HILBERT COEFFICIENTS OF PARAMETERS

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My lecture is based on the joint works [GhGHOPV1, GhGHOPV2] and [GO1, GO2] with Ghezzi, Hong, Ozeki, Phuong, and Vasconcelos.

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Then, for each $\mathfrak{m}$-primary ideal I in A, we have the Hilbert coefficients $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ of I, so that the Hilbert function of I is given by the polynomial

$$length_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

for $n \gg 0$. In my lecture we are mainly interested in the case where $I = Q$ is a parameter ideal in A and with this notation Wolmer V. Vasconcelos posed the following vanishing conjecture.

Conjecture 1 (2008, [V])**.** Assume that A is unmixed. Then A is a Cohen-Macaulay local ring, once $\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q of A.

Recall that A is said to be unmixed, if $\dim \widehat{A}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$, where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion.

First of all, we shall settle this conjecture affirmatively and then give a characterization for rings A to have $\mathrm{e}_Q^1(A) = 0$ for some (and hence every) parameter ideal Q in A. As an application we will show that A is a generalized Cohen-Macaulay ring, once the set

$$\Lambda_1(A) = \{\mathrm{e}_Q^1(A) \mid Q \text{ is a parameter ideal in } A\}$$

is finte, provided A is unmixed. This result naturally leads us to the study of the problem of when the sets

$$\Lambda_i(A) = \{\mathrm{e}_Q^1(A) \mid Q \text{ is a parameter ideal in } A\}$$

are finite for all $1 \le i \le d$, that is the question of when the number of possible Hilbert polynomials $length_A(A/Q^{n+1})$ is finite except the leading terms $\mathrm{e}_Q^0(A)\binom{n+d}{d}$. We will also discuss the constancy of the value $\mathrm{e}_Q^1(A)$ of parameter ideals Q possessing the same integral closure $\overline{Q}$, in connection with homological degrees $\mathrm{hdeg}_{\overline{Q}}(A)$. Some open problems will be given.

References

[GhGHOPV1] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong, and W. V. Vasconcelos, *Cohen–Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals*, J. London Math. Soc. (to appear).

[GhGHOPV2] Variation of the first Hilbert coefficients of parameters with a common integral closure (preprint, 2010)

[GO1] S. Goto and K. Ozeki, *The structure of Sally modules – towards a theory of non-Cohen-Macaulay cases*, Preprint 2009.

[GO2] S. Goto and K. Ozeki, *Buchsbaumness in local rings possessing constant first Hilbert coefficients of parameters*, Nagoya Math. J. (to appear).

[V] W. V. Vasconcelos, *The Chern coefficients of local rings*, Michigan Math. J. **57** (2008), 725–743.

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