Inertia-gravity waves in a stratified fluid
Geophysical applications

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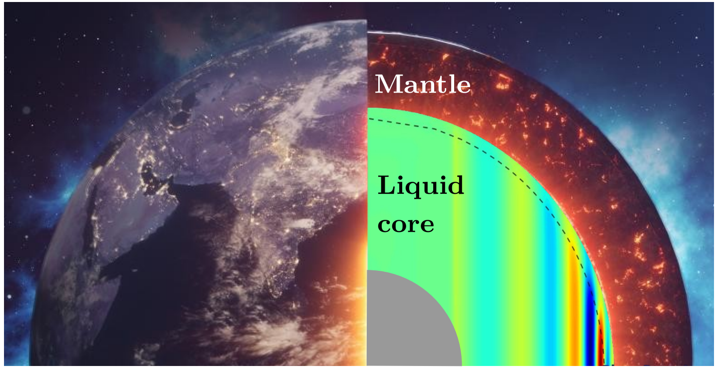
MathsInFluids (ENS Lyon), 2 February 2024

AN INTERDISCIPLINARY COLLABORATION

Myself = Geophysicist



+ **Astrophysical** flows
+ Lab. **experiments**

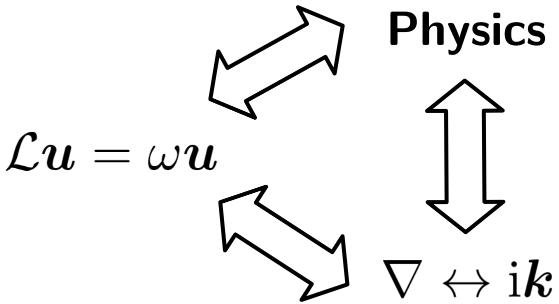
Waves in fluids

Y. Colin de Verdière

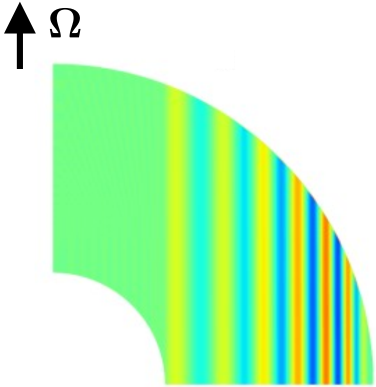
Series of papers

Vidal & Colin de Verdière, 2024, *Inertia-gravity waves in geophysical vortices*, PRSA (accepted)
Colin de Verdière & **Vidal**, 2024, *The spectrum of the Poincaré operator in an ellipsoid*, **arXiv:2305.01369**
Colin de Verdière & **Vidal**, 2024, *On gravito-inertial surface waves*, Preprint

1. **Context: from oceans to stars**
2. **Idealised model of Inertia-Gravity Modes (IGM)**
   - Quadratic eigenvalue problem
   - A few examples
3. **Ellipsoidal model for geophysical vortices**
   - Motivations
   - Pure point spectrum
4. **From free waves to turbulence?**

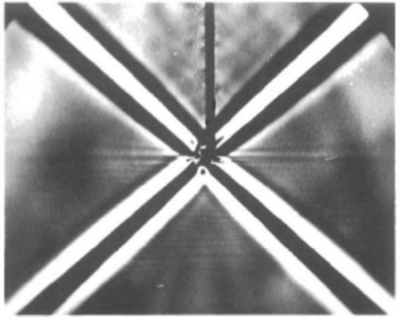


CONTEXT: FROM OCEANS TO STARS



Inertia – Gravity

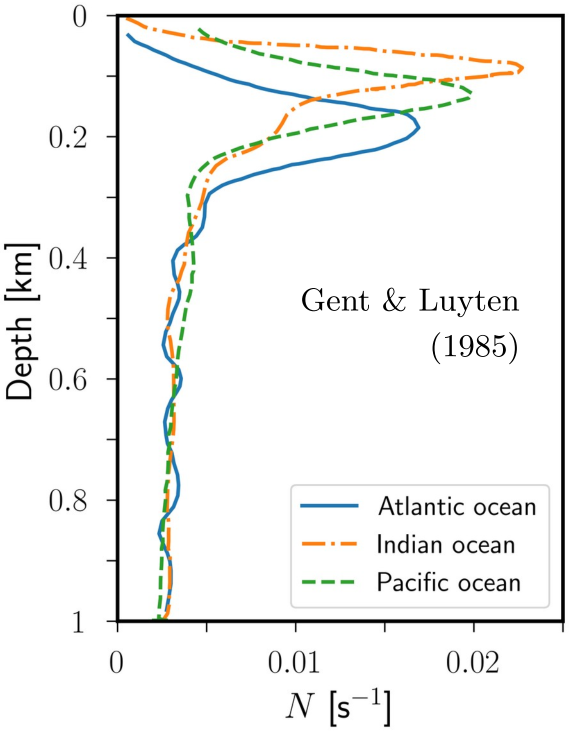
Waves (IGWs)



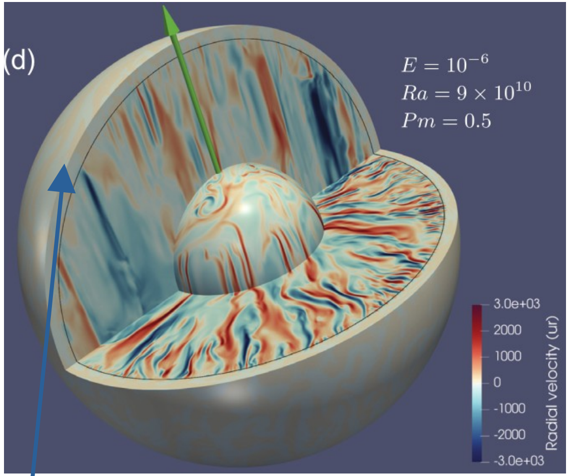
Rotation		Stratification
$2\boldsymbol{\Omega} \times \boldsymbol{v}$	**Effect** / **Restoring force**	$\rho \boldsymbol{g}$
$Ro = \dfrac{U}{\Omega L} \ll 1$	**Existence**	$N^2 = \dfrac{\boldsymbol{g}}{\rho} \cdot \left(\nabla \rho - \dfrac{\rho \boldsymbol{g}}{c_s^2} \right) \geq 0$

2 key ingredients!

Oceans



Earth/Planets

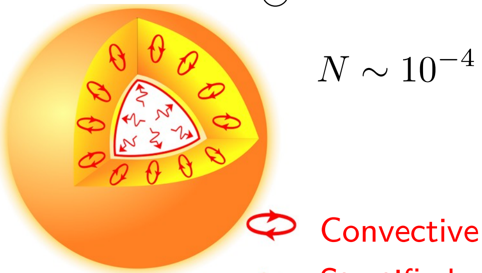


Gastine et al. (2020)

Outermost stratified layer?

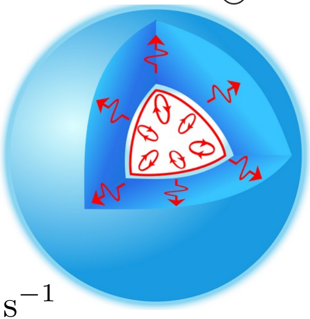
Astrophysics

$0.5 - 1.5 M_\odot$



$N \sim 10^{-4} - 10^{-3}\,\mathrm{s}^{-1}$

$> 1.5 M_\odot$

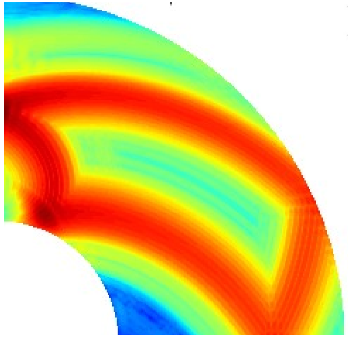


$N \sim 10^{-4} - 10^{-1}\,\mathrm{s}^{-1}$

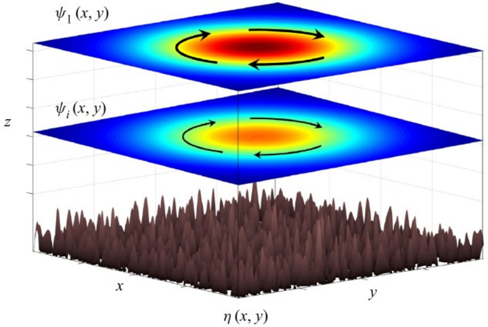
CONTEXT: FROM OCEANS TO STARS

IGWs are (likely) **ubiquitous** in natural systems : many **applications**!

Dissipation



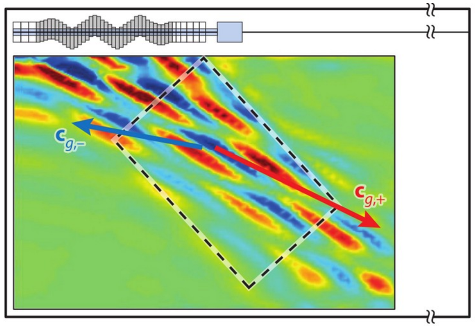
Dinstran & Rieutord (1999)



Radko (2023)

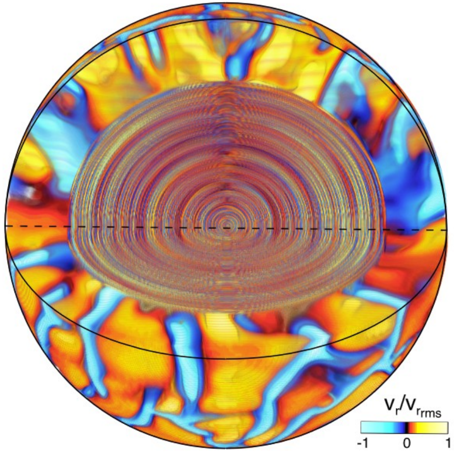
Instabilities

$$\omega_0 = \omega_1 + \omega_2$$



Bourget el al. (2013)

Transport/Mixing



Alvan (2014)

OUTLINE

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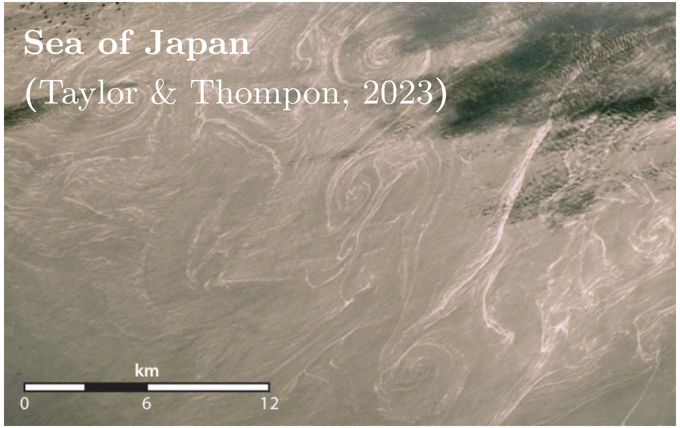
+ 

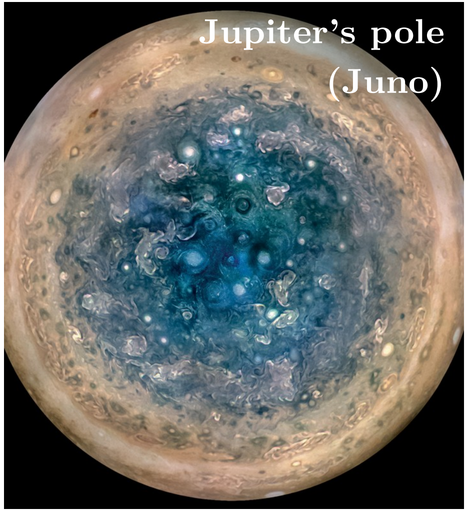
ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES

"What is [a vortex] ? It is like pornography. It is hard to defined but if you see it, you recongnise it immediately."

G. K. Vallis

$<$ m **$\sim$ km**





At all length scales!

$$\text{Vortex} \Rightarrow \text{Vorticity } \boldsymbol{\omega} = \nabla \times \boldsymbol{v} \neq \mathbf{0}$$

$$\partial_t \boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + (\rho/\rho_*)\boldsymbol{g} + \nu \nabla^2 \boldsymbol{v} + \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

$$\partial_t \rho + \boldsymbol{v} \cdot \nabla \rho_0 = \kappa \nabla^2 \rho$$

Boussinesq equations

+ Boundary Conditions (**BC**)

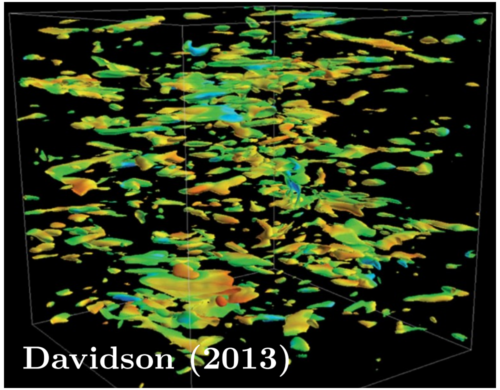
- Global **rotation**
- Density **stratification**

R >> S

Elongated vortices

or

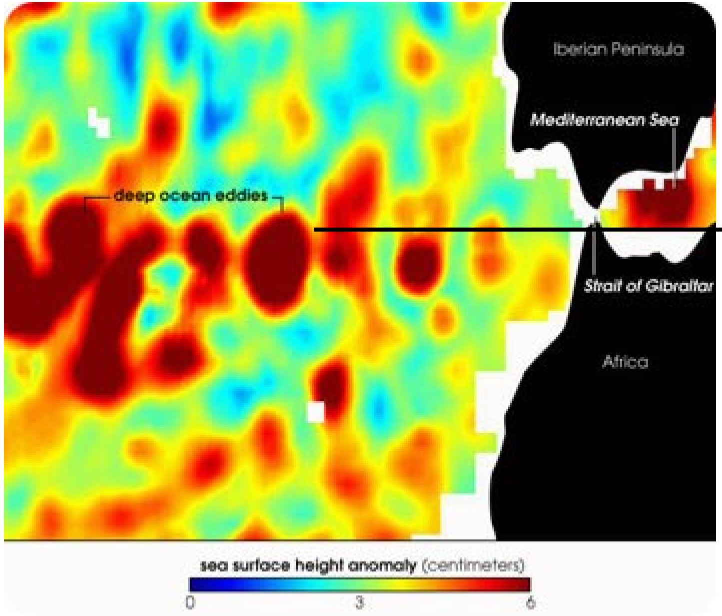
R << S



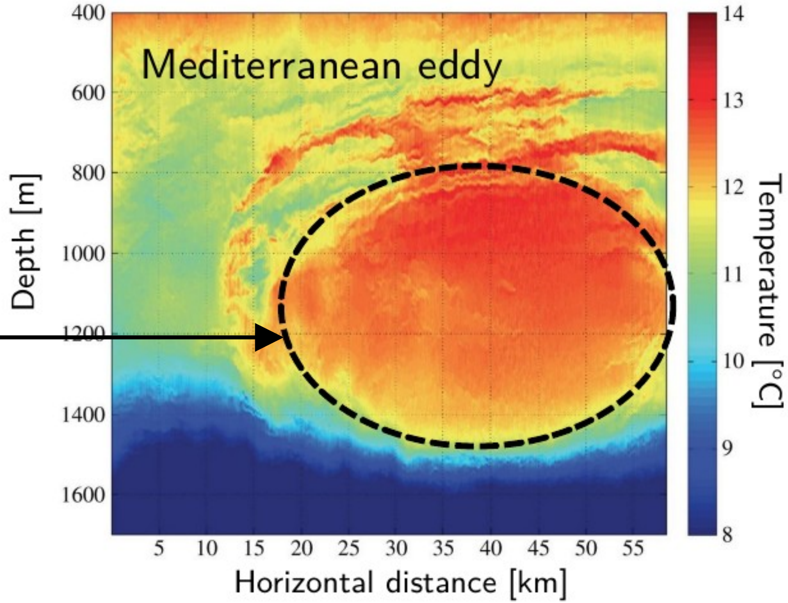
Pancake-like vortices

ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES

''Meddies''



Yan et al. (2006)

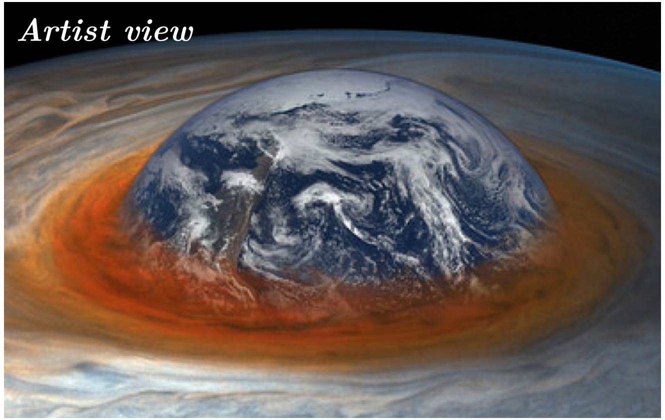


- **Anticylonic** vortices ($\rightarrow$ 50 cm.s^{-1})
- 10^9-10^{11} tons of **salt** per eddy
- Lifetime > 1 year

ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES

Jovian vortices (e.g. GRS)

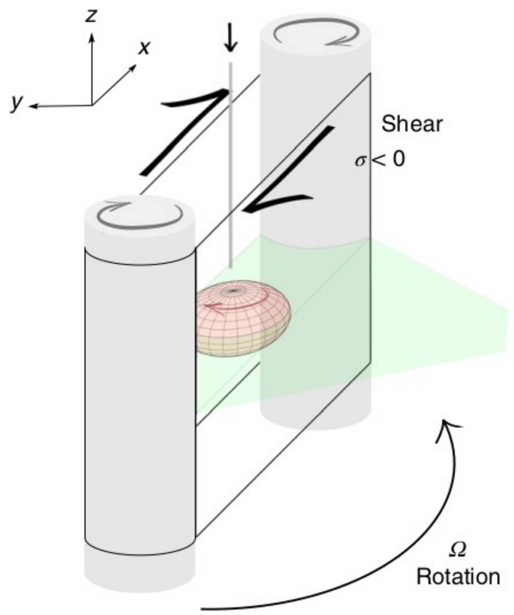
Juno - NASA - JPL-Caltech



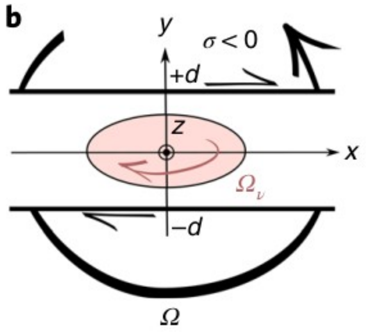
- Horizontal: 10^3 - 10^4 km
- Vertical: a **few hundred** of km
- Lifetime can exceed $O(10^2)$ years

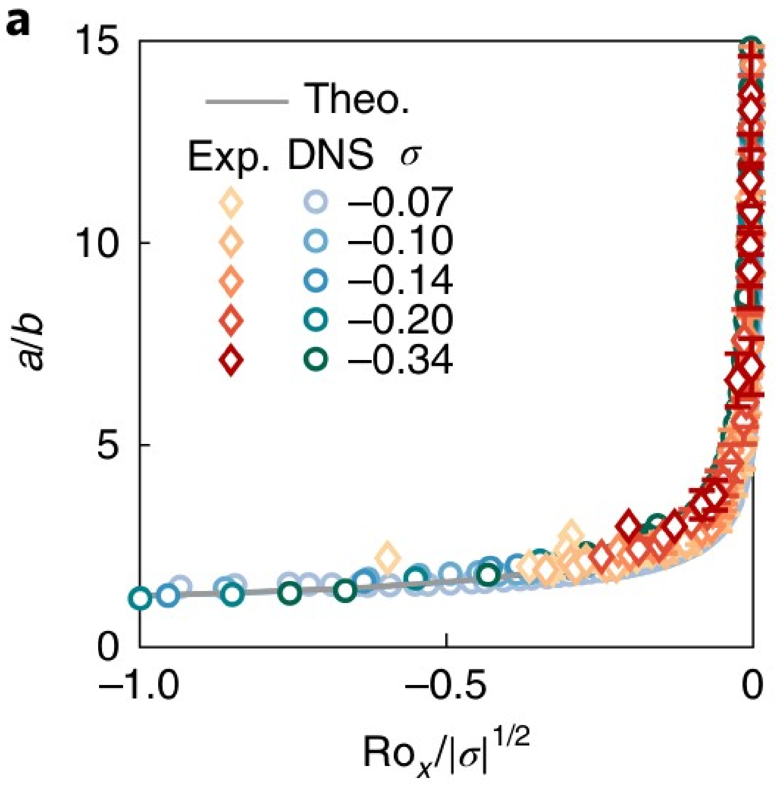
Strongly flattened

ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES

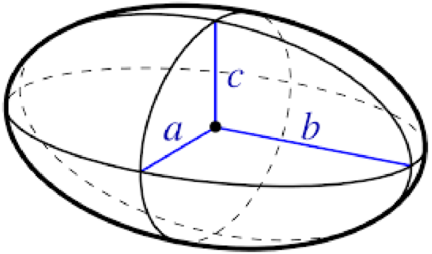


Aubert et al. (2012)
Lemasquerier et al. (2020)





Long-standing mathematical history



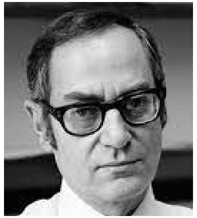
> 2020

CdV

1960 - 1990

Friedlander

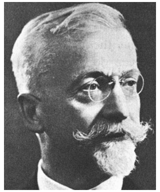
Chandra



Lebovitz

1880 - 1930

Poincaré

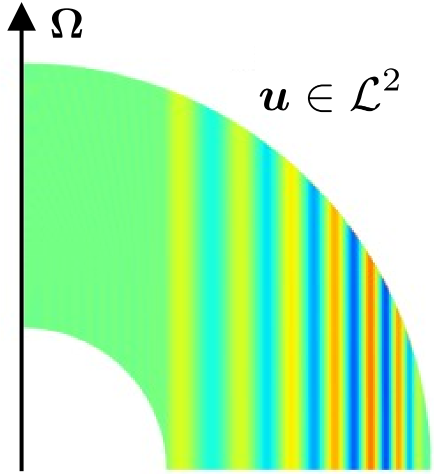


Cartan

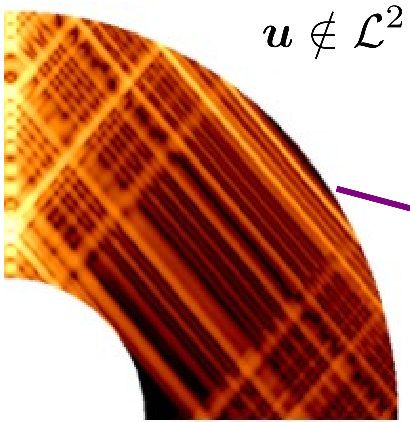
1800 - 1880

Jacobi, Dirichlet, Riemann...

ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES



Vidal (2018)



Rieutord et al. (2001)

! Shell ≠ Ellipsoid !

An **ill-posed (self-adjoint)** problem

- (Almost empty) **point** spectrum
- **Continuous** spectrum

Attractors for Two-Dimensional Waves with Homogeneous Hamiltonians of Degree 0

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LAURE SAINT-RAYMOND
Ecole Normale Supérieure de Lyon, UMPA

Abstract

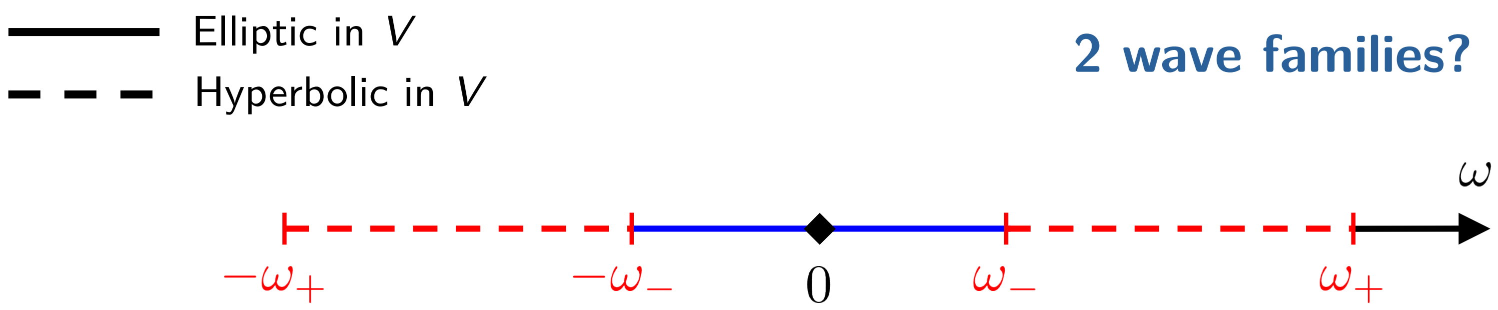
In domains with topography, inertial and internal waves exhibit interesting features. In particular, numerical and lab experiments show that, in two dimensions, for generic forcing frequencies, these waves concentrate on attractors. The goal of this paper is to analyze mathematically this behavior, using tools from spectral theory and microlocal analysis.

Principal symbol

$$\mathfrak{p} := |\boldsymbol{k}|^2\omega^2 - \left[N^2|\mathbf{1}_z \times \boldsymbol{k}|^2 + (2\boldsymbol{\Omega}\cdot\boldsymbol{k})^2\right]$$

$$\begin{cases} \omega_-^2 < \omega^2 < \omega_+^2 : & \text{Hyperbolic} \\ 0 < \omega^2 < \omega_-^2 : & \text{Elliptic} \end{cases}$$

$$2\omega_\pm^2 = [N^2 + 4|\boldsymbol{\Omega}|^2] \pm \sqrt{[N^2 + 4|\boldsymbol{\Omega}|^2]^2 - 16N^2(\boldsymbol{\Omega}\cdot\mathbf{1}_z)^2}$$



2 wave families?

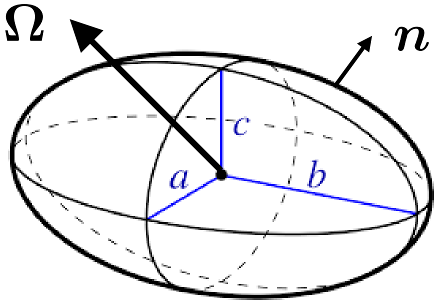
$\boldsymbol{\mathcal{V}}$: Hilbert space of square-integrable vector fields

$$\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle := \int_V \boldsymbol{u}_1^\dagger \cdot \boldsymbol{u}_2 \, \mathrm{d}V$$

$$\boldsymbol{\mathcal{V}}^0 := \{\boldsymbol{u} \in \boldsymbol{\mathcal{V}}, \nabla \cdot \boldsymbol{u} = 0 \text{ in } V, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial V} = 0\}$$

$\boldsymbol{\mathcal{P}}_n$: Vector polynomial functions whose components $\propto x^i y^j z^k$ with $i + j + k \leq n$

$$\boldsymbol{\mathcal{V}}_n^0 := \boldsymbol{\mathcal{V}}^0 \cap \boldsymbol{\mathcal{P}}_n, \quad \dim(\boldsymbol{\mathcal{V}}_n^0) = n(n+1)(2n+7)/6$$



$$(\partial V) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Poincaré & **buoyancy** operators (bounded & self-adjoints)

$$\mathrm{i}\boldsymbol{\mathcal{C}}(\boldsymbol{u}) := \mathrm{i}\mathbb{L}(2\boldsymbol{\Omega} \times \boldsymbol{u}), \quad \boldsymbol{\mathcal{K}}(\boldsymbol{u}) := \mathbb{L}(N^2 u_z \boldsymbol{1}_z), \quad \mathbb{L} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}^0$$

$$\boldsymbol{\mathcal{C}}|_{\boldsymbol{\mathcal{V}}_n^0} \subseteq \boldsymbol{\mathcal{V}}_n^0$$
$$\boldsymbol{\mathcal{K}}|_{\boldsymbol{\mathcal{V}}_n^0} \subseteq \boldsymbol{\mathcal{V}}_n^0$$
$$\oplus_n \boldsymbol{\mathcal{V}}_n^0 \text{ dense in } \boldsymbol{\mathcal{V}}^0$$

} **Pure point** spectrum in ellipsoids

Extension of the pure inertial wave problem in an ellipsoid

2015 – 2017
- Separation of variables
- Brute-force proof

→

2017
- Functional analysis in L^2
- Dimension arguments

→

$>$ **2020**

J. Fluid Mech. (2015), *vol.* 766, *pp.* 468–498. © Cambridge University Press 2015 468
doi:10.1017/jfm.2015.27

Enumeration, orthogonality and completeness of the incompressible Coriolis modes in a sphere

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[2]Institut für Geophysik, ETH, Sonneggstrasse 5, 8092 Zürich, Switzerland

GEOPHYSICAL AND ASTROPHYSICAL FLUID DYNAMICS, 2017
https://doi.org/10.1080/03091929.2017.1330412
Taylor & Francis
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Enumeration, orthogonality and completeness of the incompressible Coriolis modes in a tri-axial ellipsoid

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PHYSICAL REVIEW E **95**, 053116 (2017)

Completeness of inertial modes of an incompressible inviscid fluid in a corotating ellipsoid

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(Received 15 June 2016; revised manuscript received 5 April 2017; published 30 May 2017)

Kerswell (1993)

References

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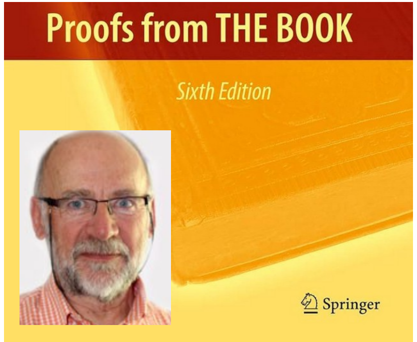
Backus, G. E., "Normal modes of small oscillation of an incompressible non-viscous fluid in a corotating rigid container," Preprint (1992).
Bayly, B. J., "Three dimensional instability of elliptical flow," *Phys. Rev. Lett.* **57**, 2160–2163 (1986).



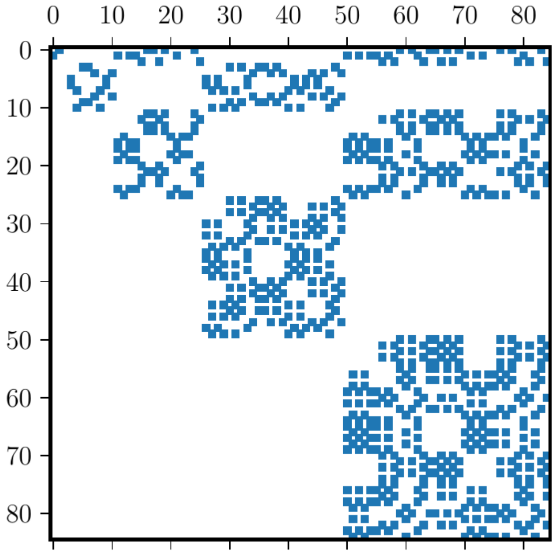
IGMs are **exact polynomials** => **Bespoke** numerical **algorithm**

- Galerkin **polynomial** basis {$\mathbf{e}_j$} in ellipsoids (Lebovitz, 1989)

$$\boldsymbol{u} = \sum_j \alpha_i \boldsymbol{e}_j, \quad \nabla \cdot \boldsymbol{e}_j = 0, \quad \boldsymbol{e}_j \cdot \boldsymbol{n}|_{\partial V} = 0$$

- Symbolic **projection method**

$$\left(-\omega^2 + \omega \mathrm{i} \boldsymbol{A}_n + \boldsymbol{B}_n\right) \boldsymbol{\alpha} = \mathbf{0} \qquad \boldsymbol{L}_n = \begin{pmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{B}_n & \mathrm{i}\boldsymbol{A}_n \end{pmatrix}$$

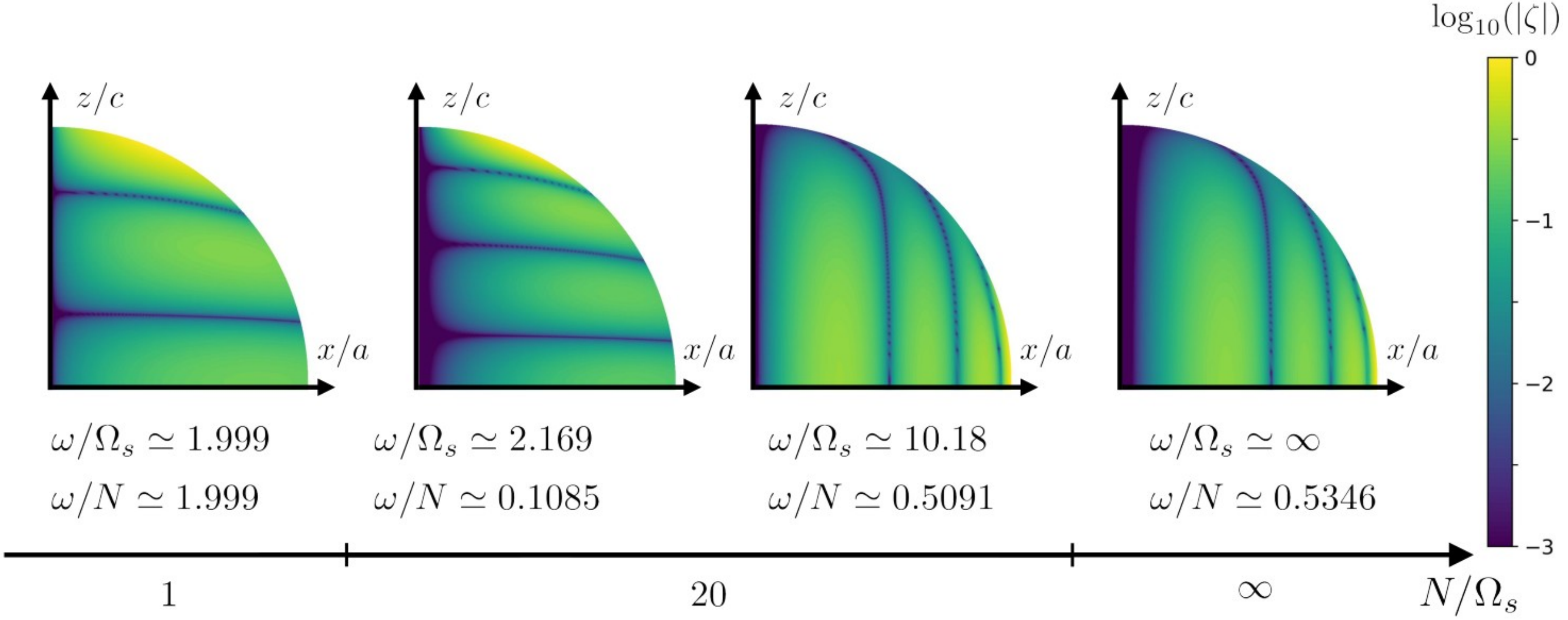


- **Numerical** solutions (exact up to machine precision)

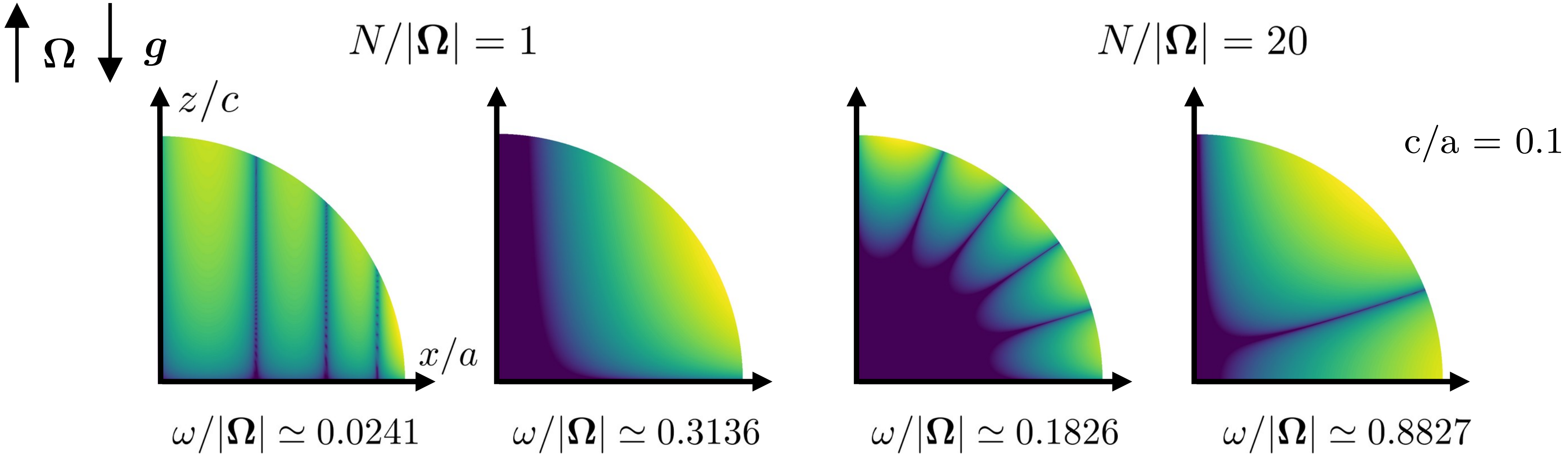
ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES

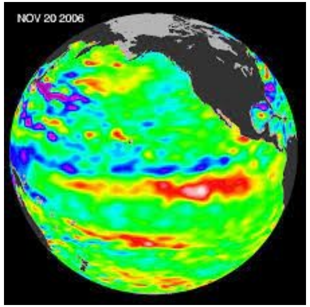
$\uparrow \boldsymbol{\Omega} \downarrow \boldsymbol{g} \quad c/a = 0.1$

No **attractor** modes!



ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES





- **Similar** properties than (coastal & equatorial) **Kelvin waves**
- Dense **essential spectrum** when $0 < |\omega| < \omega_-$

ELLIPSOIDAL MODEL FOR GEOPHYSICAL VORTICES

Asymptotic measure ($\omega_- < |\omega| < \omega_+$)

$$\int_{-\infty}^{\lambda} \mathrm{d}\pi_\infty = \frac{1}{8\pi} \mathrm{Area}(\mathfrak{S}_\lambda \cap S^2)$$

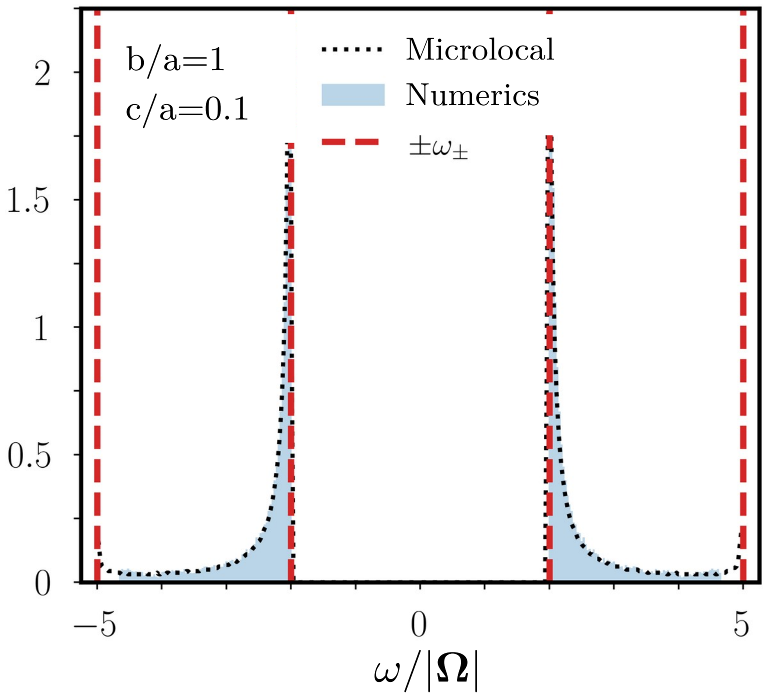
$$\mathfrak{S}_\lambda := \{\tilde{\boldsymbol{k}} \in \mathbb{R}^3 \mid \omega(\tilde{\boldsymbol{k}}) \leq \lambda\}$$

$$\tilde{\boldsymbol{k}} = (k_x/a, k_y/b, k_z/c)^\top$$

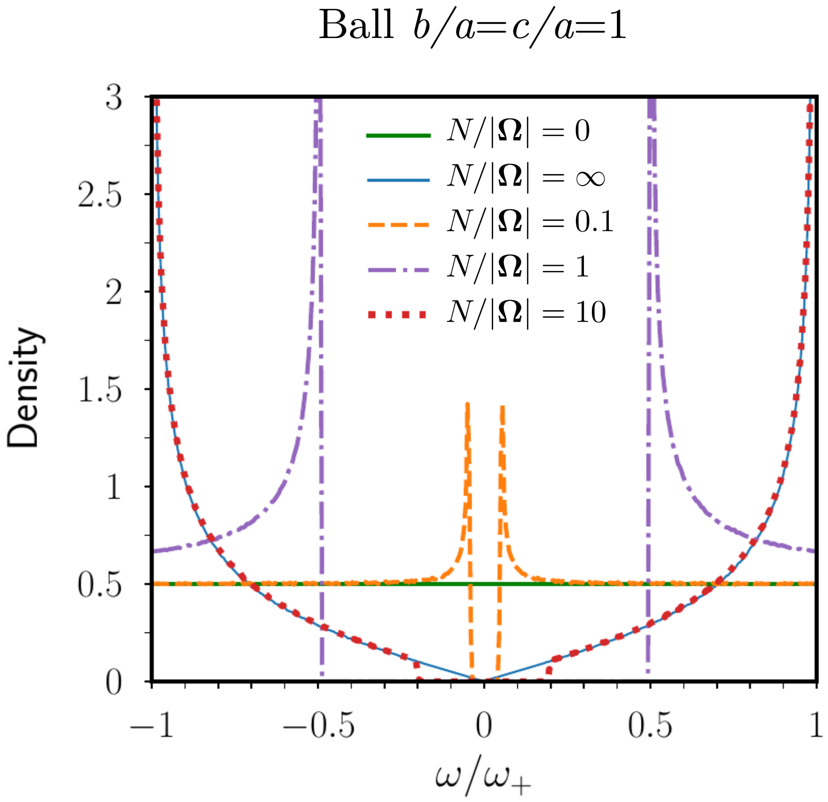
Rescaled **dispersion relation** of inertia-gravity waves

<ins>Proof</ins>: - Weyl asymptotics of the equivalent matrix operator

$$\boldsymbol{\mathcal{L}} := \begin{pmatrix} 0 & \boldsymbol{\mathcal{I}} \\ \boldsymbol{\mathcal{K}} & \mathrm{i}\boldsymbol{\mathcal{C}} \end{pmatrix}$$



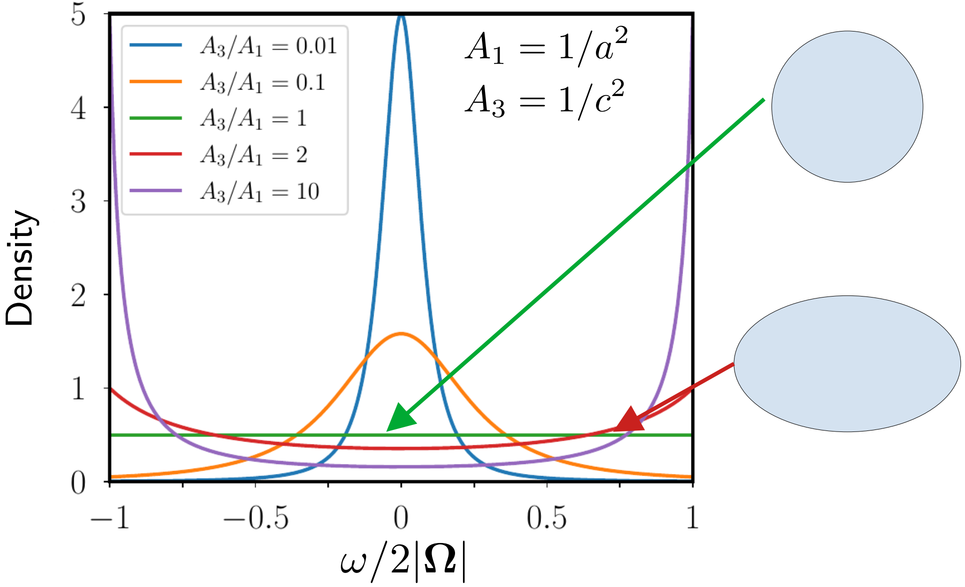
Agreement with **numerics**



Some differences with pure **inertial modes**

- **Non-uniform** distribution in the ball
- Gap size function of the orientation of $\boldsymbol{\Omega}$

$$2\omega_\pm^2 = [N^2 + 4|\boldsymbol{\Omega}|^2] \pm \sqrt{[N^2 + 4|\boldsymbol{\Omega}|^2]^2 - 16N^2(\boldsymbol{\Omega}\cdot\mathbf{1}_z)^2}$$



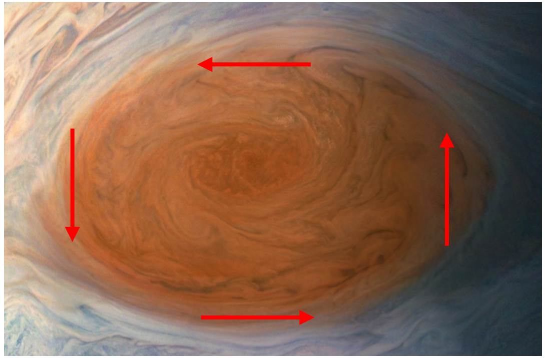
Expected for **planets** ($A_3/A_1 \sim 1$)

Geometry mismatch

Experiments / Simulations ($A_3/A_1 \sim 3$)

Shape affects the **eigenvalue distribution** (even in an ellipsoid)

OUTLINE

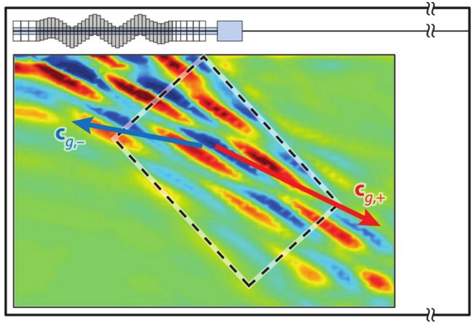


Largest-scale vortices

- Differential rotation (not of uniform-vorticity)
- Bulk (smaller-scale) turbulence

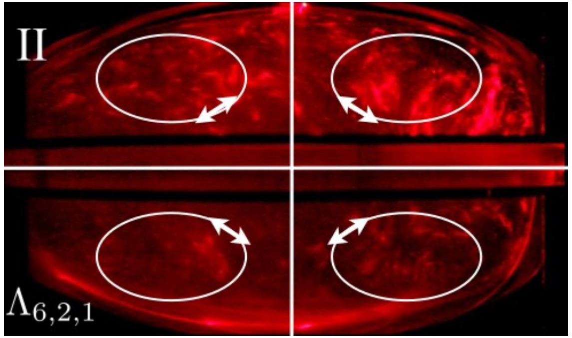
Instabilities?

- Triadic Resonant Instability (TRI)
- **Elliptical Instability** (EI)
- ...



Bourget el al. (2013)
Boury et al. (2023)

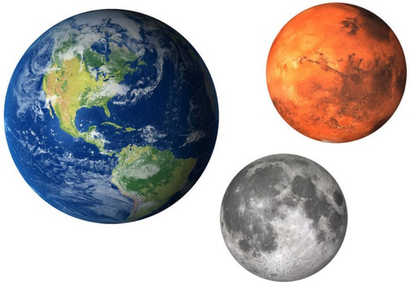
Grannan et al. (2014)

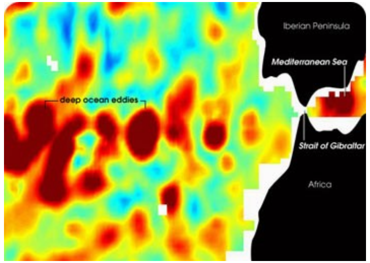


$$\omega_0 = \omega_1 \pm \omega_2$$

CONCLUSION & PERSPECTIVES

~ 0 ~ 10 ~ 100 $N/|\boldsymbol{\Omega}|$





Inertial modes

Inertia-gravity modes

Future works

- Low-frequency (Kelvin) waves in **arbitrary** geometries
- Vortex **stability** (EI, TRI?)

Perspectives

- Unstable stratification ($N^2<0$)
- **Planet's** (radial) **gravity**?