CHAPTER 8

On the Lyapunov Exponents of the Kontsevich–Zorich Cocycle

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1. Introduction

The Kontsevich–Zorich cocycle, introduced in [25], is a cocycle over the Teichmüller flow on the moduli space of holomorphic (quadratic) differentials. The study of the dynamics of this cocycle, in particular of its Lyapunov structure, has important applications to the ergodic theory of interval exchange transformations (i.e.t.'s) and related systems such as measured foliations, flows on *translation surfaces* and rational polygonal billiards (see the article by H. Masur [5] in this handbook). The Kontsevich–Zorich cocycle is a continuous-time version of a cocycle introduced by G. Rauzy [35] as a "continued fractions algorithm" for i.e.t.'s and later studied by W. Veech, in his work on the unique ergodicity of the generic i.e.t. [38], and A. Zorich [45,46] among others.

1.1. *Deviation of ergodic averages and other applications*

Zorich (see [44,46,47]) made the key discovery that typical trajectories of generic (orientable) measured foliations on surfaces of higher genus (or equivalently of generic i.e.t.'s with at least 4 intervals) deviate from the mean according to a power law with exponents determined by the Lyapunov exponents of the cocycle.

In [45] he began a systematic study of the Lyapunov spectrum of the cocycle and conjectured, on the basis of careful numerical experiments, that all of its Lyapunov exponents are non-zero and simple. He also observed that, as a consequence of the close connection between the cocycle and the Teichmüller geodesic flow, the simplicity of the top exponent, sometimes called the *spectral gap* property, is equivalent to the (non-uniform) hyperbolicity of the Teichmüller flow, which had been proved earlier by W. Veech [40].

The applications of the Kontsevich–Zorich cocycle to the dynamics of i.e.t.'s and related systems are not limited to the deviation of ergodic averages. The spectral gap property of the cocycle also plays an important role in recent results of Marmi, Moussa and Yoccoz [27,28] on the *cohomological equation* for generic i.e.t.'s, which improve on previous work of the author [19].

In a different direction, A. Avila and the author [7] have recently shown that the positivity of the second exponent (for surfaces of higher genus) implies that almost every i.e.t. which is not a rotation is weakly mixing and that the generic directional flow on the generic translation surface of higher genus is weakly mixing as well. This result answers in the affirmative a longstanding conjecture on the dynamics of i.e.t.'s. Special cases of the conjecture were earlier settled by A. Katok and A. Stepin [24] (for i.e.t.'s on 3 intervals) and W. Veech [39] (for i.e.t.'s on any number of intervals, but with special combinatorics).

1.2. *Renormalization for parabolic systems*

The role of the Kontsevich–Zorich cocycle can be explained by the somewhat vague observation that it provides a *renormalization dynamics* for i.e.t.'s (and related systems). Such systems provide fundamental examples of *parabolic* dynamics, which by definition is characterized by sub-exponential (polynomial) divergence of nearby orbits.

All systems with behavior intermediate between *elliptic*, characterized by no or "very slow" divergence of nearby orbits, and *hyperbolic*, characterized by exponential divergence of nearby orbits, can be roughly classified as parabolic. A classical example of parabolic dynamics is the horocycle flow (on a surface of constant negative curvature). For i.e.t.'s (and related systems) there is no infinitesimal divergence of orbits, but parabolic orbit divergence is produced over time by the presence of singularities.

Key generic features of parabolic dynamics include unique ergodicity, polynomial deviation of ergodic averages from the mean and presence of invariant distributional obstructions, which are not measures, to the existence of smooth solutions of the cohomological equation. The elliptic, parabolic and hyperbolic paradigms are described in depth in the survey by B. Hasselblatt and A. Katok [3] in this handbook.

Parabolic (and elliptic) systems are often studied by means of appropriate renormalization schemes which enable to understand the dynamics of the generic system in a given family through the study of an auxiliary hyperbolic system. The hyperbolic system (renormalization) can in turn be studied by means of the well-developed tools of hyperbolic theory (Lyapunov exponents, invariant manifolds, Pesin theory, Lifschitz theory).

The Teichmüller flow and the Kontsevich–Zorich cocycle (and related systems such as the Rauzy–Zorich induction [35,45] or Veech "zippered rectangles" flow [38] and the corresponding cocycles) provide an effective renormalization scheme for i.e.t.'s and related systems.

Other well-known examples of renormalization include the classical Gauss map, which renormalizes rotations of the circle, and the geodesic flow (on a surface of constant negative curvature), which renormalizes the corresponding horocycle flow.

A tentative systematic approach to renormalization for a class of parabolic flows of algebraic nature, called "pseudo-homogeneous" flows, which includes conservative flows on surfaces, classical horocycle flows and nilflows in dimension 3, has been proposed by the author in [20].

1.3. *Contents*

In this article we outline the author's proof [21] of a substantial part of the *Zorich conjecture* on the Lyapunov spectrum of the Kontsevich–Zorich cocycle.

ZORICH CONJECTURE. *The Lyapunov exponents for the canonical absolutely continuous invariant measure on any connected component of any stratum of the moduli space are all non-zero and distinct.*

In [21] we have proved that the exponents are all non-zero, hence the cocycle is by definition *non-uniformly hyperbolic*. The full Zorich conjecture, which affirms that the Lyapunov spectrum is *simple*, that is, all Lyapunov exponents are distinct, was left open in [21] in genus higher than 3. A proof based on ideas different from ours has been recently announced by A. Avila and M. Viana [8]. In this outline, we have chosen to restrict ourselves to the proof of the positivity of the second exponent (Corollary 6.3) which is easier to explain and already contains all the main ideas of our method. As we have mentioned above, this is the key property in applications to i.e.t.'s (deviations of ergodic averages, weak-mixing). In passing we give a new, rather elementary, *complete* proof of the spectral gap property (Theorem 2.2).

We then present a rather unexpected example of an $SL(2,\mathbb{R})$-invariant measure supported on a closed Teichmüller disk in genus 3 for which the second and the third Lyapunov exponents are zero (Corollary 7.4). This example shows that (in genus greater than 3) the Zorich conjecture does not hold for all $SL(2,\mathbb{R})$-invariant measures on the moduli space. The significance of this conclusion is best understood in the perspective of the ergodic theory of *rational polygonal billiards*. In fact, for the generic directional flow on a fixed rational polygonal billiard the questions on deviation of ergodic averages and weak mixing are wide open, except for special cases, as a consequence of the fact that holomorphic differentials arising from rational billiards form a zero Lebesgue measure subset of the moduli space (see the survey by H. Masur and S. Tabachnikov [6] in this handbook on the dynamics of rational polygonal billiards).

Finally, we present the bulk of our proof of a representation theorem for *Zorich cycles* (Theorem 8.2). The phase space of the Kontsevich–Zorich cocycle is a (orbifold) vector bundle over the moduli space of holomorphic (quadratic) differentials on Riemann surfaces with fiber at each holomorphic differential given by the real homology (or cohomology) of the underlying Riemann surface. This bundle is sometimes called the *real homology* (*or cohomology*) *bundle*. Zorich cycles (or cocycles) are the homology (or cohomology) classes forming the invariant stable/unstable space of the Kontsevich–Zorich cocycle. For a generic (holomorphic) quadratic differential, leaves of the horizontal/vertical measured foliation "wind around a surface" deviating from a straight line (spanned by the Schwartzman's asymptotic cycle) in the direction of Zorich cycles in the real homology of the surface (see [44,47] or [48, Appendix D]).

We prove that Zorich cycles can be represented in terms of special closed currents on the surface (in the sense of de Rham) related to the horizontal/vertical measured foliation, called *basic currents*. Basic currents for measured foliations are in turn closely related to *invariant distributions* appearing as obstructions to the existence of smooth solutions of the cohomological equation for directional flows on translation surfaces or for i.e.t.'s [19, 27,28].

1.4. *Organization*

In Section 2 we review some background on the dynamics of the Teichmüller flow on the moduli space of holomorphic (quadratic) differentials.

In Section 3 we give our definition of the Kontsevich–Zorich cocycle and state the main theorem on its Lyapunov spectrum (Theorem 3.1).

In Section 4 we derive the variational formulas which describe the evolution of cohomology classes and their norms under the action of the cocycle (Lemmas 4.2 and 4.3).

In Section 5 bounds (upper and lower) on the second Lyapunov exponent are derived from the variational formulas of Section 4. The upper bound is easily obtained and allows us to immediately prove the spectral gap property (Theorem 2.2). The proof of the lower bound is harder since there are subtle cancellations.

Following [25] we take a harmonic analysis point of view (boundary behavior of harmonic functions, Brownian motion) on the generic Teichmüller disk which happens to be an isometric copy of the Poincaré disk. In concrete terms, we compute formulas for the hyperbolic gradient and Laplacian of the norm of a (fixed) cohomology class along a Teichmüller disk (Lemma 5.2). These formulas allow us to prove a lower bound for the second exponent in terms of the lowest eigenvalue of a Hermitian form which represents a 'Hodge curvature' of the real cohomology bundle. However, we have yet to prove that such a bound is non-trivial, that is, strictly positive. In fact, the Hodge curvature is degenerate on a real analytic subvariety of codimension 2 of the moduli space of holomorphic differentials.

In Section 6 we describe such a subvariety that we have called the *determinant locus* since it coincides with the locus where the determinant of the Lie derivative of the classical period matrix along the Teichmüller flow vanishes (Lemma 6.1). The proof that the second exponent is positive on all connected components of all strata of the moduli space is reduced to the statement that no connected component of a stratum is contained in the determinant locus (Theorem 6.2). The proof of this theorem, based on asymptotic formulas for the period matrix and its Lie derivative near appropriate boundary points of the moduli space, is only sketched here. The complete argument can be found in [21, Section 4].

In Section 7 we answer in the affirmative a question asked by W. Veech on whether there exist Teichmüller disks entirely contained in the determinant locus. Our example consists of a closed Teichmüller disk in genus 3 (in the stratum of holomorphic differentials with 4 simple zeroes) generated by a non-primitive Veech surface obtained as a 2-sheeted branched cover over the square torus with 4 branching points of order 2. Such a Veech surface has appropriate symmetries, stable under the $SL(2,\mathbb{R})$-action, which imply that the Hodge curvature has the minimal rank 1 (Theorem 7.3). It follows that of the 3 non-negative exponents of the Kontsevich–Zorich cocycle only one (the trivial one) is non-zero on the corresponding closed $SL(2,\mathbb{R})$-orbit (Corollary 7.4).

Finally, in Section 8 we prove the representation theorem for Zorich cycles. The proof is based on the variational formulas of Section 4, on a Cheeger-type lower bound for the smallest eigenvalue of the flat Laplacian on a translation surface, equivalent to a Poincaré inequality for the appropriate Sobolev norms (Lemma 8.3), and on the logarithmic law for geodesic in the moduli space of holomorphic (quadratic) differentials [31].

2. Elements of Teichmüller theory

In this section we recall a few definitions and results of Teichmüller theory which are essential to understanding the material treated in later sections.

Let T_g, Q_g be the *Teichmüller spaces* of complex (conformal) structures and of holomorphic quadratic differentials on a surface of genus $g \geqslant 1$. The spaces T_g and Q_g can be roughly described as follows:

$$
\begin{aligned}
T_g &:= \{\text{complex (conformal) structures}\}/\operatorname{Diff}_0^+(M),\\
Q_g &:= \{\text{holomorphic quadratic differentials}\}/\operatorname{Diff}_0^+(M),
\end{aligned}
\tag{1}
$$

where $\mathrm{Diff}_0^+(M)$ is the group of orientation preserving diffeomorphisms of the surface M which are isotopic to the identity (equivalently, it is the connected component of the identity in the Lie group of all orientation preserving diffeomorphisms of M).

If $g \geqslant 2$, the space T_g is topologically equivalent to an open ball of real dimension $6g-6$. In fact, a theorem of L. Ahlfors, L. Bers and S. Wolpert states that T_g has a complex structure holomorphically equivalent to that of a Stein (strongly pseudo-convex) domain in $\mathbb{C}^{3g-3}$ [9, §6], or [32, Chapters 3, 4 and Appendix §6]. The space Q_g of holomorphic quadratic differentials is a complex vector bundle over T_g which can be identified to the cotangent bundle of T_g. If $g=1$, the Teichmüller space T_1 of elliptic curves (complex structures on T^2) is isomorphic to the upper half plane $\mathbb{C}^+$ and the Teichmüller space Q_1 of holomorphic quadratic differentials on elliptic curves is a complex line bundle over T_1 [32, Example 2.1.8].

Let $R_g, \mathcal{M}_g$ be the *moduli spaces* of complex (conformal) structures and of holomorphic quadratic differentials on a surface of genus $g \geqslant 1$. The spaces R_g and $\mathcal{M}_g$ can be roughly described as the quotient spaces:

$$R_g := T_g/\Gamma_g, \qquad \mathcal{M}_g := Q_g/\Gamma_g, \tag{2}$$

where Γ_g denotes the *mapping class group* $\mathrm{Diff}^+(M)/\mathrm{Diff}_0^+(M)$. If $g=1$, the mapping class group can be identified with the lattice $SL(2,\mathbb{Z})$ which acts on the upper half plane $\mathbb{C}^+$ in the standard way. The moduli space $R_1 := \mathbb{C}^+/SL(2,\mathbb{Z})$ is a non-compact finite volume surface with constant negative curvature, called the *modular surface*. The moduli space $\mathcal{M}_1$ can be identified to the cotangent bundle of the modular surface.

The *Teichmüller* (*geodesic*) *flow* is a Hamiltonian flow on $\mathcal{M}_g$, defined as the geodesic flow with respect to a natural metric on R_g called the *Teichmüller metric*. Such a metric measures the amount of *quasi-conformal distortion* between two different (equivalent classes of) complex structures in R_g. In the higher genus case, the Teichmüller metric is not Riemannian, but only *Finsler* (that is, the norm on each tangent space does not come from an Euclidean product) and, as H. Masur proved, does not have negative curvature in any reasonable sense [9, §3 (E)]. If $g=1$, the Teichmüller metric coincides with the Poincaré metric on the modular surface R_1 [32, 2.6.5], in particular it is Riemannian with constant negative curvature.

In order to obtain a more geometric description of the Teichmüller flow, we introduce below a natural action of the Lie group $SL(2,\mathbb{R})$ on Q_g (see also [4, §1.4] or [5, §3], in this handbook). This action is equivariant with respect to the action of the mapping class group, hence it passes to the quotient $\mathcal{M}_g$.

A holomorphic quadratic differential q naturally defines two transverse *measured foliations* (in the Thurston's sense [37,17]), the *horizontal* foliation $\mathcal{F}_q$ and the *vertical* foliation $\mathcal{F}_{-q}$:

$$\begin{aligned} \mathcal{F}_q &:= \left\{\mathrm{Im}\left(q^{1/2}\right)=0\right\}, \quad \text{with transverse measure } \left|\mathrm{Im}\left(q^{1/2}\right)\right|, \\ \mathcal{F}_{-q} &:= \left\{\mathrm{Re}\left(q^{1/2}\right)=0\right\}, \quad \text{with transverse measure } \left|\mathrm{Re}\left(q^{1/2}\right)\right|. \end{aligned} \tag{3}$$

Vice versa, any pair $(\mathcal{F}, \mathcal{F}^\perp)$ of transverse measure foliations determines a complex structure and a holomorphic quadratic differential q such that $\mathcal{F} = \mathcal{F}_q$ and $\mathcal{F}^\perp = \mathcal{F}_{-q}$.

Transversality for measured foliations is taken in the sense that $\mathcal{F}$ and $\mathcal{F}^{\perp}$ have a common set Σ of (saddle) singularities, have the same index at each singularity and are transverse in the standard sense on $M \setminus \Sigma$. The set Σ of common singularities coincides with the set Σ_q of zeroes of the holomorphic quadratic differential $q \equiv (\mathcal{F}, \mathcal{F}^{\perp})$.

The $SL(2,\mathbb{R})$-action on Q_g is defined as follows. Every 2×2 matrix $A \in SL(2,\mathbb{R})$ acts naturally by left multiplication on the (locally defined) pair of real-valued 1-forms $(\mathrm{Im}(q^{1/2}), \mathrm{Re}(q^{1/2}))$. The resulting (locally defined) pair of 1-forms defines a new pair of transverse measured foliations, hence a new complex structure and a new holomorphic quadratic differential $A \cdot q$.

The Teichmüller flow G_t is given by the action of the diagonal subgroup $\mathrm{diag}(e^{-t}, e^{t})$ on Q_g (on $\mathcal{M}_g$). In other terms, if we identify holomorphic quadratic differentials with pairs of transverse measured foliations as explained above, we have:

$$G_t(\mathcal{F}_q, \mathcal{F}_{-q}) := \bigl(e^{-t}\mathcal{F}_q, e^{t}\mathcal{F}_{-q}\bigr). \tag{4}$$

In geometric terms, the action of the Teichmüller flow on quadratic differentials induces a one-parameter family of deformations of the conformal structure which consist in contracting along vertical leaves (with respect to the horizontal length) and expanding along horizontal leaves (with respect to the vertical length) by reciprocal (exponential) factors.

The reader can compare the definition in terms of the $SL(2,\mathbb{R})$-action with the analogous description of the geodesic flow on a surface of constant negative curvature (such as the modular surface). In fact, if $g = 1$ the above definition reduces to the standard Lie group presentation of the geodesic flow on the modular surface: the unit sub-bundle $\mathcal{M}_1^{(1)} \subset \mathcal{M}_1$ of all holomorphic quadratic differentials of unit total area on elliptic curves can be identified with the homogeneous space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ and the geodesic flow on the modular surface is then identified with the action of the diagonal subgroup of $SL(2,\mathbb{R})$ on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

We list below, following [41,25], the main structures carried by the Teichmüller space Q_g and by the moduli space $\mathcal{M}_g$ of quadratic differentials (see also [5, §2] and [48, §4]):

(1) $\mathcal{M}_g$ is a (stratified) analytic space (orbifold); each stratum $\mathcal{M}_\kappa$ (corresponding to fixing the multiplicities $\kappa := (k_1, \ldots, k_\sigma)$ of the zeroes $\{p_1, \ldots, p_\sigma\}$ of the quadratic differentials) is $SL(2,\mathbb{R})$-invariant and, in particular, G_t-invariant.

(2) The total area function $A : \mathcal{M}_g \to \mathbb{R}^+$,

$$A(q) := \int_M |q|,$$

is $SL(2,\mathbb{R})$-invariant; hence the *unit bundle* $\mathcal{M}_g^{(1)} := A^{-1}(\{1\})$ and its strata $\mathcal{M}_\kappa^{(1)} := \mathcal{M}_\kappa \cap \mathcal{M}_g^{(1)}$ are $SL(2,\mathbb{R})$-invariant and, in particular, G_t-invariant.

Let $\mathcal{M}_\kappa$ be a stratum of *orientable* quadratic differentials, that is, quadratic differentials which are squares of holomorphic 1-forms. In this case, the natural numbers $(k_1, \ldots, k_\sigma)$ are all even.

(3) The stratum of squares $\mathcal{M}_\kappa$ has a locally affine structure modeled on the affine space $H^1(M, \Sigma_\kappa; \mathbb{C})$, with $\Sigma_\kappa := \{p_1, \ldots, p_\sigma\}$. Local charts are given by the period map $q \to [q^{1/2}] \in H^1(M, \Sigma_\kappa; \mathbb{C})$.

(4) The Lebesgue measure on the Euclidean space $H^1(M, \Sigma_\kappa; \mathbb{C})$, normalized so that the quotient torus

$$H^1(M, \Sigma_\kappa; \mathbb{C})/H^1(M, \Sigma_\kappa; \mathbb{Z} \oplus \imath\mathbb{Z})$$

has volume 1, induces an absolutely continuous $SL(2, \mathbb{R})$-invariant measure μ_κ on $\mathcal{M}_\kappa$. The conditional measure $\mu_\kappa^{(1)}$ induced on $\mathcal{M}_\kappa^{(1)}$ is $SL(2, \mathbb{R})$-invariant, hence G_t-invariant.

The ergodic theory of the Teichmüller flow begins with the natural questions whether the measure $\mu_\kappa^{(1)}$ has finite total mass and whether it is ergodic for the Teichmüller flow on $\mathcal{M}_\kappa^{(1)}$. However, it was discovered by W. Veech [41] that $\mathcal{M}_\kappa^{(1)}$ has in general several connected components. M. Kontsevich and A. Zorich [26] have been able to obtain a complete classification of the connected components of the strata. Taking this phenomenon into account, the following result holds:

THEOREM 2.1 [30,40]. *The total volume of the measure $\mu_\kappa^{(1)}$ on $\mathcal{M}_\kappa^{(1)}$ is finite and the Teichmüller geodesic flow G_t is ergodic on each connected component of $\mathcal{M}_\kappa^{(1)}$.*

Since the measure $\mu_\kappa^{(1)}$ has finite total mass, the Poincaré recurrence theorem applies. This is the core of Masur's proof [30] of the unique ergodicity for almost all i.e.t.'s and measured foliations, a statement known as the *Keane conjecture* (see the article by H. Masur [5] in this handbook on the ergodic theory of measured foliations, i.e.t.'s and translation surfaces).

Poincaré recurrence for a suitable "renormalization" flow (on the space of "zippered rectangles") is also the key idea of Veech's proof of the Keane conjecture [38]. In [40] Veech further investigated the ergodic theory of the Teichmüller flow and proved that the Teichmüller flow is *non-uniformly hyperbolic*, in the sense that all of its *Lyapunov exponents*, except one corresponding to the flow direction, are non-zero.

We recall that a Lyapunov exponent is the asymptotic exponential rate of expansion of a (tangent) vector along the orbit of a point in the phase space of a dynamical system. The Oseledec's *Multiplicative Ergodic Theorem* [34,23] establishes their existence as appropriately defined limits, for almost all points with respect to any ergodic invariant probability measure. The theory of Lyapunov exponents for *cocycles* over (smooth) dynamical systems is explained in [23, §S.1], and in the survey [1] in this handbook.

The Lyapunov spectrum (that is, the collection of Lyapunov exponents) of the Teichmüller flow with respect to any ergodic invariant probability measure μ on $\mathcal{M}_\kappa^{(1)}$ is known to have symmetries. In fact, it can be written as follows [45, §5], [25, §7], [47, §2.3]:

$$\begin{aligned} 2 &\geqslant \left(1+\lambda_2^\mu\right) \geqslant \cdots \geqslant \left(1+\lambda_g^\mu\right) \geqslant \overbrace{1=\cdots=1}^{\sigma_\kappa-1} \geqslant \left(1-\lambda_g^\mu\right) \\ &\geqslant \cdots \geqslant \left(1-\lambda_2^\mu\right) \geqslant 0 \geqslant -\left(1-\lambda_2^\mu\right) \geqslant \cdots \geqslant -\left(1-\lambda_g^\mu\right) \\ &\geqslant \underbrace{-1=\cdots=-1}_{\sigma_\kappa-1} \geqslant -\left(1+\lambda_g^\mu\right) \geqslant \cdots \geqslant -\left(1+\lambda_2^\mu\right) \geqslant -2. \end{aligned} \tag{5}$$

By the ergodicity statement of Theorem 2.1, the non-uniform hyperbolicity of the Teichmüller flow, proved by W. Veech in [40], can be formulated as follows:

THEOREM 2.2 [40]. *Let μ denote the normalized absolutely continuous $SL(2,\mathbb{R})$-invariant ergodic measure on any connected component $\mathcal{C}_\kappa^{(1)}$ of a stratum $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$ of the moduli space of orientable holomorphic quadratic differentials of unit total area. The non-negative number λ_2^μ satisfies the inequality*:

$$\lambda_2^\mu < \lambda_1^\mu = 1. \tag{6}$$

M. Kontsevich and A. Zorich have interpreted the non-negative numbers

$$\lambda_1^\mu = 1 \geqslant \lambda_2^\mu \geqslant \cdots \geqslant \lambda_g^\mu \tag{7}$$

as *Lyapunov exponents* of a cocycle over the Teichmüller flow that will be described below. This cocycle is obtained as the natural (fiber-wise linear) lift of the Teichmüller flow to an appropriate vector bundle over the moduli space. The non-negative Lyapunov exponents of the Kontsevich–Zorich cocycle turn out to be exactly the numbers in (7).

In this paper we discuss the Lyapunov spectrum and the Oseledec's splitting of this cocycle. In particular, we give a new elementary proof of the inequality (6) for *any* ergodic probability measure on a stratum of orientable holomorphic quadratic differentials (Theorem 5.1) and we outline the proof of the inequality $\lambda_2^\mu > 0$, when μ is the normalized absolutely continuous $SL(2,\mathbb{R})$-invariant ergodic measure on any connected component of a stratum of the moduli space of orientable holomorphic quadratic differentials (Corollary 6.3).

3. The Kontsevich–Zorich cocycle

M. Kontsevich (and A. Zorich) [25] have introduced a (multiplicative) 'renormalization' cocycle over the Teichmüller geodesic flow. This cocycle is a continuous-time version of a cocycle introduced by G. Rauzy [35] as a "continued fractions algorithm" for i.e.t.'s, and later studied by W. Veech, in his work on the Keane conjecture [38], and A. Zorich [45,46] among others. Zorich was motivated by the study of the asymptotic behavior in homology of (long) typical leaves of orientable measured foliations on closed surfaces of higher genus, which he initiated in [44].

Let Q_g be the Teichmüller space of holomorphic quadratic differentials on Riemann surfaces of genus $g \geqslant 2$. The *Kontsevich–Zorich cocycle* G_t^{KZ} can be defined as the quotient cocycle, with respect to the action of the mapping class group Γ_g, of the trivial cocycle

$$G_t \times \mathrm{id}\colon Q_g \times H^1(M,\mathbb{R}) \to Q_g \times H^1(M,\mathbb{R}). \tag{8}$$

The cocycle G_t^{KZ} acts on the orbifold vector bundle

$$\mathcal{H}_g^1(M,\mathbb{R}) := \big(Q_g \times H^1(M,\mathbb{R})\big)/\Gamma_g \tag{9}$$

over the moduli space $\mathcal{M}_g = Q_g/\Gamma_g$ of holomorphic quadratic differentials. The base dynamics of the Kontsevich–Zorich cocycle is the Teichmüller geodesic flow G_t on $\mathcal{M}_g$. Note that the mapping class group acts naturally on the cohomology $H^1(M,\mathbb{R})$ by pullback. We recall that the real homology $H_1(M,\mathbb{R})$ and the real cohomology $H^1(M,\mathbb{R})$ of an orientable closed surface M are endowed with a natural symplectic form (the intersection form) and are (symplectically) isomorphic by Poincaré duality.

Since the vector bundle $\mathcal{H}^1_g(M,\mathbb{R})$ has a symplectic structure, the Lyapunov spectrum of the cocycle G_t^{KZ} (with respect to any G_t-invariant ergodic probability measure μ on $\mathcal{M}_g^{(1)}$) is *symmetric*:

$$\lambda_1^\mu \geqslant \cdots \geqslant \lambda_g^\mu \geqslant 0 \geqslant -\lambda_g^\mu \geqslant \cdots \geqslant -\lambda_1^\mu. \tag{10}$$

The non-negative part of the Kontsevich–Zorich spectrum (10) coincides with the numbers (7) which appear in the Lyapunov spectrum (5) of the Teichmüller flow. This relation can be explained as follows. By Section 2 the tangent space $T\mathcal{M}_\kappa \equiv \mathcal{M}_\kappa \times H^1(M, \Sigma_\kappa; \mathbb{C})$ locally. There is a surjective map $H^1(M, \Sigma_\kappa; \mathbb{C}) \to H^1(M,\mathbb{C})$ which neglects cohomology classes dual to cycle joining two singularities. Such classes are responsible for the (trivial) part of the Lyapunov spectrum (5) consisting of $\sigma_\kappa - 1$ repeated 1's and -1's. Let then $\mathcal{H}^1_\kappa(M,\mathbb{C})$ be the bundle over the moduli space with fiber $H^1(M,\mathbb{C})$. There is the following natural isomorphism of vector bundles over $\mathcal{M}_\kappa$:

$$\mathcal{H}^1_\kappa(M,\mathbb{C}) \equiv \mathbb{C} \otimes \mathcal{H}^1(M,\mathbb{R}) \equiv \mathbb{R}^2 \otimes \mathcal{H}^1(M,\mathbb{R}), \tag{11}$$

induced by the corresponding isomorphism on the fibers. The tangent cocycle TG_t of the Teichmüller geodesic flow on $\mathcal{H}^1(M,\mathbb{C})$ can then be written in terms of the Kontsevich–Zorich cocycle:

$$TG_t = \mathrm{diag}\big(e^t, e^{-t}\big) \otimes G_t^{KZ} \quad \text{on } \mathbb{R}^2 \otimes \mathcal{H}^1(M,\mathbb{R}). \tag{12}$$

Formula (12) implies that the non-trivial Lyapunov spectrum of TG_t on $\mathcal{H}^1(M,\mathbb{C})$ can be obtained as a union of the translations of the Lyapunov spectrum of G_t^{KZ} by ± 1, hence (5) follows (see also [48, §5.7]).

We will discuss the main ideas of the proof of the following result originally conjectured by A. Zorich in [45] for the Rauzy–Veech–Zorich cocycle, a discrete-time version of the Kontsevich–Zorich cocycle, and by M. Kontsevich (and A. Zorich) in [25] for the Kontsevich–Zorich cocycle (see also [48, §5.6]):

THEOREM 3.1 [21, Theorem 8.5]. *Let μ denote the absolutely continuous $SL(2,\mathbb{R})$-invariant ergodic probability measure on any connected component $\mathcal{C}_\kappa^{(1)}$ of a stratum $\mathcal{M}_\kappa^{(1)} \subset \mathcal{M}_g^{(1)}$ of the moduli space of orientable holomorphic quadratic differentials of unit total area. The Lyapunov exponents of G_t^{KZ} over $\mathcal{C}_\kappa^{(1)}$ satisfy the inequalities*:

$$\lambda_1^\mu = 1 > \lambda_2^\mu \geqslant \cdots \geqslant \lambda_g^\mu > 0. \tag{13}$$

The inequality $\lambda_1^\mu = 1 > \lambda_2^\mu$ is the content of Veech's Theorem 2.2. We will give a complete new proof below. The other non-trivial inequality in (13) is $\lambda_g^\mu > 0$. We will describe the strategy of the proof that $\lambda_2^\mu > 0$. The full proof of the theorem for genus $g \geqslant 3$ is more complicated but it does not require substantial new ideas. The *Zorich conjecture* states that the exponents in (13) are all distinct, that is, the Lyapunov spectrum of the cocycle is *simple*. A proof of the conjecture, which yields as a corollary an independent proof of Theorem 3.1 based on completely different methods, has been recently given by A. Avila and M. Viana [8].

4. Variational formulas

The Kontsevich–Zorich cocycle can be written in the form of an O.D.E. in a fixed Hilbert space. This is accomplished as follows. Let R_q be (degenerate) Riemannian metric induced by a holomorphic quadratic differential q and let ω_q be the corresponding area form. With respect to a holomorphic local coordinate $z = x + iy$, the quadratic differential q has the form $q = \phi(z)\,dz^2$, where ϕ is a locally defined holomorphic function, and, consequently,

$$R_q = \left|\phi(z)\right|^{1/2}\left(dx^2 + dy^2\right)^{1/2}, \qquad \omega_q = \left|\phi(z)\right|\,dx \wedge dy. \tag{14}$$

The metric R_q is flat, it is degenerate at the finite set Σ_q of zeroes of q and, if q is orientable, it has trivial holonomy, hence q induces a structure of *translation surface* on M. It follows that, if q is orientable, there exists a (unique) frame $\{S, T\}$ of the tangent bundle of M over $M \setminus \Sigma_q$ with the following properties [19, §2]:

(1) The frame $\{S, T\}$ is orthonormal with respect to the Riemannian metric R_q on $M \setminus \Sigma_q$;
(2) The vector field $S[T]$ is tangent to the oriented horizontal [vertical] foliation $\mathcal{F}_q$ $[\mathcal{F}_{-q}]$ in the positive direction.

Let $L_q^2(M) := L^2(M, \omega_q)$ the space of complex-valued, square-integrable functions and $H_q^1(M)$ be the (Sobolev) subspace of functions $v \in L_q^2(M)$ such that $Sv \in L_q^2(M)$ and $Tv \in L_q^2(M)$. The flows generated by the vector fields S, T preserves the area form ω_q. In fact, the 1-forms

$$\imath_S\omega_q = \mathrm{Im}\left(q^{1/2}\right) \quad \text{and} \quad \imath_T\omega_q = -\,\mathrm{Re}\left(q^{1/2}\right) \tag{15}$$

are closed and the Lie derivatives

$$\begin{aligned} \mathcal{L}_S\omega_q &= d\imath_S\omega_q + \imath_S d\omega_q = 0, \\ \mathcal{L}_T\omega_q &= d\imath_T\omega_q + \imath_T d\omega_q = 0. \end{aligned} \tag{16}$$

Hence, the vector fields S, T yield densely defined anti-symmetric (in fact, essentially skew-adjoint) operators on the Hilbert space $L_q^2(M)$. In addition, these operators commute in the following sense. Let $(\cdot,\cdot)_q$ denote the inner product in $L^2_q(M)$. For all functions $v_1, v_2 \in H^1_q(M)$,

$$(Sv_1, Tv_2)_q = (Tv_1, Sv_2)_q. \tag{17}$$

In conclusion, there is a well-defined action of the commutative Lie algebra $\mathbb{R}^2$ on $L^2_q(M)$ by essentially skew-adjoint operators [19].

The above properties are not surprising since, with respect to a local canonical (holomorphic) coordinate $z = x + \imath y$ at a point $p \in M \setminus \Sigma_q$, the holomorphic quadratic differential $q = dz^2$, the metric R_q is Euclidean, the area form $\omega_q = dx \wedge dy$ and the vector fields $S = \partial/\partial x$, $T = \partial/\partial y$. The formulas for S, T in a neighbourhood of a zero $p \in \Sigma_q$ of even order $k \geqslant 2$ are given in [19, (2.7)].

A key idea in [19,21] is to consider the *Cauchy–Riemann operators* determined by an orientable quadratic differential.

LEMMA 4.1 [19, Proposition 3.2]. *Let q be an orientable quadratic differential on M. The Cauchy–Riemann operators*

$$\partial^{\pm}_q := \frac{S \pm iT}{2} \tag{18}$$

with (dense) domain $H^1_q(M) \subset L^2_q(M)$ are closed and have closed range of finite codimension equal to the genus of M. Let $\mathcal{M}^{\pm}_q \subset L^2_q(M)$ be the subspaces of meromorphic, respectively, anti-meromorphic, functions. The following orthogonal splittings hold:

$$L^2_q(M) = \mathrm{Ran}\big(\partial^+_q\big) \oplus \mathcal{M}^-_q = \mathrm{Ran}\big(\partial^-_q\big) \oplus \mathcal{M}^+_q. \tag{19}$$

The spaces $\mathcal{M}^{\pm}_q$ consist of all meromorphic, respectively anti-meromorphic, functions with poles at Σ_q of orders bounded above in terms of the multiplicities of the points $p \in \Sigma_q$ as zeroes of the quadratic differential q. The complex dimension of $\mathcal{M}^{\pm}_q$ can therefore be computed by the Riemann–Roch theorem and it is equal to the genus of M. By (17) the adjoint operators $(\partial^{\pm}_q)^*$ are extensions of the operators $-\partial^{\mp}_q$. It follows that the kernels of $(\partial^{\pm}_q)^*$ are the subspaces $\mathcal{M}^{\mp}_q$, respectively, hence the splitting (19) follows immediately by Hilbert space theory.

(Absolute) real cohomology classes on M can be represented in terms of meromorphic (or anti-meromorphic) functions in $L^2_q(M)$. In fact, by the theory of Riemann surfaces [16, III.2], any $c \in H^1(M,\mathbb{R})$ can be represented as the real part of a holomorphic differential h on M. Let q be an orientable holomorphic quadratic differential on M and let $q^{1/2}$ a holomorphic square root of q. The quotient $h/q^{1/2}$ is a meromorphic function on M with poles at the set Σ_q of zeroes of q. A computation shows that $m^+ = h/q^{1/2} \in L^2_q(M)$, hence $m^+ \in \mathcal{M}^+_q$. The following *representation* of cohomology classes therefore holds:

$$c \in H^1(M,\mathbb{R}) \leftrightarrow c = \mathrm{Re}\big[m^+ \cdot q^{1/2}\big], \quad m^+ \in \mathcal{M}^+_q. \tag{20}$$

The map $\mathcal{M}_q^+ \to H^1(M,\mathbb{R})$ given by the representation (20) is bijective and it is in fact *isometric* if $\mathcal{M}_q^+$ is endowed with the Euclidean structure induced by $L_q^2(M)$ and $H^1(M,\mathbb{R})$ with the *Hodge product* relative to the complex structure of the Riemann surface M_q carrying the holomorphic quadratic differential $q \in \mathcal{M}_g$.

Let $q \in \mathcal{Q}_\kappa^{(1)}$ and $c \in H^1(M,\mathbb{R})$. Let $q_t := G_t(q)$ be the orbit of q under the Teichmüller flow and $c_t := G_t^{KZ}(c)$ the orbit of c under the Kontsevich–Zorich cocycle. Let M_t the Riemann surface carrying $q_t \in \mathcal{Q}_\kappa^{(1)}$. By (20),

$$c_t = \mathrm{Re}\big[m_t^+ \cdot q_t^{1/2}\big] \in H^1(M_t,\mathbb{R}), \tag{21}$$

where $m_t^+ \in \mathcal{M}_t^+$, the space of meromorphic function on M_t which are in $L_q^2(M)$. At this point, we have to make the following crucial remark. By the very definition of the Teichmüller flow G_t, the area form ω_t of the metric R_t induced by the quadratic differential q_t is *constant*. Hence the Hilbert space $L_q^2(M)$ is invariant under the action of the Teichmüller flow on $\mathcal{Q}_\kappa^{(1)}$. Let $\mathcal{M}_t^\pm \subset L_q^2(M)$ be the subspaces of meromorphic, respectively, anti-meromorphic, functions on the Riemann surface M_t. Such spaces are, respectively, the kernels of the adjoints of the Cauchy–Riemann operators $\partial_t^\mp$, related to the holomorphic quadratic differential q_t. By Lemma 4.1, the dimension of $\mathcal{M}_t^\pm$ is constant equal to the genus $g \geqslant 1$ of M. It can be proved that $\{\mathcal{M}_t^\pm \mid t \in \mathbb{R}\}$ are smooth families of g-dimensional subspaces of the fixed Hilbert space $L_q^2(M)$.

Let $\pi_q^\pm : L_q^2(M) \to \mathcal{M}_q^\pm$ denote the orthogonal projection onto the finite-dimensional subspace of meromorphic, respectively anti-meromorphic, functions. It follows immediately from (19) that, for every $u \in L_q^2(M)$, there exist functions $v^\pm \in H_q^1(M)$ such that

$$u = \partial_q^+ v^+ + \pi_q^-(u) = \partial_q^- v^- + \pi_q^+(u). \tag{22}$$

Let $\pi_t^\pm : L_q^2(M) \to \mathcal{M}_t^\pm$ denote the orthogonal projections in the (fixed) Hilbert space $L_q^2(M)$. By definition, the projections $\pi_t^\pm$ coincide with the projections $\pi_q^\pm$ for $q = q_t$, for any $t \in \mathbb{R}$.

LEMMA 4.2 [21, Lemma 2.1]. *The Kontsevich–Zorich cocycle is described by the following variational formulas*:

$$\begin{cases} m_t^+ = \partial_t^+ v_t + \pi_t^-(m_t^+), \\ \frac{d}{dt} m_t^+ = \partial_t^- v_t - \overline{\pi_t^-(m_t^+)}. \end{cases} \tag{23}$$

PROOF. By the definition (4) of the Teichmüller flow G_t, the quadratic differential $q_t := G_t(q)$ and the related Cauchy–Riemann operators $\partial_t^\pm$ can be explicitly written in terms of q and of corresponding frame $\{S, T\}$. In fact, we have $\mathrm{Re}(q_t^{1/2}) \equiv e^t\,\mathrm{Re}(q^{1/2})$, $\mathrm{Im}(q_t^{1/2}) \equiv e^{-t}\,\mathrm{Re}(q^{1/2})$ and

$$S_t \equiv e^{-t} S, \qquad T_t \equiv e^t T, \qquad \partial_t^\pm \equiv \frac{e^{-t} S \pm i e^t T}{2}; \tag{24}$$

hence, by straightforward computations,

$$\frac{d}{dt} q_t^{1/2} \equiv \overline{q_t}^{1/2}, \qquad \frac{d}{dt} \partial_t^{\pm} \equiv -\partial_t^{\mp}. \tag{25}$$

Equation (23) in the statement of the lemma follows from the formulas (25) by a computation based on the following two remarks. First, since the function m_t^+ is meromorphic on the Riemann surface M_t, it satisfies the equation $\partial_t^+ m_t^+ \equiv 0$ in the weak sense in $L_q^2(M)$. It follows that, by taking a time derivative,

$$-\partial_t^- m_t^+ + \partial_t^+ \left(\frac{d}{dt} m_t^+ \right) \equiv 0. \tag{26}$$

Second, by the definition (8) of the cocycle G_t^{KZ}, the one-parameter family of cohomology classes $c_t := G_t^{KZ}(c)$ is locally constant, that is, $c_t \equiv c \in H^1(M, \mathbb{R})$. It follows that the time derivative of the 1-form $\mathrm{Re}(m_t^+ q_t^{1/2})$ is equal to zero in $H^1(M, \mathbb{R})$, hence it is an exact form. There exists therefore a function $U_t \in H^1(M)$ such that

$$\mathrm{Re}\left[\left(\frac{d}{dt} m_t^+ + \overline{m_t^+} \right) q_t^{1/2} \right] = dU_t. \tag{27}$$

A straightforward computation based on formulas (26), (27) and on the splittings (22) for $q = q_t$, applied to the functions $m_t^+ \in \mathcal{M}_t^+ \subset L_q^2(M)$ and $dm_t^+/dt \in L_q^2(M)$, concludes the argument. In fact, the splitting in the first line of (23) is simply the first splitting in (22) for $q = q_t$, applied to the function $m_t^+ \in \mathcal{M}_t^+ \subset L_q^2(M)$. It is therefore an identity which determines the function $v_t \in H_q^1(M)$ up to an additive constant. The second line is a formula for the derivative dm_t^+/dt written in terms of the second splitting in (22) for $q = q_t$. □

An immediate consequence of Lemma 4.2 is the following result on the variation of the Hodge norm of cohomology classes under the action of the Kontsevich–Zorich cocycle. Let $B_q : L_q^2(M) \times L_q^2(M) \to \mathbb{C}$ be the complex bilinear form given by

$$B_q(u, v) := \int_M uv\omega_q, \quad \text{for all } u, v \in L_q^2(M). \tag{28}$$

LEMMA 4.3 [21, Lemma 2.1′]. *The variation of the Hodge norm* $\|c_t\|$, *which coincides with the* L_q^2*-norm* $|m_t^+|_0$ *under the identification* (21), *is given by the following formulas*:

$$\begin{aligned} &\text{(a)} \quad \frac{d}{dt} |m_t^+|_0^2 = -2\,\mathrm{Re}\, B_q\big(m_t^+\big) = -2\,\mathrm{Re} \int_M \big(m_t^+\big)^2 \omega_q, \\ &\text{(b)} \quad \frac{d^2}{dt^2} |m_t^+|_0^2 = 4 \left\{ \big|\pi_t^-\big(m_t^+\big)\big|_0^2 - \mathrm{Re} \int_M \big(\partial_t^+ v_t\big)\big(\partial_t^- v_t\big)\omega_q \right\}. \end{aligned} \tag{29}$$

PROOF. The formulas (29) can be immediately deduced from (23) by taking into account the G_t-invariance of the inner product in $L^2_q(M)$ and the orthogonality of the splittings (19), (22) for $q = q_t$. □

5. Bounds on the exponents

Lemma 4.2 immediately implies Veech's Theorem 2.2. In fact, we have

THEOREM 5.1 [21, Corollary 2.2]. *Let μ denote any ergodic G_t-invariant probability measure on the moduli space $\mathcal{M}_g^{(1)}$ of orientable holomorphic quadratic differentials of unit total area. The Lyapunov exponents of the Kontsevich–Zorich cocycle G_t^{KZ} with respect to the ergodic measure μ satisfy the following inequality*:

$$\lambda_1^\mu = 1 > \lambda_2^\mu. \tag{30}$$

PROOF. By formula (a) in (29),

$$\frac{d}{dt}\log\left|m_t^+\right|_0^2 = -2\frac{\operatorname{Re} B_q(m_t^+)}{|m_t^+|_0^2}. \tag{31}$$

Since by the Schwarz inequality,

$$\left|B_q\left(m_t^+\right)\right| = \left|\left(m_t^+, \overline{m_t^+}\right)_q\right| \leqslant \left|m_t^+\right|_0^2, \tag{32}$$

Equation (31) implies that the upper Lyapunov exponent

$$\lambda_1^\mu := \limsup_{T\to\pm\infty}\frac{1}{T}\log\left|m_T^+\right|_0 \leqslant 1. \tag{33}$$

Moreover, the 1-dimensional subspace of complex constant functions is invariant under the flow of Equation (23), since for $m_t^+ \in \mathbb{C}$, the function $v_t \equiv 0$ and the orthogonal projection $\pi_t^-(m_t^+) \equiv m_t^+ \in \mathbb{C}$. By the definition of the isomorphism (20), this corresponds to the fact that the plane $E_q \subset H^1(M, \mathbb{R})$ generated by the cohomology classes $\{\operatorname{Re}(q^{1/2}), \operatorname{Im}(q^{1/2})\}$ is invariant under the cocycle G_t^{KZ}. The Lyapunov exponents of G_t^{KZ} restricted to this plane are ± 1, as it can be seen directly from the definition or by the formula (31) in the case of purely real or purely imaginary constant functions. Hence $\lambda_1^\mu = 1$. The exponent λ_2^μ is the top Lyapunov exponent of G_t^{KZ} on the bundle with fiber $H^1(M_q, \mathbb{R})/E_q$. Under the isomorphism (20), the vector space $H^1(M_q, \mathbb{R})/E_q$ is represented by meromorphic functions with *zero average* (orthogonal to constant functions). It can be seen that the subspace of zero average meromorphic functions is invariant under the flow of Equation (23). Let

$$\Lambda^+(q) := \max\left\{\frac{|B_q(m^+)|}{|m^+|_0^2}\,\Big|\, m^+ \in \mathcal{M}_q^+ \setminus \{0\},\ \int_M m^+\omega_q = 0\right\}. \tag{34}$$

By averaging (31) over the interval $[0, T]$, taking the upper limit and applying the Birkhoff ergodic theorem with respect to the G_t-invariant measure μ to the r.h.s., we have that, if $m_0^+ \in \mathcal{M}_q^+$ has zero average, for μ-almost all $q \in \mathcal{M}_\kappa$,

$$\limsup_{T\to+\infty} \frac{1}{T} \log \left|m_T^+\right|_0 \leqslant \int_{\mathcal{M}_\kappa} \Lambda^+(q)\, d\mu(q). \tag{35}$$

Since, by the Schwarz inequality (32), $\Lambda^+(q) \leqslant 1$ for all $q \in \mathcal{M}_\kappa$, it is sufficient to prove that $\Lambda^+(q) < 1$ on a positive measure set. In fact, $\Lambda^+(q) = 1$ if and on only if there exists a *non-zero* meromorphic function with zero average $m^+ \in \mathcal{M}_q^+$ such that $|(m^+, \overline{m^+})_q| = |m^+|_0^2$. A well-known property of the Schwarz inequality then implies that there exists $\lambda \in \mathbb{C}$ such that $m^+ = \lambda\, \overline{m^+}$. However, it cannot be so, since in that case m^+ would be meromorphic and anti-meromorphic, hence constant, and by the zero average condition it would be zero. We have therefore proved that $\Lambda^+(q) < 1$ for all $q \in \mathcal{M}_\kappa$. The argument is completed. □

The proof of *lower bounds* on the Lyapunov exponents of the Kontsevich–Zorich cocycle relies on the formula (29), (b), for the second derivative. Unfortunately, the r.h.s of the formula contains two terms and, while the first is at least clearly non-negative, the sign of second appears to be oscillating in a way difficult to control. In order to overcome this difficulty, we follow an idea of [25] which consists in averaging over the orbits of the circle group $SO(2,\mathbb{R})$ in the stratum $\mathcal{M}_\kappa$.

Let $SL(2,\mathbb{R})\, q$ be an orbit of $SL(2,\mathbb{R})$ in $\mathcal{M}_\kappa$. For almost all $q \in \mathcal{M}_\kappa$, the quotient $SL(2,\mathbb{R})\, q/SO(2,\mathbb{R})$ is a copy of the Poincaré disk, in the sense that it is an immersed two-dimensional disk on which the Teichmüller metric reduces to the standard Poincaré metric (with curvature -4). Such a disk is called a *Teichmüller disk* (see [32, 2.6.5]).

The *hyperbolic Laplacian* of the Hodge norm of a cohomology class on a Teichmüller disk can be computed as follows. We write formula (29), (b), for all quadratic differentials in a $SO(2,\mathbb{R})$-orbit, we then average with respect to the Haar measure on $SO(2,\mathbb{R})$. The averaging eliminates the 'bad' second term in the r.h.s. of formula (29), (b) (the oscillation is canceled!).

LEMMA 5.2 [21, Lemma 3.2]. *The following formulas hold for the hyperbolic gradient ∇_h and the hyperbolic Laplacian Δ_h of the norm of a cohomology class on a Teichmüller disk*:

$$\begin{aligned} &\text{(a)} \quad \nabla_h \left|m_z^+\right|_0^2 = -2\big(\operatorname{Re} B_q\big(m^+\big), \operatorname{Im} B_q\big(m^+\big)\big), \\ &\text{(b)} \quad \Delta_h \left|m_z^+\right|_0^2 = 8\left|\pi_q^-\big(m^+\big)\right|_0^2. \end{aligned} \tag{36}$$

Hence, by a straightforward calculation,

$$\Delta_h \log\left|m_z^+\right|_0 = 4\frac{|\pi_z^-(m_z^+)|_0^2}{|m_z^+|_0^2} - 2\frac{|B_q(m_z^+)|^2}{|m_z^+|_0^4} \geqslant 2\frac{|\pi_z^-(m_z^+)|_0^2}{|m_z^+|_0^2}. \tag{37}$$

An analysis of the solutions of the hyperbolic Poisson equation, combined with the Oseledec's theorem on the existence of Lyapunov exponents and Birkhoff ergodic theorem, leads to the following lower bound. Let

$$\Lambda^-(q) := \min\left\{ \frac{|\pi_q^-(m^+)|_0^2}{|m^+|_0^2} \,\Big|\, m^+ \in \mathcal{M}_q^+ \setminus \{0\} \right\}. \tag{38}$$

THEOREM 5.3 [21, Theorem 3.3]. *Let μ be any G_t-ergodic $SL(2,\mathbb{R})$-invariant probability measure on $\mathcal{M}_\kappa^{(1)}$. The second Lyapunov exponent λ_2^μ of the Kontsevich–Zorich cocycle with respect to the measure μ, satisfies the following lower bound*:

$$\lambda_2^\mu \geqslant \int_{\mathcal{M}_\kappa^{(1)}} \Lambda^-(q)\, d\mu(q). \tag{39}$$

Theorem 5.3 shows that to be able to prove that $\lambda_2^\mu > 0$ it is sufficient to prove that the non-negative continuous function $\Lambda^- : \mathcal{M}_g^{(1)} \to \mathbb{R}$ is strictly positive at some $q \in \operatorname{supp}(\mu) \subset \mathcal{M}_\kappa$. Hence we are led to consider the locus $\{\Lambda^- = 0\}$ in the moduli space $\mathcal{M}_g^{(1)}$.

6. The determinant locus

Let π_q^- be as above the orthogonal projection on the subspace $\mathcal{M}_q^- \subset L_q^2(M)$ of anti-meromorphic functions. Let H_q be the non-negative definite Hermitian form on the subspace $\mathcal{M}_q^+ \subset L_q^2(M)$ defined as follows. For all $(m_1^+, m_2^+) \in \mathcal{M}_q^+ \times \mathcal{M}_q^+$,

$$\mathrm{H}_q\big(m_1^+, m_2^+\big) := \big(\pi_q^-\big(m_1^+\big), \pi_q^-\big(m_2^+\big)\big)_q. \tag{40}$$

The non-negative number $\Lambda^-(q)$ is by definition the *smallest eigenvalue* of the Hermitian form H_q. The locus $\{\Lambda^- = 0\}$ coincides therefore with the set of quadratic differentials for which the Hermitian form H_q is degenerate, that is, represented by a $g \times g$ Hermitian matrix with zero determinant.

There is a close relation between the Hermitian form H_q and the derivative of the classical *period matrix* along the Teichmüller trajectory in the moduli space determined by the quadratic differential q on M.

Let us recall the definition of the period matrix. Let M be a marked Riemann surface of genus $g \geqslant 2$ and let $\{a_1, b_1, \ldots, a_g, b_g\} \subset H_1(M,\mathbb{Z})$ be a *canonical homology basis* (see [16, III.1]), characterized by the property that, for all $i, j \in \{1, \ldots, g\}$,

$$a_i \cap a_j = b_i \cap b_j = 0 \quad \text{and} \quad a_i \cap b_j = \delta_{ij}. \tag{41}$$

In other terms, a canonical homology basis is a symplectic basis with respect to the symplectic structure on the real homology $H_1(M,\mathbb{R})$ given by the (*algebraic*) *intersection* *form* $\cap$. Let $\{\theta_1, \dots, \theta_g\}$ be the dual basis of the space of holomorphic (Abelian) differentials on M, characterized by the conditions $\theta_i(a_j) = \delta_{ij}$, for all $i, j \in \{1, \dots, g\}$. The $g \times g$ complex matrix Π given by

$$\Pi_{ij}(M) := \int_{b_j} \theta_i, \quad i, j \in \{1, \dots, g\}, \tag{42}$$

is the *period matrix* of the marked Riemann surface M. The period matrix yields a holomorphic mapping $\Pi : T_g \to \mathfrak{S}_g$ on the Teichmüller space of Riemann surfaces with values in the *Siegel space* $\mathfrak{S}_g$ of $g \times g$ complex symmetric matrices with positive definite imaginary part.

Let $q \in Q_g^{(1)}$ be a holomorphic quadratic differential on the Riemann surface M_q. Let $(M_t, q_t) := G_t(M_q, q)$, for $t \in \mathbb{R}$, be the Teichmüller orbit of (M_q, q) in the Teichmüller space $Q_g^{(1)}$. The equation

$$\det\left[\frac{d}{dt}\Pi(M_t)\bigg|_{t=0}\right] = 0 \tag{43}$$

defines a real analytic hypersurface $D_g^{(1)} \subset Q_g^{(1)}$ of real codimension 2. In other words, the hypersurface $D_g^{(1)}$ is the locus where the derivative of the period matrix in the direction of the Teichmüller flow is degenerate.

It is immediate to see that Equation (43), hence the locus $D_g^{(1)}$, is invariant under change of marking on M, that is, invariant under the action of the mapping class group Γ_g. It follows that the projection $\mathcal{D}_g^{(1)} := D_g^{(1)}/\Gamma_g$ of $D_g^{(1)}$ into the moduli space $\mathcal{M}_g^{(1)}$ is well defined. The real analytic hypersurface $\mathcal{D}_g^{(1)} \subset \mathcal{M}_g^{(1)}$ of real codimension 2 was introduced in [21, §4], and called the *determinant locus*. The following lemma holds.

LEMMA 6.1 [21, Lemma 4.1]. *The locus* $\{\Lambda^- = 0\} \subset \mathcal{M}_g^{(1)}$ *coincides with the determinant locus* $\mathcal{D}_g^{(1)}$.

PROOF. Let $\{m_1^+, \dots, m_g^+\}$ be an orthonormal basis of $\mathcal{M}_q^+ \subset L_q^2(M)$. The (symmetric) matrix $B(q)$ of the projection operator $\pi_q^- : \mathcal{M}_q^+ \to \mathcal{M}_q^-$, with respect to the bases $\{m_1^+, \dots, m_g^+\} \subset \mathcal{M}_q^+$ and $\{\overline{m_1^+}, \dots, \overline{m_g^+}\} \subset \mathcal{M}_q^-$, and the Hermitian non-negative matrix $H(q)$ of the Hermitian form H_q, with respect to the basis $\{m_1^+, \dots, m_g^+\}$, are given by the following formulas:

$$\begin{aligned} B_{ij}(q) &= B_q\big(m_i^+, m_j^+\big) = \big(m_i^+, \overline{m_j^+}\big)_q, \\ H(q) &= B(q)^* B(q) = \overline{B(q)} B(q). \end{aligned} \tag{44}$$

The quotients $\phi_i^+ := \theta_i / q^{1/2}$ are meromorphic functions on M_q with poles at Σ_q, which belong to the space $L_q^2(M)$. The system $\{\phi_1^+, \dots, \phi_g^+\}$ is a basis of the space $\mathcal{M}_q^+$.

The infinitesimal deformation of the complex structure of the Riemann surface M_q induced by the Teichmüller flow in the direction of the quadratic differential $q \in Q_g^{(1)}$ can be represented by a canonical *Beltrami differential* $\mu_q := |q|/q$, hence by Rauch's formula [22, Proposition A.3]:

$$\frac{d}{dt}\Pi_{ij}(M_t)\bigg|_{t=0} = \int_M \theta_i\theta_j\mu_q = \int_M \phi_i^+\phi_j^+\omega_q = B_q(\phi_i^+, \phi_j^+). \tag{45}$$

Since $\{\phi_1^+, \dots, \phi_g^+\}$ is a basis of $\mathcal{M}_q^+$, there exists a non-singular $g \times g$ complex matrix $C(q)$ such that

$$\phi_i^+ = \sum_{j=1}^{g} C_{ij}(q)m_j^+ \quad \text{and} \quad C(q)C(q)^* = \mathrm{Im}(\Pi). \tag{46}$$

In fact, by [16, III.2.3],

$$\begin{aligned}(\phi_i^+, \phi_j^+)_q &= \frac{i}{2}\int_M \theta_i \wedge \overline{\theta_j} \\ &= \frac{i}{2}\sum_{k=1}^{g}\left\{\int_{a_k}\theta_i\int_{b_k}\overline{\theta_j} - \int_{b_k}\theta_i\int_{a_k}\overline{\theta_j}\right\} = \mathrm{Im}(\Pi_{ij}).\end{aligned} \tag{47}$$

By (45) and (46),

$$\begin{aligned}\left|\det\left(\frac{d}{dt}\Pi_{ij}(M_t)\bigg|_{t=0}\right)\right| &= \left|\det C(q)B(q)C(q)^t\right| = \left|\det C(q)\right|^2\left|\det B(q)\right| \\ &= \det \mathrm{Im}(\Pi)\big[\det H(q)\big]^{1/2}.\end{aligned} \tag{48}$$

Since $\mathrm{Im}(\Pi)$ is positive definite, the Hermitian form H_q is degenerate, hence $\Lambda^-(q) = 0$, if and only if $q \in \mathcal{D}_g^{(1)}$. □

The geometry of the determinant locus, in particular with respect to the foliation of the moduli space $\mathcal{M}_g^{(1)}$ by orbits of the $SL(2,\mathbb{R})$-action, plays an important role in the study of Lyapunov exponents of the Kontsevich–Zorich cocycle (and of the Teichmüller flow). We have proved the following non-trivial result:

THEOREM 6.2 [21, Theorem 4.5]. *Let $\mathcal{M}_\kappa^{(1)}$ be any stratum of the moduli space of orientable holomorphic quadratic differentials. No connected component of $\mathcal{M}_\kappa^{(1)}$ is contained in the determinant locus. In fact, the following stronger result holds. Let*

$$\Lambda_1(q) \equiv 1 \geqslant \Lambda_2(q) \geqslant \cdots \geqslant \Lambda_g(q) \geqslant 0 \tag{49}$$

be the eigenvalues of the Hermitian form H_q *in decreasing order. Let* $\mathcal{C}_\kappa^{(1)}$ *denote any connected component of* $\mathcal{M}_\kappa^{(1)}$. *We have*:

$$\sup_{q\in\mathcal{C}_\kappa^{(1)}} \Lambda_i(q) = 1, \quad \textit{for all } i \in \{1,\dots,g\}. \tag{50}$$

The proof of Theorem 6.2 shows that the supremum of the (continuous) functions Λ_i is achieved at a certain kind of *boundary points* of the moduli space which can be found in the closure of any connected component of any stratum. The argument is based on asymptotic expansions for the period matrix (and its derivatives) [18, Chap. III], [29,43], [21, §4].

The simplest and most intuitive choice of the appropriate boundary points is the disjoint sums of g tori with $2g-2$ paired punctures. At these points, the period matrix and its derivative along the Teichmüller flow are diagonal with all diagonal entries different from zero. It follows that the Hermitian form H_q is non-degenerate. In fact, it is immediate to see that $\Lambda_1 = \cdots = \Lambda_g = 1$. Riemann surfaces pinched along $g-1$ (separating) cycles homologous to zero converge to boundary points of that type.

Unfortunately, quadratic differentials on such pinched surfaces cannot in general belong to a stratum with a zero of high multiplicity as the pinching parameters converge to zero. In order to overcome this difficulty and treat all strata, we have considered a different type of boundary points. Such points are given by meromorphic quadratic differentials on Riemann spheres with $2g$ paired punctures, having poles of order 2 with strictly positive real residues at all punctures, equal at paired punctures (the residue of a quadratic differential at a pole $p \in M$ is the standard residue of the holomorphic 1-form $z\,\phi(z)\,dz$ with respect to a holomorphic coordinate $z : M \to \mathbb{C}$ such that $z(p) = 0$ and $q = \phi(z)\,dz^2$).

A basic step of the proof of Theorem 6.2 consists in constructing in every connected component of every stratum $\mathcal{M}_\kappa$ of the moduli space a family of quadratic differentials on Riemann surfaces pinched along a set of g distinct closed regular trajectories spanning a *Lagrangian subspace* in homology. The limit of any such family as the pinching parameters converge to zero is a meromorphic quadratic differential on a Riemann sphere of the type just described. The period matrix and its derivative converge to a diagonal matrix only in the *projective* sense, but this is enough for the proof.

As a corollary of Theorems 5.3 and 6.2, we obtain

COROLLARY 6.3 [21, Corollary 4.5′]. *Let* μ *be the normalized absolutely continuous invariant measure on any connected component* $\mathcal{C}_\kappa^{(1)}$ *of a stratum* $\mathcal{M}_\kappa^{(1)}$ *of the moduli space of orientable holomorphic quadratic differentials of unit total area. The second Lyapunov exponents of* G_t^{KZ} *over* $\mathcal{C}_\kappa^{(1)}$ *is strictly positive*, *in fact*

$$\lambda_2^\mu \geqslant \int_{\mathcal{M}_\kappa^{(1)}} \Lambda^-(q)\,d\mu(q) > 0. \tag{51}$$

The proof of Theorem 3.1 is complete only if $g=2$. If $g \geqslant 3$, the complete proof of the theorem is based on formulas similar to (37) for the logarithm of the k-volume of k-dimensional isotropic subspaces of $H^1(M,\mathbb{R})$, for all $k \in \{1,\dots,g\}$. Unfortunately, only in the case $k=g$ these computations yield a closed formula for the Lyapunov exponents, that is, independent of the Oseledec's splitting of the real cohomology bundle $\mathcal{H}^1_\kappa(M,\mathbb{R})$. As a consequence, the complete proof of Theorem 3.1 is rather convoluted and beyond the scope of this paper. In the case $k=g$ we find a somewhat different version of a formula discovered by M. Kontsevich and A. Zorich:

THEOREM 6.4 ([25] and [21, Corollary 5.3]). *Let μ be the normalized absolutely continuous invariant measure on any connected component $\mathcal{C}^{(1)}_\kappa$ of a stratum $\mathcal{M}^{(1)}_\kappa$ of the moduli space of orientable holomorphic quadratic differentials of unit total area. The Lyapunov exponents of G^{KZ}_t over $\mathcal{C}^{(1)}_\kappa$ satisfy the following formula*:

$$\lambda^\mu_1+\cdots+\lambda^\mu_g=\int_{\mathcal{M}^{(1)}_g}\big(\Lambda_1(q)+\cdots+\Lambda_g(q)\big)\,d\mu(q). \tag{52}$$

We remark that, since $\Lambda_1(q)\equiv\lambda^\mu_1=1$, the above formula yields a closed formula for the sum $\lambda^\mu_2+\cdots+\lambda^\mu_g$, hence for the second exponent λ^μ_2 if $g=2$. We do not know of any other closed formulas for single exponents or partial sums of them if $g\geqslant 3$.

Kontsevich (and Zorich) [25] have conjectured that the sums of the Lyapunov exponents (52) are rational numbers for all connected components of all strata. These numbers are conjecturally related to the Siegel–Veech constants which arise in counting problems for embedded flat cylinders or saddle-connections on translation surfaces [42,13]. Siegel–Veech constants can in turn be computed (exactly!) by formulas expressing them in terms of the volumes of connected components of strata [14,15] (see the article by A. Eskin [2] in this handbook on counting problems, Siegel–Veech constants and volumes of strata).

7. An example

The problem of describing the intersections of $SL(2,\mathbb{R})$-orbits of quadratic differentials with the determinant locus $\mathcal{D}^{(1)}_g\subset\mathcal{M}^{(1)}_g$ is in general open. Since $\mathcal{D}^{(1)}_g$ is by its very definition invariant under the action of the circle subgroup $SO(2,\mathbb{R})$, this problem can be reduced to the one of describing the intersection of the projection $\mathcal{D}^{(1)}_g/SO(2,\mathbb{R})$ of the determinant locus with Teichmüller disks inside the quotient space $\mathcal{M}^{(1)}_g/SO(2,\mathbb{R})$.

The determinant locus has real codimension 2 while Teichmüller disks have dimension 2, hence it is natural to expect that the intersection with a *generic* disk be either empty or a discrete (possibly countable) set. In many cases, it is immediate to see that the intersection is non-empty. Examples of Teichmüller disks with non-empty intersection are provided by quadratic differentials with symmetries.

W. Veech asked whether there exists a Teichmüller disk (in the moduli space of *orientable* quadratic differentials) entirely contained in the projection of the determinant locus. We will show below that the answer to this question is affirmative by exhibiting an example in genus $g=3$. It should be remarked that we can prove that the answer to Veech's question is negative in genus $g=2$.

The idea behind our example is to consider (orientable) holomorphic quadratic differentials with appropriate symmetries which are stable under the $SL(2,\mathbb{R})$-action. We are able to answer a refined version of Veech's question which has immediate consequences for the Lyapunov exponents of the Kontsevich–Zorich cocycle. We introduce a natural filtration

$$\mathcal{R}_g^{(1)}(1) \subset \mathcal{R}_g^{(1)}(2) \subset \cdots \subset \mathcal{R}_g^{(1)}(g-1) = \mathcal{D}_g^{(1)} \tag{53}$$

of the determinant locus $\mathcal{D}_g^{(1)}$ by the sets

$$\mathcal{R}_g^{(1)}(k) := \big\{ q \in \mathcal{M}_g^{(1)} \mid \Lambda_{k+1}(q) = \cdots = \Lambda_g(q) = 0 \big\}. \tag{54}$$

It is immediate to see that $\mathcal{R}_g^{(1)}(k)$ is a real analytic subvariety of the moduli space (described by the vanishing of all minors of order $k+1$ of the derivative of the period matrix along the Teichmüller flow), invariant under the action of the circle group, for all $k \in \{1, \ldots, g-1\}$.

We will describe below a closed $SL(2,\mathbb{R})$-orbit contained not only in the determinant locus $\mathcal{D}_3^{(1)}$ but in the smaller locus $\mathcal{R}_3^{(1)}(1)$. We do not know whether there are similar examples in any genus $g \geqslant 3$.

The relevance of the locus $\mathcal{R}_g^{(1)}(1)$ is given by the following vanishing result for the Lyapunov exponents of the Kontsevich–Zorich cocycle:

COROLLARY 7.1. *Let μ be an $SL(2,\mathbb{R})$-invariant ergodic probability measure on the moduli space $\mathcal{M}_g^{(1)}$. If* $\operatorname{supp}(\mu) \subset \mathcal{R}_g^{(1)}(1)$, *then*

$$\lambda_2^{\mu} = \cdots = \lambda_g^{\mu} = 0. \tag{55}$$

PROOF. It can be proved that the Kontsevich–Zorich formula (52) holds for any $SL(2,\mathbb{R})$-invariant ergodic probability measure on $\mathcal{M}_g^{(1)}$. Hence the result follows. □

We are unable to prove by our methods stronger vanishing results, based on conditions of type $\operatorname{supp}(\mu) \subset \mathcal{R}_g(k)$ for $k > 1$.

Let $q \in Q_g^{(1)}$ be a holomorphic (orientable) quadratic differential with a non-trivial group $\operatorname{Aut}(q)$ of symmetries. The group $\operatorname{Aut}(q) \subset \operatorname{Aut}(M_q)$ is defined as the subgroup formed by all automorphisms $a \in \operatorname{Aut}(M_q)$ such that $a^*(q) = q$. There is a natural unitary action (by pull-back) of $\operatorname{Aut}(q)$ on the finite-dimensional Euclidean space $\mathcal{M}_q^+ \subset L_q^2(M)$ of meromorphic functions.

For any $a \in \operatorname{Aut}(q)$, let $\{m_1^+(a), \ldots, m_g^+(a)\}$ be an orthonormal basis of eigenvectors and let $\{u_1(a), \ldots, u_g(a)\}$ the corresponding eigenvalues for the unitary operator induced by a on $\mathcal{M}_q^+$. Let $B^a(q)$ be the matrix of the projection operator $\pi_q^- : \mathcal{M}_q^+ \to \mathcal{M}_q^-$, with respect to the bases $\{m_1^+(a), \ldots, m_g^+(a)\} \subset \mathcal{M}_q^+$ and $\{\overline{m_1^+}(a), \ldots, \overline{m_g^+}(a)\} \subset \mathcal{M}_q^-$, that is

$$B_{ij}^a(q) = B_q\big(m_i^+(a), m_j^+(a)\big) = \int_M m_i^+(a) m_j^+(a) \omega_q. \tag{56}$$

For any $I, J \subset \{1, \ldots, g\}$ with $\#(I) = \#(J)$, let $B_{I,J}^a(q)$ be the minor of the matrix $B^a(q)$ with entries $B_{ij}^a(q)$ for $i \in I$ and $j \in J$.

LEMMA 7.2. *Let $q \in Q_g^{(1)}$ be a holomorphic quadratic differential with a non-trivial group* $\mathrm{Aut}(q)$ *of symmetries. For any $a \in \mathrm{Aut}(q)$,*

$$\prod_{i \in I} \prod_{j \in J} u_i(a) u_j(a) \neq 1 \quad \Longrightarrow \quad \det B^a_{I,J}(q) = 0. \tag{57}$$

PROOF. Since $a \in \mathrm{Aut}(q)$, by (56) and by change of variables, we have

$$B^a_{ij}(q) = \int_M a^* m_i^+(a) a^* m_j^+(a) \omega_q = u_i(a) u_j(a) B^a_{ij}(q). \tag{58}$$

The result follows. □

Let Q_0 be the stratum of *meromorphic* quadratic differentials with 4 simple poles on the (punctured) Riemann sphere $\mathbb{P}^1(\mathbb{C})$. The corresponding moduli space $\mathcal{M}_0^{(1)}$ of meromorphic quadratic differentials with unit total area consists of a single $SL(2, \mathbb{R})$-orbit.

Let $\kappa = (1, 1, 1, 1)$ and let $\mathcal{M}_\kappa$ the stratum of holomorphic differentials (on Riemann surfaces of genus $g = 3$) with 4 simple zeroes. Let $V \subset \mathcal{M}_\kappa$ be subvariety of all orientable quadratic differentials obtained as the pull-back of a meromorphic quadratic differential $q_0 \in Q_0$ by a 4-sheeted branched covering, branched over the 4 poles of q_0 (with branching order equal to 4 at each pole).

The subvariety $V^{(1)} = V \cap \mathcal{M}_\kappa^{(1)}$ consists of a single closed $SL(2, \mathbb{R})$-orbit. In fact, it can be described as the (closed) $SL(2, \mathbb{R})$-orbit of the (*non-primitive*) *Veech surface* obtained as a 2-sheeted branched cover of the torus $\mathbb{C}/(\mathbb{Z} \oplus \imath\mathbb{Z})$, branched over the 4 half-integer points $(\mathbb{Z}/2 \oplus \imath\mathbb{Z}/2)/(\mathbb{Z} \oplus \imath\mathbb{Z})$ (see the article by P. Hubert and T. Schmidt [4] in this handbook on the theory of Veech surfaces).

THEOREM 7.3. *The closed $SL(2, \mathbb{R})$-orbit $V^{(1)} \subset \mathcal{M}_\kappa^{(1)}$ is entirely contained in the locus* $\mathcal{R}_3(1)$.

PROOF. Let $q \in V^{(1)}$. By definition there exists a 4-sheeted branched covering $z : M_q \to \mathbb{P}^1(\mathbb{C})$, branched over 4 (distinct) points $x_1, \ldots, x_4 \in \mathbb{P}^1(\mathbb{C})$ and a meromorphic quadratic differential q_0 on $\mathbb{P}^1(\mathbb{C})$, with 4 simple poles at the points $x_1, \ldots, x_4$ such that $q = z^*(q_0)$. The Riemann surface M_q is a genus 3 surface determined by the algebraic equation:

$$w^4 = (z - x_1)(z - x_2)(z - x_3)(z - x_4). \tag{59}$$

The group $\mathrm{Aut}(M_q)$ of all automorphisms of the Riemann surface M_q is cyclic of order 4, generated by the automorphism $a : M_q \to M_q$ given by

$$a(z, w) = (z, \imath w). \tag{60}$$

The divisors of the meromorphic functions z, w and of the meromorphic differential dz are of the following form:

$$(z) = \frac{P_1 P_2 P_3 P_4}{Q_1 Q_2 Q_3 Q_4}, \qquad (w) = \frac{X_1 X_2 X_3 X_4}{Q_1 Q_2 Q_3 Q_4},$$
$$(dz) = \frac{X_1^3 X_2^3 X_3^3 X_4^3}{Q_1^2 Q_2^2 Q_3^2 Q_4^2}, \tag{61}$$

where $z^{-1}\{0\} = \{P_1, \ldots, P_4\}$, $z^{-1}\{\infty\} = \{Q_1, \ldots, Q_4\}$ and $X_1, \ldots, X_4$ are the branching points of the covering $z : M_q \to \mathbb{P}^1(\mathbb{C})$. It follows that the differentials

$$\theta_1 := \frac{dz}{w^2}, \qquad \theta_2 := \frac{dz}{w^3}, \qquad \theta_3 := \frac{zdz}{w^3}, \tag{62}$$

form a basis of the space of holomorphic differentials on M_q which diagonalizes the action of the group $\mathrm{Aut}(M_q)$ on the vector space of holomorphic differentials on M_q. In fact, by (60) and (62), the action of the automorphism $a \in \mathrm{Aut}(M_q)$ on the basis (62) is diagonal with eigenvalues -1 (with multiplicity 1) and $\imath = \sqrt{-1}$ (with multiplicity 2):

$$a^*(\theta_1) = -\theta_1, \qquad a^*(\theta_2) = \imath\theta_2, \qquad a^*(\theta_3) = \imath\theta_3. \tag{63}$$

The orientable quadratic differentials $q \in V^{(1)}$ is therefore equal to θ_1^2 (up to multiplication by a non-zero complex number) and the spectrum of the action of $a \in \mathrm{Aut}(q)$ on the space $\mathcal{M}_q^+ \subset L_q^2(M)$ of meromorphic functions consists of the eigenvalues

$$u_1(a) = 1, \qquad u_2(a) = -\imath, \qquad u_3(a) = -\imath. \tag{64}$$

It follows that $q \in \mathcal{R}_3^{(1)}(1)$. In fact, by Lemma 7.2 all entries $B_{ij}^a(q) = 0$ for all $(i, j) \neq (1, 1)$, hence the matrix $B^a(q)$ and, consequently, the Hermitian form H_q have rank 1. The argument is concluded. □

By Corollary 7.1, we have

COROLLARY 7.4. *The normalized $SL(2, \mathbb{R})$-invariant measure μ supported on the closed $SL(2, \mathbb{R})$-orbit $V^{(1)}$ is an $SL(2, \mathbb{R})$-invariant ergodic probability measure on $\mathcal{M}_3^{(1)}$ such that*

$$\lambda_2^\mu = \lambda_3^\mu = 0. \tag{65}$$

8. Invariant sub-bundles

By Oseledec's theorem [34], [1, §5], for almost all holomorphic quadratic differentials $q \in \mathcal{M}_\kappa^{(1)}$, the fiber $H^1(M_q, \mathbb{R})$ of the cohomology bundle $\mathcal{H}_\kappa^1(M, \mathbb{R})$ has a direct splitting

$$H^1(M_q, \mathbb{R}) = E_q^+ \oplus E_q^- \oplus E_q^0, \tag{66}$$

where E_q^+, E_q^- and E_q^0 are the subspaces of cohomology classes with, respectively, strictly positive, strictly negative and zero Lyapunov exponent. Since the cohomology bundle has a symplectic structure E_q^+ and E_q^- are isotropic subspaces of the same dimension. In fact, according to Theorem 3.1, $E_q^0 = \{0\}$ and E_q^+, E_q^- are Lagrangian. We will not rely below on this result, hence the results of this section will be independent of the non-uniform hyperbolicity of the Kontsevich–Zorich cocycle.

The homology cycles in the Poincaré dual of E_q^+, E_q^- are called (following I. Nikolaev and E. Zhuzhoma [33, §7.9.3]) the *Zorich cycles* for the horizontal, respectively, vertical, measured foliation of the quadratic differential q. Zorich cycles for an orientable measured foliations $\mathcal{F}$ are a generalization of the *Schwartzman's asymptotic cycle* which coincides with the Poincaré dual of the cohomology class of the closed 1-form $\eta_{\mathcal{F}}$ such that $\mathcal{F} := \{\eta_{\mathcal{F}} = 0\}$.

In fact, by unique ergodicity, the Schwartzman's cycle yields the direction of the leading term in the asymptotic behavior in homology of a typical leaf of a generic orientable measured foliation on a surface of genus $g \geqslant 1$, while Zorich cycles yield the direction of the first g terms (under the hypothesis that the cocycle is non-uniformly hyperbolic) as the length of the leaf gets large. The remainder in this asymptotics, that is, the distance in homology of the typical leaf from the space of all Zorich's cycles, stays uniformly bounded (see [44,47] or [48, Appendix D]).

We will outline below the proof of a *representation theorem* which states that all Zorich cycles (or rather the corresponding dual cohomology classes) can be represented in terms of *currents* of order 1 satisfying certain properties with respect to the measured foliation $\mathcal{F}$.

A *basic current* (of dimension 1) for a measured foliation $\mathcal{F}$ (with singularities at a finite set $\Sigma_{\mathcal{F}} \subset M$) is a 1-dimensional current C (in the sense of G. de Rham [12], that is, a continuous functional on the vector space of smooth 1-forms with compact support) on $M \setminus \Sigma_{\mathcal{F}}$ which satisfies the vanishing conditions

$$\imath_X C = \mathcal{L}_X C = 0, \tag{67}$$

for all smooth vector fields X with compact support in $M \setminus \Sigma_{\mathcal{F}}$ tangent to the leaves of the foliation $\mathcal{F}$. (The operation of contraction $\imath_X$ and Lie derivative $\mathcal{L}_X$ are extended to currents in the standard distributional sense [36, Chapter IX, §3].)

Basic currents are a distributional generalization of basic forms, a well-known notion in the geometric theory of foliations. Since M has dimension 2, a current of dimension 1 satisfying (67) is closed, hence it represents, by the generalized de Rham theorem (see [12, Theorem 12], or [36, Chapter IX, §3, Theorem I]) a cohomology class in $H^1(M \setminus \Sigma_{\mathcal{F}}, \mathbb{R})$.

Let $q \in Q_\kappa^{(1)}$ be an orientable quadratic differential. Let $\mathcal{B}_{\pm q}(M)$ be, respectively, the space of basic currents for the measured foliations $\mathcal{F}_{\pm q}$ (we recall that $\mathcal{F}_q$ is the horizontal foliation and $\mathcal{F}_{-q}$ the vertical foliation). Let $\{S, T\}$ be the orthonormal frame of the tangent bundle described in Section 4 and $\{\eta_T, \eta_S\}$ be the dual frame of the cotangent bundle, which is defined by

$$\eta_T := -\imath_T \omega_q = \operatorname{Re}\left(q^{1/2}\right), \qquad \eta_S := \imath_S \omega_q = \operatorname{Im}\left(q^{1/2}\right). \tag{68}$$

For the statement of the representation theorem, the notion of order of a current, taken with respect to a scale of *Sobolev spaces*, is crucial. Let Σ_q be the set of the zeroes of q.

A current on $M \setminus \Sigma_q$ has *order* $r \in \mathbb{N}$ if it extends to a continuous functional on the Sobolev space $H_q^r(M)$ of all L_q^2 forms with L_q^2 derivatives (with respect to the vector fields S, T) up to order r. We remark that under this definition the order of a current is not uniquely defined. In fact, a current of order r has also order r' for all $r' \geqslant r$.

Let $\mathcal{B}_{\pm q}^r(M) \subset \mathcal{B}_{\pm q}(M)$ be the subsets of basic currents of order r. There is a close relation between basic currents (of order r) and *invariant distributions* (of order r). An S-invariant, respectively T-invariant, distribution (of order r) is a distributional solution $\mathcal{D}$ (of order r) of the equation

$$S\mathcal{D} = 0, \quad \text{respectively} \quad T\mathcal{D} = 0. \tag{69}$$

We have proved in [19] that invariant distributions of finite order for the vector field S, respectively T, yield a complete system of obstructions to the existence of smooth solutions u to the *cohomological equation*

$$Su = f, \quad \text{respectively} \quad Tu = f, \tag{70}$$

in the following sense. There exists $\gamma > 1$ such that for almost all quadratic differentials $q \in \mathcal{M}_\kappa^{(1)}$ and for any function $f \in H_q^r(M)$ which belongs to the kernel of all S-invariant, respectively T-invariant, distributions of order r, the cohomological equation $Su = f$, respectively $Tu = f$, has a solution $u \in H_q^s(M)$ for all $s < r - \gamma$ (finite loss of derivatives).

The following result describes the relation between basic currents and S-invariant, T-invariant distributions:

LEMMA 8.1 [21, Lemma 6.6]. *A current $C \in \mathcal{B}_q^r(M)$, respectively $C \in \mathcal{B}_{-q}^r(M)$, if and only if $C = \mathcal{D} \cdot \eta_S$, respectively $C = \mathcal{D} \cdot \eta_T$, where $\mathcal{D}$ is an S-invariant, respectively a T-invariant, distribution of order $r \in \mathbb{N}$.*

The main result of this section states that, for almost all $q \in \mathcal{M}_\kappa^{(1)}$, the Poincaré dual of every Zorich cycle is the cohomology class of a basic current of *order* 1. It can be proved that the natural cohomology maps

$$\mathcal{B}_{\pm q}^1(M) \to H^1(M \setminus \Sigma_q, \mathbb{R})$$

are injective and their images $H_{\pm q}^{1,1}(M, \mathbb{R})$ satisfy the inclusions

$$H_{\pm q}^{1,1}(M, \mathbb{R}) \subset H^1(M, \mathbb{R}) \subset H^1(M \setminus \Sigma_q, \mathbb{R}).$$

We can finally state the representation theorem for Zorich cycles:

THEOREM 8.2 [21, Theorem 8.3]. *For almost all $q \in \mathcal{M}_\kappa^{(1)}$, we have*

$$E_q^+ = H_q^{1,1}(M, \mathbb{R}), \qquad E_q^- = H_{-q}^{1,1}(M, \mathbb{R}). \tag{71}$$

(*The Poincaré duals of Zorich cycles for a generic orientable measured foliation $\mathcal{F}$ are represented by basic currents for $\mathcal{F}$ of Sobolev order* 1).

We will outline below the proof of the main part of Theorem 8.2, that is, the inclusions $E_q^{\pm} \subset H^{1,1}_{\pm q}(M,\mathbb{R})$. The argument is based on the following *Cheeger-type estimate* for the constant in the *Poincaré inequality* (equivalently, for the first non-trivial eigenvalue of the Laplace–Beltrami operator of the flat metric R_q induced by the quadratic differential q on M).

The *Dirichlet form* of the metric R_q, introduced in [19, (2.6)], is defined as the Hermitian form on the Hilbert space $L^2_q(M)$ given by

$$\mathcal{Q}(u,v) := (Su, Sv)_q + (Tu, Tv)_q. \tag{72}$$

The domain of the Dirichlet form $\mathcal{Q}$ is the Sobolev space $H^1_q(M) \equiv H^1(M)$ of functions $u \in L^2_q(M)$ such that $Su, Tu \in L^2_q(M)$.

LEMMA 8.3 [21, Lemma 6.9]. *There is a constant $K_{g,\sigma} > 0$ such that the following holds. Let $q \in Q_g^{(1)}$ be a holomorphic (orientable) quadratic differential, let Σ_q be the set of its zeroes and let $\sigma := \#(\Sigma_q)$. Denote by $\|q\|$ the R_q-length of the shortest geodesic segment with endpoints in Σ_q. Then, for any $v \in H^1_q(M)$, the following inequality holds*:

$$\left| v - \int_M v\,\omega_q \right|_0 \leqslant \frac{K_{g,\sigma}}{\|q\|}\mathcal{Q}(v,v)^{1/2}. \tag{73}$$

The proof of Lemma 8.3 follows closely Cheeger's proof (see [11] or [10, Chapter III, D.4]) for the case of a smooth Riemann metric. The degenerate (or singular) character of the metric R_q at the finite set Σ_q does not affect Cheeger's argument. Moreover, we are able to give an explicit estimate of Cheeger's *isoperimetric constant* in terms of the quantity $\|q\|$.

PARTIAL PROOF OF THEOREM 8.2. We prove the inclusion $E_q^+ \subset H_q^{1,1}(M,\mathbb{R})$. The inclusion $E_q^- \subset H^{1,1}_{-q}(M,\mathbb{R})$ can be proved by a similar argument.

Let $q \in Q_\kappa^{(1)}$ be any Oseledec regular point of the Kontsevich–Zorich cocycle and let $c_t := G_t^{KZ}(c)$, $t \in \mathbb{R}$, be the orbit under the cocycle of a cohomology class $c \in H^1(M,\mathbb{R})$.

Let $\mathcal{M}_t^+$ be the space of meromorphic functions, with respect to the complex structure induced by the quadratic differential $q_t := G_t(q) \in Q_\kappa^{(1)}$, which belong the space $L^2_{q_t}(M)$.

According to the representation formula (20), for each $t \in \mathbb{R}$ there exists a function $m_t^+ \in \mathcal{M}_t^+$ such that

$$c_t = \mathrm{Re}\big[m_t^+ q_t^{1/2}\big]. \tag{74}$$

Since the L^2_q norm is invariant under the action Teichmüller flow on the Teichmüller space, the space $\mathcal{M}_t^+ \subset L^2_q(M)$ for all $t \in \mathbb{R}$, and it can be proved that the map $t \to m_t^+ \in L^2_q(M)$ is smooth.

There exist a measurable function $K_1 > 0$ on $\mathcal{M}_\kappa^{(1)}$ and an exponent $0 < \lambda < 1$ such that, if $c \in E_q^+$, the Hodge norm

$$\|c\|_{q_t} = |m_t^+|_0 \leqslant K_1(q)|m_0^+|_0 \exp(-\lambda|t|), \quad t \leqslant 0. \tag{75}$$

Since $c_t \equiv c \in H^1(M, \mathbb{R})$ (by the definition (8) of G_t^{KZ}), there exists a unique zero average function $U_t \in L_q^2(M)$ such that

$$dU_t = \mathrm{Re}\big[m_t^+ q_t^{1/2}\big] - \mathrm{Re}\big[m_0^+ q^{1/2}\big]. \tag{76}$$

It follows that, by the variational formula (23), the function U_t satisfies the following Cauchy problem in $L_q^2(M)$:

$$\begin{cases} \frac{d}{dt} U_t = 2\,\mathrm{Re}(v_t), \\ U_0 = 0 \end{cases} \tag{77}$$

(if the function $v_t \in H^1(M)$ in (23) is chosen with zero average).

For any (orientable) quadratic differential $q \in Q_\kappa^{(1)}$, by the commutativity property (17) of the vector fields S, T, the Dirichlet form can be written as

$$\mathcal{Q}(v, v) = |\partial_q^\pm v|_0^2, \quad \text{for all } v \in H_q^1(M)$$

(where $\partial_q^\pm$ are the Cauchy–Riemann operators introduced in Section 4).

Since the function $v_t \in H_q^1(M) \equiv H^1(M)$ in (23) is chosen with zero average, by the Poincaré inequality Lemma 8.3 and by the orthogonality of the decomposition (19), (22) for $q = q_t$, we have

$$|v_t|_0 \leqslant K_{g,\sigma}\|q_t\|^{-1}|\partial_t^+ v_t|_0 \leqslant K_{g,\sigma}\|q_t\|^{-1}|m_t^+|_0, \tag{78}$$

where $\|q_t\|$ denotes as above the length of the shortest geodesic segment with endpoints in the set of zeroes of the quadratic differential q_t with respect to the induced metric.

It follows, by formulas (75), (77) and (78), that there exists a measurable function $K_2 > 0$ on $\mathcal{M}_\kappa^{(1)}$ such that, if $c \in E_q^+$,

$$\left|\frac{d}{dt} U_t\right|_0 \leqslant 2|v_t|_0 \leqslant K_2(q)|m_0^+|_0\|q_t\|^{-1}\exp(-\lambda|t|), \quad t \leqslant 0. \tag{79}$$

Since $U_0 = 0$, by Minkowski's integral inequality, formula (79) implies the following estimate:

$$|U_t|_0 \leqslant K_2(q)|m_0^+|_0 \int_0^{|t|} e^{-\lambda|s|}\|q_s\|^{-1}\,ds, \quad t \leqslant 0. \tag{80}$$

By the *logarithmic law* for the Teichmüller geodesic flow on the moduli space, proved by H. Masur in [31], the following estimate holds for almost all quadratic differentials $q \in \mathcal{M}_\kappa^{(1)}$ (see [31, Proposition 1.2]):

$$\limsup_{t\to\pm\infty} \frac{-\log\|q_t\|}{\log|t|} \leqslant \frac{1}{2}. \tag{81}$$

It follows that, for almost all $q \in \mathcal{M}_\kappa^{(1)}$, the integral in formula (80) converges as $t \to -\infty$, hence the family of functions $\{U_t \mid t \leqslant 0\}$ is uniformly bounded in the Hilbert space $L^2_q(M)$.

Let $U \in L^2_q(M)$ be any weak limit of U_t as $t \to -\infty$ (which exists since bounded subsets of separable Hilbert spaces are sequentially weakly compact). By contraction of the identity (76) with the vector field S and by taking the limit as $t \to -\infty$, we have

$$\begin{aligned} SU_t &= -\operatorname{Re}\left(m_0^+\right) + e^t \operatorname{Re}\left(m_t^+\right), \quad t \leqslant 0; \\ SU &= -\operatorname{Re}\left(m_0^+\right). \end{aligned} \tag{82}$$

The identities in (82) hold in the sense of distributions. It follows by a straightforward computation that there exists a distribution $\mathcal{D}$ such that

$$dU = -\operatorname{Re}\left[m_0^+ q^{1/2}\right] + \mathcal{D}\cdot\eta_S. \tag{83}$$

In fact, $dU = SU\,\eta_T + TU\eta_S$, hence by (68) the identity (83) holds with $\mathcal{D} := TU - \operatorname{Im}(m_0^+)$. Since $U \in L^2_q(M)$ the distribution $\mathcal{D}$ has Sobolev order 1 and the current $C := \mathcal{D}\cdot\eta_S$ is a basic current of order 1 for the horizontal foliation $\mathcal{F}_q$ representing the cohomology class $c \in E_q^+$.

In fact, it is immediate by (83) that C is closed and represents $c = \operatorname{Re}[m_0^+ q^{1/2}]$. Finally, C is basic for $\mathcal{F}_q = \{\eta_S = 0\}$ since, if X is any vector fields tangent to $\mathcal{F}_q$ on $M \setminus \Sigma_q$, we have in the distributional sense:

$$\begin{aligned} \imath_X C &= \mathcal{D}\cdot\imath_X\eta_S = \mathcal{D}\cdot 0 = 0, \\ \mathcal{L}_X C &= \imath_X dC + d\imath_X C = 0. \end{aligned} \tag{84}$$

Otherwise, since C is closed and by a standard formula $dC = S\mathcal{D}\cdot\omega_q$, the distribution $\mathcal{D}$ is S-invariant, hence C is basic for $\mathcal{F}_q$ by Lemma 8.1. □

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