (resp. intersecting) for all n and converge to a pair (L_∞, L'_∞) of asymptotic lines, then the product $I_n I'_n$ is hyperbolic (resp. real elliptic) and $I_\infty I'_\infty$ is 2-step unipotent. As a consequence the elliptic classes that are products of one involution and a 2-step unipotent isometry lie in the closure of the set of classes that can be written as products of one involution and a hyperbolic or real elliptic isometry. For that reason we will only consider pairs (I, A) where I is an involution and A is hyperbolic or real elliptic. The result is the following.

Proposition 10 *(1) An elliptic isometry is the product of one central involution and two complex symmetries if and only if its angle pair lies in the shaded region $\mathcal{E}_{++-}$ depicted on Figure 9.*
(2) An elliptic isometry is the product of three complex symmetries if and only if its angle pair lies in the shaded region $\mathcal{E}_{+++}$ depicted on Figure 10.

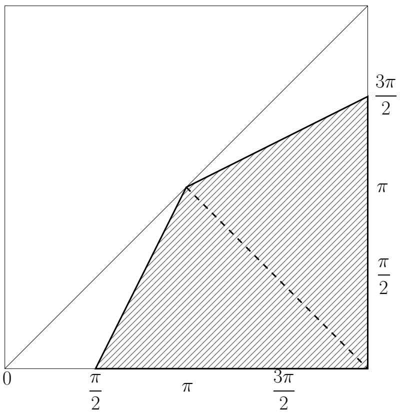

Figure 9: $\mathcal{E}_{++-}$: angle pairs of regular elliptic products of one central involution and two complex symmetries

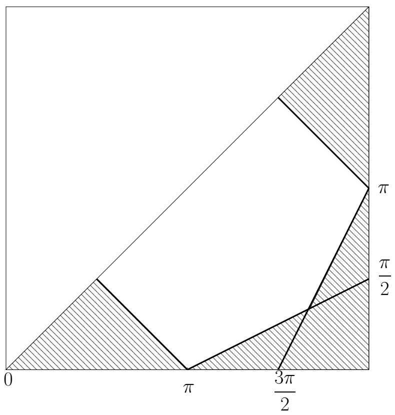

Figure 10: $\mathcal{E}_{+++}$: angle pairs of regular elliptic products of three complex symmetries

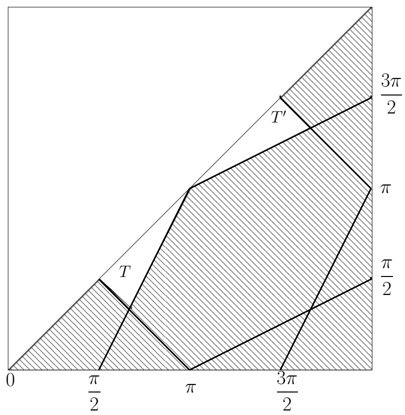

Figure 11: $\mathcal{E}_{++-} \cup \mathcal{E}_{+++}$: Regular elliptic classes that aren't products or three involutions are those in the interior of one of the two triangles T and T'

From Proposition 10, we obtain the following by applying Remark 4.

Corollary 4 *An elliptic isometry $E \in \mathrm{PU}(2,1)$ is a product of three involutions if and only if its angle pair lies outside the two open triangles T and T' given by their vertices as follows.*

$$T : (\pi, \pi), (2\pi/3, \pi/3), (\pi/2, \pi/2) \quad T' : (\pi, \pi), (5\pi/3, 4\pi/3)(3\pi/2, 3\pi/2)$$

The two triangles T and T' are pictured on Figure 11.

To prove Proposition 10 we will separate the cases, first studying products of an involution an a hyperbolic isometry, then products of an involution and a real elliptic isometry.

6.1 Products of an involution and a hyperbolic isometry

Applying the strategy of Section 4 we first need to describe the reducible walls. We consider pairs (I_1, A) where I_1 is an involution and A is hyperbolic. There are two cases.

- First assume I_1 is a complex symmetry. If $I_1 A$ is regular elliptic it has no boundary fixed point and thus the only possible common fixed point in $\mathbb{C}P^2$ for I_1 and A is the point polar to the complex axis of A. This implies that the axis of A and the mirror of I_1 are either equal or orthogonal. In the first case, $I_1 A$ is loxodromic, as I_1 fixes pointwise the axis of A. Therefore, the mirror of I_1 must be orthogonal to the complex axis of A. In particular I_1 acts on the complex axis of A as a half-turn.
- If I_1 is a central involution, its eigenvectors are either of positive or negative type (but not of null type), and so reducibility means that the fixed point of I_1 belongs to the complex axis of A (which is thus perserved by I_1).

In both cases, we see that I_1 preserves the complex axis of A and acts on it by a half-turn. We use the ball model of $\mathrm{H}^2_{\mathbb{C}}$, with Hermitian form $\mathrm{diag}(1, 1, -1)$. We can normalize so that lifts of I_1, A and $I_1 A$ to SU(2,1) have the form

$$I_1 = \begin{bmatrix} -1 & & \\ & \varepsilon & \\ & & -\varepsilon \end{bmatrix}, \text{ with } \varepsilon = \pm 1,\ A = \begin{bmatrix} 1 & \\ & \tilde{A} \end{bmatrix} \text{ and } I_1 A = \begin{bmatrix} -1 & \\ & \tilde{B} \end{bmatrix}. \tag{24}$$

In (24), I_1 is a central involution when $\varepsilon = -1$ and a complex symmetry when $\varepsilon = 1$. As A is hyperbolic, the 2×2 matrix $\tilde{A}$ has spectrum $\{r, 1/r\}$ for some $r > 1$. Similarly $I_1 A$ is elliptic and thus $\tilde{B}$ has eigenvalues $e^{i\alpha}$ (of positive type) and $e^{i\beta}$ (of negative type) for some $\alpha, \beta \in [0, 2\pi)$. The determinant of $I_1 A$ is equal to 1, and therefore we have $\alpha + \beta = \pi\,[2\pi]$, that is $\alpha + \beta = \pi$ or $\alpha + \beta = 3\pi$. The angle pair of $I_1 A$ is given by

$$\theta_C = \alpha - \beta \text{ and } \theta_N = \pi - \beta,$$

where θ_C is the rotation angle of $I_1 A$ in the complex axis of A (the common preserved complex line), and θ_N is the rotation angle in the normal direction. Using the conditions on the sum $\alpha + \beta$, we see that the angle pair $\{\theta_C, \theta_N\}$ of $I_1 A$ satisfies one of the following two relations

$$\theta_C = 2\theta_N - \pi \quad \text{if} \quad \alpha + \beta = \pi, \tag{25}$$
$$\theta_C = 2\theta_N + \pi \quad \text{if} \quad \alpha + \beta = 3\pi. \tag{26}$$

We denote by $\tilde{s_1}$ and $\tilde{s_2}$ the segments given by (25) and (26) for $0 < \theta_C < 2\pi$ (see Figure 12).

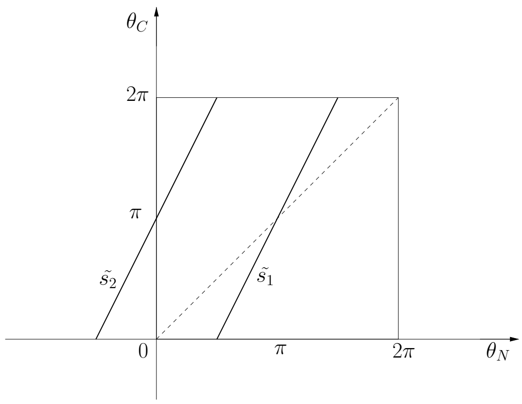

Figure 12: The two segments $\tilde{s_1}$ and $\tilde{s_2}$

Proposition 11 *Let A be a hyperbolic isometry with fixed conjugacy class.*
(1) If I_1 is a central involution such that (I_1, A) is reducible, then the possible angle pairs for the product $I_1 A$

when it is elliptic are the points of $\tilde{s_1}$ (see Figure 12).
(2) If I_1 is a complex symmetry such that (I_1, A) is reducible, then the possible angle pairs for the product I_1A when it is elliptic are the points of $\tilde{s_2}$ (see Figure 12).

Proof. We know already that in both cases, the restriction of I_1 to the axis of A is a half-turn. As a consequence, we can apply Proposition 2 to the restrictions of I_1 and A.

1. If I_1 is a central involution, decompose A as a product I_2I_3 of two central involutions with fixed points on the (real) axis of A. Clearly, if I_1 coincides with I_2 or I_3, the product I_1A is a central involution, and in this case $\theta_C = \theta_N = \pi$. This point is the midpoint of $\tilde{s_1}$. Deforming this configuration, Proposition 2 shows that any point on $\tilde{s_1}$ can be obtained by a reducible product of three central involutions.

2. If I_1 is a complex symmetry, we decompose A as a product I_2I_3 of two complex symmetries and obtain in the case where $I_1 = I_2$ or $I_1 = I_3$ that I_1A is also a complex symmetry. In this case $\theta_C = \pi$ and $\theta_N = 0$, which gives the midpoint of $\tilde{s_2}$. By a similar argument any point on (25) with $0 < \theta_C < 2\pi$ can be obtained by a reducible product of three complex symmetries.

Now consider three central involutions (I_1, I_2, I_3) with fixed points in a common complex line L. The triple product $I_1I_2I_3$ acts on L as a half-turn if and only if at least two of the I_k's are equal. In that case, $I_1I_2I_3$ is a central involution.This proves in particular that a complex symmetry cannot be a product I_1A where the the pair (I_1, A) is reducible, I_1 is a central involution and A is hyperbolic.

By a similar argument, a central involution cannot be a product I_1A where (I_1, A) is reducible, I_1 is a complex reflection of order two and A is hyperbolic.

If a point of $\tilde{s_1}$ were a reducible product of a complex symmetry and a hyperbolic, then by Proposition 2, we could deform it continuously to obtain the midpoint of $\tilde{s_1}$. This contradicts the previous discussion. A similar argument shows that no point of $\tilde{s}_2$ can be a reducible product of a central involution and a hyperbolic map. □

The following corollary describes the reducible walls. It is obtained in a straightforward way from Proposition 11 by projecting the two segments $\tilde{s_1}$ and $\tilde{s_2}$ onto the lower half of the square by reduction modulo 2π and symmetry about the diagonal.

Corollary 5 *Let $\mathcal{C}_1$ be a conjugacy class of involutions, $\mathcal{C}_2$ be a hyperbolic conjugacy class, and $(I_1, A) \in \mathcal{C}_1 \times \mathcal{C}_2$ be a reducible pair such that I_1A is elliptic.*

1. *If $\mathcal{C}_1$ is the class of central involutions, then the angle pair of I_1A can take any value on the two segments $\left[\left(\frac{\pi}{2}, 0\right), (\pi, \pi)\right]$ and $\left[(\pi, \pi), \left(2\pi, \frac{3\pi}{2}\right)\right]$.*
2. *If $\mathcal{C}_1$ is the class of complex symmetries, then the angle pair of I_1A can take any value on the two segments $\left[(\pi, 0), \left(2\pi, \frac{\pi}{2}\right)\right]$ and $\left[\left(\frac{3\pi}{2}, 0\right), (2\pi, \pi)\right]$.*

These segments are the thicker ones on Figures 13 and 14. We can now describe all elliptic classes that are obtained as a product of an involution and a hyperbolic map.

Proposition 12 *(1) An elliptic isometry is the product of a central involution and a hyperbolic isometry if and only if its angle pair lies in the dashed polygon depicted on Figure 13.*
(2) An elliptic isometry is the product of a complex symmetry and a hyperbolic isometry if and only if its angle pair lies in the dashed polygon depicted on Figure 14.

Proof. For each of the reducible walls, Corollary 2 tells us that at least one of the chambers bounded by this wall is full. In the case where I_1 is a central involution, Corollary 1 tells us that the two chambers bounded by a piece of the diagonal are empty (see Figure 13). Therefore the third chamber must be full. In the case where I_1 is a complex symmetry, Corollary 1 tells us that the chamber bounded by the diagonal is empty. Applying Corollary 2 at the intersection point of the reducible walls tells us that the three other chambers are full. □

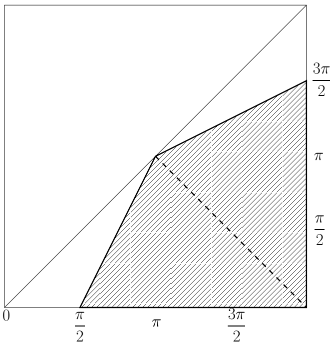

Figure 13: Elliptic classes that are products of one central involution and a hyperbolic map

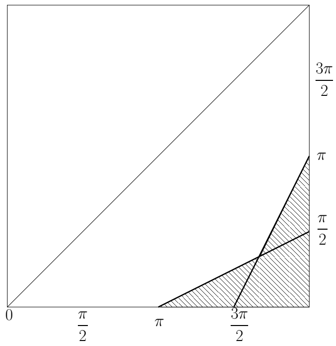

Figure 14: Elliptic classes that are products of one complex symmetry and a hyperbolic map

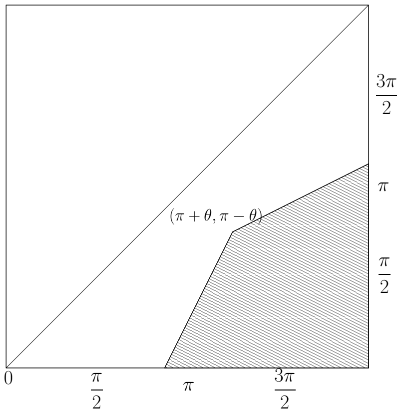

Figure 15: Elliptic classes that are products of one central involutions and a real elliptic map with angle pair $\{2\pi-\theta,\theta\}$

6.2 Products of a central involution and a real elliptic isometry

Central involution has angle pair $\{\pi,\pi\}$; for such pairs the reducible walls are as follows.

Proposition 13 *Let (I_1, A) be a reducible pair, where I_1 is a central involution and A is real elliptic with angle pair $\{2\pi-\theta,\theta\}$ for some fixed $\theta\in[0,\pi]$. The possible angle pairs for the product I_1A (when it is elliptic) are the two segments* $\Big[\Big(0,\dfrac{\pi+3\theta}{2}\Big),\Big(\pi+\theta,\pi-\theta\Big)\Big]$ *and* $\Big[\Big(\pi+\theta,\pi-\theta\Big),\Big(2\pi,\dfrac{3(\pi-\theta)}{2}\Big)\Big]$.

These two segments are depicted on Figure 15.

Proof. We are now dealing with the product map on the product of two elliptic conjugacy classes. The general situation in that case has been described in Section 4.4. In the case we are interested in the angle pair of I_1 is (π,π). In particular, the two totally reducible points are equal and the spherical reducible wall is reduced to the point $(\pi+\theta,\pi-\theta)$. Similarly, there are only two hyperbolic reducible segments, which, in the notation of Section 4.4 are C_{13} and C_{14}, with

$$\theta_1=\pi,\ \theta_2=\pi,\ \theta_3=2\pi-\theta \text{ and } \theta_4=\theta.$$

From the precise description of C_{13} and C_{14} given by (22), we see that these segments are those emanating from the point $(\pi+\theta,\pi-\theta)$ with slope 2 and 1/2, and connecting it respectively to the horizontal and vertical edges of the square (see Figure 15). □

We can now describe the intersection of the product map with $\mathcal{E}$ in this case. It is given in the following corollary, which is straightforward from Proposition 13 by applying the results of Section 4 (in particular, Theorem 3, Corollary 1 and Corollary 2).

Corollary 6 *The possible angle pairs for the product of a central involution and a real elliptic map with angle pair* $\{2\pi-\theta,\theta\}$ *are those points in the (convex) polygon with vertices* $\left(0,\frac{\pi+3\theta}{2}\right)$, $\left(\pi+\theta,\pi-\theta\right)$, $\left(2\pi,\frac{3(\pi-\theta)}{2}\right)$ *and* $(2\pi,0)$.

The following proposition is straightforward by taking the union of all polygons described in Corollary 6 when θ varies from 0 to π.

Proposition 14 *An elliptic element is the product of a central involution and two complex symmetries with intersecting mirrors if and only if its angle pairs belongs to the convex polygon with vertices* $(\pi/2,0)$, (π,π), $(2\pi,3\pi/2)$ *and* $(2\pi,0)$.

Observe that this polygon is the same as the one for the product of a central involution and a hyperbolic map.

6.3 Products of a complex symmetry and a real elliptic isometry

We now have a complex symmetry, with angle pair $\{\pi,0\}$, and we fix a real elliptic conjugacy class. Reducible walls are again otained by applying the results described in Section 4.4 with, this time

$$\theta_1=\pi,\ \theta_2=0,\ \theta_3=2\pi-\theta \text{ and } \theta_4=\theta.$$

Proposition 15 *Let* (R_1,A) *be a reducible pair, where* R_1 *is a complex symmetry and* A *is a real elliptic with angle pair* $\{2\pi-\theta,\theta\}$ *for some fixed* $\theta\in[0,\pi]$. *The possible angle pairs for the product* R_1A *(when it is elliptic) are as follows.*

1. *The two totally reducible vertices are the projections to* $\mathcal{E}$ *of the points in* $\mathbb{R}^2$ *with coordinates* $(3\pi-\theta,\theta)$ *and* $(2\pi-\theta,\pi+\theta)$.
2. *If* (R_1,A) *is spherical reducible the possible angle pairs are the points on the slope* -1 *segments in* $\mathcal{E}$ *connecting the projections to* $\mathcal{E}$ *of the two points* $(\theta,\pi-\theta)$ *and* $(2\pi-\theta,\pi+\theta)$.
3. *If* (R_1,A) *is hyperbolic reducible, the possible angle pairs are the segments* s_3 *and* s_4*, that are respectively the projections to* $\mathcal{E}$ *of two segments* $\tilde{s_3}$, $\tilde{s_4}$ *in* $\mathbb{R}^2$ *given by*

$$\tilde{s_3}=\left[\left(\theta,3\pi-\theta\right),\left(\frac{3\theta-\pi}{2},2\pi\right)\right] \text{ and } \tilde{s_4}=\left[\left(2\pi-\theta,\pi+\theta\right),\left(\frac{5\pi-3\theta}{2},2\pi\right)\right]. \qquad (27)$$

The aspect of the segment s_3 in the chart Δ is made more explicit in the table given by Figure 16. The wall s_4 is obtained from s_3 by the symmetry about the anti-diagonal of the square $[0,2\pi]^2$, given by $(x,y)\longmapsto(2\pi-y,2\pi-x)$. This symmetry corresponds to conjugating the pair (R_1,A) by an anti-holomorphic map, which preserves both conjugacy classes of R_1 and A ans is therefore an expected symmetry of the set of possible angle pairs. The reducible segments are depicted for various values of θ on Figures 17 to 20. The two totally reducible vertices lie respectively on the lines with equations $x+y=\pi$ and $x+y=3\pi$. For all values of θ, the spherical reducible segment appears disconnected in the affine chart of $\mathcal{E}$ and is contained in the union of the latter lines. The aspect of the hyperbolic reducible segments in the chart Δ depends on the value of $\theta\in[0,\pi)$.

From Corollary 1 in Section 4, it is straightforward to pass from the description of the reducible walls to the description of the image.

Corollary 7 *The full chambers for elliptic products of one complex symmetry and a real elliptic map with angle pair* $\{2\pi-\theta,\theta\}$ *are exactly those not containing an open segment of the diagonal in their closure.*

The full chambers for various values of θ are represented as shaded on Figures 17 to 20.

Value of θ	$0 \leqslant \theta < \pi/3$	$\theta = \pi/3$	$\pi/3 < \theta < \pi/2$	$\pi/2 \leqslant \theta \leqslant \pi$
Coordinates of v_1	$(\pi - \theta, \theta)$	$(\pi - \theta, \theta)$	$(\pi - \theta, \theta)$	$(\theta, \pi - \theta)$
Coordinates of v_2	$\left(2\pi, \frac{3\theta - \pi}{2}\right)$	$(0, 0)$	$\left(\frac{3\theta - \pi}{2}, 0\right)$	$\left(\frac{3\theta - \pi}{2}, 0\right)$
s_3 disconnected in chart	YES	NO	NO	NO
s_3 bounces on diagonal	NO	NO	YES	NO
slope	1/2 close to v_1 2 close to v_2	1/2	1/2 close to v_1 2 close to v_2	2

Figure 16: Aspect of the hyperbolic reducible segment s_3 depending on the value of θ. The vertices v_1 and v_2 are the endpoints of s_3, v_1 being the totally reducible point. The other segment s_4 is obtained from s_3 by symmetry about the anti-diagonal of the square.

7 Loxodromic and parabolic triple products

In this section we examine which parabolic isometries are obtained as triple products of involutions. We will do this by considering specific configurations of involutions; this will also gives us an alternate proof of the fact that any loxodromic isometry is a product of three involutions of any kind.

7.1 Ideal triangles and null-type eigenvalues of triple products of involutions

The following facts are classical (we refer the reader to Chapter 7 of [G1] for details). Let $\tau = (p_1, p_2, p_3)$ be a non-degenerate ideal triangle (meaning that $p_i \neq p_j$ for all pairs (i, j)). The *Cartan invariant* of τ is defined as

$$\alpha(\tau) = \arg\left(-\langle \mathbf{p}_1, \mathbf{p}_2\rangle\langle \mathbf{p}_2, \mathbf{p}_3\rangle\langle \mathbf{p}_3, \mathbf{p}_1\rangle\right), \tag{28}$$

where the vectors $\mathbf{p}_i$ are lifts to $\mathbb{C}^3$ of the vertices p_i of τ; $\alpha(\tau)$ is independent of the choice of lifts.

Proposition 16 *The Cartan invariant takes its values in $[-\pi/2, \pi/2]$, and it classifies ideal triangles up to holomorphic isometry. Moreover, $\alpha(\tau) = 0$ (resp. $\pm\pi/2$) if and only if τ is contained in a real plane (resp. a complex line).*

Ideal triangles appear naturally when considering triples of involutions with product fixing a boundary point.

Lemma 4 *Let (I_1, I_2, I_3) be a triple of holomorphic involutions of $\mathrm{H}^2_{\mathbb{C}}$. The following conditions are equivalent.*
(1) The triple product $I_1 I_2 I_3$ fixes a point on the boundary of $\mathrm{H}^2_{\mathbb{C}}$.
(2) There exists an ideal triangle $\tau = (p_1, p_2, p_3)$ such that I_k exchanges p_{k-1} and p_{k+1} (indices taken mod 3).

Proof. Let p_2 be a fixed point of $I_1 I_2 I_3$ in $\partial\mathrm{H}^2_{\mathbb{C}}$. Define p_1 and p_2 by $p_1 = I_3(p_2)$ and $p_3 = I_2(p_1)$. Then $\tau = (p_1, p_2, p_3)$ is satisfactory. □

Note that in general, the triangle τ may be degenerate if the involutions I_k have common boundary fixed points. For instance, if I_1, I_2 and I_3 are complex symmetries about three lines that share a common point in

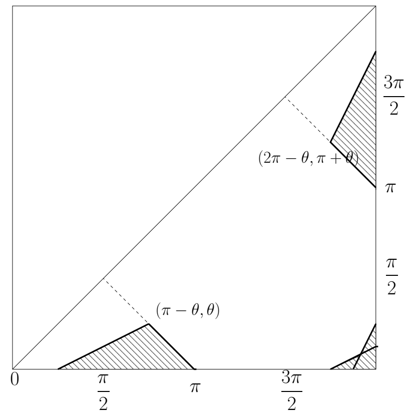

Figure 17: Elliptic products one complex symmetry and a real elliptic for $0 < \theta < \pi/3$

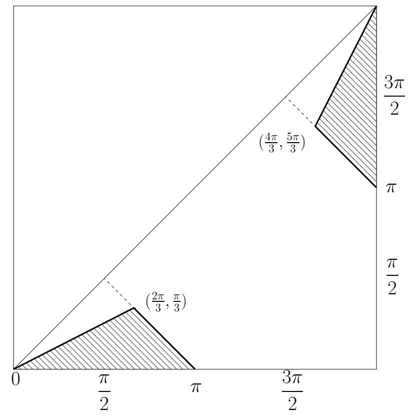

Figure 18: Elliptic products one complex symmetry and a real elliptic for $\theta = \pi/3$.

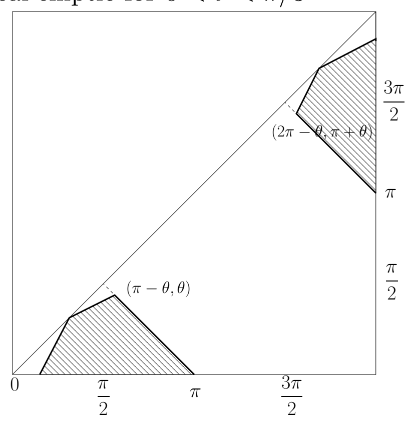

Figure 19: Elliptic products one complex symmetry and a real elliptic for $\pi/3 < \theta \leqslant \pi/2$

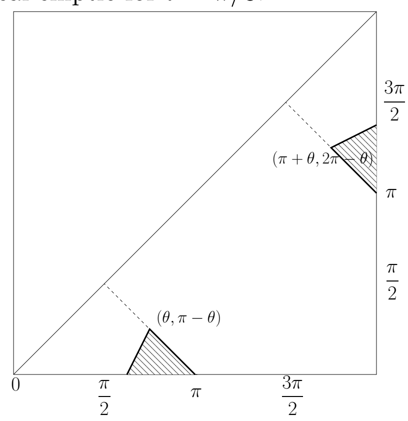

Figure 20: Elliptic products one complex symmetry and a real elliptic for $\pi/2 < \theta \leqslant \pi$

$\partial \mathrm{H}^2_{\mathbb{C}}$, then τ is reduced to a point. We now consider the case where τ isn't degenerate. Let τ be such a triangle, with Cartan invariant α. We chose lifts to $\mathbb{C}^3$ of the vertices, denoted $\mathbf{p}_i$, satisfying

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \langle \mathbf{p}_2, \mathbf{p}_3 \rangle = -1 \text{ and } \langle \mathbf{p}_3, \mathbf{p}_1 \rangle = -e^{i\alpha}. \tag{29}$$

We denote by σ_i the geodesic connecting p_{i-1} and p_{i+1} (indices taken mod. 3). In terms of the lifts given in (29), these geodesics are parametrized as follows (for $t \in \mathbb{R}$):

$$\sigma_1(t) = e^{t/2}\mathbf{p}_2 + e^{-t/2}\mathbf{p}_3 \quad \sigma_2(t) = e^{t/2}\mathbf{p}_3 + e^{-t/2}e^{i\alpha}\mathbf{p}_1 \quad \sigma_3(t) = e^{t/2}\mathbf{p}_1 + e^{-t/2}\mathbf{p}_2. \tag{30}$$

For $k = 1, 2, 3$, I_k exchanges the endpoints of σ_k, and thus it fixes a unique point on it, denoted $\sigma_k(t_k)$. The following Lemma shows that the triple (I_1, I_2, I_3) is completely determined by the involution type of each of the I_k's, and the three parameters $(t_1, t_2, t_3) \in \mathbb{R}^3$ of their fixed points on σ_1, σ_2 and σ_3.

Lemma 5 *Let t_k be the parameter of the fixed point on σ_k of the involution I_k $(k = 1, 2, 3)$. Then I_k is given by $I_k(Z) = -Z + \varepsilon_k \langle Z, \mathbf{n}_k \rangle \mathbf{n}_k$, where $\varepsilon_k = 1$ when I_k is a complex symmetry about a line, and $\varepsilon_k = -1$ when I_k is a central involution and*

$$\mathbf{n}_1 = e^{t_1/2}\mathbf{p}_2 - \varepsilon_1 e^{-t_1/2}\mathbf{p}_3 \quad \mathbf{n}_2 = e^{t_2/2}\mathbf{p}_3 - \varepsilon_2 e^{-t_2/2}e^{i\alpha}\mathbf{p}_1 \quad \mathbf{n}_3 = e^{t_3/2}\mathbf{p}_1 - \varepsilon_3 e^{-t_3/2}\mathbf{p}_2. \tag{31}$$

Proof. Given any pair (p, q) of boundary points in $\mathrm{H}^2_{\mathbb{C}}$, connected by a geodesic γ, any isometric I involution exchanging p and q preserves γ, and acts on γ as a half-turn. Now, if I is a central involution, then it is completely determined by its fixed point, which can be any point on γ. If I is a complex symmetry, its mirror is orthogonal to the complex line spanned by p and q, and it intersects γ at the unique fixed point of I on γ. In both cases, the type of I and its fixed point on γ determine I completely. To obtain the above expressions, note that $\langle \mathbf{n}_k, \mathbf{n}_k\rangle = 2\varepsilon_k$, so that the involution I_k defined in the statement has the right nature. Moreover by direct computation, we see that I_k given above fixes $\sigma_k(t_k)$ in all cases, and exchanges p_{k-1} with p_{k+1}. Note that when $\varepsilon_k = -1$, the $\mathbf{n}_k$ is in fact $\sigma_k(t_k)$. □

These expressions allow us to compute the eigenvalue of $I_1I_2I_3$ associated to $\mathbf{p}_2$.

Proposition 17 *The eigenvalue of $I_1I_2I_3$ associated to $\mathbf{p}_2$ is equal to $-\varepsilon_1\varepsilon_2\varepsilon_3 e^{t_1+t_2+t_3-i\alpha}$*

Proof. Using the expressions in Lemma 5, it is straightforward to verify that:
$I_3(\mathbf{p}_2) = -\varepsilon_3 e^{t_3}\mathbf{p}_1$, $I_2(\mathbf{p}_1) = -\varepsilon_2 e^{t_2-i\alpha}\mathbf{p}_3$ and $I_1(\mathbf{p}_3) = -\varepsilon_1 e^{t_1}\mathbf{p}_2$. □

In particular, this observation gives another point of view on Proposition 9, that says that a loxodromic isometry is a triple product of any type.

Proof. [Alternative proof of Proposition 9] Let λ be a complex number with modulus $|\lambda| > 1$. First, it is always possible to find three real numbers t_1, t_2 and t_3 such that $|\lambda| = e^{t_1+t_2+t_3}$. Having fixed such values of the $t_i's$, it is possible to find a value of $\alpha \in [-\pi/2, \pi/2]$ such that $-\varepsilon_1\varepsilon_2\varepsilon_3 e^{t_1+t_2+t_3-i\alpha}$ is equal to λ, up to multiplication by a cube root of unity. In view of Section 3.3, this means that the triple product $I_1I_2I_3$ can belong to any loxodromic conjugacy class. □

7.2 Screw-parabolic triple products as limits of elliptic triple products

We will need the following simple facts. Assume (E_n) is a sequence of elliptic elements with angle pairs (α_n, β_n), converging to a limit $E_\infty \neq Id$. Then E_∞ is either parabolic or elliptic. In the case where $\lim \alpha_n = 0$ and $\lim \beta_n = \beta_\infty \neq 0$, then E_∞ is either special elliptic with angle pair $(0, \beta_\infty)$, or a screw-parabolic map with rotation angle β_∞. If $\beta_\infty = 0$, then the limit is unipotent parabolic, but it can be of any unipotent type.

Proposition 18 *(1) Every screw-parabolic isometry is a product of three central involutions.*
(2) Every screw-parabolic isometry which is not half-turn parabolic is a product of a complex symmetry and two central involutions.

Proof. (1) Fix a hyperbolic conjugacy class $\mathcal{C}$. In Section 6.1, we described the possible elliptic conjugacy classes of a product HI, with $H \in \mathcal{C}$ and I an involution.

Assume that I is a central involution, so that HI is a product of three central involutions (recall H can be written as a product of two central involutions). The possible elliptic conjugacy classes for the product HI are depicted on Figure 13. The boundary segments of this chambers are of two types.

- Reducible walls correspond to reducible pairs (H, I).
- One horizontal segment and one vertical one on the boundary of the square, given respectively by $h = \{(\theta, 0), \pi/2 \leqslant \theta \leqslant 2\pi\}$ and $v = \{(2\pi, \theta), 0 \leqslant \theta \leqslant 3\pi/2\}$ (see Figure 13).

Consider a point on one of the two segments h and v, which is not a reducible point, that is not an intersection point of one of the reducible walls with h or v. As the image of the product map is closed, this point represents the conjugacy class of a product HI as above. However, if it corresponded to an elliptic conjugacy class, it would be special elliptic, and thus by Lemma 2 the pair (H, I) would be reducible. Therefore the product HI can only be parabolic in that case. Moreover, its rotation angle can take any value θ such that $0 < \theta < 2\pi$.

(2) To prove the second item, we proceed along the same lines. We fix a hyperbolic class, so that any element in it is a product of two central involutions, and then we consider the polygon which is the image of the product map of one hyperbolic element and a complex symmetry. This polygon is depicted on Figure 14. By the same argument as for the first item, every screw parabolic isometry of which rotation angle appears on the non-reducible boundary of the image polygon can be written as a product of two central involution

and a complex symmetry. The non-reducible boundary of the image polygon is formed by the two segments $\{(\theta,0),\pi<\theta<2\pi\}$ and $\{(2\pi,\theta),0<\theta<\pi\}$. In turn, we obtain this way every screw-parabolic element except for half-turn ones for which we cannot decide yet. □

7.3 Half-turn and unipotent parabolic isometries

We now study separately the remaining parabolic conjugacy classes: unipotent and half-turn parabolic isometries. To decide whether or not a given parabolic isometry is a product of three involutions, we will consider pairs (P,I) where P is parabolic and I an involution and decide if PI is a product of two involutions using the results of Section 3.4.

A. 3-step unipotent isometries

Proposition 19 *A 3-step unipotent map is the product of three holomorphic involutions of any type.*

Proof. By Remark 4, it suffices to prove that a 3-step unipotent is a both a triple product of type $(-,-,-)$ and $(+,-,-)$.

We first consider the case of three central involutions, that is $(-,-,-)$. We know from Proposition 1 that a parabolic map in the Poincaré disk is a product of three half-turns. Consider such a configuration of half-turns, and embed the Poincaré disk into $\mathrm{H}^2_{\mathbb{C}}$ as a real plane, mapping the three half-turns to central involutions. Each of the central involutions preserves the real plane. As a result, we obtain a parabolic element in PU(2,1) that preserves a real plane. It is thus 3-step unipotent (see Section 3.3.2).

For the second case when two of the I_k's are central involutions and the third one is a complex symmetry, we go back to Lemma 5 and Proposition 17. In that case we see that $\varepsilon_1\varepsilon_2\varepsilon_3$ is equal to -1 . Therefore the null-type eigenvalue of $I_1I_2I_3$ is equal to $-e^{-i\alpha}$. If the product is unipotent, then this eigenvalue must be equal to a cube root of 1. The only possiblities are $\alpha=\pm\pi/3$. In particular the ideal triangle Δ is not contained a real plane. An example of such a triple of involution can thus not be as simple as for triples of central involutions. We thus proceed by giving an example of a pair (P,I) where P is 3-step unipotent and I is a complex symmetry, with PI hyperbolic. This shows that P is a product of a hyperbolic isometry and a complex symmetry, which is what we need. We take P and I as follow, in the Siegel model.

$$P=T_{[1,0]}=\begin{bmatrix}1 & -\sqrt{2} & -1\\ 0 & 1 & \sqrt{2}\\ 0 & 0 & 1\end{bmatrix},\quad I=\frac{1}{8}\begin{bmatrix}-6 & i\sqrt{6} & 1/2\\ -4i\sqrt{6} & 4 & -i\sqrt{6}\\ 8 & 4i\sqrt{6} & -6\end{bmatrix}.$$

The involution I is the complex symmetry about the line polar to the positive vector $\begin{bmatrix}1/4 & -i\sqrt{6}/2 & 1\end{bmatrix}^T$. By a direct computation, we see that $\mathrm{tr}(PI)=4e^{2i\pi/3}$. Therefore $e^{-2i\pi/3}PI$ has trace 4. It is thus hyperbolic, and can be written as a product of two central involutions. □

B. 2-step unipotent isometries

Proposition 20 *A 2-step unipotent isometry can be written as a product of three complex symmetries, but cannot be written as a triple product of any other kind.*

To prove Proposition 20, we will use the following.

Lemma 6 *Any pair (P,I) where P is 2-step unipotent and I is an involution is conjugate in Isom($\mathrm{H}^2_{\mathbb{C}}$) to a pair given in the Siegel model by (P,I_u) or (P,I_∞), where*

$$P=\begin{bmatrix}1 & 0 & i\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\ \text{and}\ I_u=\begin{bmatrix}0 & 0 & u\\ 0 & -1 & 0\\ u^{-1} & 0 & 0\end{bmatrix}\ \text{for some}\ u\neq 0,\ \text{or}\ I_\infty=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\tag{32}$$

Note that when the involution I_u is a complex symmetry (resp. a central involution) when $u > 0$ (resp. $u < 0$). The involution I_∞ is a complex symmetry that fixes the fixed point of P.

Proof. Given such a pair (P, I), we can always conjugate by an isometry it so that P is given by the matrix given in (6). Then we still have the freedom of conjugating I be an element normalizing P in SU(2,1). In particular, we may conjugate I by any Heisenberg translation $T_{[z,t]}$ as in (12).

1. First assume I is a central involution. The fixed point of P is q_∞. Writing $I(q_\infty) = [w, s]$ in Heisenberg coordinates and conjugating the pair (P, I) by $T_[-w, -s]$ gives an involution that exchanges q_∞ and the origin of the Heisenberg group, which has the form I_u with $u < 0$.

2. If I is a complex symmetry that does not fix q_∞, then we do the same, and obtain I_u with this time $u > 0$.

3. Finally, if I is a complex symmetry fixing q_∞, let $[w, s]$ be another fixed point of I in $\partial \mathrm{H}^2_{\mathbb{C}}$. Then conjugating by $T_{[-w,-s]}$ gives a complex symmetry with mirror the complex line connecting q_∞ to the origin of the Heisenberg group. This is I_∞. □

Proof. [Proof of Proposition 20] In view of Lemma 6, we only need to consider the pair (P, I_u) of (P, I_∞) as in (32). By a straightforward computation, we have:

$$\mathrm{tr}(PI_u) = -1 + \frac{i}{u} \text{ and } \mathrm{tr}(PI_\infty) = -1. \tag{33}$$

Any lift to SU(2,1) of a hyperbolic isometry has trace of the form $xe^{2ik\pi/3}$, where $x > 3$ and $k \in \{0, 1, 2\}$. This shows that none of the above quantities can be the trace of a hyperbolic isometry. Since hyperbolic isometries are products of two central involutions, this proves that P is not a triple product of type $(-,-,-)$ or $(+,-,-)$.

We still need to consider the $(+,+,+)$ and $(+,+,-)$ types, i.e. triple products where at least two of the involutions are complex symmetries. If the mirrors of the complex symmetries are ultraparallel, then their product is hyperbolic, and we fall in the previous case. We thus need to determine when a product PI as above can be real elliptic or 3-step unipotent. First, the above discussion applies, and the trace of PI still has real part equal to -1. In turn PI cannot be unipotent, as any lift to SU(2,1) of a unipotent isometry has trace 3ω, where ω is a cube root of 1.

If PI is real elliptic, its trace must be of the form $xe^{2ik\pi/3}$ with $x \in [-1, 3)$ and $k = 0, 1, 2$. Considering (33), we see that the only possible pairs are

$$(P, I_u) \text{ with } u = \pm\sqrt{3}^{-1} \text{ or } (P, I_\infty).$$

- If $u = -\sqrt{3}^{-1}$, I_u is a central involution. Computing the eigenvalues and eigenvectors of PI_u in that case we see that the angle pair of PI_u is $\{5\pi/3, 4\pi/3\}$. Thus PI_u is not real elliptic, and cannot be a product of two complex symmetries.

- Assume $u = \sqrt{3}^{-1}$. In this case I_u is a complex symmetry. Similary, we see that the angle pair of the product is $\{5\pi/3, \pi/3\}$. This means that PI_u is real elliptic, and thus can be written as a product of two complex symmetry.

- Consider now the pair (P, I_∞). In this case we see that

$$PI_\infty = \begin{bmatrix} -1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

  which is half-turn parabolic. By Proposition 4, it is not a product of two involutions.

The only possibility is thus that P is a product of three complex symmetries. □

C. Half-turn parabolics

Proposition 21 *A half-turn parabolic isometry*
(1) can be written as a product of three complex symmetries,
(2) cannot be written as a product of a complex symmetry and two central involutions.

Proof. Let (I_1, I_2, I_3) be a triple of involutions of one of the above types, and $\Delta = (p_1, p_2, p_3)$ be the ideal triangle associated to the fixed point $p_2 \in \mathrm{H}^2_{\mathbb{C}}$ of $I_1I_2I_3$, as in Section 7.1. For these triples of involutions the product $\varepsilon_1\varepsilon_2\varepsilon_3$ is equal to 1. Going back to Proposition 17, we see that if τ is non-degenerate, then the eigenvalue associated to p_2 is $-e^{-i\alpha}$, where α is the Cartan invariant of τ. Now, the null-type eigenvalue of a half-turn parabolic element $P \in SU(2,1)$ is one of -1, $-e^{2i\pi/3}$ or $-e^{-2i\pi/3}$. As the Cartan invariant belongs to $[-\pi/2, \pi/2]$ the only possibilty is $\alpha = 0$. This implies that τ is contained in a real plane, and this real plane is preserved by the triple product $I_1I_2I_3$. But a half-turn parabolic map doesn't preserve any real plane. This discussion shows that the triangle Δ must be degenerate.

To verify the first part, it suffices to consider a triple of complex symmetries whose mirrors all have a common point on $\partial\mathrm{H}^2_{\mathbb{C}}$. For such a configuration, the triple product is half-turn parabolic as soon as the three complex lines are distinct. For example, the product of the three reflections I_1, I_2, I_3 with mirrors polar to:

$$\mathbf{n}_1 = \begin{bmatrix} \frac{-i}{2} \\ 1 \\ 0 \end{bmatrix}, \mathbf{n}_2 = \begin{bmatrix} -\frac{1+i}{2} \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{n}_3 \begin{bmatrix} \frac{-1}{2} \\ 1 \\ 0 \end{bmatrix}, \tag{34}$$

gives a triple product equal to

$$\begin{bmatrix} -1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{35}$$

The discussion at the beginning of this proof shows that p_2 is the fixed point of the half-turn parabolic triple product $I_1I_2I_3$ with, say, I_3 a complex symmetry, and I_1 and I_2 central involutions. Since two central involutions and a complex symmetry cannot have a common boundary fixed point, the only possiblity is that two points exactly among p_2, $p_1 = I_3p_2$ and $p_3 = I_1p_2$ are equal. Assume $p_3 = p_2$. This means that p_2 is a fixed point of I_3, and the associated eigenvalue is equal to $-\omega$ with ω a cube root of 1. As I_3 fixes $p_3 = p_2$, I_1 and I_2 both exchange p_1 and p_2. In particular, the product I_1I_2 is hyperbolic and fixes p_2, and the associated eigenvalue is equal to some $r\omega'$ for $r > 1$ and ω' a cube root of unity. This implies that the product $I_1I_2I_3$ has eigenvalue associated to p_2 equal to $-r\omega\omega'$, and thus it is half-turn loxodromic. The other cases are similar. □

Parabolic conjugacy class	$(+,+,+)$	$(+,+,-)$	$(+,-,-)$	$(-,-,-)$
Screw-parabolic with $\theta \neq \pi$	yes	yes	yes	yes
Half-turn parabolic	yes	yes	no	yes
2-step unipotent	yes	no	no	no
3-step unipotent	yes	yes	yes	yes

Figure 21: Triple product types of parabolic isometries

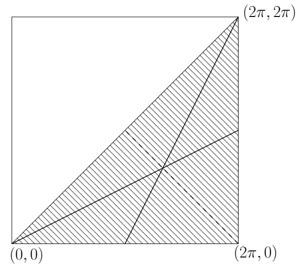

Figure 22: Every regular elliptic isometry is the product of two hyperbolic isometries

8 Involution and commutator length

8.1 Involution length

We now prove Theorem 1, stated in the introduction: the involution length of PU$(2, 1)$ is 4.

Proof. [Proof of Theorem 1] It only remains to prove that those elements in PU(2,1) that are not products of two or three central involutions, are products of four central involutions. This leaves :

1. The regular elliptics whose angle pair does not lie in the shaded polygon of Figure 13.
2. Non-regular elliptic isometries (complex reflections about lines and about points with arbitrary rotation angles).
3. 2-step unipotent parabolic isometries.

For the first part, it suffices to prove that any regular elliptic map is a product of two hyperbolic isometries. To do so, we fix two hyperbolic conjugacy classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and apply the strategy of Section 4. If (A, B) is a reducible pair in $\mathcal{C}_1 \times \mathcal{C}_2$ with elliptic product, then only a positive type vector can be a common eigenvector for A and B. In particular, the product AB has a lift to SU(2,1) of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{C} \end{bmatrix},$$

where $\tilde{C}$ has eigenvalues $\{e^{i\alpha}, e^{i\beta}\}$, where $e^{i\alpha}$ has positive type, $e^{i\beta}$ has negative type and α, β lie in $[0, 2\pi)$. The rotation angles of AB are θ_C (in the complex line preserved by A and B) and θ_N (in the normal direction). Applying the same arguments as in Sections 5 and 6 we see that the two rotation angles of AB satisfy

$$\theta_C = 2\theta_N \mod 2\pi, \text{ with } \theta_N \in [2, 2\pi) \tag{36}$$

This implies by projecting to the lower triangle of the square $[0, 2\pi]^2$ that the reducible walls are the two segments given by

$$r_1 = \Big[(0,0), (2\pi, \pi)\Big] \text{ and } r_2 = \Big[(\pi, 0), (2\pi, 2\pi)\Big] \tag{37}$$

Considering configurations of two hyperbolic isometries preserving a common real plane, we see that all angle pairs $(\theta, 2\pi - \theta)$ are obtained by irreducible configurations (these pairs form the dashed segment on Figure 22). This implies that all regular elliptic isometries are products of four central involutions.

Lets us now consider non regular elliptics. First, complex reflections about points have angle pairs of the form $\{\theta, \theta\}$ lying on the diagonal of Figure 22. As the image of the product map is closed, they are obtained as limits of regular elliptic products of two hyperbolic maps. Secondly, we know from Proposition 10 that for every regular elliptic element E with angle pair $\{\pi + \theta, \pi\}$, there exists a triple of involutions (I_1, I_2, I_3) such that $E = I_1 I_2 I_3$ (note the pair $\{\pi + \theta, \pi\}$ lies in $\mathcal{E}_{++-}$) on Figure 13). Now, consider the central involution I_4 about the fixed point of E. The product $I_1 I_2 I_3 I_4$ has angle pair $\{2\pi + \theta, 2\pi\} \sim \{\theta, 0\}$. This shows that $I_1 I_2 I_3 I_4$ is a complex reflection, and that any complex reflection can be obtained this way. Finally, we consider 2-step parabolics. We know from Sections 7.2 and 7.3 that any half-turn parabolic P is a product of three involutions. Writing $P = I_1I_2I_3$, call I_4 the complex symmetry about the complex line preserved by P. Then the product $I_1I_2I_3I_4$ is 2-step parabolic; for example when

$$P = \begin{bmatrix} -1 & 0 & -it \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } I_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

we obtain $PI_4 = T_{[0,1]}$. □

Note that the first part of the proof showed the following result:

Proposition 22 *Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be three conjugacy classes in* $\mathrm{PU}(2,1)$*, two of them hyperbolic and one regular elliptic. Then there exists $(A, B, C) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ such that $ABC = \mathrm{Id}$.*

In fact, the following stronger statement follows by combining this with Proposition 6:

Proposition 23 *Let $\mathcal{C}_3$ be a regular elliptic conjugacy class in* $\mathrm{PU}(2,1)$*. There exists an open subset of $\mathcal{L} \times \mathcal{L}$, containing $\mathcal{H} \times \mathcal{H}$ (and depending explicitly on $\mathcal{C}_3$) such that for any $\mathcal{C}_1, \mathcal{C}_2$ in this subset, there exists $(A, B, C) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3$ such that $ABC = \mathrm{Id}$.*

We can now prove Theorem 2, stated in the introduction: the involution length of PU(n,1) is at most 8 for all $n \geqslant 3$. The proof is done by combining the ingredients of Theorem 1 and Theorem 3.1 of [GT].

Proof. [Proof of Theorem 2] Let $A \in$ PU(n,1) be a holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$. First assume that A is elliptic, i.e. belongs to a copy of U(n) (identified to P(U(n) × U(1)), for example via the embedding $U \mapsto \mathrm{P}((U, 1))$). Given any element $\tilde{B} \in \mathrm{U}(2)$ with $\det(B) = \det(A)^{-1}$, we extend $\tilde{B}$ to an element B of U(n,1) as follows:

$$B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & I_{n-1} \end{bmatrix}.$$

Then AB belongs to $\mathrm{SU}(n) \times \{1\}$, so by Theorem 3.1 and Lemma 3.3 of [GT] it is a product of at most four involutions of $\mathrm{U}(n) \times \{1\}$.

The matrix B corresponds to an elliptic isometry preserving a copy of $\mathrm{H}^2_{\mathbb{C}}$ in $\mathrm{H}^n_{\mathbb{C}}$. Its rotation angles are $\{\theta_1, \theta_2, 0, \cdots, 0\}$, where θ_1 and θ_2 are the rotation angles of $\tilde{B}$. The only constraint on θ_1 and θ_2 is that $e^{i(\theta_1+\theta_2)} = \det(A)^{-1}$. But every line of the form $\theta_1 + \theta_2 = \mathrm{C}$ intersects the region $\mathcal{E}_{++-} \cup \mathcal{E}_{+++}$ representing elliptic conjugacy classes which are triple products of involutions (see Figure 11 and Proposition 10). Therefore we can choose θ_1, θ_2 in such a way that $\tilde{B}$, resp. B, is a product of three involutions in PU(2,1), resp. in PU(n,1) (again, under the embedding of U(2) as $\mathrm{P}(\mathrm{U}(2) \times \{1\})$). Therefore AB is a product of at most 7 involutions.

Now if A is not elliptic, there exists an involution $I \in$ PU(n,1) such that IA is elliptic. Indeed, pick any point $x_0 \in \mathrm{H}^n_{\mathbb{C}}$, so that $Ax_0 \neq x_0$, and let I be the central involution about the midpoint of (x_0, Ax_0). Then IA fixes x_0, therefore IA is a product of at most 7 involutions and A is a product of at most 8 involutions. □

8.2 Commutator length

Theorem 5 *Every holomorphic isometry of $\mathrm{H}^2_{\mathbb{C}}$ is a commutator of holomorphic isometries.*

In fact we get a slightly more precise statement, Proposition 24 below, using the following definition:

Definition 2 *A pair (A, B) is $\mathbb{C}$-decomposable if there exist three complex involutions (I_1, I_2, I_3) such that $A = I_1I_2$ and $B = I_3I_2$.*

Note that this definition is slightly more general than in [W2], where the involutions were required to be complex symmetries.

Proposition 24 *For any element C in PU(2,1), there exists a $\mathbb{C}$-decomposable pair (A, B) such that $[A, B] = C$.*

Proof. It suffices to show that every element in PU(2,1) has a square root which is a product of three involutions. Indeed, if I_1, I_2 and I_3 are involutions, then we have $(I_1 I_2 I_3)^2 = [I_1 I_2, I_3 I_2]$.
(1) This is clear for loxodromic isometries, as the square root of a loxodromic map is loxodromic and thus is a product of three complex symmetries.
(2) Every screw- or 2-step unipotent parabolic isometry has a square root which is screw parabolic, thus a product of thee complex symmetries. The square root of a 3-step unipotent isometry is also 3-step unipotent, and thus a product of three complex symmetries.
(3) Let E be an elliptic element with angle pair $\{\theta_1, \theta_2\}$. Its square roots are those elliptic elements with angle pairs $\{\theta_1/2 + n\pi, \theta_2/2 + m\pi\}$, where m and n are 0 or 1. This implies in particular that every elliptic element has a square root which is regular elliptic with angle pair in $\mathcal{E}_{+++} \cup \mathcal{E}_{++-}$ (see Figures 13 and 14), and hence is a triple product of involutions. □

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