Continuum modeling and numerical resolution for elastoviscoplastic fluids

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Conservation equations

Mass and momentum:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{v}) = 0$$
$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v} \right) - \mathbf{div}\,\boldsymbol{\sigma} = \boldsymbol{f}$$

Notations:

- ρ : mass density
- $\boldsymbol{v}$: velocity field
- $\boldsymbol{\sigma}$: Cauchy stress tensor
- $\boldsymbol{f}$: external forces, given

2 equations, 3 unknowns : $\rho, \boldsymbol{v}, \boldsymbol{\sigma}$
$\Rightarrow$ add a **constitutive** equation for $\boldsymbol{\sigma}$ vs $\rho, \boldsymbol{v}$

Example 1: compressible Navier-Stokes

$$\boldsymbol{\sigma} = -p(\rho)\boldsymbol{I} + 2\eta D(\boldsymbol{v}) + \lambda(\mathrm{div}\,\boldsymbol{v})\boldsymbol{I}$$

where

$p(\rho)$: pressure function

η, λ : viscosities s.t. $\lambda + 2\eta/d > 0$, d = spatial dimension

$D(\boldsymbol{v}) = \mathbf{sym}(\nabla \boldsymbol{v})$

examples

$p(\rho) = c\rho^2$: shallow water

$\quad\;\; = c\rho^\gamma$: adiabatic gas, $\gamma > 1$

well-posed system

[1993, P. L. Lions]

[2007-2022, D. Bresch & al.] non-constant $\eta(\rho), \lambda(\rho)$

Example 2: incompressible Navier-Stokes

$$\rho = cte > 0 \quad \text{and} \quad \boldsymbol{\sigma} = -p\boldsymbol{I} + 2\eta D(\boldsymbol{v})$$

where

p : unknown pressure field

$\eta > 0$: viscosity

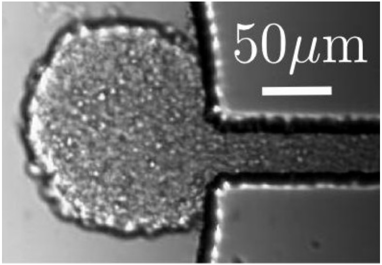
2 unknowns : $p, \boldsymbol{v}$ s.t.

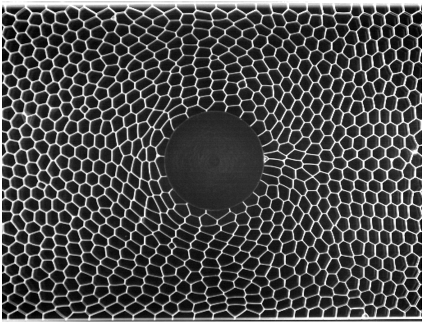
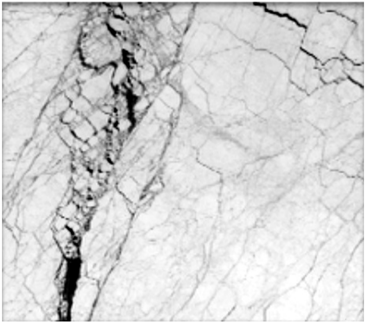
$$\begin{cases} \operatorname{div} \boldsymbol{v} = 0 \\ \rho \left(\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v} \right) - \mathbf{div}\,(-p\boldsymbol{I} + 2\eta D(\boldsymbol{v})) = \boldsymbol{f} \end{cases}$$

well-posed system

[1924, J. Leray] when $d = 2$

My motivations

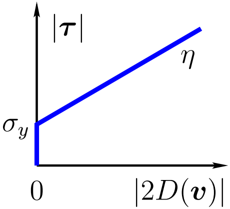
Outline

1. Viscoplasticity: Bingham
2. Viscoelasticity: Oldroyd
3. Elastoviscoplastic fluid
4. Frictional effect & damage
5. Conclusion & perspectives

1. Viscoplastic fluid: Bingham (1922)

Cauchy stress tensor:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$$



with

$$|\boldsymbol{\tau}| \leqslant \sigma_y \quad \text{when } D(\boldsymbol{v}) = 0$$

$$\boldsymbol{\tau} = 2\eta D(\boldsymbol{v}) + \sigma_y \frac{D(\boldsymbol{v})}{|D(\boldsymbol{v})|} \quad \text{when } D(\boldsymbol{v}) \neq 0$$

when $\sigma_y = 0 \quad \Longrightarrow$ incompressible Navier-Stokes

Coupling with conservation equations

(P): *find* $(\boldsymbol{\tau}, \boldsymbol{v}, p)$ *such that*

$$\begin{cases} |\boldsymbol{\tau}| \leqslant \sigma_y & \text{when } D(\boldsymbol{v}) = 0 \\ \boldsymbol{\tau} = 2\eta D(\boldsymbol{v}) + \sigma_y \dfrac{D(\boldsymbol{v})}{|D(\boldsymbol{v})|} & \text{when } D(\boldsymbol{v}) \neq 0 \\ \rho \left(\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v} \right) - \mathbf{div}\, \boldsymbol{\tau} + \nabla p = \boldsymbol{f} \\ \mathrm{div}\, \boldsymbol{v} = 0 \\ I.C. + B.C. \end{cases}$$

Numerical resolution

Common assumption:

$(\boldsymbol{v}.\nabla)\boldsymbol{v}$: inertia, neglected, slow flow

Minimization of a convex non-differentiable dissipation potential:

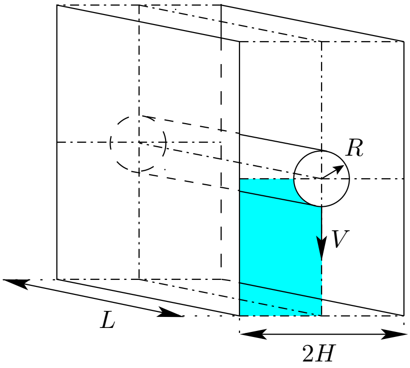
$$\min_{\boldsymbol{v}\in\ker(\mathbf{div})} \int_\Omega 2\eta\,|D(\boldsymbol{v})|^2 + \sigma_y\,|D(\boldsymbol{v})| - \boldsymbol{f}.\boldsymbol{v}$$

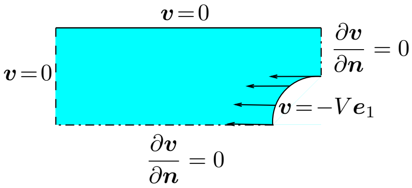
Present : combination of two methods:

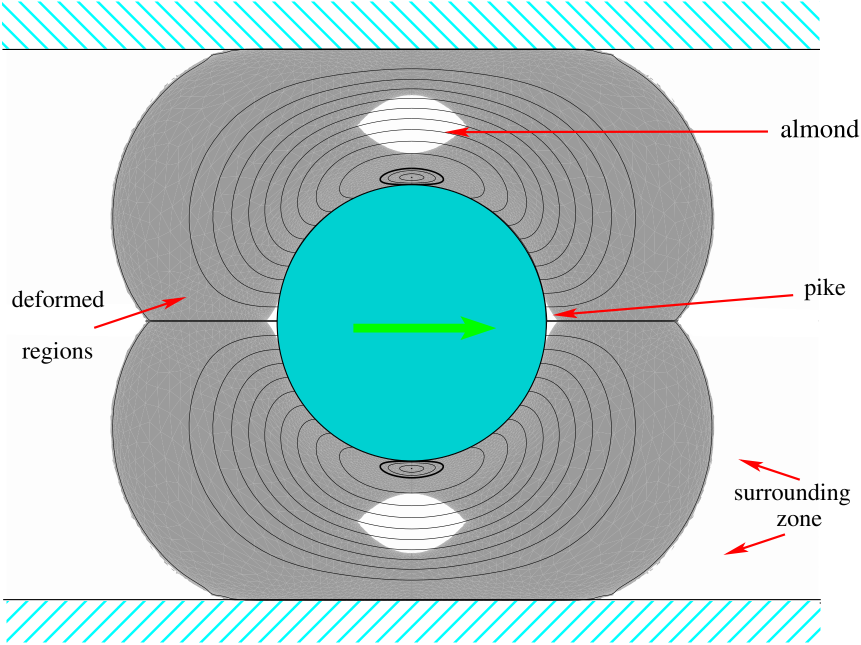
- augmented Lagrangian method [Glowinski, Lions & Trémolières, 1981]
- automatic anisotropic mesh adaptation [Hecht, 1997]

[Roquet & Saramito, 2003]

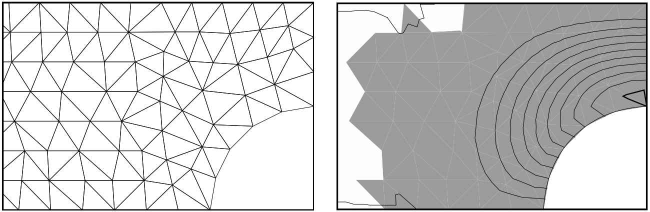
Example: flow around a cylinder





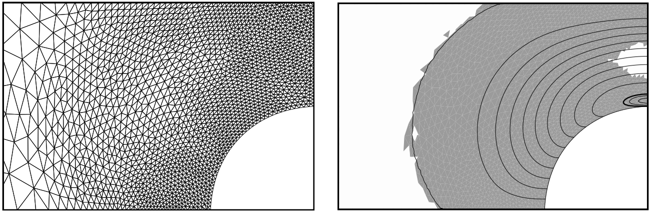

almond
deformed
regions
pike
surrounding
zone

0



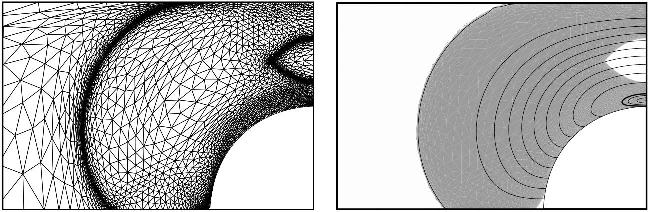
$N = 539$

1

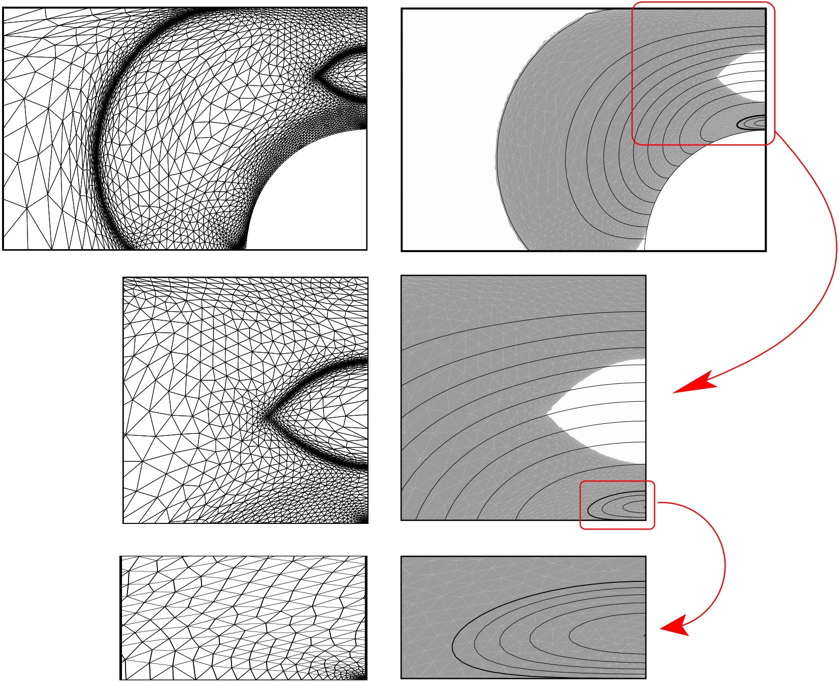


$N = 15\,466$

9

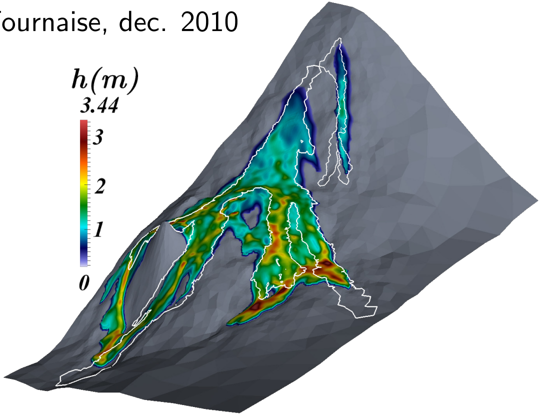


$N = 41\,955$



Application: volcanic lava flow

Piton de la Fournaise, dec. 2010



[Bernabeu, Saramito & Smutek, 2016]

2. Viscoelastic fluid: Oldroyd-B (1950)

Cauchy stress tensor

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\eta_0 D(\boldsymbol{v}) + \boldsymbol{\tau}$$

with

$$\frac{\overset{\nabla}{\boldsymbol{\tau}}}{G} + \frac{\boldsymbol{\tau}}{\eta} = 2D(\boldsymbol{v})$$

objective tensor derivative

$$\overset{\nabla}{\boldsymbol{\tau}} = \frac{\partial \boldsymbol{\tau}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{\tau} - \nabla \boldsymbol{v}\,\boldsymbol{\tau} - \boldsymbol{\tau}\,\nabla \boldsymbol{v}^T$$

when $1/G = 0 \quad \Longrightarrow$ incompressible Navier-Stokes

Coupling with conservation equations

(P): *find* $(\boldsymbol{\tau}, \boldsymbol{v}, p)$ *such that*

$$\left\{\begin{array}{l} \dfrac{\overset{\nabla}{\boldsymbol{\tau}}}{G} + \dfrac{\boldsymbol{\tau}}{\eta} - 2D(\boldsymbol{v}) = 0 \\ \rho\left(\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}\right) - \mathbf{div}\big(-p\boldsymbol{I} + 2\eta_0 D(\boldsymbol{v}) + \boldsymbol{\tau}\big) = \boldsymbol{f} \\ \mathrm{div}\,\boldsymbol{v} = 0 \\ I.C. + B.C. \end{array}\right.$$

Existence results

[1985, Renardy] stationnary

[1987-1990, C. Guillopé & J. C. Saut] local ; global for small data

[1998, Fernández-Cara, Guillén & Ortega] improve, still local

[2000, Lions & Masmoudi] **global, but** corotational tensor derivative

[2004, Molinet & Talhouk] improve, still local

[2007, D. Hu & T. Lelièvre] new energy estimate

[2011, Masmoudi] **global, but** FENE-P model, not Oldroyd-B

[2012, Constantin & Kliegl] **global, but** additional stress diffusion

[2021, Thomases & Renardy] recent review paper

Change of variable

- Conformation tensor

$$\boldsymbol{c} = \boldsymbol{I} + \frac{\boldsymbol{\tau}}{G} \; > \; 0$$ [Hulsen, 1990]

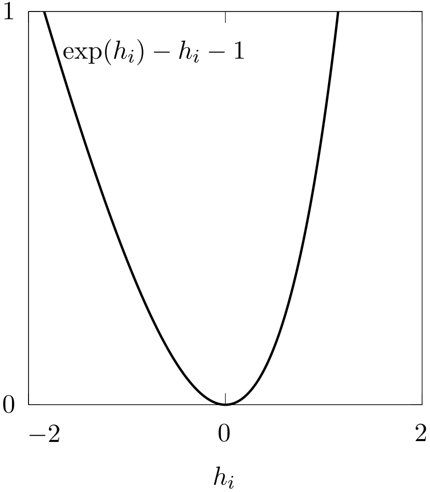
- Log-conformation tensor

$$\boldsymbol{h} = \log \boldsymbol{c}$$ [Fattal & Kupfermann, 2005]

$$\stackrel{\text{déf}}{=} Q(\mathrm{diag}_i \log c_i) Q^T$$

$\Longrightarrow$ usefull for both theory & computations

Energy



$$E(t) = \int_\Omega \frac{\rho}{2}|\boldsymbol{v}|^2 + \frac{G}{2}\mathrm{tr}\left(\exp \boldsymbol{h} - \boldsymbol{h} - \boldsymbol{I}\right)$$

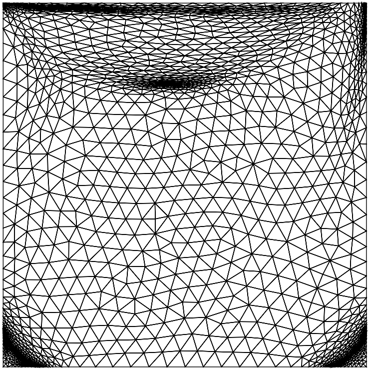
Energy decrease

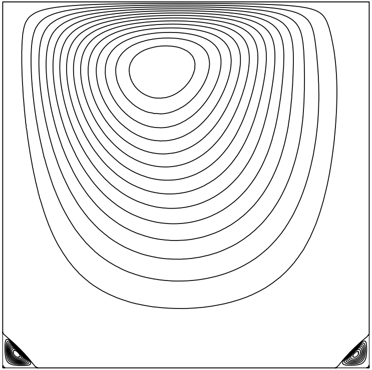
$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\boldsymbol{v}|^2 + \frac{G}{2}\mathrm{tr}\left(\exp \boldsymbol{h} - \boldsymbol{h} - \boldsymbol{I}\right) \\
&+ \int_\Omega 2\eta_0|D(\boldsymbol{v})|^2 + \frac{G^2}{2\eta}\mathrm{tr}\left(\exp \boldsymbol{h} - \boldsymbol{h} - \boldsymbol{I}\right) \\
&\leqslant\ 0, \quad \forall t > 0
\end{aligned}
$$

$$\Longrightarrow E'(t) \leqslant 0$$

[2007, D. Hu & T. Lelièvre]

Numerical resolution



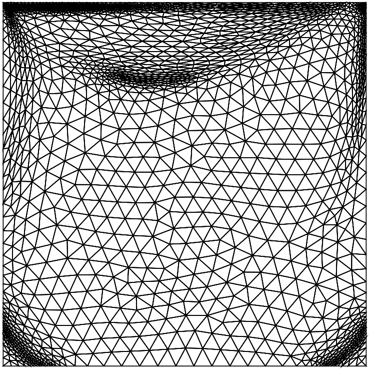


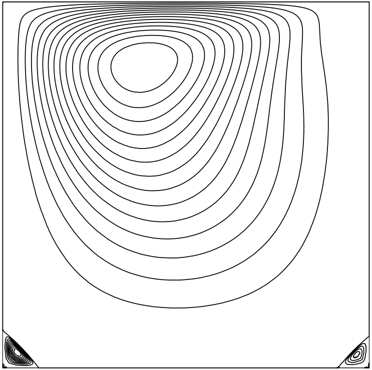
$$We \stackrel{\text{déf}}{=} \eta V/(GL) = 1$$

- Log-conformation tensor : $\boldsymbol{h} = \log \boldsymbol{c}$
- divergence-free FEM : $\boldsymbol{h}, \boldsymbol{v}, p \in P_{1,d} \times P_2 \cap C^0 \times P_{1,d}$
- non-smooth exact Newton method
- mesh adaptation

[Saramito, 2014]

Numerical resolution



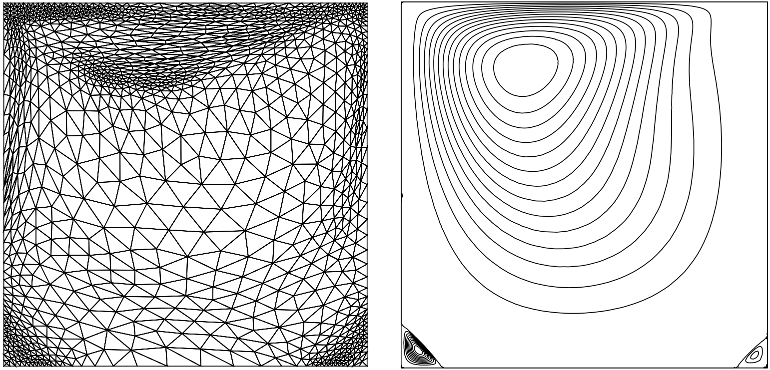


$$We \stackrel{\text{déf}}{=} \eta V/(GL) = 2$$

- Log-conformation tensor : $\boldsymbol{h} = \log \boldsymbol{c}$
- divergence-free FEM : $\boldsymbol{h}, \boldsymbol{v}, p \in P_{1,d} \times P_2 \cap C^0 \times P_{1,d}$
- non-smooth exact Newton method
- mesh adaptation

[Saramito, 2014]

Numerical resolution



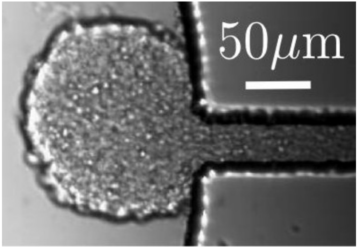
$$We \stackrel{\text{déf}}{=} \eta V/(GL) = 3$$

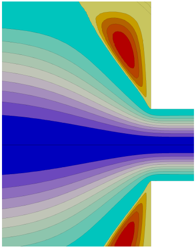
- Log-conformation tensor : $\boldsymbol{h} = \log \boldsymbol{c}$
- divergence-free FEM : $\boldsymbol{h}, \boldsymbol{v}, p \in P_{1,d} \times P_2 \cap C^0 \times P_{1,d}$
- non-smooth exact Newton method
- mesh adaptation

[Saramito, 2014]

Application: biology

Living tissue aspirated in a constriction :



Quantitative comparisons: work in progress

[Tlili, Graner & Delanoë-Ayari, Development, 2022]

3. Elastoviscoplastic fluid

Cauchy stress tensor

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\eta_0 D(\boldsymbol{v}) + \boldsymbol{\tau}$$

and

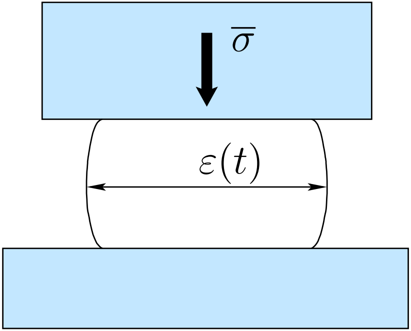
$$\frac{\overset{\nabla}{\boldsymbol{\tau}}}{G} + \max\left(0,\ 1 - \frac{\sigma_y}{|\boldsymbol{\tau}|}\right)\frac{\boldsymbol{\tau}}{\eta} = 2D(\boldsymbol{v})$$

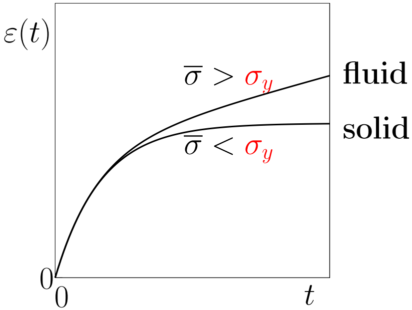
when

- $\sigma_y = 0$: Oldroyd fluid
- $1/G = 0$: Bingham fluid
- $1/G = \sigma_y = 0$: incompressible Navier-Stokes

[Saramito, JNNFM, 2007]

Example: creeping test





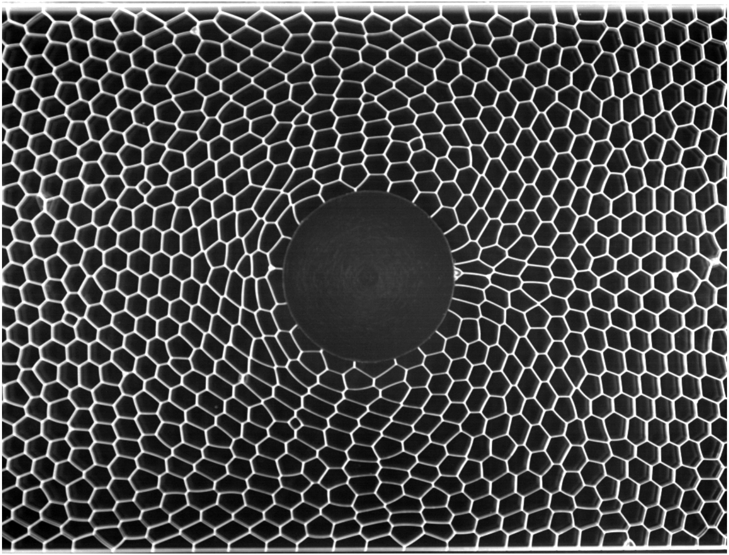
material = viscoelastic **solid**, then viscoelastic **fluid**

Coupling with conservation equations

(P): *find* $(\boldsymbol{\tau}, \boldsymbol{v}, p)$ *such that*

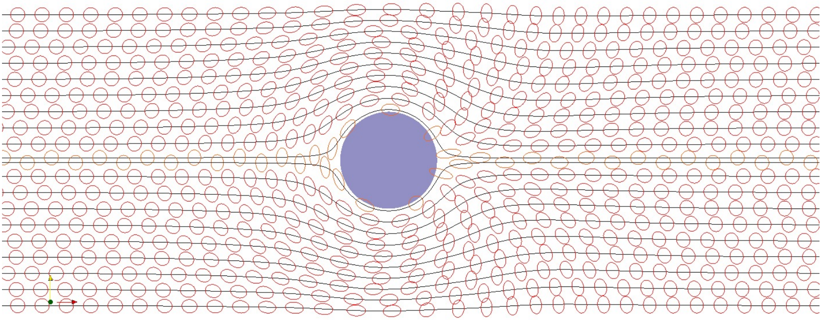
$$\begin{cases} \dfrac{\overset{\nabla}{\boldsymbol{\tau}}}{G} + \max\left(0,\ 1 - \dfrac{\sigma_y}{|\boldsymbol{\tau}|}\right) \dfrac{\boldsymbol{\tau}}{\eta} - 2D(\boldsymbol{v}) = 0 \\ \rho\left(\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}\right) - \mathbf{div}\big(-p\boldsymbol{I} + 2\eta_0 D(\boldsymbol{v}) + \boldsymbol{\tau}\big) = \boldsymbol{f} \\ \mathrm{div}\,\boldsymbol{v} = 0 \\ I.C. + B.C. \end{cases}$$

Example: liquid foam around an obstacle

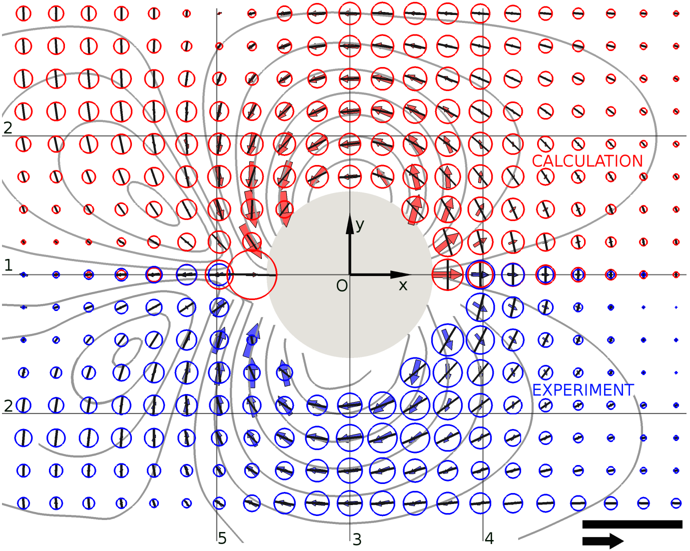


[Dollet & Graner, JFM, 2007]

Numerical resolution

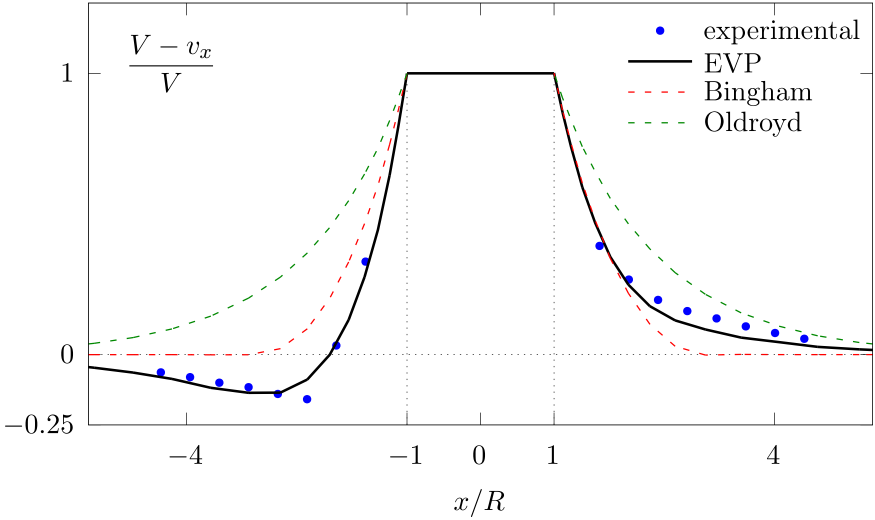


Computation vs experiment



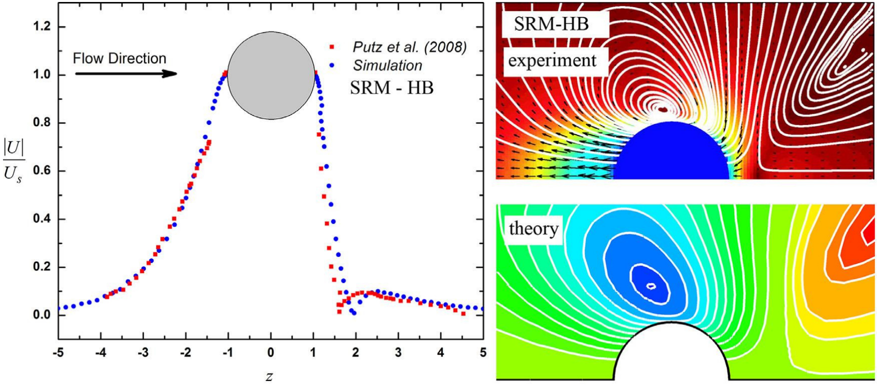
[Cheddadi, Saramito, Dollet, Raufaste & Graner, EPJE, 2011]

Velocity along the axis: negative wake



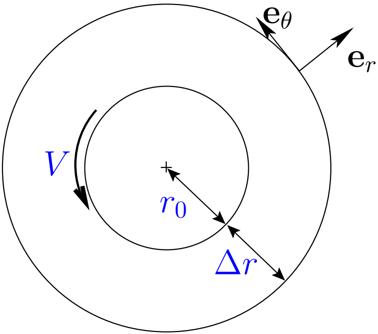
[Cheddadi & Saramito, JNNFM, 2013]

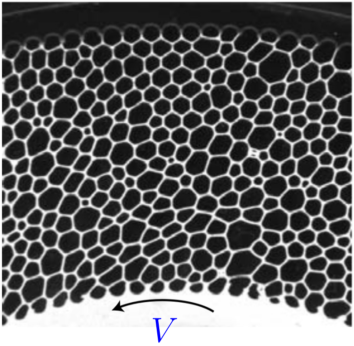
Moving sphere in a carbopol gel



[Fraggedakis, Dimakopoulos & Tsamopoulos, JNNFM, 2016]

Example: Couette Geometry





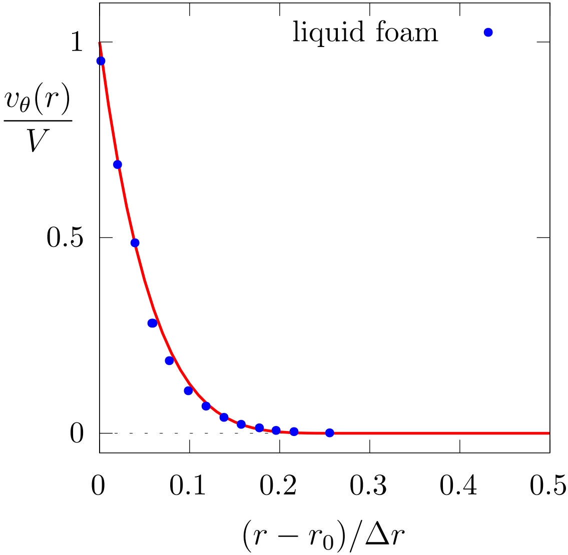
$$\mathbf{v} = \begin{pmatrix} 0 \\ v_\theta(r) \end{pmatrix} \qquad \tau = \begin{pmatrix} \tau_{rr}(r) & \tau_{r\theta}(r) \\ \tau_{\theta r}(r) & \tau_{\theta\theta}(r) \end{pmatrix}$$

Two dimensionless numbers

$$We = \frac{\eta V}{G \, \Delta r} \quad \text{viscoelasticity}$$

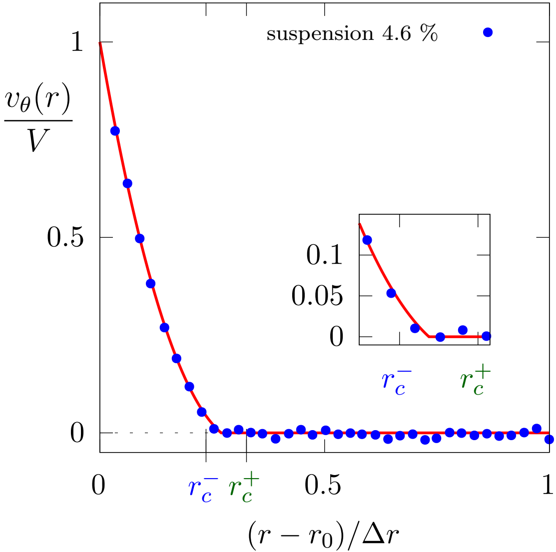
$$Bi = \frac{\sigma_y \, \Delta r}{\eta V} \quad \text{viscoplasticity}$$

Observation: (1) smooth profile



experiment: [Debrégeas et al., PRL, 2001]

Observation: (2) abrupt profile



experiment: [Coussot et al., PRL, 2002]

10 years of questioning

year	profil	authors	material
2001	smooth	*Debrégeas* et al.	liquid foam
2002	abrupt	*Coussot* et al.	suspensions, emulsions
2003	abrupt	*Salmon* et al.	wormlike micelles
2004	abrupt	*Lauridsen* et al.	liquid foam
2006	abrupt	*Gilbreth* et al.	liquid foam
2008	abrupt	*Dennin* et al.	liquid foam

$\Longrightarrow$ are profils abrupts ?

10 years of questioning

year	profil	authors	material
2001	smooth	*Debrégeas* et al.	liquid foam
2002	abrupt	*Coussot* et al.	suspensions, emulsions
2003	abrupt	*Salmon* et al.	wormlike micelles
2004	abrupt	*Lauridsen* et al.	liquid foam
2006	abrupt	*Gilbreth* et al.	liquid foam
2008	abrupt	*Dennin* et al.	liquid foam
2008	smooth	*Kätgert* et al.	liquid foam
2010	smooth	*Coussot* et al.	suspensions, emulsions
2010	smooth	*Ovarlez* et al.	liquid foam
2010	smooth	*Kätgert* et al.	liquid foam

$\Longrightarrow$ are profiles smooth ?

$\Longrightarrow$ computations + theoretical interpretation

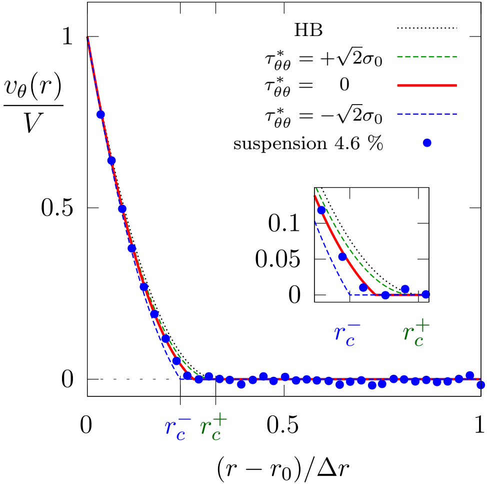
Initial stress influence ?

$$\tau(t=0) = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} \\ \tau_{\theta r} & \tau_{\theta\theta} \end{pmatrix} \Leftarrow \begin{pmatrix} 0 & 0 \\ 0 & \tau^*_{\theta\theta} \end{pmatrix}$$

Three extremal cases:

$\tau^*_{\theta\theta}$	
0	no pre-stress
$+\sqrt{2}\,\sigma_y$	pre-stress
$-\sqrt{2}\,\sigma_y$	pre-stress

Computation: (1) abrupt profile

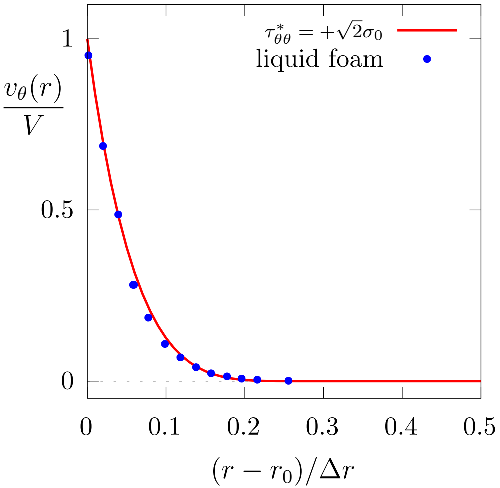


computation: $\textit{We} = 0.026$, $\textit{Bi} = 27$, $n = 1$

Cheddadi, Saramito, Graner, JoR, 2012

experiment: *Coussot et al., PRL, 2002*

Computation: (2) smooth profile



computation: $We = 0.035$, $Bi = 10$, $n = 1/3$

Cheddadi, Saramito, Graner, JoR, 2012

experiment: *Debrégeas et al., PRL, 2001*

Discussion

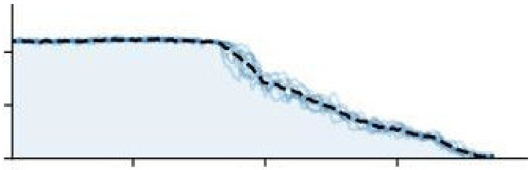
- "*To eliminate transient effects, we ran the experiment a full round before taking data.*"
  [Debrégeas, Tabuteau & di Meglio, 2001]

- [Coussot et al., 2002]: not presheared

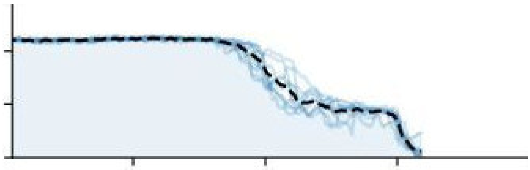
- "*Before any measurement, the Laponite sample is presheared for* 1 *min at* $+1500\ s^{-1}$ *and for* 1 *min at* $-1500\ s^{-1}$ *to erase most of the sample history. We checked that this procedure leads to reproducible results over a few hours.*"
  [Gibaud, Barentin & Manneville, 2008]

- "*The preshear protocol prior to the experiment may strongly influence the results.*"
  [Divoux, Fardin, Manneville & Lerouge, 2016]

4. Friction μ



without cohesion $\sigma_y = 0$
e.g. dry granular

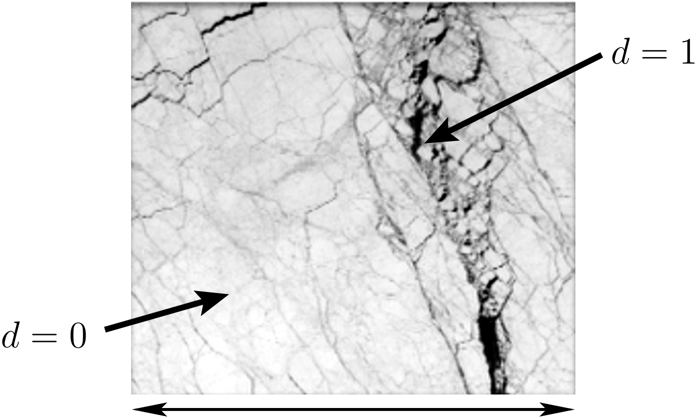


with cohesion $\sigma_y \neq 0$
e.g. wet granular

[Abramian Staron Lagrée 2022]

Damage

$$0 \leqslant d(t, \boldsymbol{x}) \leqslant 1$$



59 km : spot satelite

$$1 - d = \frac{E}{E_0} \rightsquigarrow$$ structural parameter

[Kachanov 1958]

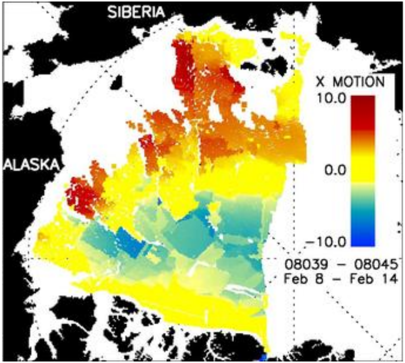
Present motivation: sea ice motion

[Polar region atlas, CIA, 1978]

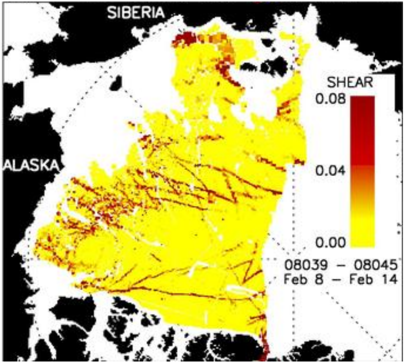
- sea ice drift:
  due to ocean currents and wind
- accelerated:
  due to global heating

Sea ice motion

velocity v_x



deformation rate $|D(\boldsymbol{v})|$



satellite observation [Kwok 2010]

$\Longrightarrow$ piecewise constant velocity
localization, fracture network

$\Longrightarrow$ piecewise solid behavior
global fluid behavior

Continuous modeling from thermodynamics

Why thermodynamics ?

- it has a physical meaning
- it automatically leads to constitutive equations
- it easily leads to energy decrease proofs

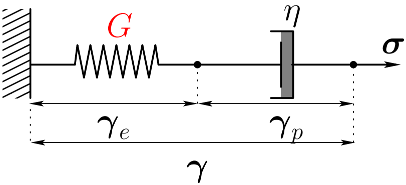
  it avoids fields to tend to infinity...
- it leads to efficient numerical methods

framework: standard generalized materials
[Germain 1973 ; Moreau 1974 ; Halphen & Nguyen 1975 ; Saramito 2016]

large defs. extension: [Saramito 2023]

Example: Maxwell viscoelatic (VE) fluid

- **state variables** : γ, γ_p
  rate variables : $\dot{\gamma}, \dot{\gamma}_p$
  where $\dot{\gamma} = \partial_t \gamma + \boldsymbol{v}.\nabla\gamma$



- **functions**

$$\psi(\gamma, \gamma_p) = \frac{G}{\rho}|\gamma - \gamma_p|^2 \quad : \text{ free energy}$$

$$\phi(\dot{\gamma}, \dot{\gamma}_p) = \mathscr{I}_{\ker(\mathrm{tr})}(\dot{\gamma}) + \eta\,|\dot{\gamma}_p|^2 \ : \text{ dissipation potential}$$

$\Rightarrow$ **constitutive equations** : automatically derived

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\gamma}} + \frac{\partial\phi}{\partial\dot{\boldsymbol{\gamma}}} = -p\boldsymbol{I} + \boldsymbol{\tau}$$

$$0 = \rho\frac{\partial\psi}{\partial\boldsymbol{\gamma}_p} + \frac{\partial\phi}{\partial\dot{\boldsymbol{\gamma}}_p} \iff \frac{\dot{\boldsymbol{\tau}}}{G} + \frac{\boldsymbol{\tau}}{\eta} = 2\dot{\gamma}$$

where $\boldsymbol{\tau} = 2G\gamma_e =$ elastic stress

From kinematics, replace: $\dot{\gamma} \to D(\boldsymbol{v})$ and $\dot{\boldsymbol{\tau}} \to \overset{\nabla}{\boldsymbol{\tau}}$

Here: BEVP model development

- **variables**

γ	:	deformation tensor
γ_p	:	its irreversible part
d	:	damage

$\dot{\gamma}, \dot{\gamma}_p, \dot{d}$: rate variables

- **functions** : still to develop

$\psi(\gamma, \gamma_p, d)$	:	Helmholtz free energy
$\phi(\dot{\gamma}, \dot{\gamma}_p, \dot{d})$	:	dissipation potential

⇒ **constitutive equations** : automatically derived

$$
\begin{aligned}
\boldsymbol{\sigma} &= \rho \frac{\partial \psi}{\partial \boldsymbol{\gamma}} + \frac{\partial \phi}{\partial \dot{\boldsymbol{\gamma}}} \\
0 &= \rho \frac{\partial \psi}{\partial \boldsymbol{\gamma}_p} + \frac{\partial \phi}{\partial \dot{\boldsymbol{\gamma}}_p} \\
0 &= \rho \frac{\partial \psi}{\partial d} + \frac{\partial \phi}{\partial \dot{d}}
\end{aligned}
$$

BEVP construction: Hooke's elasticity

- variables : $\boldsymbol{\gamma}, \boldsymbol{\gamma}_p$, d

- Helmholtz free energy

$$\psi(\boldsymbol{\gamma}, \boldsymbol{\gamma}_p, d) = \frac{G(d)}{\rho}|\boldsymbol{\gamma} - \boldsymbol{\gamma}_p|^2 + \frac{\lambda(d)}{2\rho}\mathrm{tr}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_p)^2$$

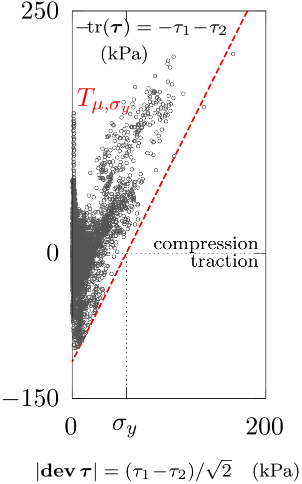
$$\Longrightarrow \quad \boldsymbol{\tau} = \rho\frac{\partial\psi}{\partial\boldsymbol{\gamma}}$$

$$= \ 2G(d)\boldsymbol{\gamma}_e + \lambda(d)\,\mathrm{tr}(\boldsymbol{\gamma}_e)\boldsymbol{I} \qquad \text{elastic stress}$$

with
$$G(d) = \frac{E(d)}{2(1+\nu(d))}$$
$$\lambda(d) = \frac{E(d)\ \nu(d)}{(1+\nu(d))(1-2\nu(d))} \qquad \text{Lamé coeffs}$$

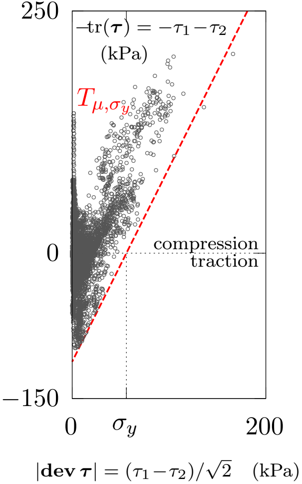
and
$$E(d) = (1-d)E_0 \qquad \text{by the Kachanov's definition of } d$$
$$\nu(d) = \nu_0 + (\nu_1 - \nu_0)d \qquad \text{increasing Poisson ratio, } \nu_1 \geqslant \nu_0$$

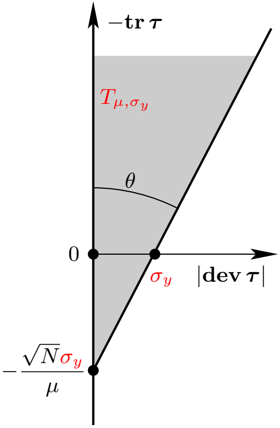
Friction & cohesion



data from sea ice: [Weiss Schulson, J Phys D, 2009]

Friction & cohesion : Drucker-Prager plasticity





fit:

$\mu \approx 1/\sqrt{2}$

$\sigma_y \approx 58$ Pa

dependency:

$$\sigma_y(d) = (1-d)\sigma_{y0}$$
$$\eta(d) = (1-d)\eta_0$$

T_{μ,σ_y} = translated Drucker-Prager cone

$$= \left\{ \boldsymbol{\tau} \in \mathbb{R}_s^{N\times N} \;;\; |\mathbf{dev}\,\boldsymbol{\tau}| \leqslant \sigma_y - \frac{\mu}{\sqrt{N}}\,\mathrm{tr}\,\boldsymbol{\tau} \right\}$$

Damage: embeded cone

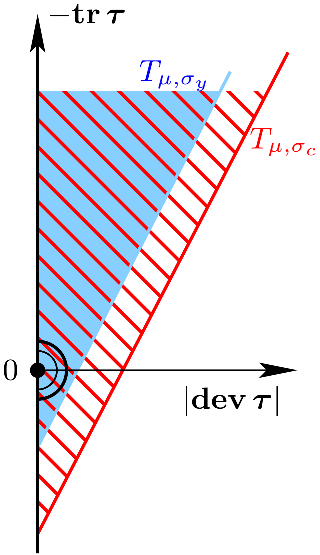
Damage evolution:

$$\dot{d} = 0 \qquad \text{when } \boldsymbol{\tau} \in T_{\mu,\sigma_c}$$
$$\dot{d} > 0 \qquad \text{otherwise}$$

Two embedded DP cones

$\sigma_y < \sigma_c \;\Rightarrow\; T_{\mu,\sigma_y} \subset T_{\mu,\sigma_c}$

$\Rightarrow$ plasticity, then damage



BEVP construction: merge

Dissipation potential

$$\phi(\dot{\gamma}, \dot{\gamma}_p, \dot{d}) = \mathscr{I}_{\ker(\mathrm{tr})}(\dot{\gamma}) + \eta_0 \, |\dot{\gamma}|^2 + \underbrace{\eta \, |\dot{\gamma}_p|^2 + \left(\mathscr{I}_{-T_{\mu,\sigma_y}}\right)^*(\dot{\gamma}_p)}_{\phi_p(\dot{\gamma}_p)} + \phi_d(\dot{d})$$

Main results

- ϕ convex : satisfies the second principle of thermodynamics
- by construction : satisfies the Onsager symmetry principle

[Saramito, JNNFM, 2021]

Problem statement

(P): *find* $(d, \gamma_e, \boldsymbol{v}, p)$ *such that*

$$\begin{cases} \dot{d} - \nabla\phi_d^*(d, \gamma_e) = 0 \\ \overset{\triangledown}{\gamma}_e + \nabla\phi_p^*(d, \gamma_e) - D(\boldsymbol{v}) = 0 \\ \rho\dot{\boldsymbol{v}} - \mathbf{div}\,\boldsymbol{\sigma} = \boldsymbol{f} \\ \mathrm{div}\,\boldsymbol{v} = 0 \\ +B.C. + I.C. \end{cases}$$

with

$$\begin{aligned} \boldsymbol{\sigma} &= -p\boldsymbol{I} + 2\eta_0 D(\boldsymbol{v}) + \boldsymbol{\tau} &= \text{Cauchy stress} \\ \boldsymbol{\tau} &= 2G(d)\,\gamma_e + \lambda(d)\,(\mathrm{tr}\,\gamma_e)\boldsymbol{I} &= \text{elastic stress} \end{aligned}$$

Nonlinear Oldroyd-like + kinetic eqn for d
→ classical structure

[Saramito, JNNFM, 2021]

Notations

- viscoelasticity

$$\overset{\triangledown}{\gamma}_e + \nabla\phi_p^*(\boldsymbol{\tau}) = D(\boldsymbol{v})$$

where

$$\nabla\phi_p^*(\boldsymbol{\tau}) = \frac{\kappa_{\sigma_y}(\boldsymbol{\tau})}{2\,(1+\mu^2)\,\eta}\left(\boldsymbol{\tau} - \frac{\xi_{\mu,\sigma_y}(\boldsymbol{\tau})}{\sqrt{N}\,\mu}\boldsymbol{I}\right)$$

Notations

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- damage

$$\dot{d} = \nabla\phi_d^*(d)$$

where

$$\nabla\phi_d^*(d) = \frac{\kappa_{\sigma_c}(\boldsymbol{\tau})\ (1-d)}{2\,(1+\mu^2)\,\eta_d}\ Y$$

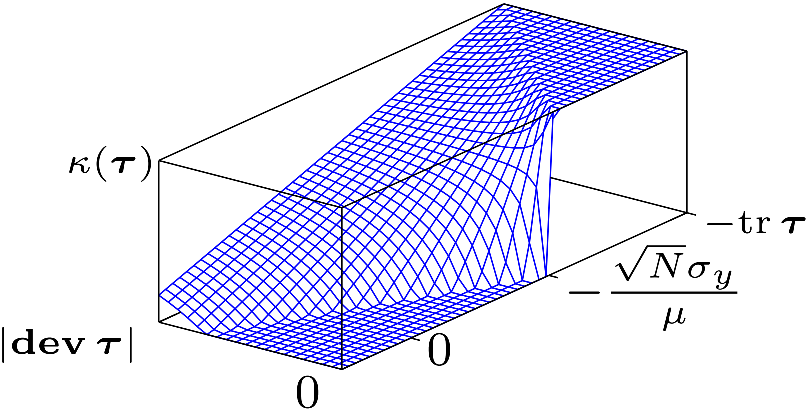
$$Y = -\left\{2G'(d)\gamma_e + \lambda'(d)\,\mathrm{tr}(\gamma_e)\boldsymbol{I}\right\}:\gamma_e$$

$$= \text{strain energy release rate}$$

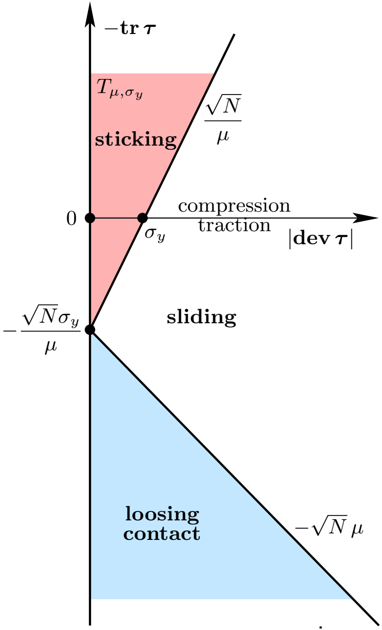
The $\kappa(\boldsymbol{\tau})$ function

$$\kappa(\boldsymbol{\tau}) = \begin{cases} 1+\mu^2 & \text{when } -\mu^2|\mathbf{dev}\,\boldsymbol{\tau}| \geqslant \sigma_y - \dfrac{\mu}{\sqrt{N}}\,\mathrm{tr}\,\boldsymbol{\tau} \\ 1 - \dfrac{\sigma_y - \dfrac{\mu}{\sqrt{N}}\,\mathrm{tr}\,\boldsymbol{\tau}}{|\mathbf{dev}\,\boldsymbol{\tau}|} & \text{when } -\mu^2|\mathbf{dev}\,\boldsymbol{\tau}| < \sigma_y - \dfrac{\mu}{\sqrt{N}}\,\mathrm{tr}\,\boldsymbol{\tau} < |\mathbf{dev}\,\boldsymbol{\tau}| \\ 0 & \text{otherwise} \end{cases}$$

The $\kappa(\boldsymbol{\tau})$ function: elevation view

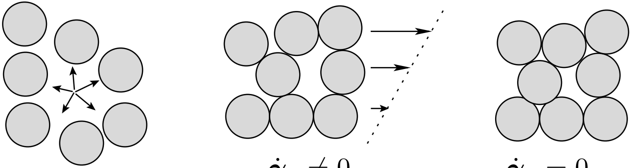


The $\kappa(\boldsymbol{\tau})$ function: top view



The $\kappa(\boldsymbol{\tau})$ function: cut along the trace axis

losing contact | **sliding** | **sticking**



$\dot{\gamma}_p \neq 0$ (sliding) | $\dot{\gamma}_p = 0$ (sticking)

$$\boldsymbol{\tau} = 2\eta\dot{\gamma}_p + \frac{\sigma_y \boldsymbol{I}}{\sqrt{N}\,\mu} \qquad\qquad \boldsymbol{\tau} \neq 2\eta\dot{\gamma}_p + \frac{\sigma_y \boldsymbol{I}}{\sqrt{N}\,\mu}$$

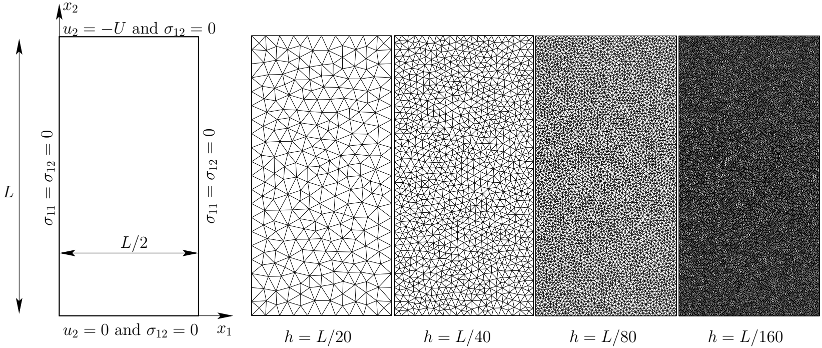
linear fluid | nonlinear fluid | solid

$-\mu^2|\mathbf{dev}\,\boldsymbol{\tau}|$ $\quad 0 \quad$ $|\mathbf{dev}\,\boldsymbol{\tau}|$ $\quad \sigma_y - \dfrac{\mu\,\mathrm{tr}\,\boldsymbol{\tau}}{\sqrt{N}}$

$\Longrightarrow$ micro-structural interpretation

granular matter : 3 states = gaz, liquid, solid

Example: uniaxial compression

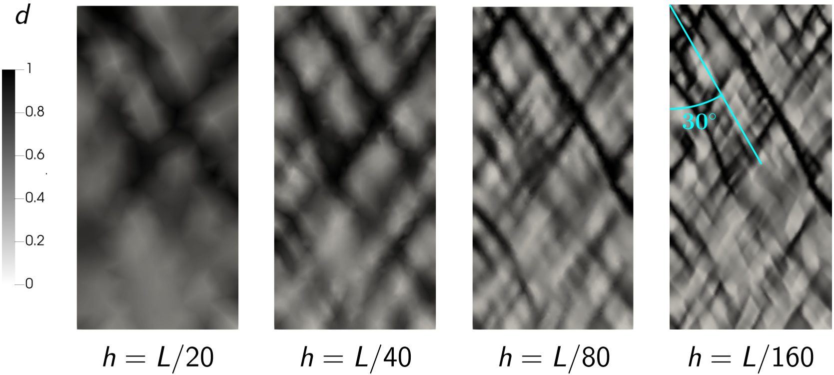


Uniform random heterogeneity

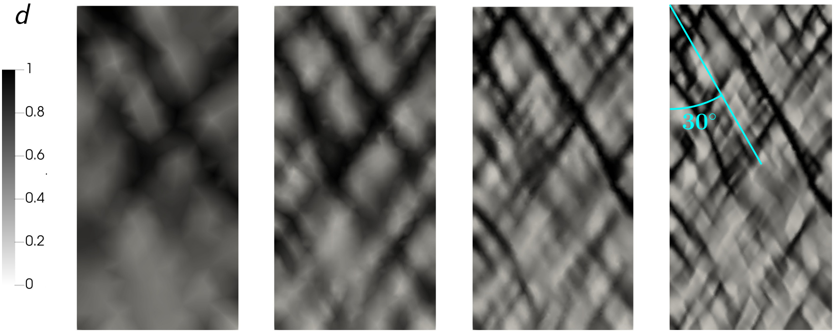
$$
\begin{aligned}
\sigma_{y0}(\boldsymbol{x}) &= \bar{\sigma}_{y0}\,(1 + 0.3\,\chi(\boldsymbol{x})) \\
\sigma_c(\boldsymbol{x}) &= \bar{\sigma}_c\,(1 + 0.3\,\chi(\boldsymbol{x})) \\
\chi(\boldsymbol{x}) &\in [-1, 1]
\end{aligned}
$$

$\rightarrow$ breaks symmetry

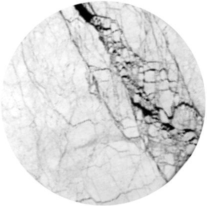
Damage value



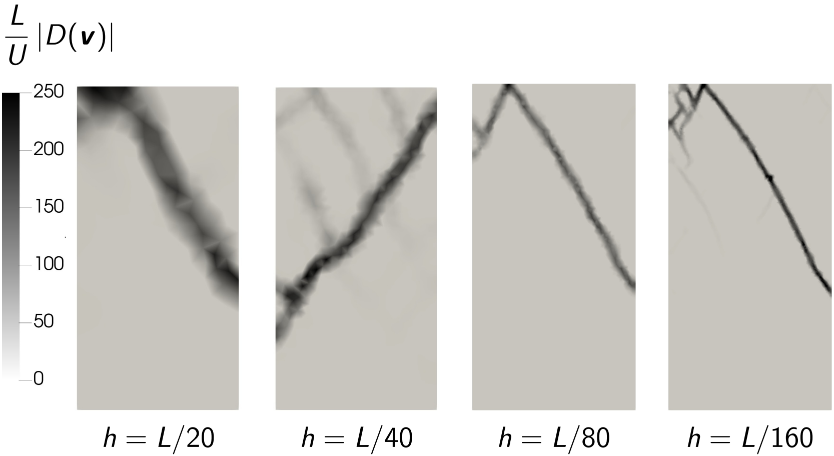
Damage value



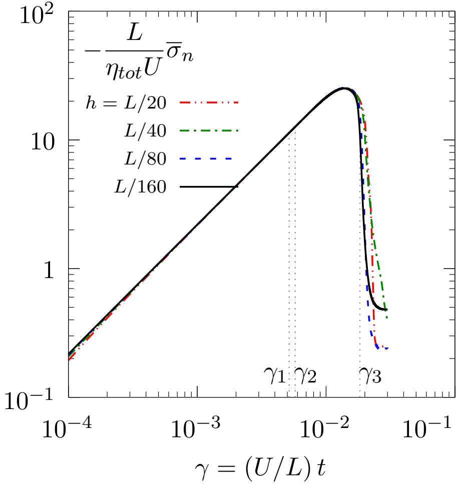
$h = L/20$ $\quad h = L/40$ $\quad h = L/80$ $\quad h = L/160$



Deformation rate



Averaged normal stress on top boundary



$$\overline{\sigma}_n(t) = \frac{2}{L}\int_{top} \sigma_{yy}\,\mathrm{d}x$$

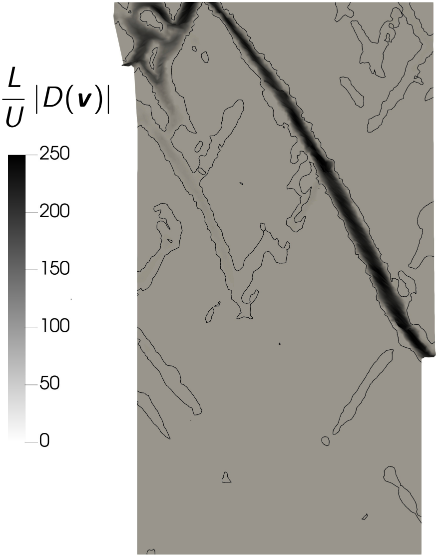
Four flow regimes

γ_1: first plastic event

γ_2: first damage

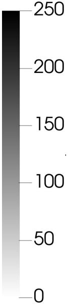
γ_3: post-failure

Post-failure: deformed geometry & yield surfaces

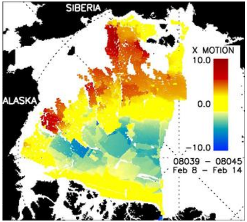


Post-failure: deformed geometry & yield surfaces

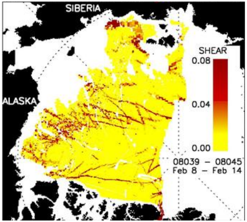
$\frac{L}{U}|D(\boldsymbol{v})|$

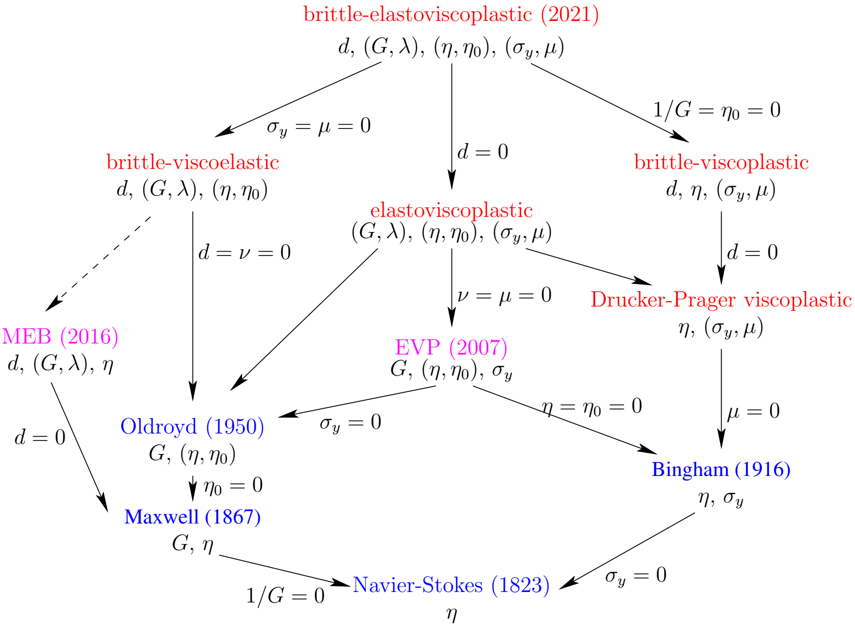


velocity v_x



deformation rate $|D(\boldsymbol{v})|$




brittle-elastoviscoplastic (2021)
d, (G, λ), (η, η_0), (σ_y, μ)
$\sigma_y = \mu = 0$
$d = 0$
$1/G = \eta_0 = 0$
brittle-viscoelastic
d, (G, λ), (η, η_0)
brittle-viscoplastic
d, η, (σ_y, μ)
elastoviscoplastic
(G, λ), (η, η_0), (σ_y, μ)
$d = \nu = 0$
$d = 0$
$\nu = \mu = 0$
Drucker-Prager viscoplastic
η, (σ_y, μ)
MEB (2016)
d, (G, λ), η
EVP (2007)
G, (η, η_0), σ_y
$\eta = \eta_0 = 0$
$\mu = 0$
$d = 0$
Oldroyd (1950)
G, (η, η_0)
$\sigma_y = 0$
Bingham (1916)
η, σ_y
$\eta_0 = 0$
Maxwell (1867)
G, η
$1/G = 0$
Navier-Stokes (1823)
η
$\sigma_y = 0$

5. Conclusion

- EVP & BEVP models: 3D, frame invariance, second principle
  few parameters, all measurable: G, σ_y, μ

- EVP: quantitative predictions on complex geometry
  $\Rightarrow$ negative wake
  model - experiment $\leqslant 5\%$
  pre-stress protocol clarification

- BEVP: new Drucker-Prager VP fluid = Bingham + friction
  friction: microstructural interpretation

Perspectives

- EVP: biological tissues, embryogenesis

- BEVP: sea ice, climate change, earth cracks, granular matter

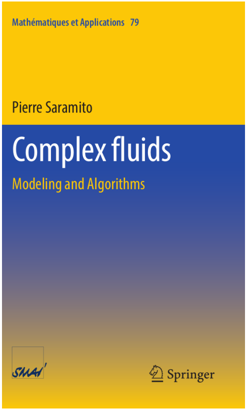
More reading

papers:

Cheddadi, Saramito, Dollet, Raufaste, Graner, EPJE, 2011

Dansereau, Weiss, Saramito, Lattes Cryosphere, 2016

Saramito, JNNFM, 2021, 2007



books: Saramito, *Complex fluids*
Springer, 2016

Saramito, *Continuous modeling from thermodynamics*
Springer, 2023, to appear

code: Saramito, 2022
Rheolef FEM C++ library
Free software: GPL licence
http://www-ljk.imag.fr/membres/Pierre.Saramito/rheolef