


Star and convex regular polyhedra by Origami.

Build polyhedra by Origami.]

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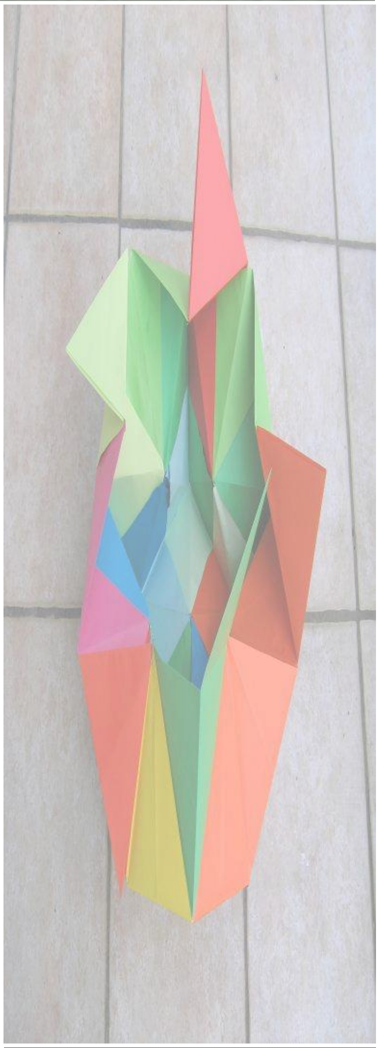
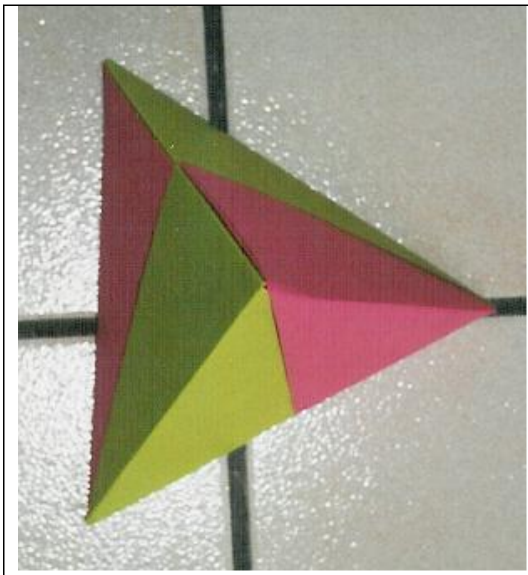
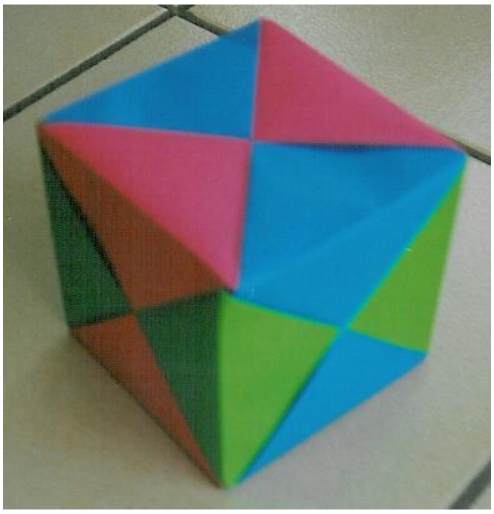
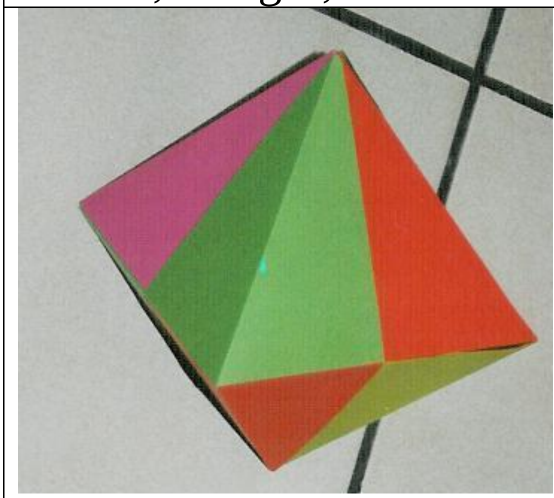
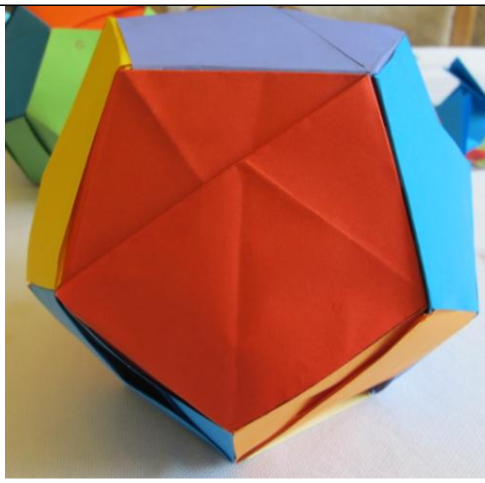
Alice Morales



2009

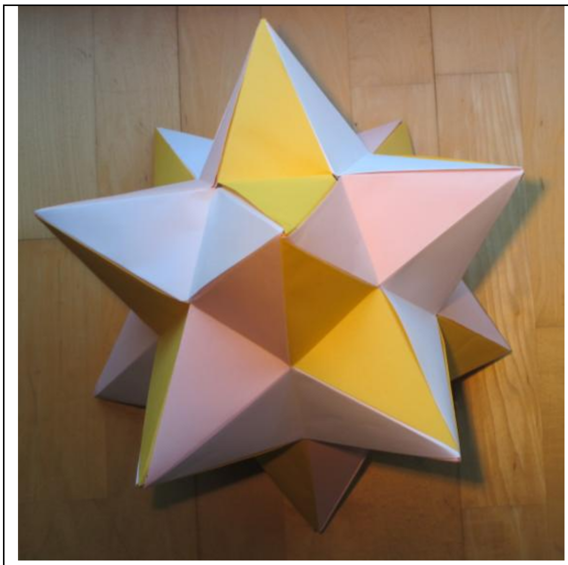
EDITION MORALES

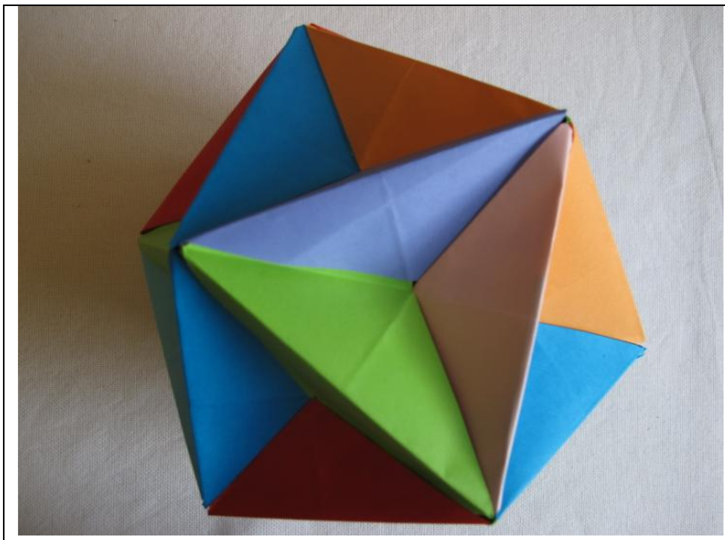
I) Table of convex regular Polyhedra

 **Tetrahedron** 2 triangular pieces of distinct type (A+B) 4 faces, 6 edges, 4 vertices	 **Cube** 6 square pieces of the same type, 6 faces, 12 edges, 8 vertices
 **Octahedron** 4 triangular pieces of the same type, 8 faces, 12 edges, 6 vertices	 **Dodecahedron** 12 pentagons 12 faces, 30 edges, 20 vertices

Icosahedron 10 triangular pieces (5A+5B) 20 faces, 30 edges, 12 vertices	

II) Table of star regular Polyhedra (non convex)

 **The small star dodecahedron** 30 pieces of the same type: 12 faces (star pentagons called pentagrams), 30 edges, 20 vertices	 **The great star dodecahedron** 30 pieces of the same type: 12 faces (star pentagons called pentagrams), 30 edges, 20 vertices



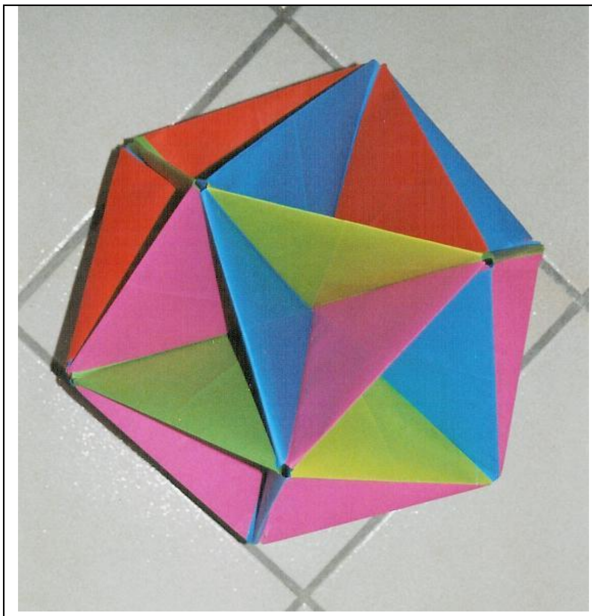
The great dodecahedron
30 pieces of the same type:
12 faces (star pentagons called pentagrams), 30 edges, 20 vertices

The great icosahedron
12 pieces of the same type,
20 faces(equilateral triangles), 30 edges, 12 vertices

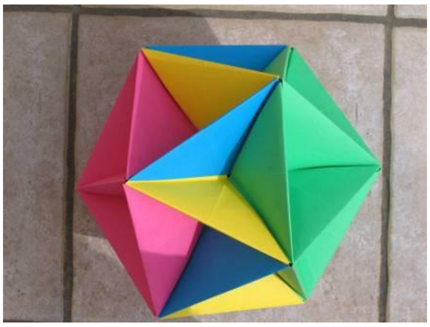
III) Some Polyhedra that should be named regular done with equilateral triangles

The triangular small star dodecahedron
30 pieces of the same type:
60 faces, 90 edges, 32 vertices

The triangular great star dodecahedron
30 pieces of the same type:
60 faces, 90 edges, 32 vertices

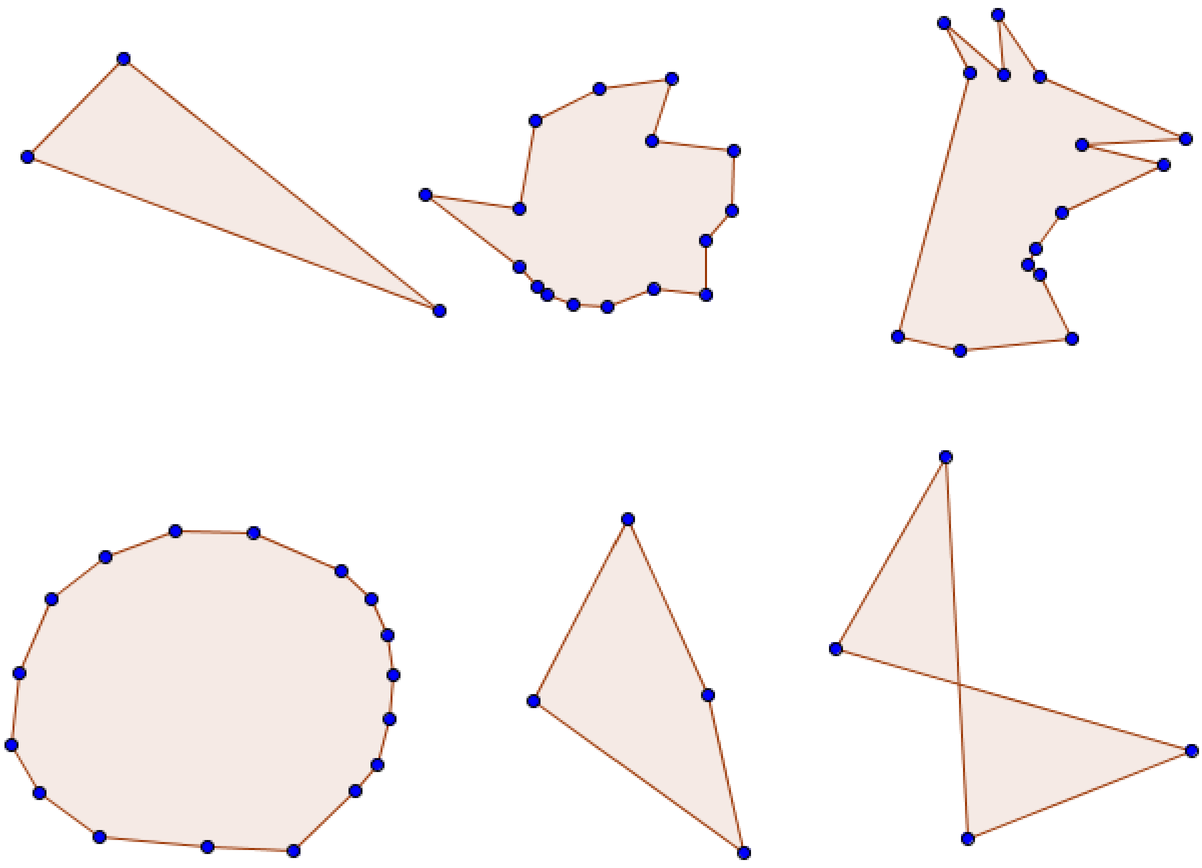
 **<u>The triangular great dodecahedron</u>** 30 pieces of the same type: 60 faces, 90 edges, 32 vertices	 **<u>The triangular great icosahedron</u>** 30 pieces of the same type:
	**<u>The great star dodecahedron inversed or the third stellation of the icosahedron</u>** 30 pieces of the same type: **60 faces, 90 edges, 32 vertices**

IV) Some Polyhedra that should be named regular done with rectangle triangles

<u>The great star dodecahedron inversed or the third stellation of the icosahedron</u> 30 pieces of the same type: **60 faces, 90 edges, 32 vertices** 	**<u>The small star dodecahedron Dodecahedron rectangle</u>** 30 square pieces of the same type: **60 faces, 90 edges, 32 vertices** 	**<u>Le great dodecahedron rectangle</u>** 30 square pieces of the same type: 60 faces, 90 edges, 32 vertices 
<u>The rectangular great icosahedron</u> 120 square pieces of the same type: **180 faces, 270 edges, 92 vertices** 	**<u>Rectangular Final Stellation of the icosahedron</u>** 90 square pieces of the same type. 	**<u>The Epcot's ball</u>** **270 square pieces** of the same type. 

v) Some geometry of polygons

A polygon is a figure in the plane bounded by a sequence of segments of lines, called edges or sides, a point common to the extremities of two edges is a corner. We give some examples:



We can see that we can distinguish such polygons, first by the number of vertices, then by the number of sides, then by the shape. Let me be more precise:

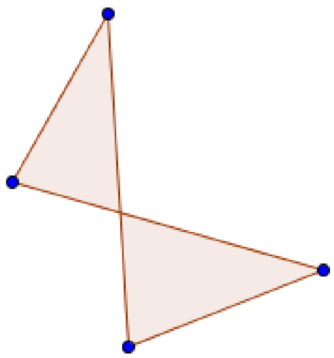
The number of sides is very important:

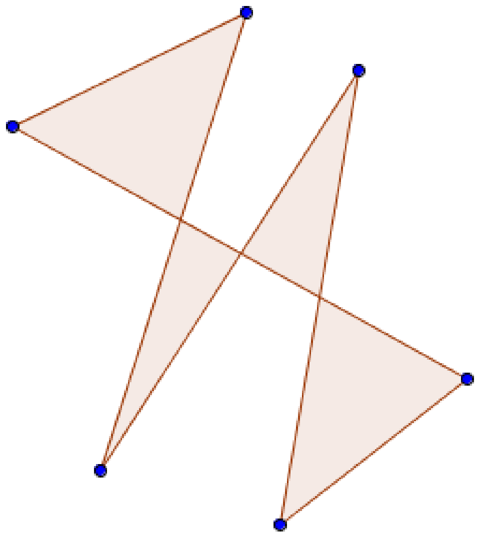
- Triangle or Polygon with three sides,
- Quadrilateral or Polygon with four sides,
- Pentagon or Polygon with five sides,

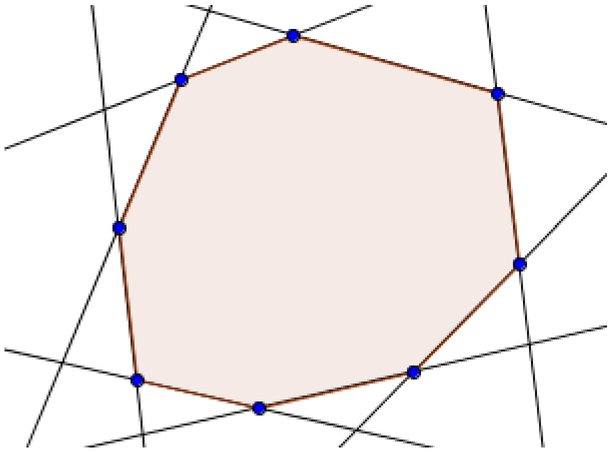
- Hexagon or Polygon with six sides,
- Heptagon or Polygon with seven sides,
- Octagon or Polygon with eight sides,
- Enneagon or Polygon with nine sides,
- Decagon or Polygon with ten sides,
- Hendecagon or Polygon with eleven sides,
- Dodecagon or Polygon with twelve sides,
- Icosagon or Polygon with twenty sides.

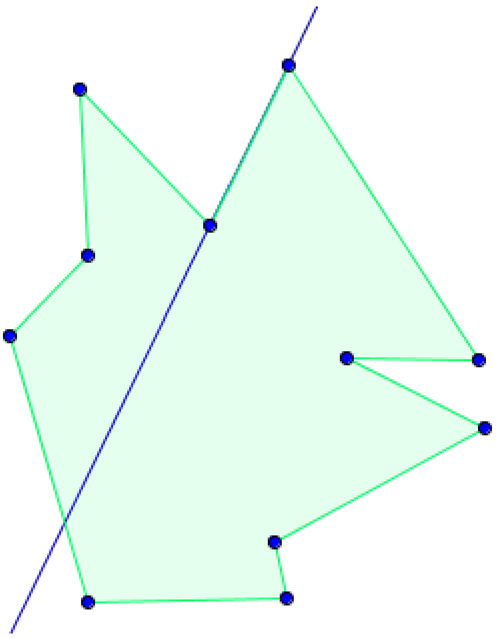
Looking for the shape of the figure we can distinguish:

- The convex polygons: Each side of the polygon is on a line, this line divides the plane into two regions, and we say that the polygon is convex if it is not divided into two polygons by every line containing one side.
- The **star polygons**:
  Two sides meet in a point which is not an extremity.

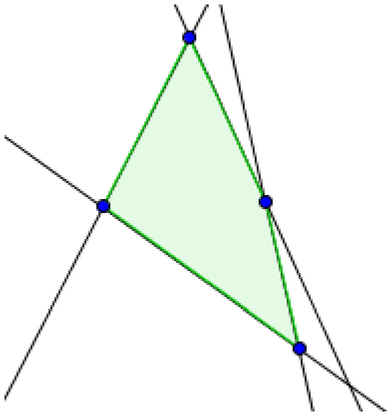


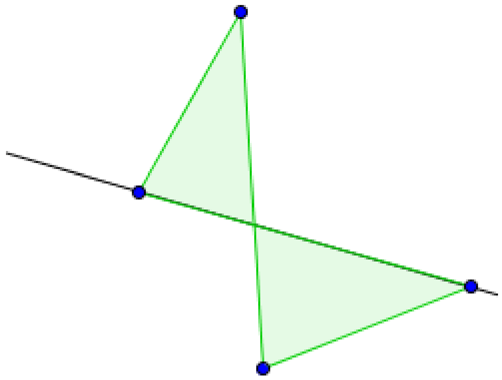






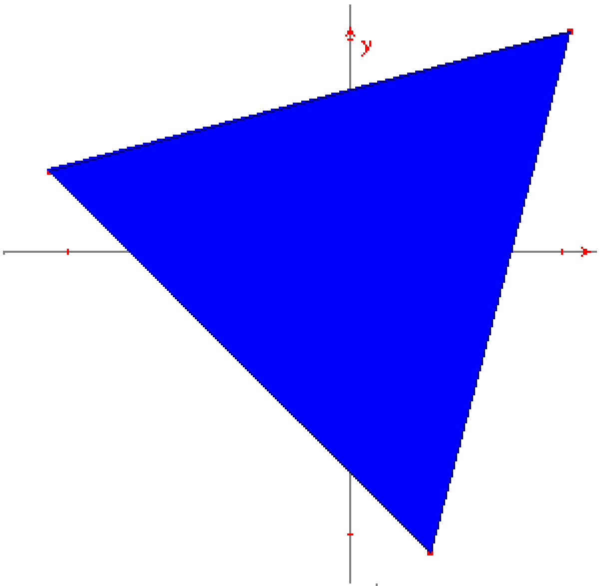
The polygon on the left is convex, but the one on the right is a star polygon.

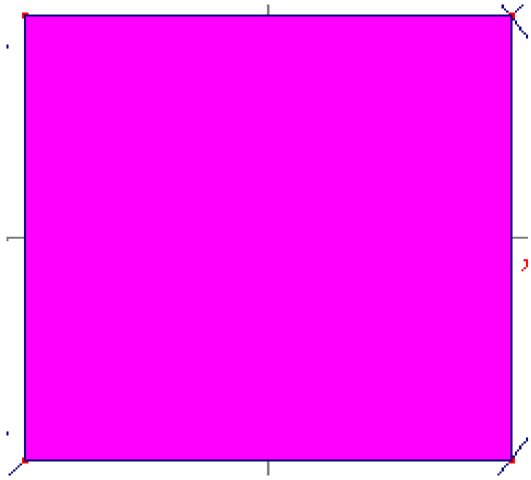


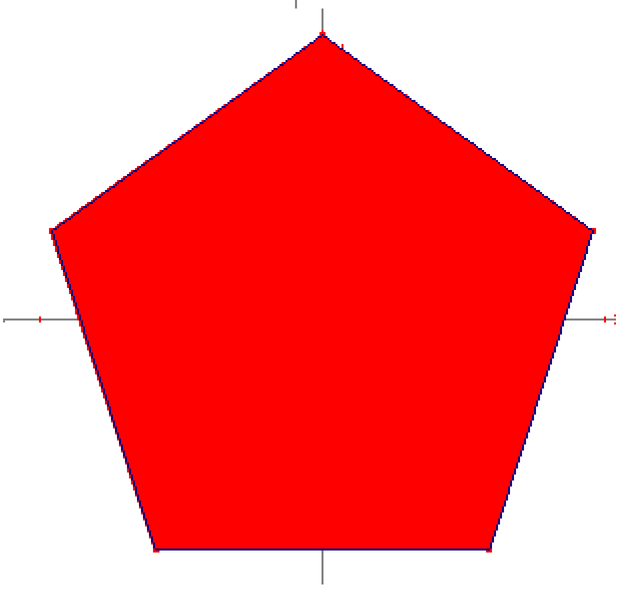


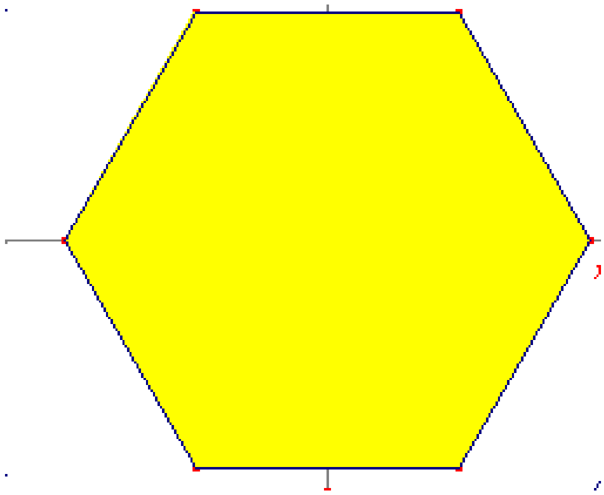
- The **<u>regular polygons</u>**. Usually a regular polygon is a convex **<u>polygon</u>** with all the <u>angles</u> of the same measure and all sides are of the same length, but there are also regular star polygons.

VI) The convex regular polygons

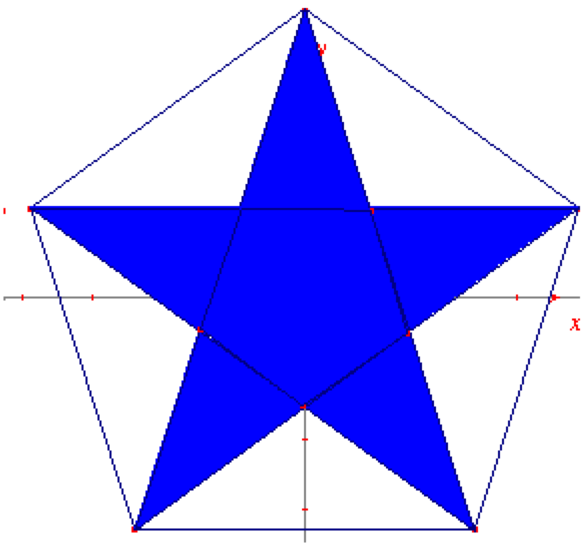


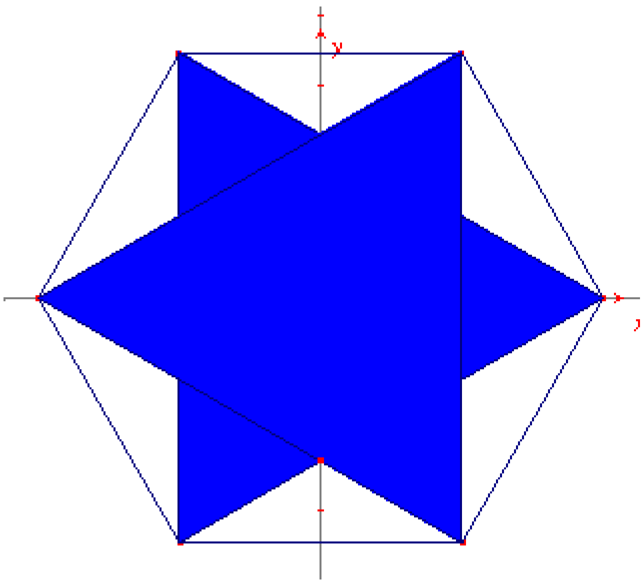


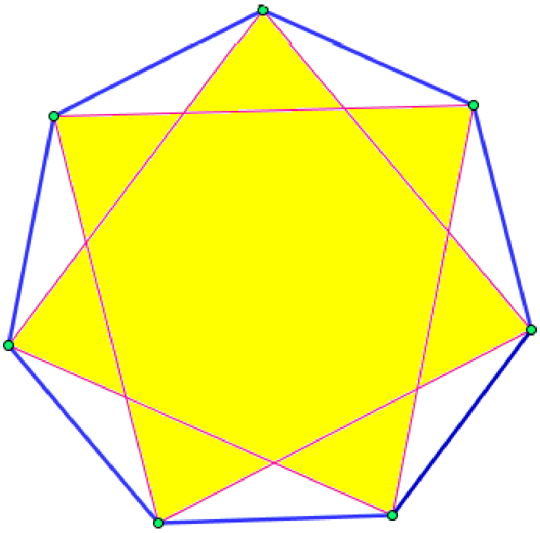
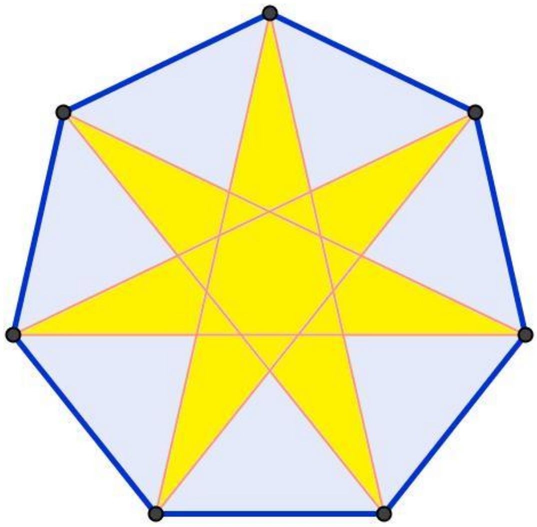


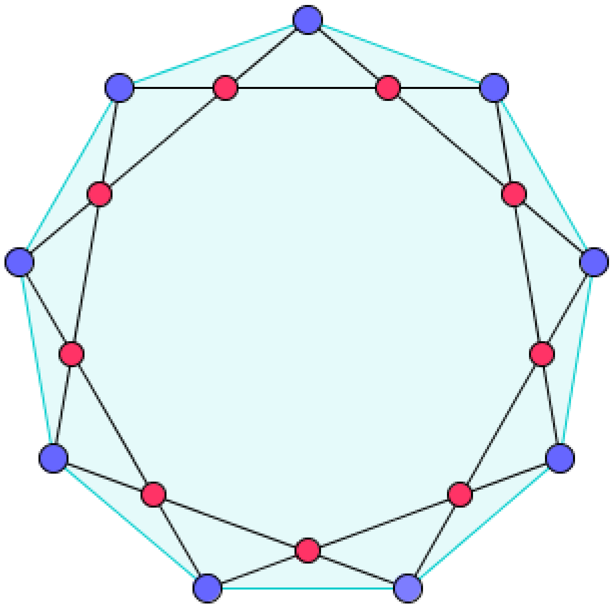
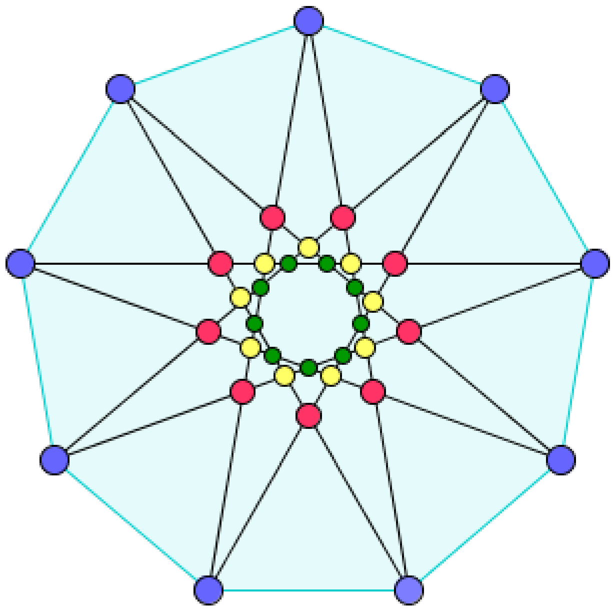


VII) The star regular polygons





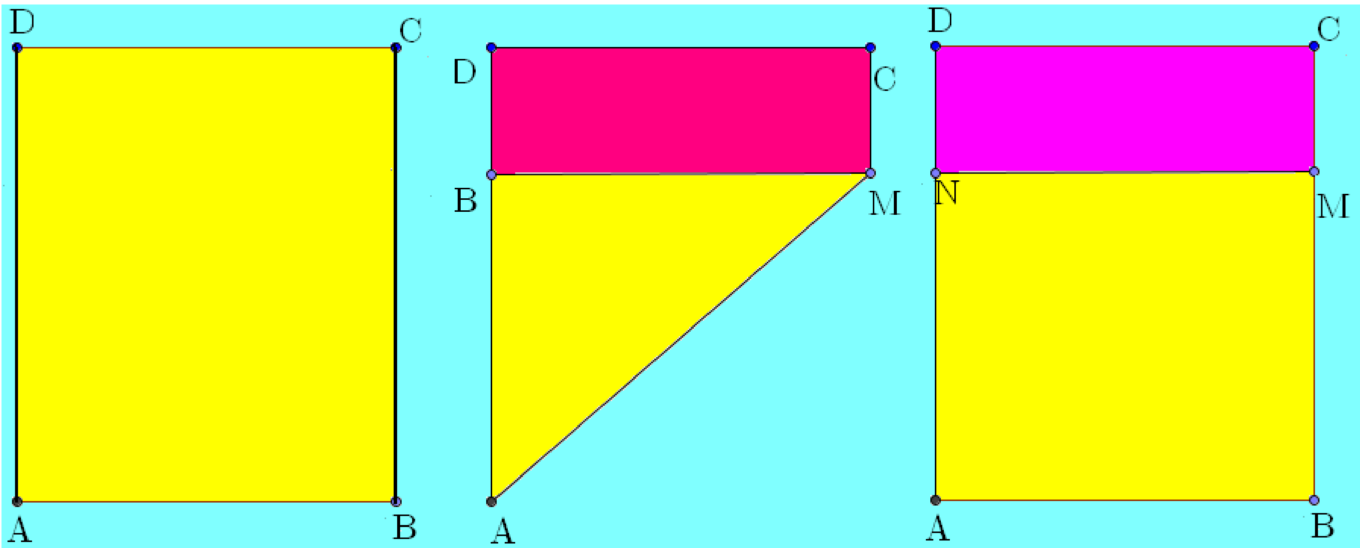



The first one is a star pentagon, it is also called pentagram, and denoted by {5/2}. The second one is a star hexagon and denoted by {6/2}. The third one is a star heptagon but if you start with one vertex and go to another vertex by skipping the next vertex, you draw a side of a star heptagon, and so on, you can see that you turn twice around the center of the heptagon, for this reason this is denoted by {7/2}, and the last one is denoted by {7/3}. The two last are the star enneagon {9/2} and {9/4}. Another explanation for this terminology {9/4} is that we have four kinds of points in the enneagon when the sides meet; here they are colored with distinct colors.

VIII) Regular convex polygons by folding paper

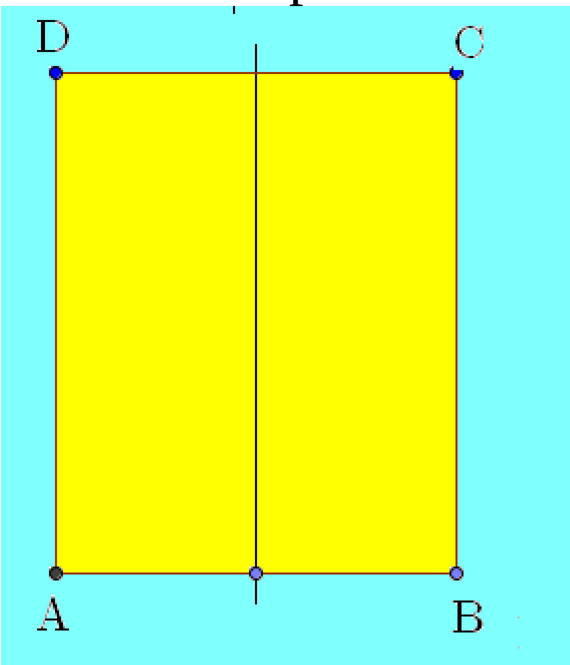
1) The square

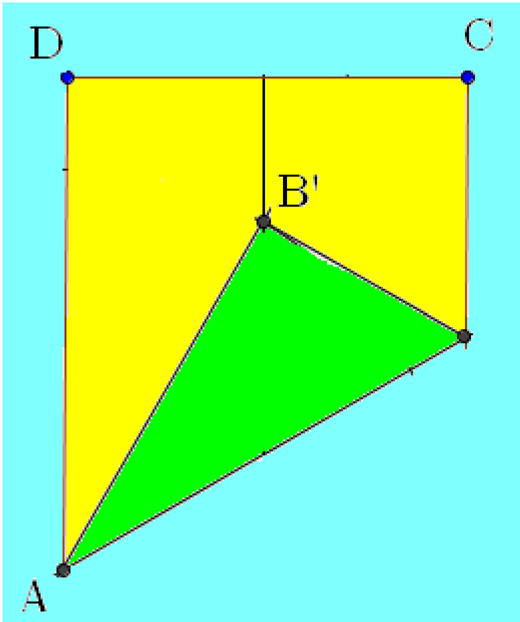
Take any rectangular sheet of paper (for example A4 or A3) ABCD put the side AB on the side AD, fold the sheet to determine the point M and N, fold the sheet of paper long the segment (MN). The figure ABMN is a square.



2) The equilateral triangle

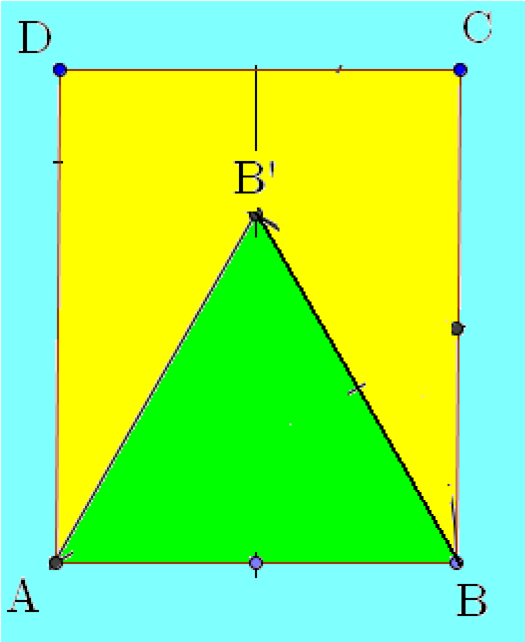
Take any rectangular sheet of paper (for example A4 or A3) ABCD, fold it in the middle along AB, you get a folding line, fold the sheet of paper such that the point B on the side AB will be on the folding line. We can determine the point B'.

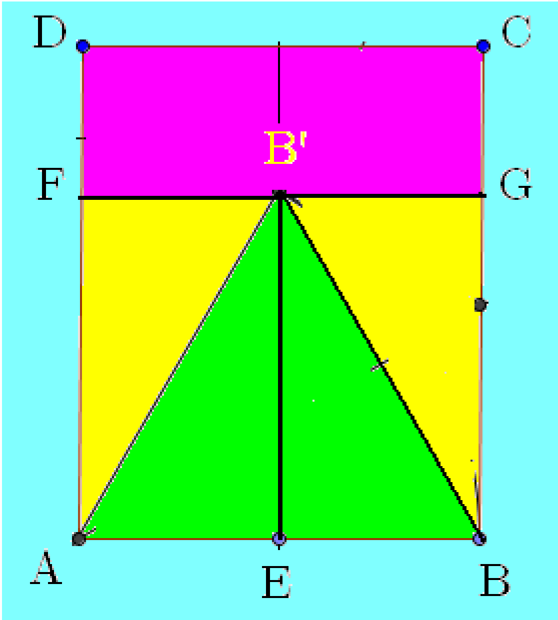




Fold along the line AB', the triangle ABB' is equilateral.

Make a bronze sheet of paper: In order to make regular Polyhedra by folding paper, we will need special dimensions on a sheet of paper; people working on Origami call it a bronze rectangle or bronze sheet of paper. A bronze sheet of paper is a rectangular sheet with width l and length $l\sqrt{3}$. The rectangles AEB'F and EBGB' are bronze rectangles.





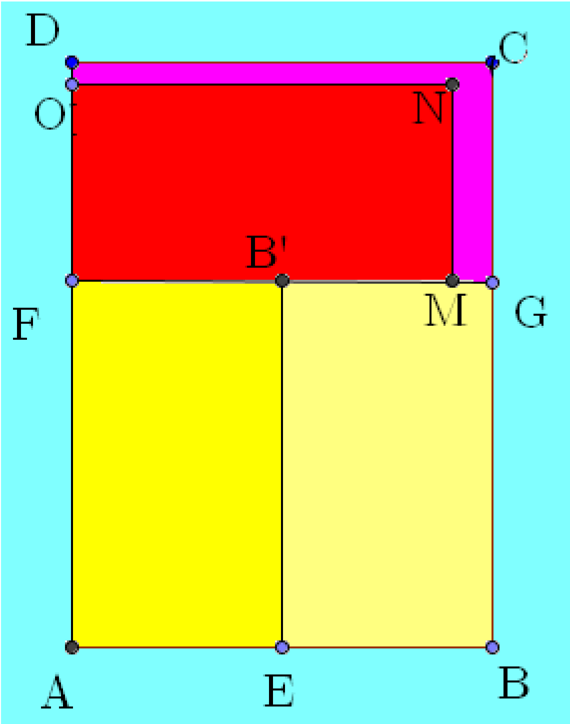
Example

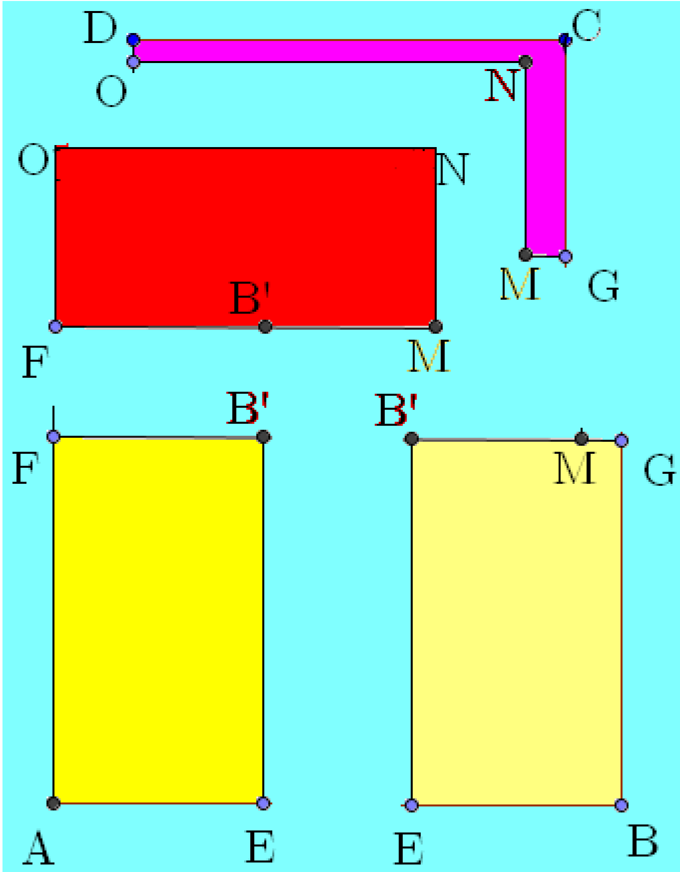
If we have a sheet of paper A4: 210x297 mm.

We can get 3 bronze rectangles of the same dimensions, the following are bronze rectangles AEB'F, EBGB' and FMNO.

We give the lengths in order to cut an A4 sheet of paper.

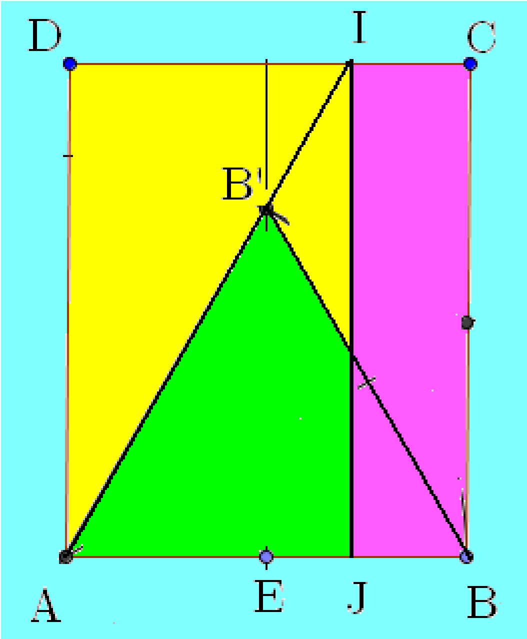
AE=105mm, AF=182mm, FO=105mm, OD=10mm, FM=182mm, MG=22mm.

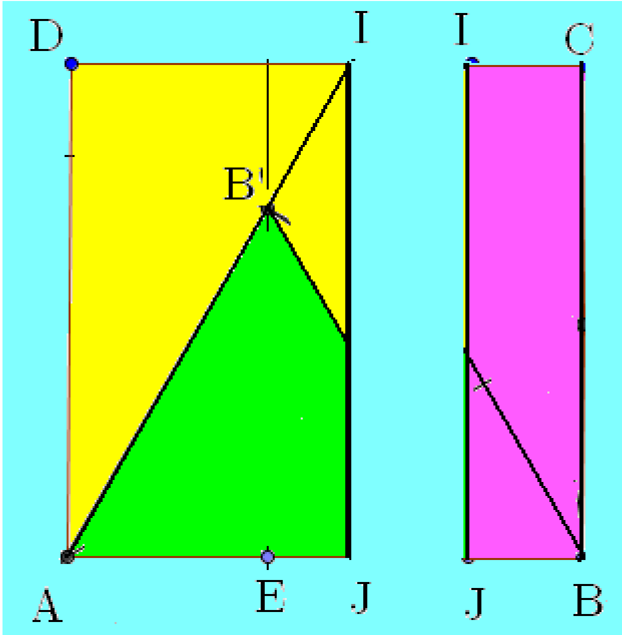




Given any rectangular sheet of paper we explain now how to get the biggest bronze rectangle contained in this sheet:

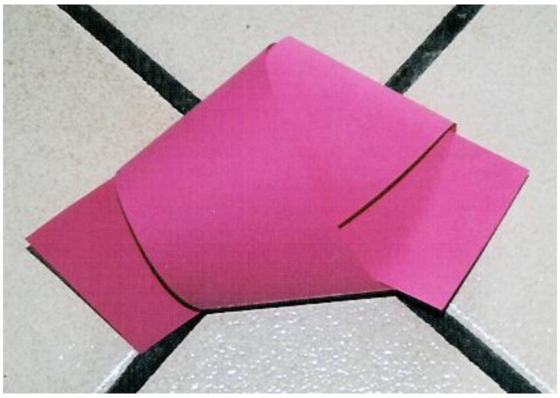
Proceed as before, then fold the sheet of paper along the line (AB’) and determine the point I, so we can determine the point J by folding. Cut the sheet along the line (JI). The rectangle AJID is a bronze rectangle. The rectangle JBCI is a band of paper or ribbon, which we will use to make a regular pentagon.

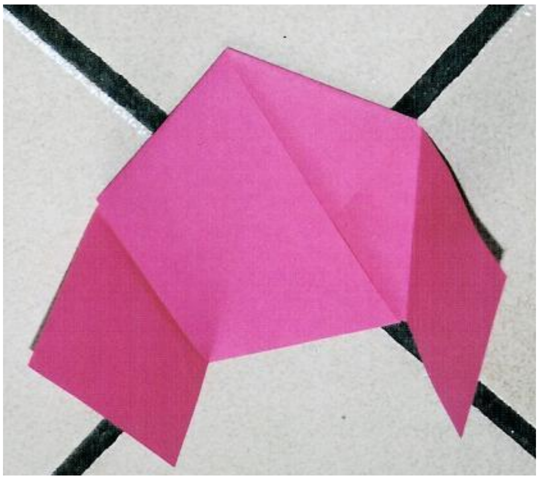




3) The regular pentagon

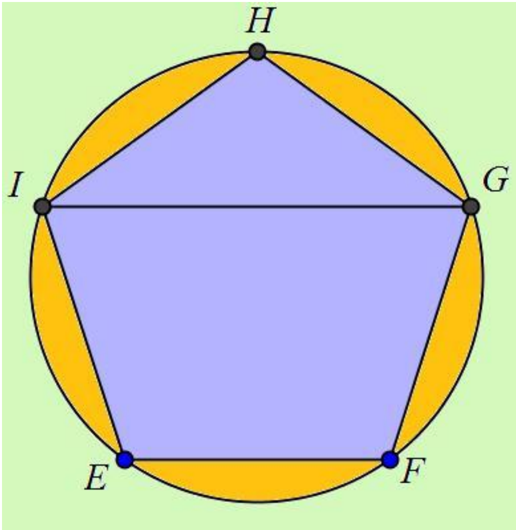
Take a ribbon of paper ABCD; take care that the length AB should be at least 8 times the width BC. Make a knot with the ribbon and pull on the two extremities until the limit position. Use your finger nails to fold the ribbon. As you can see we have a regular pentagon with two tabs.



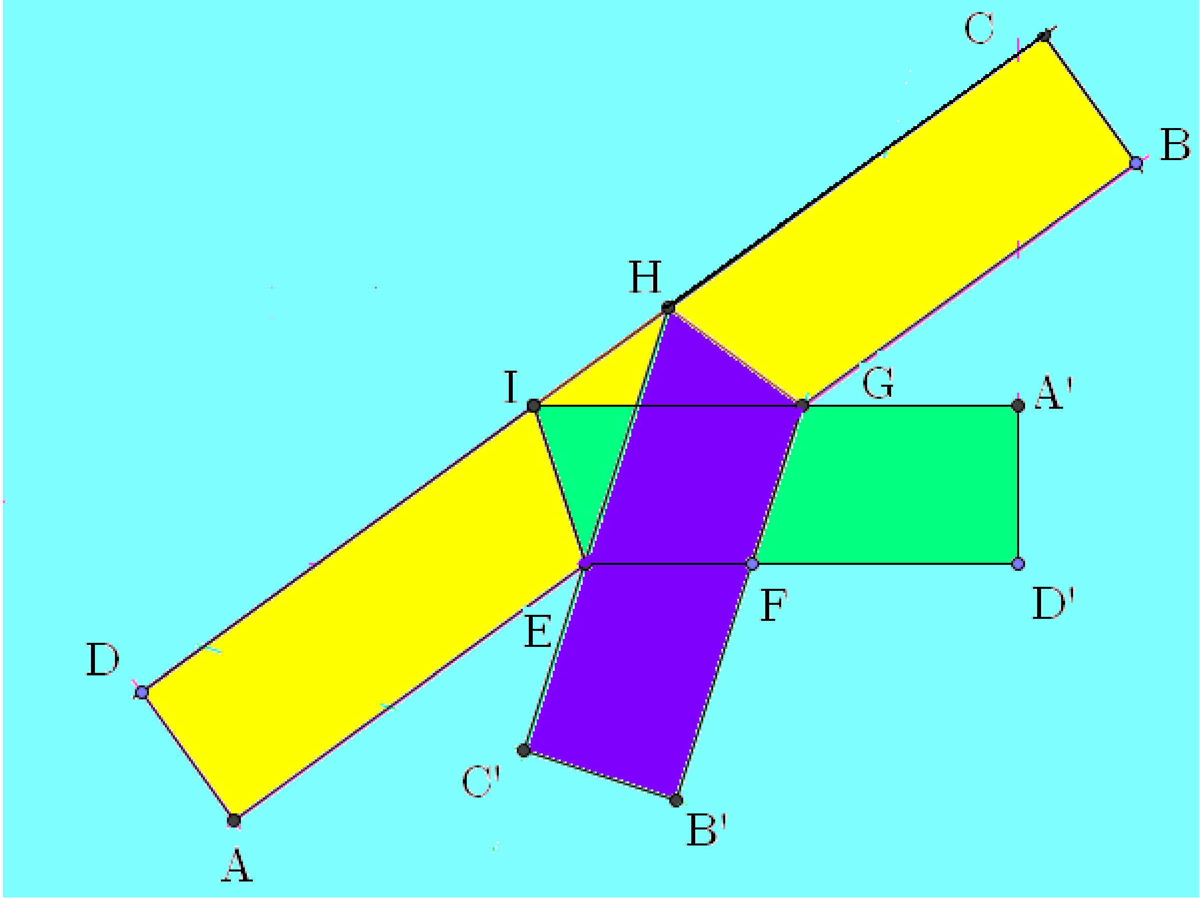


Now we explain why we get a regular pentagon:

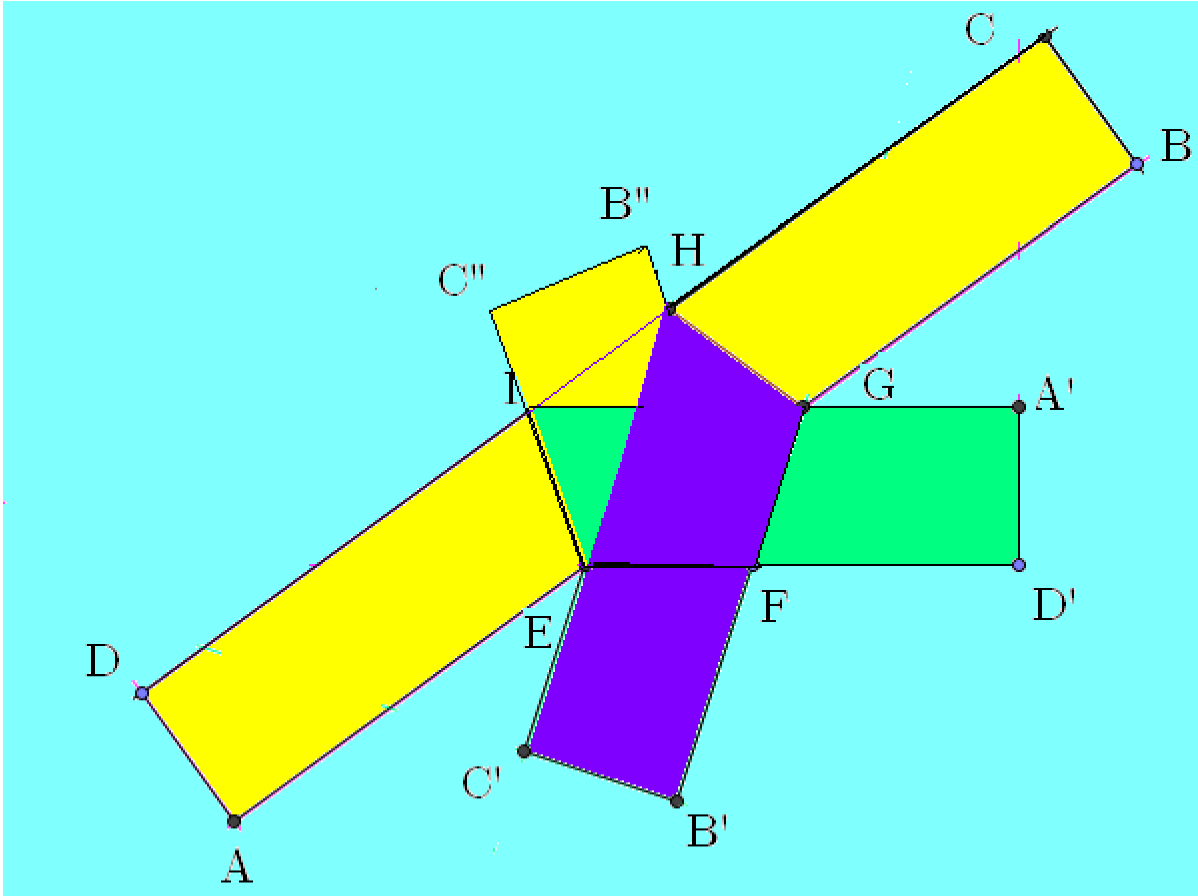
Remark that in the regular pentagon EFGHI the line EF is parallel to the line GI. This was proved by Euclid two thousand years ago.



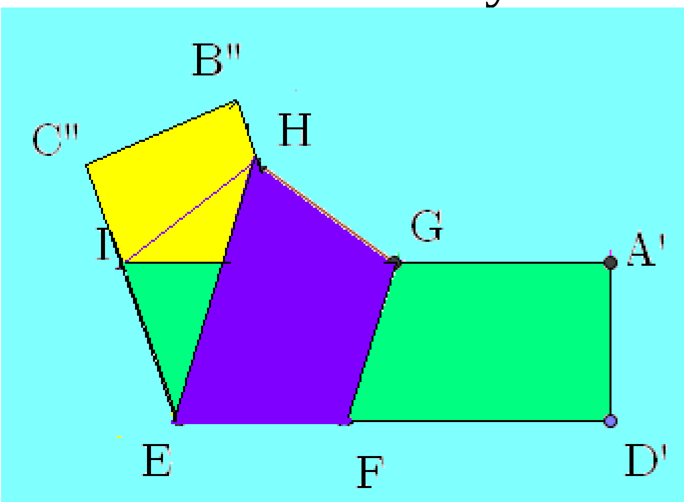
Take a regular pentagon EFGHI, we draw a rectangle ABCD in the following way, the line (AB) contains the points E and G. The line (CD) contains the points H and I. To fold the rectangle ABCD along the line (GH) means to make an orthogonal symmetry with axis the line (GH). The quadrilateral BCHG is transformed by this symmetry into the quadrilateral C’B’GH (drawn in mauve). In the same way the quadrilateral AEID is transformed into the quadrilateral D’A’IE (drawn in green) by the symmetry of axis the line (EI).



In order to get the knot which produces a regular pentagon from the ribbon ABCD, we still need to transform the quadrilateral C’B’FE by the symmetry of axis the line (EF) into the quadrilateral EFB’’C’’.



We erase all auxiliary lines and you can see the knot:



4) A few of vocabulary about Polyhedra

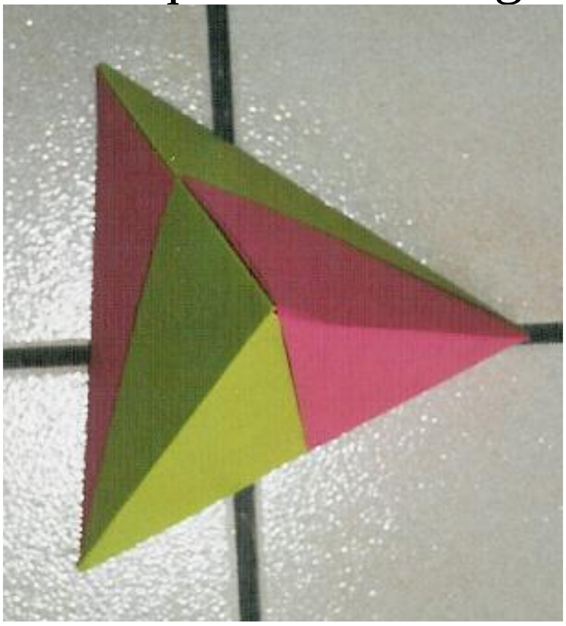
A polyhedron is a volume in the space bounded by polygons (**faces of the polyhedron**) which are in distinct planes. Two faces meet in segments of line (**edges of the polyhedron**); a meeting point of several edges (or faces) is called a **vertex**.

Take one face of a polyhedron, this face is contained in a plane, this plane divide the space in two regions. <u>**A polyhedron is convex**</u> if for each one of its faces the polyhedron is completely contained in one of these two regions. In this picture we show some convex and non convex Polyhedra:

5) Making regular Polyhedra by folding paper

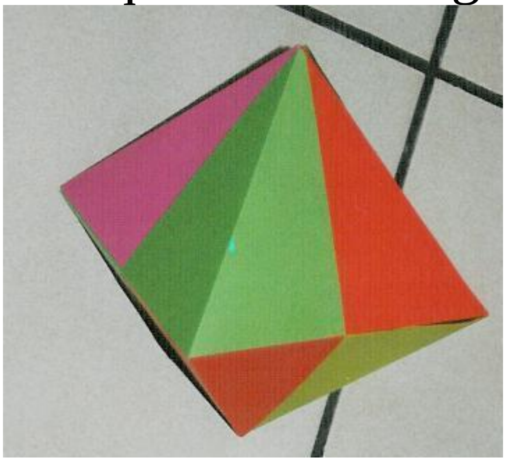
We start with the convex regular Polyhedra. A polyhedron is regular convex if it is convex and all its faces are convex regular polygons of the same type an dimensions (congruent). Two thousand years ago the ancient Greeks have discovery that there are exactly five convex regular Polyhedra:

1. **<u>The Tetrahedron</u>** has 4 faces; each face of the Tetrahedron is an equilateral triangle.

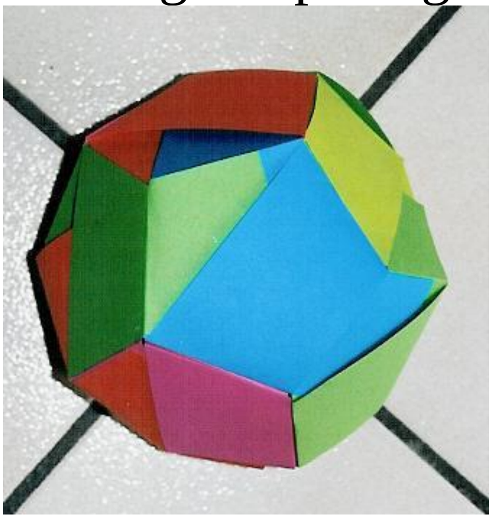


2. **<u>The Cube</u>** has 6 faces; each face of the Cube is a square.

3. **The Octahedron** has 8 faces; each face of the Octahedron is an equilateral triangle.



4. **The Dodecahedron** has 12 faces; each face of the Dodecahedron is a regular pentagon.

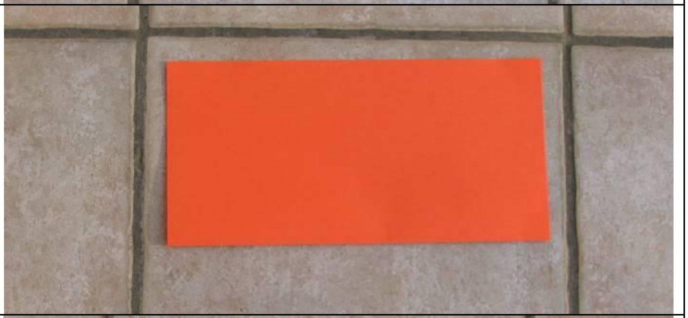


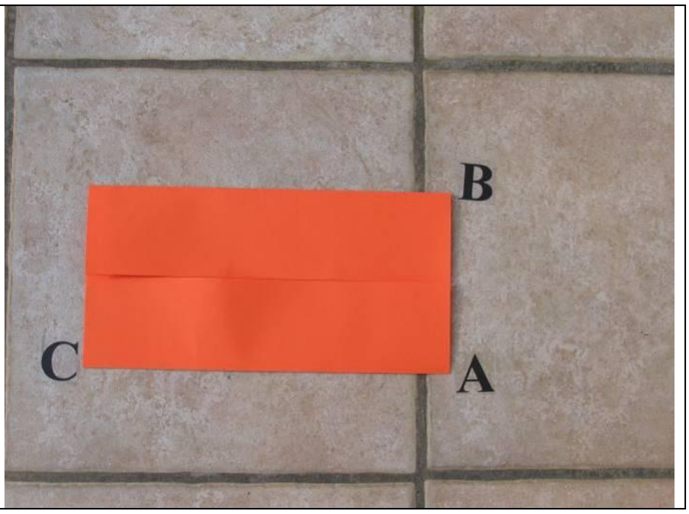
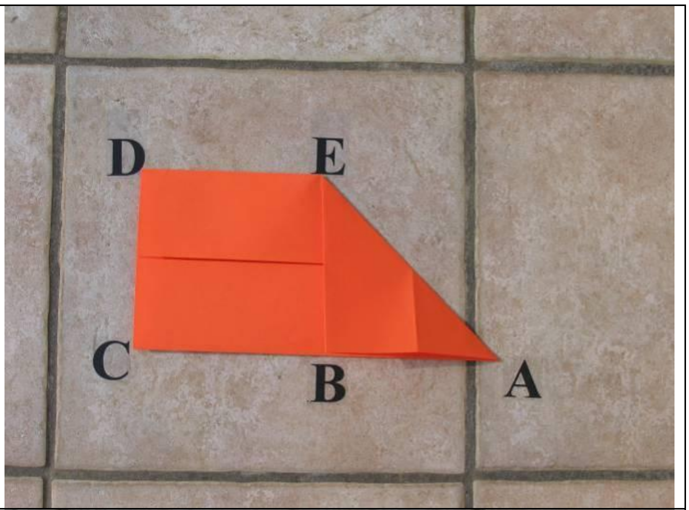
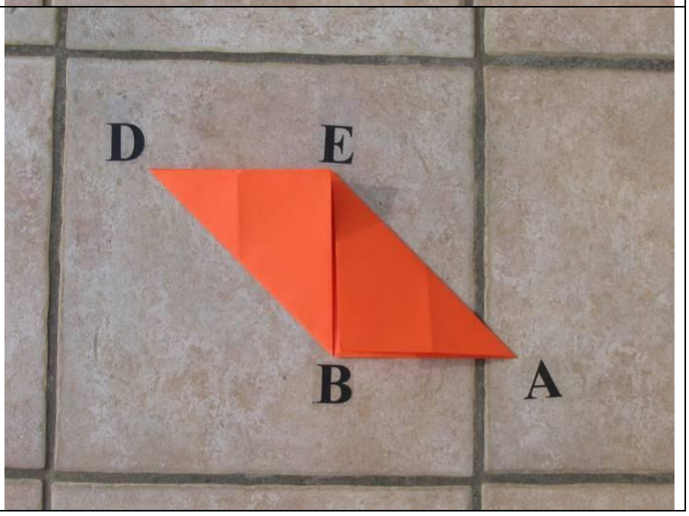
5. **The Icosahedron** has 20 faces; each face of the Icosahedron is an equilateral triangle.

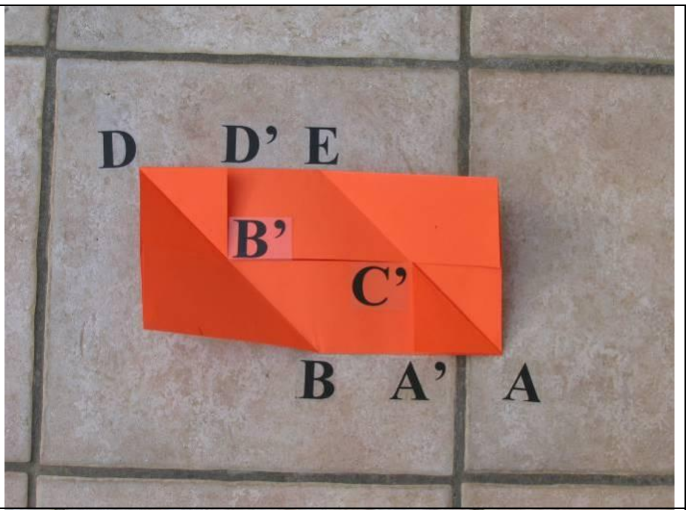
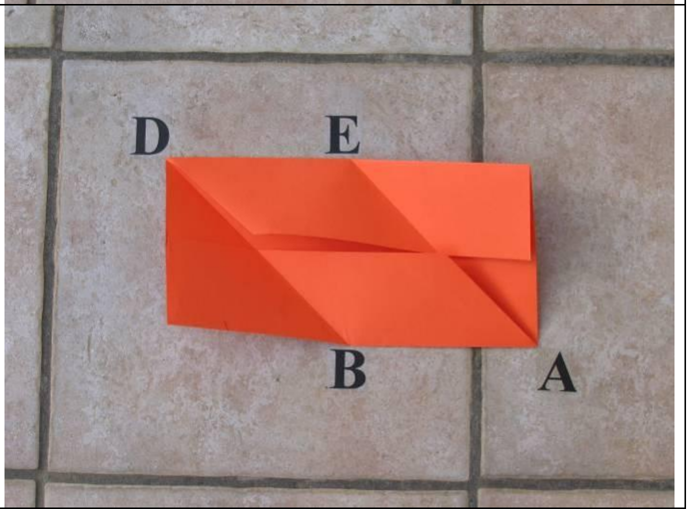
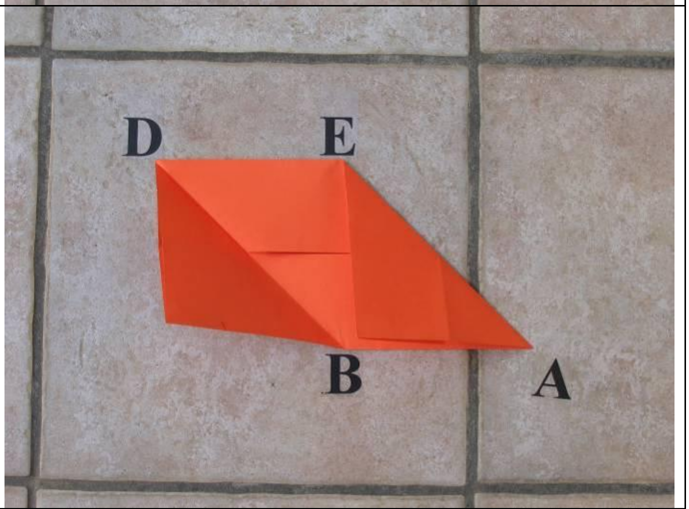
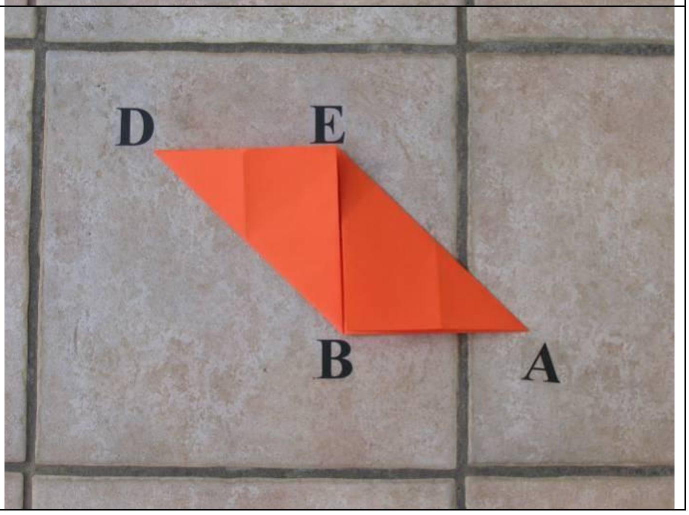
IX) Making a cube by folding paper

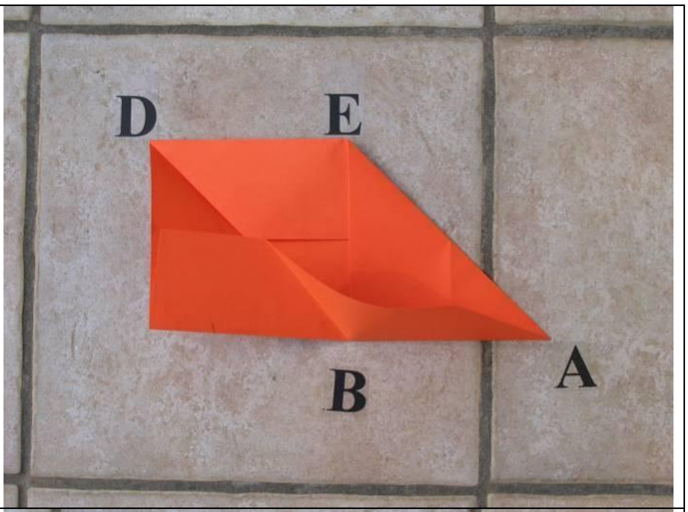
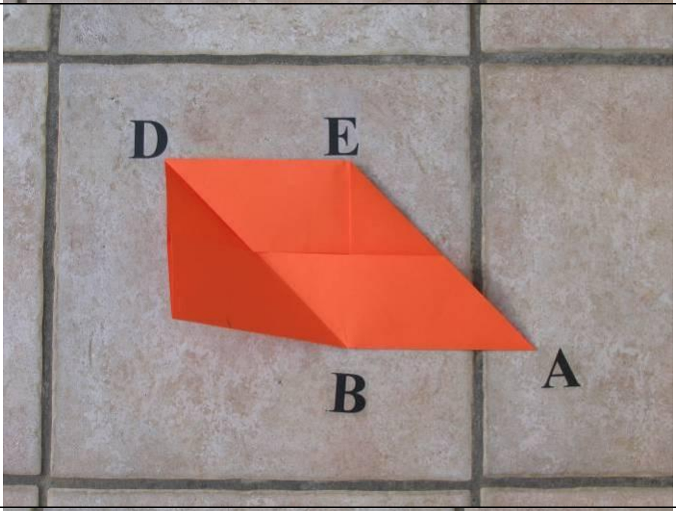
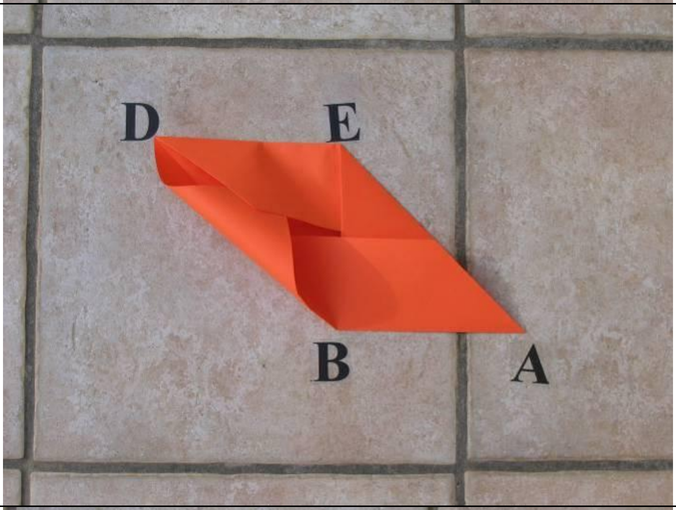
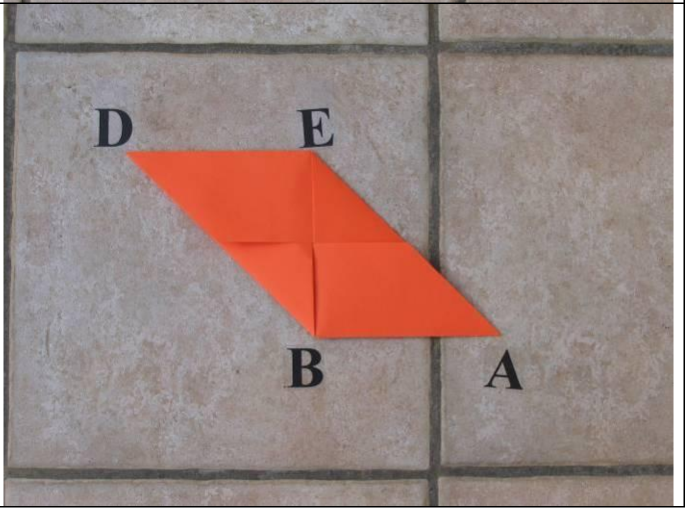
We need 6 square sheet of paper of the same dimensions. First we show how to fold a square sheet of paper in order to get the basic square module.

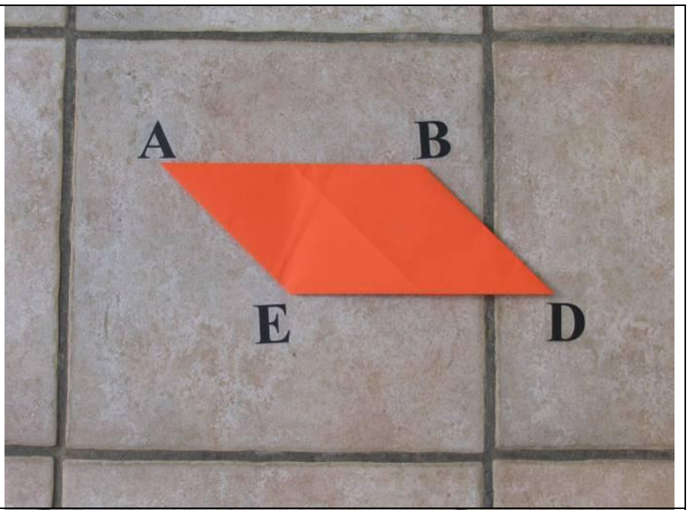
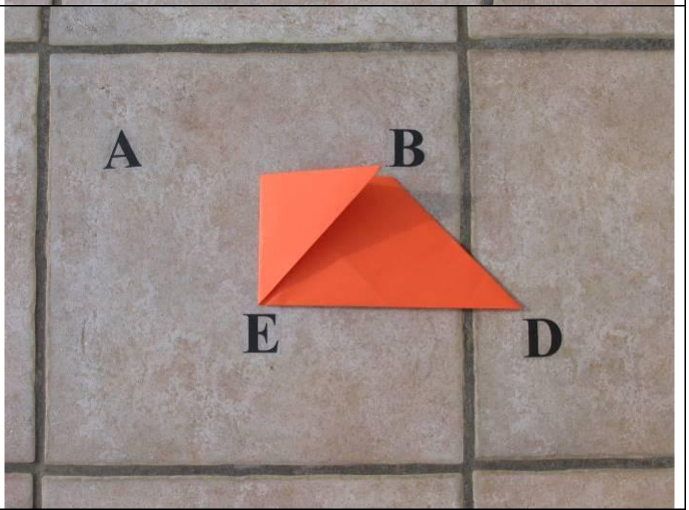
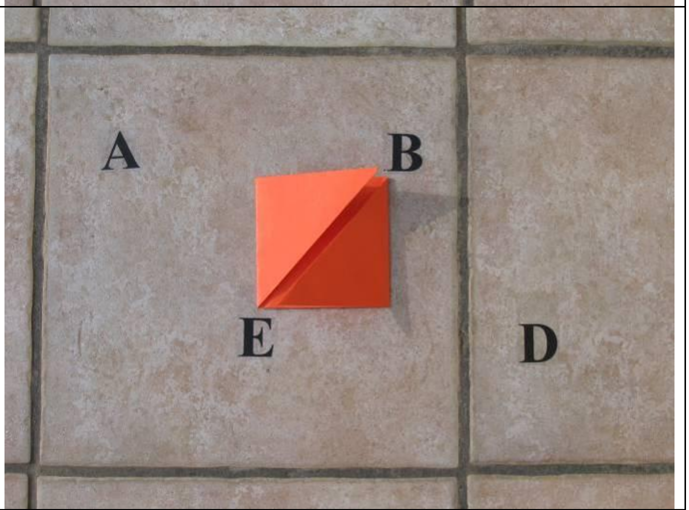
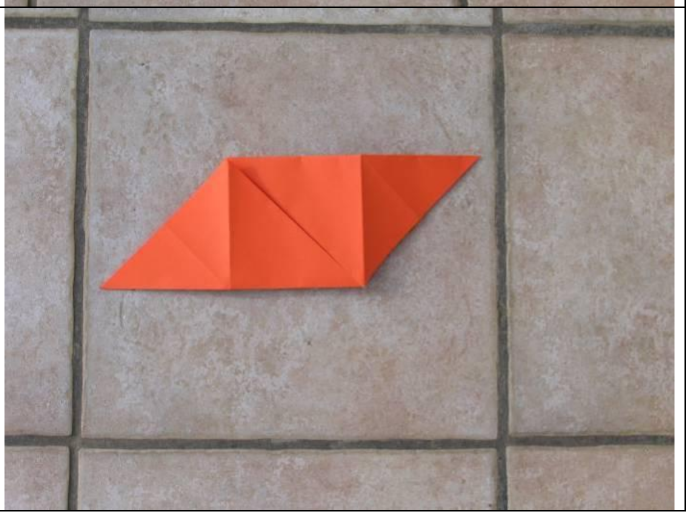
1) Making a basic square module

Take a square sheet of paper.	
Fold it in two equal parts.	
Fold each part in two equal parts. Here we present the first folding	

Here is the second fold.	
Put the right vertical side [AB] on the down horizontal side [CA].	
Put the left vertical side [CD] on the up horizontal side [DE].	

Open it. We can see two small triangles A’AC and B’D’D.	
Fold these small triangles A’AC and B’D’D in the interior (hide them).	
Fold again the right triangle. We get the triangle BAE.	
Fold again the left triangle. We get the triangle BED.	

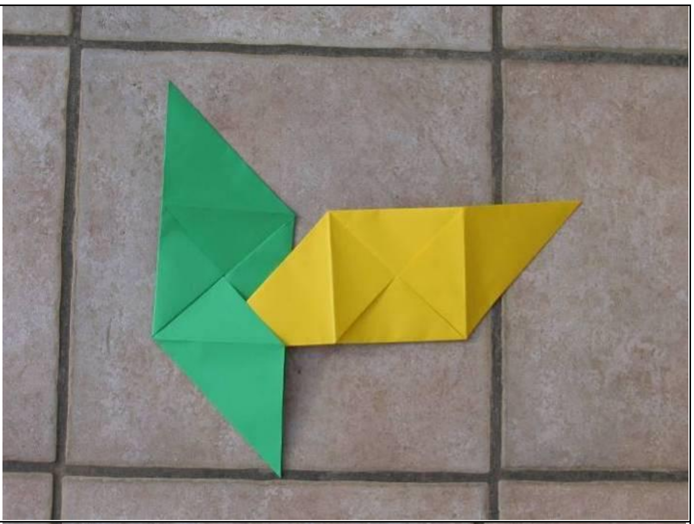
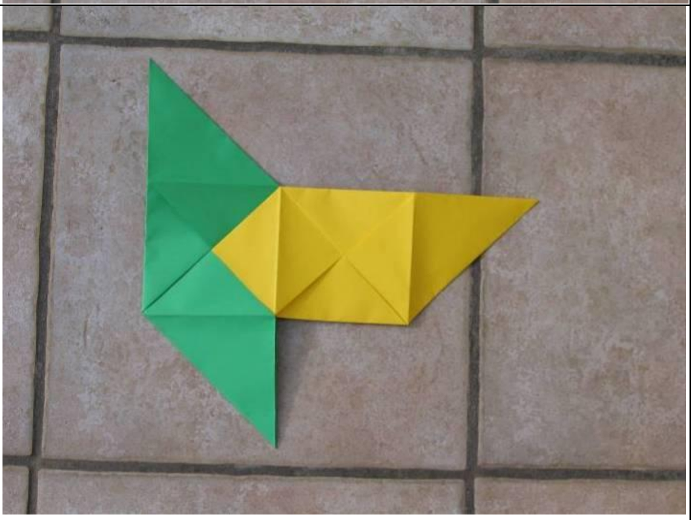
Open it. Insert the triangle BAE into the fold.	
You can see the outcome.	
Insert the left triangle into the fold.	
You can see the outcome. Please remark two slots, each one cross the other in a right angle.	

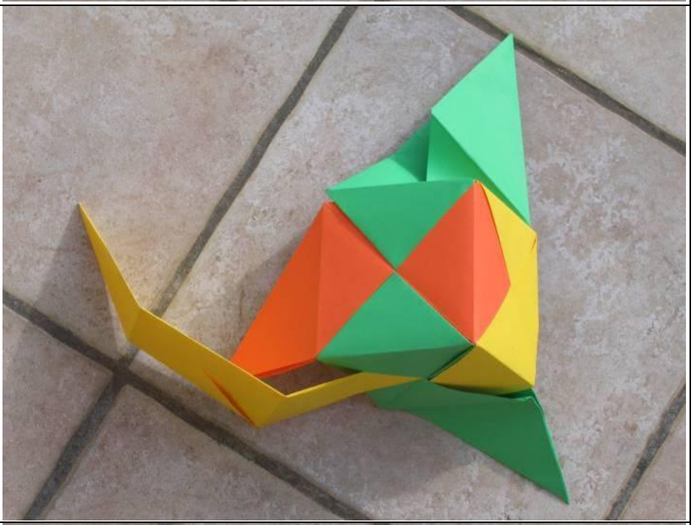
<table>
<tr><td>Turnover (we cannot see the slots).</td><td>
</td></tr>
<tr><td>Put the point A over the point B, we get a right triangle.</td><td>
</td></tr>
<tr><td>Put the point D over the point E, we get a right triangle.</td><td>
</td></tr>
<tr><td>Open it and turn over. Here is the outcome. We can see a spot in the diagonal of the square. The two triangles will be called tabs. This is the basic square module.</td><td></td></tr>
</table>

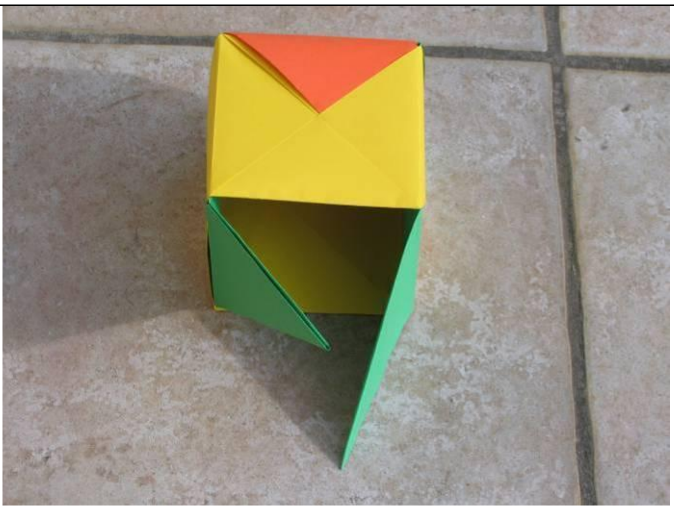
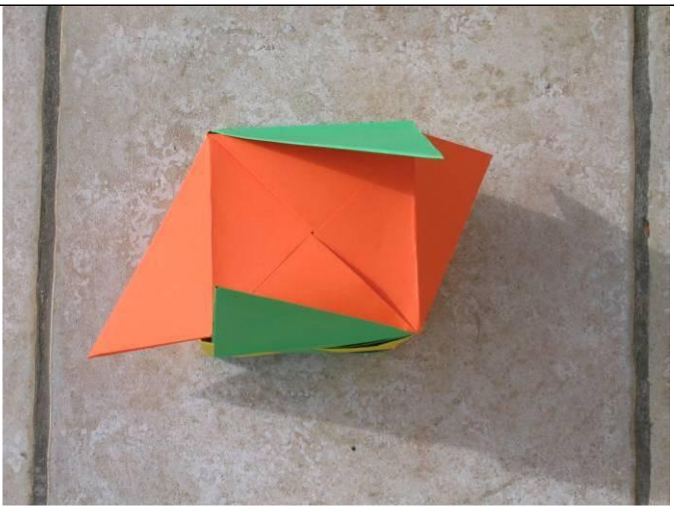
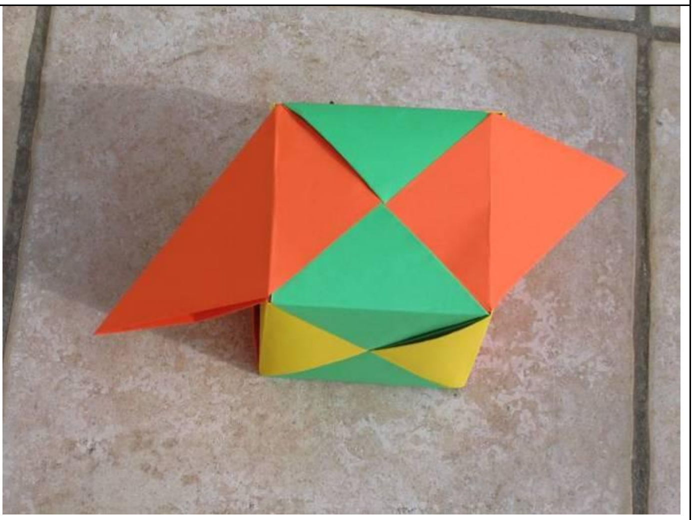
We can also produce a symmetric or chiral basic square module. The process is similar, only the third step change: Fold making a right angle the upper left corner onto the bottom horizontal side.	
Fold in the same way by putting the bottom right corner onto the horizontal up side.	
All other steps are similar: Here is the outcome: **the symmetric of chiral basic square module.**	

2) Assembly the cube

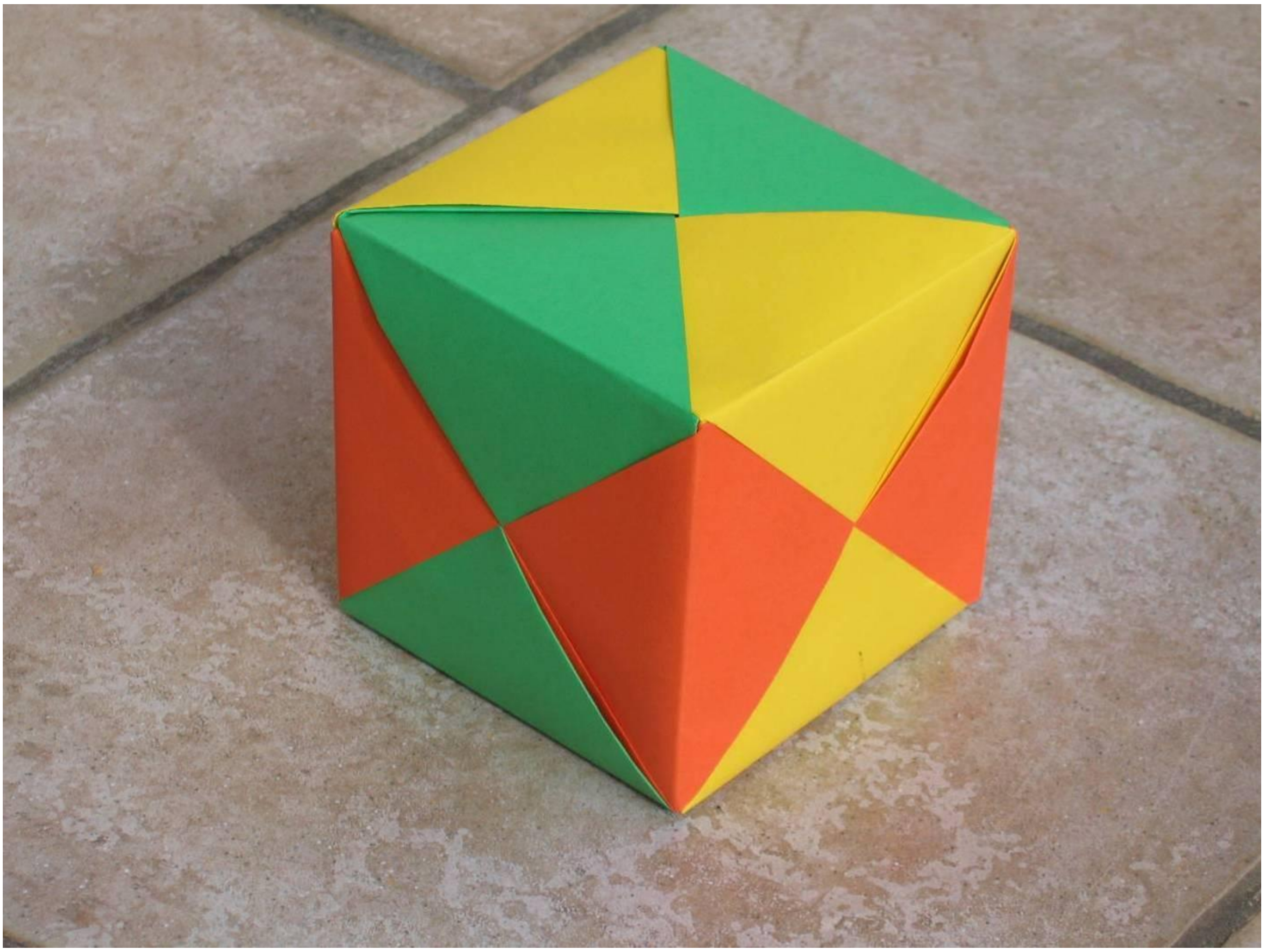
We take six square sheets of paper, and we fold it in order to have six identical basic square modules. **Before starting take care to put over all together, and check that you don't have chiral modules. If you mix chiral modules with normal modules you cannot make a cube.**

Slot together: Insert a tab yellow into a green slot.	
See the outcome.	
Insert a tab of the second yellow basic module into the second green slot. The first face of the cube is finished.	

Insert a tab of the green basic module into the slot of the orange square module. Then insert the right tab of the orange square module into the slot of the yellow square module.	
Insert a tab of the second green basic module into the slot of the orange square module.	
Insert a tab of the orange basic module into the slot of the yellow square module. Then insert the tab of the yellow basic module into the slot of the green square module.	

The outcome is an open cube with 5 faces and two tabs of the green module.	
Put the last orange module as in the picture.	
Insert the tabs of the green module into the slots of the orange module.	

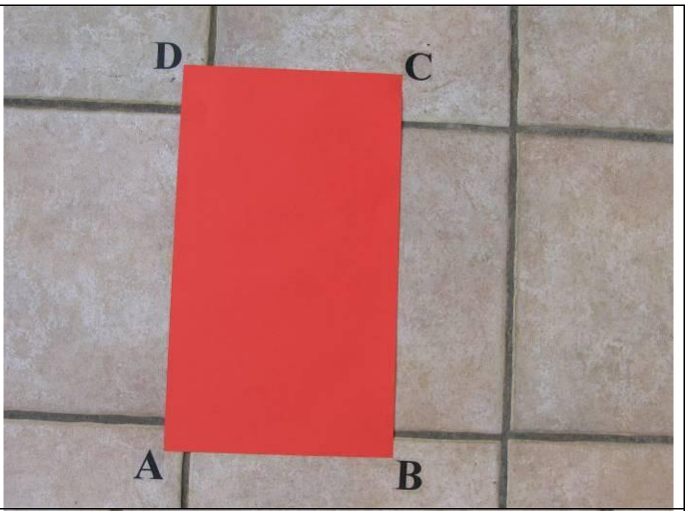
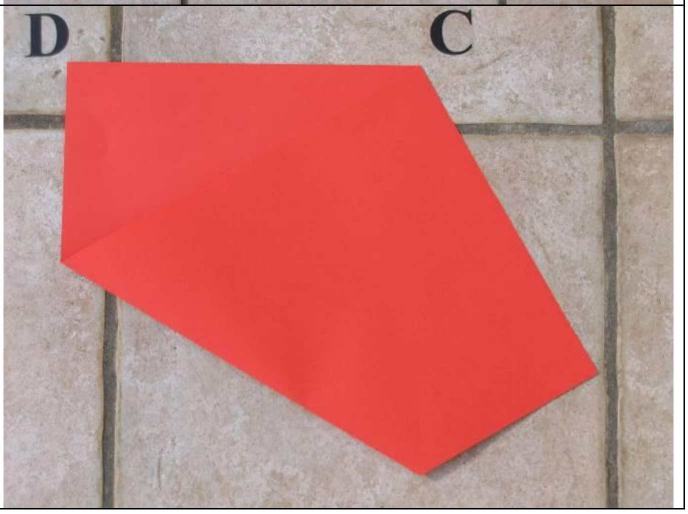
Insert the tabs of the orange module into the slots of the yellow module.	

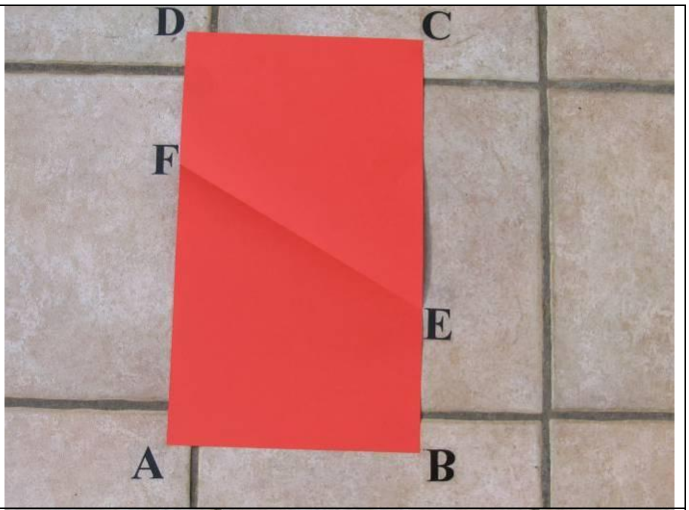
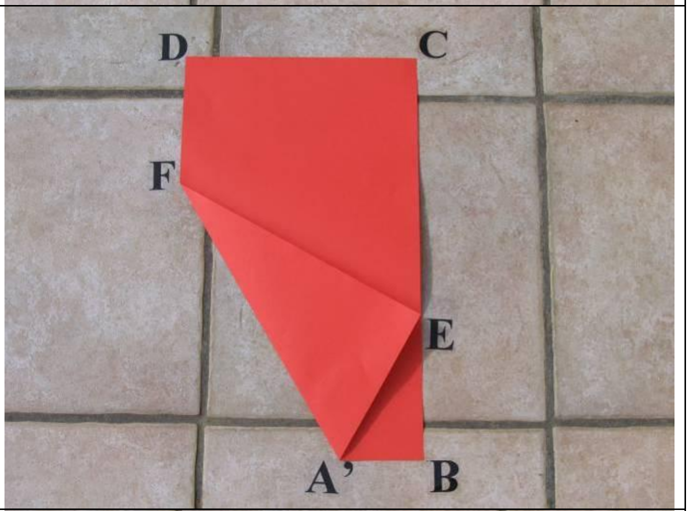
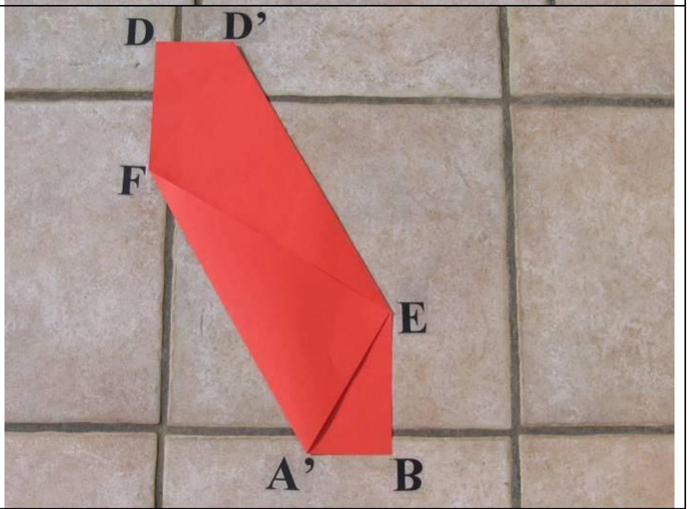
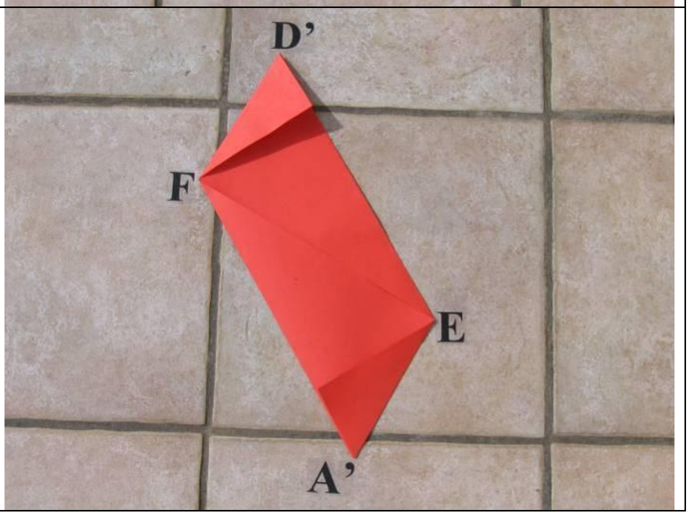


x) Making regular Polyhedra with equilateral triangles as faces

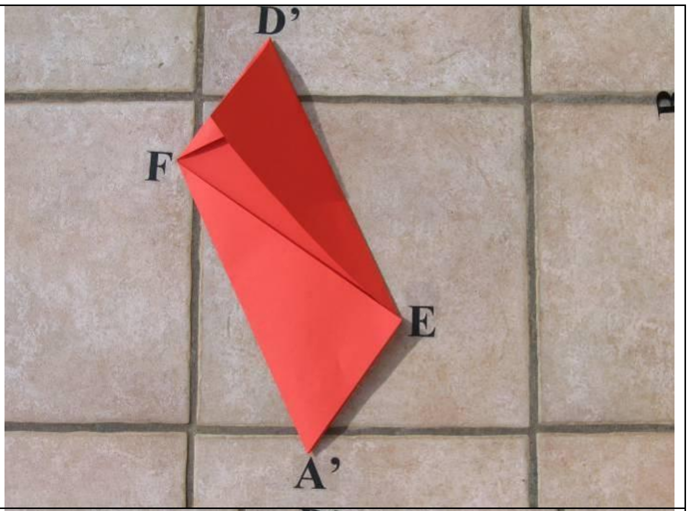
We will work with **bronze rectangular sheets** of paper.
At first we will show how to fold this sheet of paper in order to get the **basic triangle module**, and then we will show how to get the symmetric or chiral **basic triangle module**.

1) Folding the basic triangle module

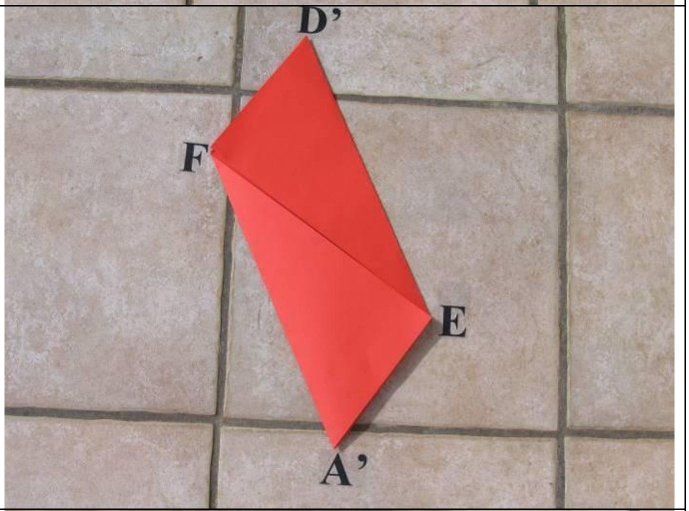
Take a **bronze rectangular sheet** of paper.	
Put the left bottom corner of the sheet of paper over the right top corner, fold using your nails.	

Unfold; we can see a line (EF).	
Put the left bottom corner A over the point E, by using the line (EF) as guide. Fold with the nails along A'F.	
Put the right upper corner C over the point F, by using the line (EF) as guide. Fold with the nails along D'E.	
Fold the small triangles D'DF and A'BE.	

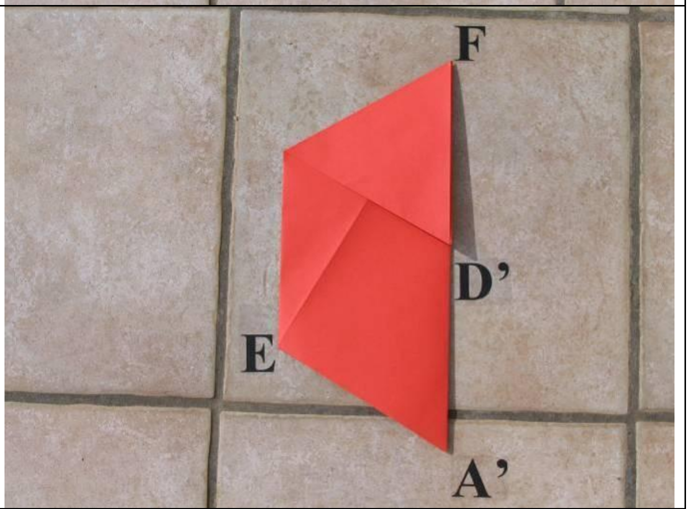
Put inside the two small triangles.



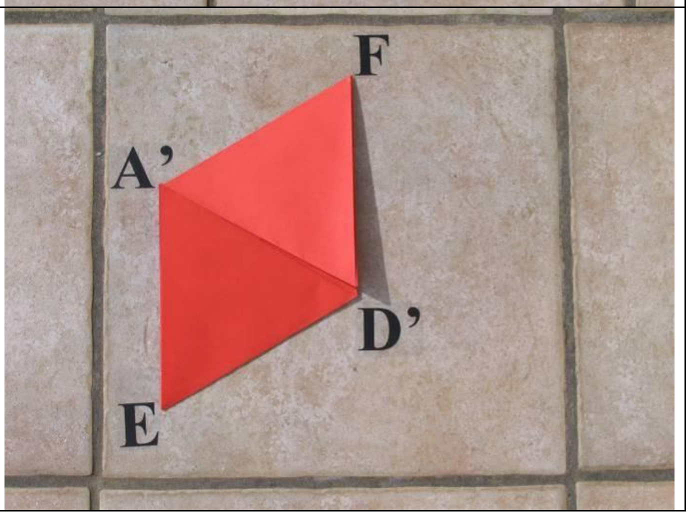
The outcome is a parallelogram with one slot in the diagonal.

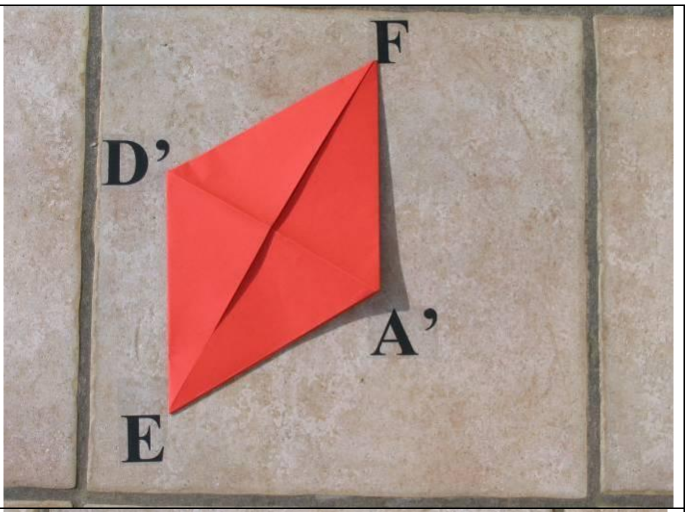
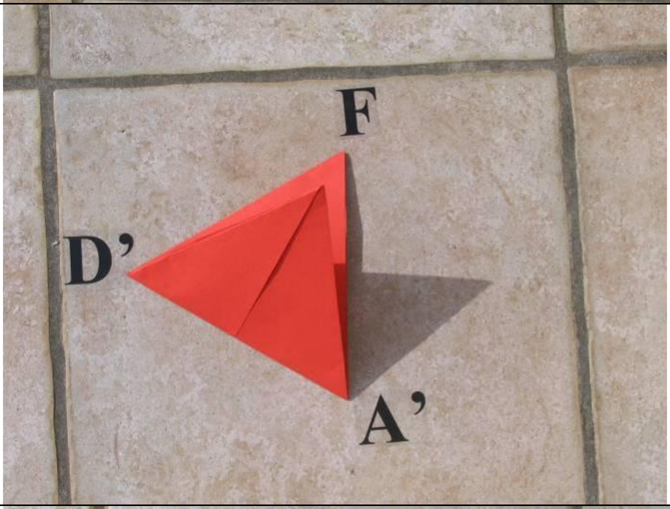
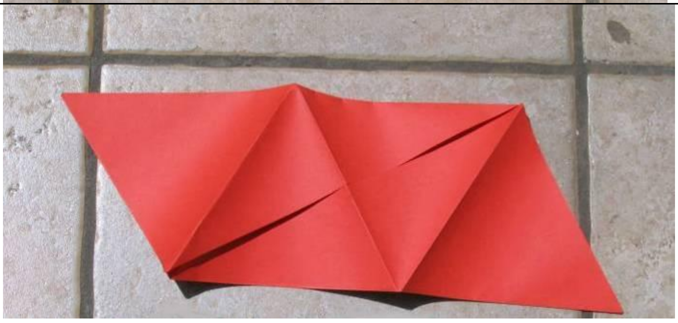


Turn over the sheet of paper, (we cannot see the slot). Put the segment [D’F] over the segment [A’F]. We get an equilateral triangle.

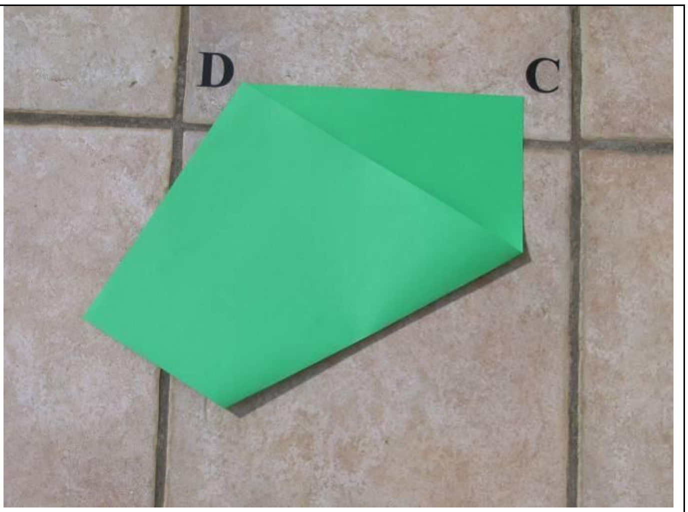
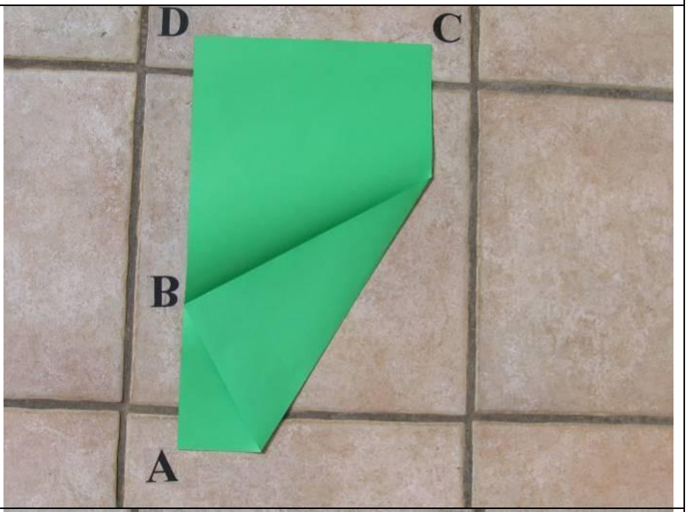
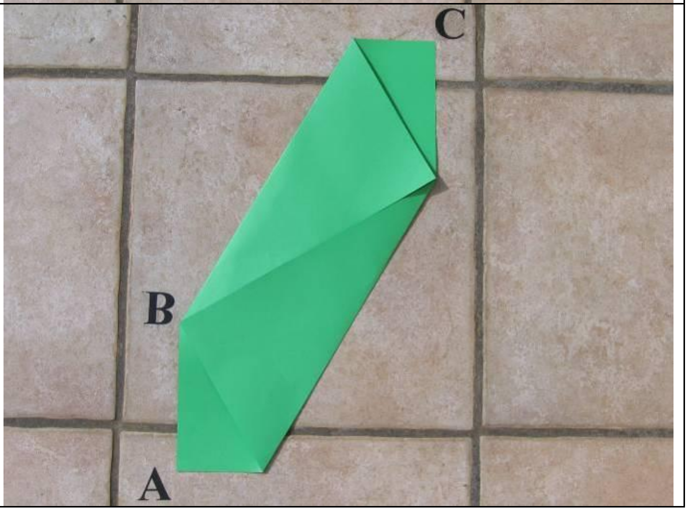


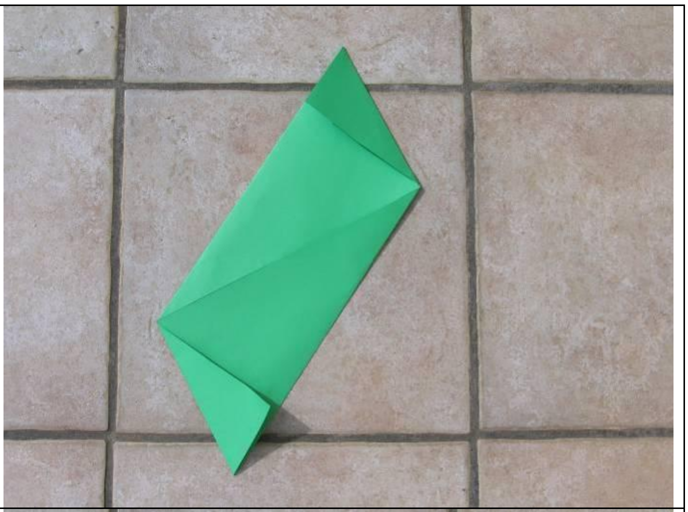
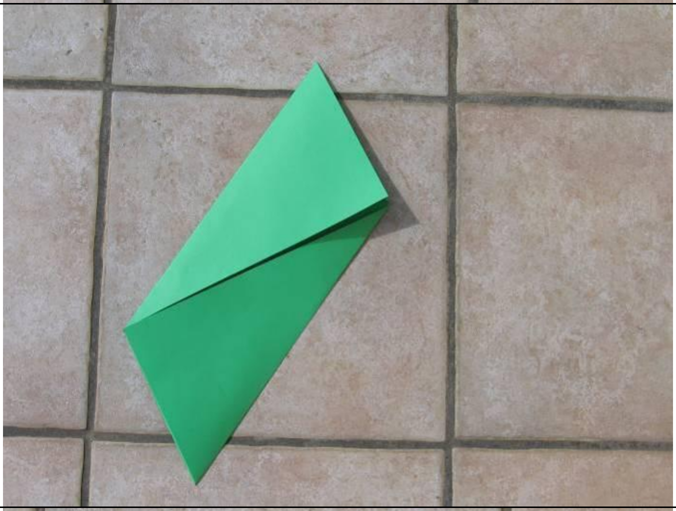
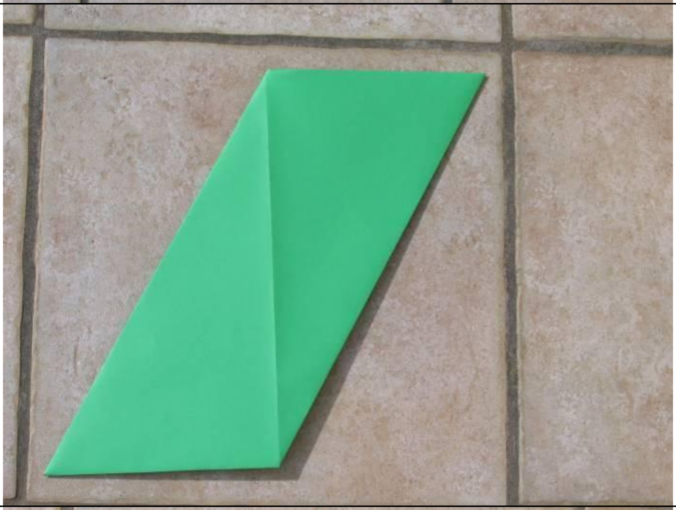
Put the segment [EA’] over the left vertical side. We get a second equilateral triangle, and one rhombus.

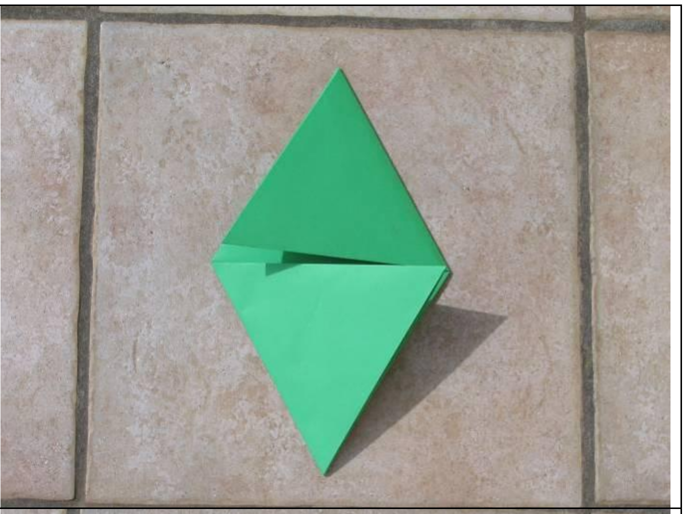
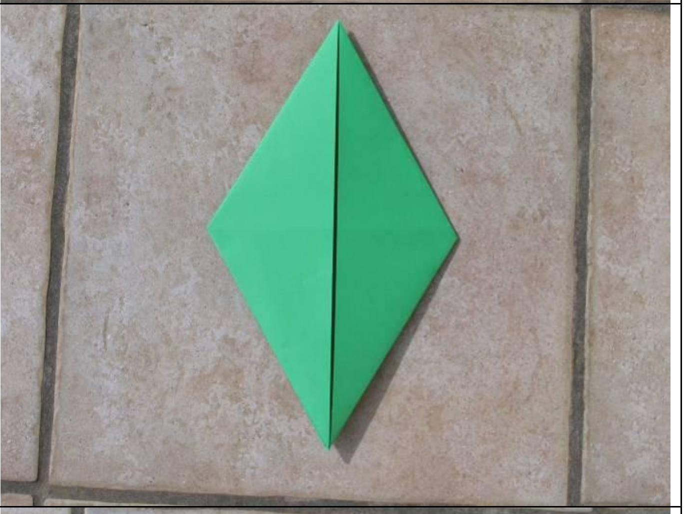
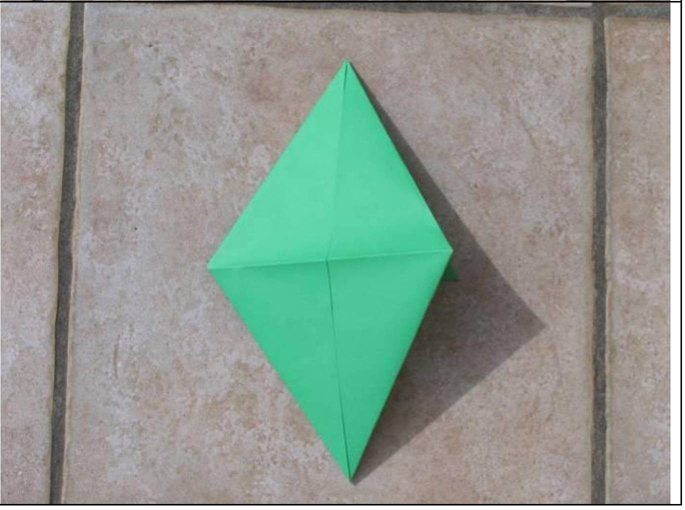


Turn the piece.	
We fold the rhombus along the segment [A'D'] so that the slot is apparent. For star Polyhedra we will fold in such a way that the slot is hidden.	
Open the piece. We have divided the parallelogram into four equilateral triangles.	 **Basic triangle module**

2) Folding the symmetric basic triangle module

<u>Symmetric basic triangle module</u>: Only the first step change. Other steps are similar. Put the right bottom corner over the left upper corner. Use your nails to fold.	
Unfold; put the right bottom corner B on the vertical side [AD] by using the fold line as guide. Fold using your nails.	
Put the left upper corner D over the vertical side by using the fold line as guide.	

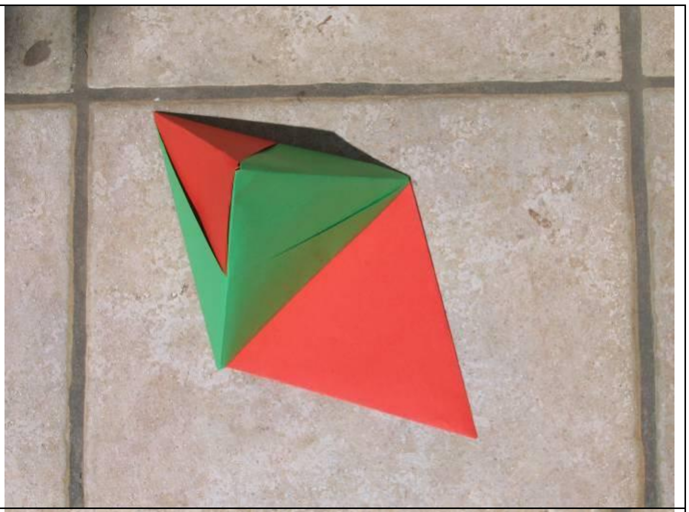
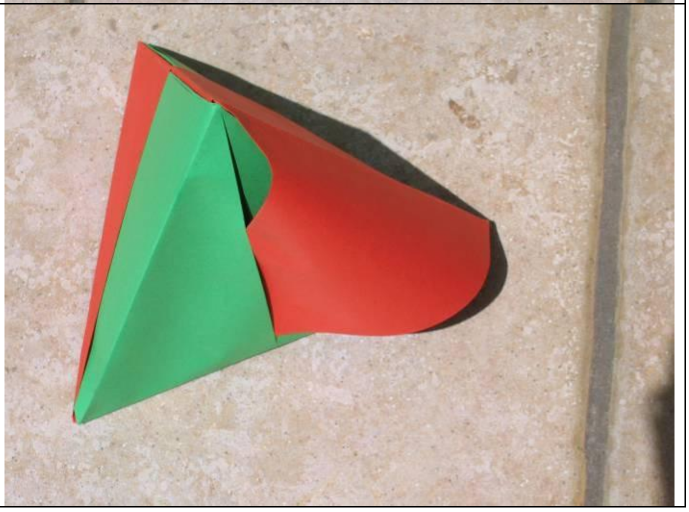
Fold the two small triangles in the top and the bottom.	
Hide the two small triangles. We get a parallelogram.	
Turn over the piece.	
Put the upper side over the left side.	

Put the bottom side over the right side.	
Turn over the piece. We get a rhombus with slop in the largest diagonal.	
Fold the rhombus along the shortest diagonal. **This is the symmetric or chiral triangle module.**	

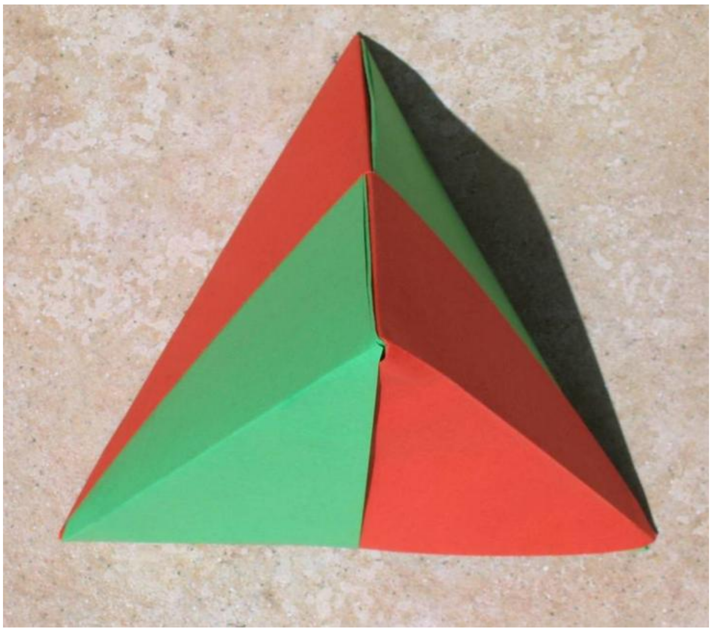
3) Assembly the Tetrahedron

We must have two bronze rectangular sheet of paper.

To make the Tetrahedron we need a triangle module (red) and a chiral triangle module (green).	
<u>We show how to assembly a triangle module (red) and a chiral triangle module.</u> Put the two pieces (you must see the slots); Insert the triangle (tab) of the green module into the slot of the red module.	
Insert the triangle (tab) of the red module into the slot of the green module, and insert the second triangle (tab) of the green module into the second slot of the red module.	

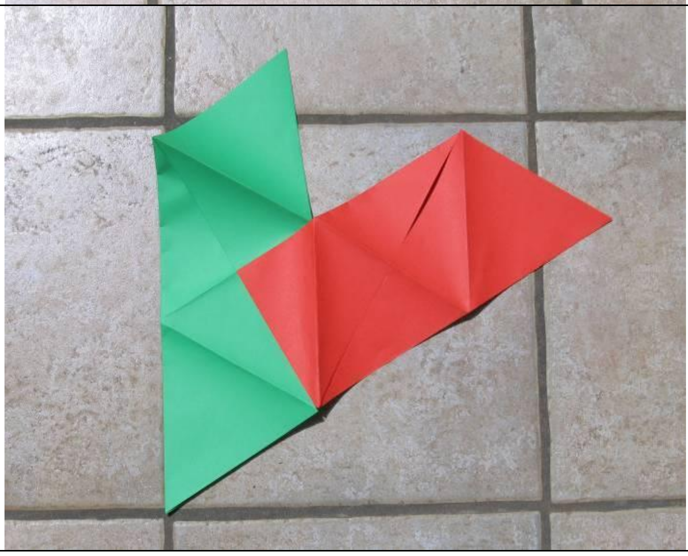
The Tetrahedron is almost finished.	
Insert the triangle (tab) of the red module into the slot of the green module.	

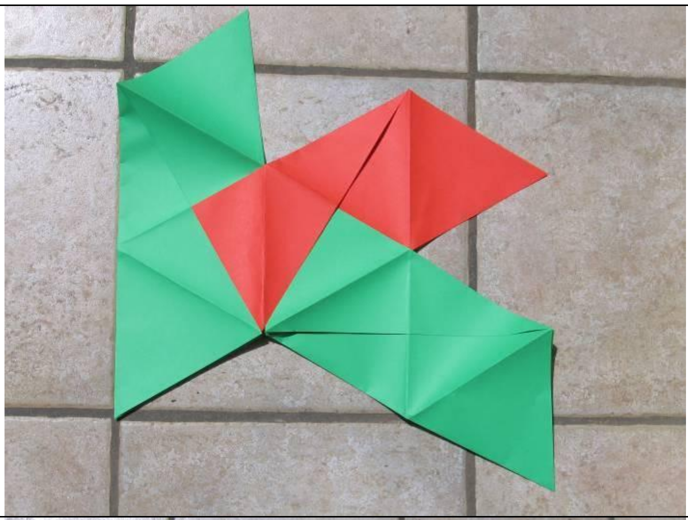
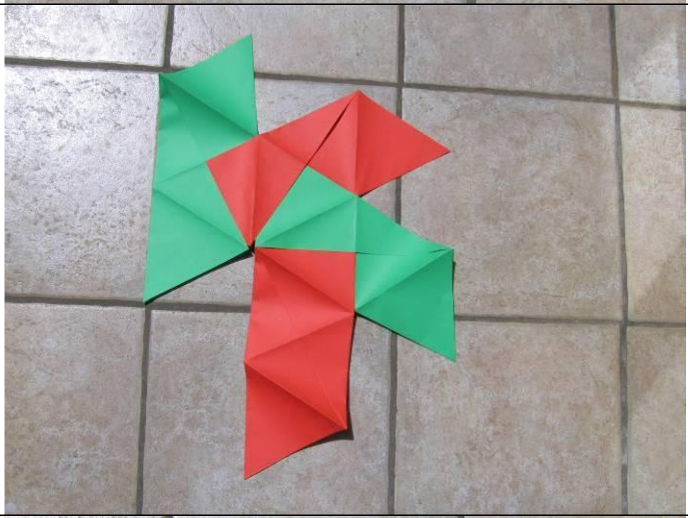
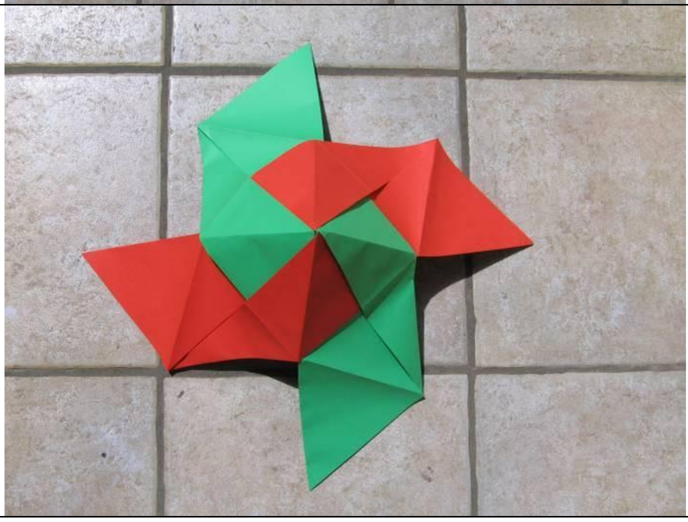
The outcome is the Tetrahedron, it has four faces of the same type and we cannot distinguish one face from other. If you rotate the Tetrahedron you cannot see the difference.

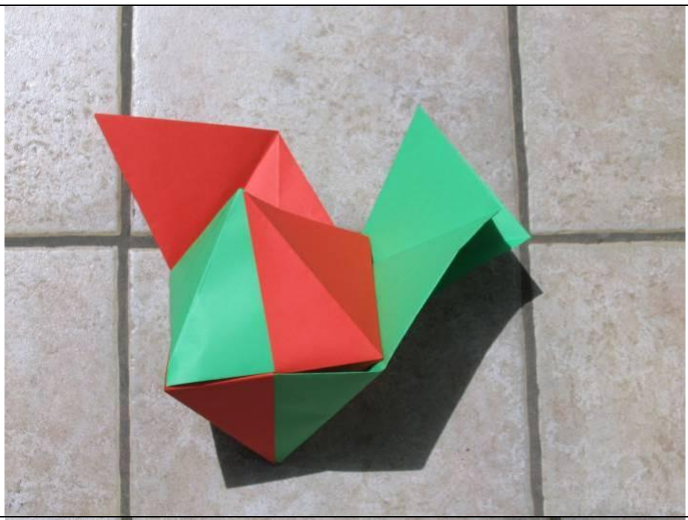
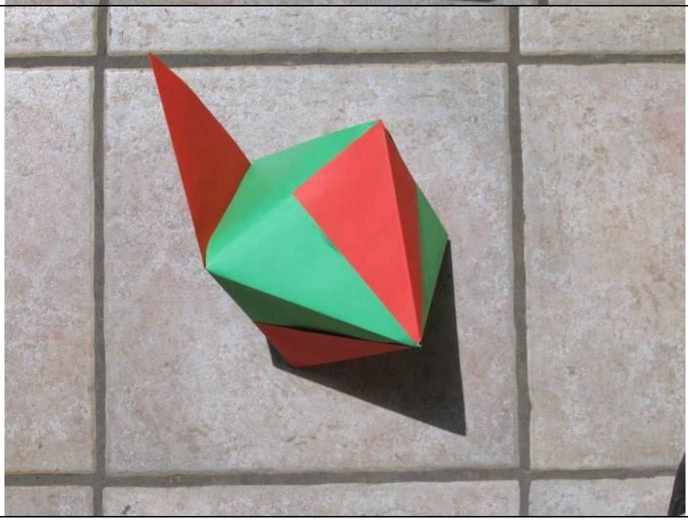


4) Assembly the Octahedron

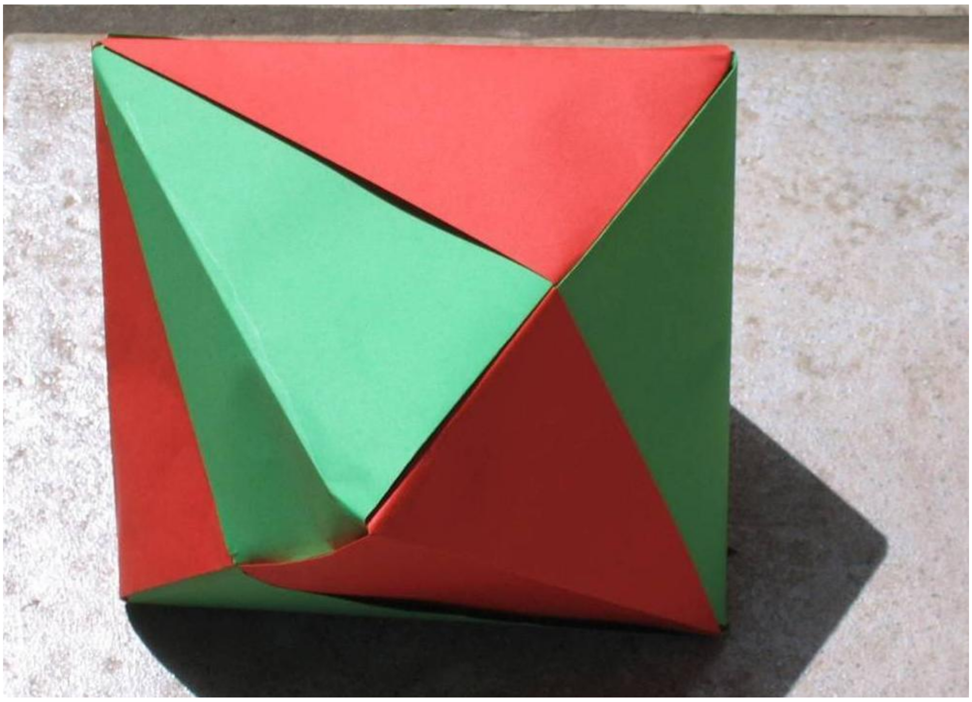
<u>The Octahedron</u> is convex regular polyhedron with 8 faces; each face is an equilateral triangle. We need 4 bronze rectangular sheet of paper of the same dimensions.

We need four triangle modules: two red and two green. Overlap the modules to check that we don't mix with chiral modules.	
<u>We show how to assembly a triangle module (red) and a green triangle module.</u> Put the two pieces (you must see the slots); Insert the triangle (tab) of the red module into the slot of the green module. Glue together.	

Insert the triangle (tab) of the green module into the slot of the red module. Glue together.	
Insert the triangle (tab) of the red module into the slot of the green module. Glue together.	
Insert the triangle (tab) of the green module into the slot of the red module. Glue together. We get a pyramid with four faces.	

With the rest tabs we will make a pyramid with four faces.	
Insert the triangle (tab) of the red module into the slot of the green module.	
Insert the triangle (tab) of the green module into the slot of the red module. We have a last red tab that we insert into the last green slot.	

The outcome is the Octahedron, as you can see all the faces are of the identical. Rotate the Octahedron by taking it by two opposite vertices. For this reason the Octahedron is regular.

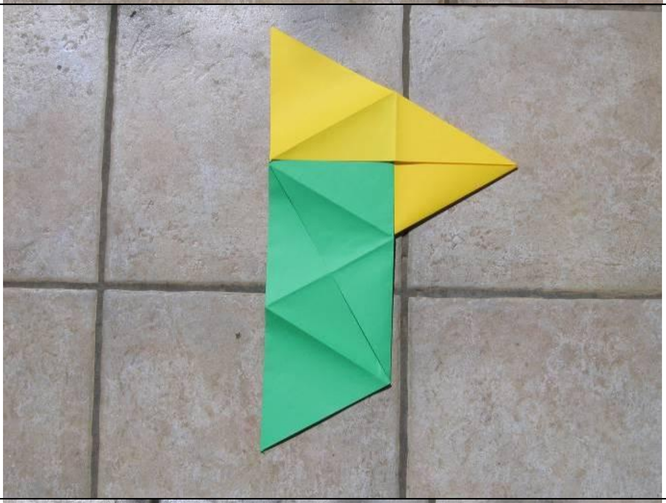


5) Assembly the Icosahedron

The Icosahedron is a convex regular polyhedron with twenty faces; each face is an equilateral triangle.

We need ten bronze rectangular sheet of paper:

Five triangles modules and five chiral triangle modules.

We take five triangular modules of distinct colors: orange, yellow, blue, red and green.	
Put two triangle modules (you must see the slots); insert the triangle (tab) of the green module into the slot of the yellow triangle module. Glue together.	
Insert the triangle (tab) of the red module into the slot of the green module. Glue together.	

Insert the triangle (tab) of the blue module into the slot of the red module. Glue together.	
Insert the triangle (tab) of the orange module into the slot of the blue module. Glue together.	
Insert the triangle (tab) of the yellow module into the slot of the orange module. Glue together. We get a pyramid with five faces (each face is bicolor).	
Repeat the above steps with the five chiral triangle modules. So we get a second pyramid with five faces (each face is bicolor). You can see the two pyramids.