Shallow water as a gauge theory

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Abstract

Based on [1] and [2].

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1 Shallow water

We consider a $(2+1)$-dimensional spacetime with coordinates $(t, \boldsymbol{x})$. The height and horizontal velocity of the fluid are described respectively by $h(t, \boldsymbol{x})$ and $\boldsymbol{u}(t, \boldsymbol{x})$. In the Eulerian picture, we define the covariant derivative

$$D_t \coloneqq \partial_t + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) = \partial_t + u^i \partial_i. \tag{1.1}$$

Spatial indices are lowered and raised with the Euclidean metric δ_{ij}. The shallow water dynamics is then described by[1]

$$D_t h + h(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = 0, \qquad D_t h + h\partial_i u^i = 0, \tag{1.2a}$$

$$D_t \boldsymbol{u} + g\boldsymbol{\nabla} h - f{*}\boldsymbol{u} = 0, \qquad D_t u^i + g\partial^i h - f\varepsilon^{ij} u_j = 0, \tag{1.2b}$$

where the dual $*\boldsymbol{u}$ has components $(*u)^i = \varepsilon^{ij} u_j = (u_2, -u_1)$ and is such that $**\boldsymbol{u} = -\boldsymbol{u}$. Here g is the gravitational acceleration and f is the Coriolis parameter. We consider them as spacetime constants, so that they satisfy $\partial_\mu g = 0 = \partial_\mu f$. Introducing the 2-dimensional relative vorticity $\zeta = \boldsymbol{\nabla} \cdot *\boldsymbol{u} = \varepsilon^{ij} \partial_i u_j$ and the absolute vorticity $\xi \coloneqq \zeta + f$, we can rewrite these two equations as

$$\partial_t h + \boldsymbol{\nabla} \cdot (h\boldsymbol{u}) = 0, \qquad \partial_t h + \partial_i (h u^i) = 0, \tag{1.3a}$$

$$\partial_t \xi + \boldsymbol{\nabla} \cdot (\xi \boldsymbol{u}) = 0, \qquad \partial_t \xi + \partial_i (\xi u^i) = 0. \tag{1.3b}$$

Equation (1.3a) follows immediately from (1.2a), while (1.3b) follows from the contraction of (1.2b) with $\boldsymbol{\nabla} \cdot *$ and the use of the 2-dimensional identity $\boldsymbol{\nabla} \cdot \big((\boldsymbol{u} \cdot \boldsymbol{\nabla})(*\boldsymbol{u})\big) = \boldsymbol{\nabla} \cdot (\zeta \boldsymbol{u})$. In coordinates this reduces to $\varepsilon^{ij} \partial_i u^k \partial_k u_j = \zeta \partial_i u^i$.

Equation (1.3a) is the conservation of mass. The conserved quantity associated with equation (1.3b) is the circulation

$$\Gamma \coloneqq \int_{\mathcal{S}} \mathrm{d}^2 x\, \xi = \int_{\mathcal{S}} \mathrm{d}^2 x\, (\zeta + f) = \int_{\mathcal{S}} \mathrm{d}^2 x \left(\varepsilon^{ij} \partial_i u_j + \frac{1}{2} f \partial_i x^i \right) = \int_{\mathcal{S}} \mathrm{d}^2 x\, \varepsilon^{ij} \partial_i \left(u_j + \frac{1}{2} f x^k \varepsilon_{kj} \right). \tag{1.4}$$

The fact that this conserved quantity as actually a total derivative, and can therefore be rewritten using Stokes' theorem as a contour integral over $\mathcal{C} = \partial \mathcal{S}$, suggests that it is the charge of a gauge theory. This is precisely what we want to show and study.

1.1 Gauge theory formulation

Equations (1.3) are both conservation equations $\partial_\mu J^\mu = \partial_t J^0 + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$, for currents given respectively by $J^\mu = (J^0, \boldsymbol{J}) = (h, h\boldsymbol{u})$ and $\tilde{J}^\mu = (\tilde{J}^0, \tilde{\boldsymbol{J}}) = (\xi, \xi \boldsymbol{u})$. One can note in particular that

$$\tilde{J}^\mu = q J^\mu, \qquad q = \frac{\xi}{h}, \tag{1.5}$$

[1] We will often go back and forth between index-full and index-free notation.

where q is the so-called (Rossby, or shallow water) potential vorticity. One can then deduce from this relation and from the conservation of the two currents that $D_t q \mathrel{\hat{=}} 0$ (note that we use the SW EOMs to obtain this). Let us now introduce the pairs of $(2+1)$-dimensional magnetic and electric fields

$$B := J^0 = h, \qquad \boldsymbol{E} := *\boldsymbol{J} = h*\boldsymbol{u}, \qquad E^i := \varepsilon^{ij} J_j = h\varepsilon^{ij} u_j, \tag{1.6a}$$

$$\tilde{B} := \tilde{J}^0 = \xi, \qquad \tilde{\boldsymbol{E}} := *\tilde{\boldsymbol{J}} = \xi*\boldsymbol{u}, \qquad \tilde{E}^i := \varepsilon^{ij} \tilde{J}_j = \xi\varepsilon^{ij} u_j, \tag{1.6b}$$

These definitions imply in particular that the velocity profile is $\boldsymbol{u} = -*\boldsymbol{E}/B = -*\tilde{\boldsymbol{E}}/\tilde{B}$, and also that $\partial_i(E^i/B) = \zeta$. With this, the conservation equations take the form of Bianchi identities, i.e.

$$\partial_\mu J^\mu = 0 = \partial_\mu \tilde{J}^\mu \qquad \Leftrightarrow \qquad \partial_t B - \boldsymbol{\nabla} \cdot *\boldsymbol{E} = 0 = \partial_t \tilde{B} - \boldsymbol{\nabla} \cdot *\tilde{\boldsymbol{E}}. \tag{1.7}$$

Let us now write the electric and magnetic fields in terms of two U(1) gauge fields as

$$B = \varepsilon^{ij}\partial_i A_j, \qquad E_i = \partial_t A_i - \partial_i A_0, \qquad J^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu A_\rho, \tag{1.8a}$$

$$\tilde{B} = \varepsilon^{ij}\partial_i \tilde{A}_j, \qquad \tilde{E}_i = \partial_t \tilde{A}_i - \partial_i \tilde{A}_0, \qquad \tilde{J}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{A}_\rho. \tag{1.8b}$$

These gauge fields will now be used to build a Lagrangian.

1.1.1 Lagrangian

Let us now forget completely about all the information introduced above, and consider completely arbitrary gauge fields $(A_\mu, \tilde{A}_\mu)$ and the associated electric and magnetic fields (E_i, B) and $(\tilde{E}_i, \tilde{B})$. By definition we then automatically have (1.7), but at this point these equations do not imply anything physical, since we have not provided a map between the electromagnetic fields and the physical fields (h, u^i) of the SW model.

In order to continue however, we need to assume the relation (1.6a) for the non-tilde sector. This then automatically implies the conservation equation (1.3a). The other half of the identification, namely (1.6b), will be obtained by combining three EOMs coming from our Lagrangian. Consider therefore the Lagrangian (note however that we are here in the Eulerian fluid picture)

$$\begin{aligned} L[A,\tilde{A}] &= \frac{1}{2}\left(\frac{E^2}{B} - gB^2\right) + fA_0 - \varepsilon^{\mu\nu\rho}A_\mu\partial_\nu\tilde{A}_\rho \\ &= \frac{1}{2}\left(\frac{E^2}{B} - gB^2\right) + fA_0 - \varepsilon^{\mu\nu\rho}A_\mu\partial_\nu\alpha\partial_\rho\beta, \end{aligned} \tag{1.9}$$

where $E^2 = \boldsymbol{E}\cdot\boldsymbol{E}$, and where in the second line we have used the so-called Clebsch parametrization $\tilde{A}_\mu = \partial_\mu\chi + \alpha\partial_\mu\beta$. One should note that the first two terms in the Lagrangian are

$$\frac{E^2}{2B} - \frac{1}{2}gB^2 = \frac{1}{2}hu^2 - \frac{1}{2}gh^2, \tag{1.10}$$

and therefore represent kinetic energy minus potential energy. As explained in details in [1], these two terms are not sufficient to produce the correct EOMs, and they need to be supplemented by the auxiliary fields $\tilde{A}_\mu$. The variation of the Lagrangian is

$$
\begin{aligned}
\delta L &= \frac{E^i}{B}\delta E_i - \left(\frac{E^2}{2B^2} + gB\right)\delta B + f\delta A_0 - \varepsilon^{\mu\nu\rho}\delta A_\mu \partial_\nu \tilde{A}_\rho - \varepsilon^{\mu\nu\rho} A_\mu \big(\partial_\nu \delta\alpha \partial_\rho \beta + \partial_\nu \alpha \partial_\rho \delta\beta\big) \\
&= \left[\partial_i \left(\frac{E^i}{B}\right) + f - \tilde{B}\right]\delta A_0 - \left[\partial_t \left(\frac{E^i}{B}\right) + \varepsilon^{ij}\partial_j \left(\frac{E^2}{2B^2} + gB\right) - \varepsilon^{ij}\tilde{E}_j\right]\delta A_i + \varepsilon^{\mu\nu\rho}\partial_\mu A_\nu \big(\partial_\rho \alpha \delta\beta - \delta\alpha \partial_\rho \beta\big) \\
&\quad + \partial_t \left(\frac{E^i}{B}\delta A_i\right) - \partial_i \left(\frac{E^i}{B}\delta A_0 + \left(\frac{E^2}{2B^2} + gB\right)\varepsilon^{ij}\delta A_j\right) + \partial_\mu \Big(\varepsilon^{\mu\nu\rho} A_\nu \big(\delta\alpha \partial_\rho \beta - \partial_\rho \alpha \delta\beta\big)\Big).
\end{aligned}
\tag{1.11}
$$

The EOMs are

$$
\begin{aligned}
\delta A_0 \quad &\Rightarrow \quad \partial_i \left(\frac{E^i}{B}\right) + f - \tilde{B} = \zeta + f - \tilde{B} = \xi - \tilde{B} = 0, && \text{(1.12a)} \\
\delta A_i \quad &\Rightarrow \quad \partial_t \left(\frac{E^i}{B}\right) + \varepsilon^{ij}\partial_j \left(\frac{E^2}{2B^2} + gB\right) - \varepsilon^{ij}\tilde{E}_j = 0, && \text{(1.12b)} \\
\delta\alpha \quad &\Rightarrow \quad \varepsilon^{\mu\nu\rho}\partial_\mu A_\nu \partial_\rho \beta = J^\mu \partial_\mu \beta = 0, && \text{(1.12c)} \\
\delta\beta \quad &\Rightarrow \quad \varepsilon^{\mu\nu\rho}\partial_\mu A_\nu \partial_\rho \alpha = J^\mu \partial_\mu \alpha = 0, && \text{(1.12d)}
\end{aligned}
$$

where for the second one we have used the identity $u^k \partial_j u_k - \zeta \varepsilon_{jk} u^k = u^k \partial_k u_j$. The first EOM tells us that $\tilde{B} = \xi$. The last two EOMs imply that $\partial_\mu \alpha$ and $\partial_\mu \beta$ are both orthogonal to J^μ, which in turn means that $\varepsilon^{\mu\nu\rho}\partial_\nu \alpha \partial_\rho \beta \propto J^\mu$. Since $\varepsilon^{\mu\nu\rho}\partial_\nu \alpha \partial_\rho \beta = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{A}_\rho = \tilde{J}^\mu$, we therefore conclude that $\tilde{J}^\mu \propto J^\mu$. Explicitly, we can expand the last two EOMs as

$$
\begin{aligned}
J^0 \partial_t \beta + J^i \partial_i \beta = 0 \quad &\Rightarrow \quad \partial_t \beta = -\frac{J^i}{J^0}\partial_i \beta, && \text{(1.13a)} \\
J^0 \partial_t \alpha + J^i \partial_i \alpha = 0 \quad &\Rightarrow \quad \partial_t \alpha = -\frac{J^i}{J^0}\partial_i \alpha. && \text{(1.13b)}
\end{aligned}
$$

Using the definition $\tilde{J}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{A}_\rho$ we finally find

$$
\begin{aligned}
\tilde{J}^0 &= \varepsilon^{ij}\partial_i \tilde{A}_j = \tilde{B} = \xi, && \text{(1.14a)} \\
\tilde{J}^i &= \varepsilon^{ij}(\partial_j \tilde{A}_0 - \partial_t \tilde{A}_j) \\
&= \varepsilon^{ij}(\partial_j \alpha \partial_t \beta - \partial_t \alpha \partial_j \beta) \\
&= \frac{J^k}{J^0}\varepsilon^{ij}(\partial_k \alpha \partial_j \beta - \partial_j \alpha \partial_k \beta) \\
&= \frac{J^i}{J^0}\varepsilon^{jk}\partial_j \alpha \partial_k \beta \\
&= \frac{J^i}{J^0}\tilde{B} \\
&= \xi u^i, && \text{(1.14b)}
\end{aligned}
$$

where we have used a non-trivial 2-dimensional identity in the middle. Using this result, the second EOM finally becomes the shallow water equation

$$
\partial_t \left(\frac{E^i}{B}\right) + \varepsilon^{ij}\partial_j \left(\frac{E^2}{2B^2} + gB\right) - \varepsilon^{ij}\tilde{E}_j = \varepsilon^{ij}\Big(D_t u_j + g\partial_j h - f\varepsilon_{jk} u^k\Big) = 0. \tag{1.15}
$$

1.1.2 Hamiltonian

To obtain the Lagrangian in Hamiltonian form, we first compute the momenta to find

$$P^i := \frac{\partial L}{\partial \partial_t A_i} = \frac{E^i}{B}, \qquad \tilde{P}^i := \frac{\partial L}{\partial \partial_t \tilde{A}_i} = -\varepsilon^{ij} A_j, \tag{1.16}$$

and then write

$$\begin{aligned} L &= \frac{E^i}{2B}(\partial_t A_i - \partial_i A_0) - \frac{1}{2} g B^2 + f A_0 - A_0 \tilde{B} + A_i \varepsilon^{ij} \tilde{E}_j \\ &= \frac{P^i}{2}(\partial_t A_i - \partial_i A_0) - \frac{1}{2} g B^2 + f A_0 - A_0 \tilde{B} + A_i \varepsilon^{ij} \tilde{E}_j \\ &= P^i \partial_t A_i + A_0 \big(\partial_i P^i + f - \tilde{B}\big) - \frac{1}{2}\big(BP^2 + gB^2\big) - \partial_i (A_0 P^i) + A_i \varepsilon^{ij} \tilde{E}_j \\ &= P^i \partial_t A_i + A_0 \big(\partial_i P^i + f - \tilde{B}\big) - \frac{1}{2}\big(BP^2 + gB^2\big) - \partial_i \big(A_0 P^i + \tilde{A}_0 \tilde{P}^i\big) + \tilde{P}^i \partial_t \tilde{A}_i + \tilde{A}_0 \partial_i \tilde{P}^i, \end{aligned} \tag{1.17}$$

where in the last line we have also performed the Legendre transform for the tilde sector using $A_i \varepsilon^{ij} \tilde{E}_j = \tilde{P}^i (\partial_t \tilde{A}_i - \partial_i \tilde{A}_0)$ (note also that $B = -\partial_i \tilde{P}^i$).

1.1.3 Global symmetries

Let us consider the global transformations $\delta_\epsilon A_\mu = 0$ and $\delta_\epsilon(\alpha, \beta) = (\epsilon\alpha, -\epsilon\beta)$ with $\partial_\mu \epsilon = 0$. Using (1.11), one can then easily see that

$$\delta_\epsilon L = \text{EOM}\, \delta_\epsilon \Phi + \partial_\mu \theta^\mu[\delta_\epsilon] = -\varepsilon^{\mu\nu\rho} \partial_\mu A_\nu \partial_\rho (\epsilon\alpha\beta) + \partial_\mu \theta^\mu[\delta_\epsilon] = \partial_\mu \Big(\theta^\mu[\delta_\epsilon] - \epsilon\alpha\beta\varepsilon^{\mu\nu\rho} \partial_\nu A_\rho\Big). \tag{1.18}$$

This gives a conserved Noether current

$$N^\mu = \alpha\beta\varepsilon^{\mu\nu\rho} \partial_\nu A_\rho = \alpha\beta J^\mu, \qquad \partial_\mu N^\mu \doteq 0. \tag{1.19}$$

The associated Noether charge is the time component

$$N^0 = \alpha\beta J^0 = \alpha\beta B, \tag{1.20}$$

and for consistency one can check using the EOMs (1.13) that

$$\begin{aligned} \partial_t N^0 &= (\partial_t \alpha\beta + \alpha \partial_t \beta) J^0 + \alpha\beta \partial_t J^0 \\ &\doteq -\frac{J^i}{J^0}(\partial_i \alpha\beta + \alpha \partial_i \beta) J^0 + \alpha\beta \partial_t J^0 \\ &= -J^i \partial_i (\alpha\beta) + \alpha\beta \partial_t J^0 \\ &= -\partial_i (\alpha\beta J^i) + \alpha\beta \partial_\mu J^\mu \\ &= -\partial_i (\alpha\beta J^i). \end{aligned} \tag{1.21}$$

1.1.4 Gauge symmetries

Let us consider as in [2] the gauge transformations $\delta_\lambda A_\mu=\partial_\mu\lambda$ and $\delta_\lambda\tilde{A}_\mu=0$. They act on the Lagrangian as

$$\delta_\lambda L=\partial_t(\lambda f)-\partial_\mu\big(\varepsilon^{\mu\nu\rho}\lambda\partial_\nu\tilde{A}_\rho\big)=\partial_t(\lambda f)-\partial_\mu\big(\lambda\tilde{J}^\mu\big)=-\partial_t(\lambda\zeta)-\partial_i(\lambda\xi u^i)=\partial_\mu b^\mu. \tag{1.22}$$

Writing the variation of the Lagrangian as $\delta L=\text{EOM}\,\delta\Phi+\partial_\mu\theta^\mu[\delta]$, we also find

$$\begin{aligned}\text{EOM}\,\delta_\lambda\Phi&=\left[\partial_i\left(\frac{E^i}{B}\right)+f-\tilde{B}\right]\partial_t\lambda-\left[\partial_t\left(\frac{E^i}{B}\right)+\varepsilon^{ij}\partial_j\left(\frac{E^2}{2B^2}+gB\right)-\varepsilon^{ij}\tilde{E}_j\right]\partial_i\lambda\\&=\big(\partial_t\tilde{B}-\varepsilon^{ij}\partial_i\tilde{E}_j\big)\lambda-\partial_i\left(\left[\partial_t\left(\frac{E^i}{B}\right)+\varepsilon^{ij}\partial_j\left(\frac{E^2}{2B^2}+gB\right)-\varepsilon^{ij}\tilde{E}_j\right]\lambda\right)\\&=\big(\partial_t\tilde{B}-\varepsilon^{ij}\partial_i\tilde{E}_j\big)\lambda-\partial_i\left(\lambda\varepsilon^{ij}\big(D_tu_j+g\partial_jh-f\varepsilon_{jk}u^k\big)\right).\end{aligned} \tag{1.23}$$

This has the expected form

$$\text{EOM}\,\delta_\lambda\Phi=(\text{Noether identities})-\partial_\mu C^\mu, \tag{1.24}$$

with $C^\mu\mathrel{\hat{=}}0$. Here the Noether identity leading to the vanishing of the bulk term is the second Bianchi identity in (1.7) and $C^0=0$. The Noether current is then defined by $N^\mu:=\theta^\mu[\delta_\lambda]-b^\mu$, and we find the components

$$N^0=\frac{E^i}{B}\partial_i\lambda+\lambda\zeta=\partial_i(\lambda\varepsilon^{ij}u_j), \tag{1.25a}$$

$$N^i=-\frac{E^i}{B}\partial_t\lambda-\left(\frac{E^2}{2B^2}+gB\right)\varepsilon^{ij}\partial_j\lambda+\lambda\xi u^i \tag{1.25b}$$

$$=\lambda\varepsilon^{ij}\left(D_tu_j+g\partial_jh-f\varepsilon_{jk}u^k\right)-\partial_t(\lambda\varepsilon^{ij}u_j)-\varepsilon^{ij}\partial_j\left[\left(\frac{u^2}{2}+gh\right)\lambda\right]. \tag{1.25c}$$

As expected, the Noether current satisfies $\partial_\mu N^\mu=\partial_\mu C^\mu\mathrel{\hat{=}}0$, and can be written as

$$N^\mu=C^\mu+\partial_\nu Q^{\mu\nu}, \tag{1.26}$$

with

$$Q^{0i}=-Q^{i0}=\lambda\varepsilon^{ij}u_j,\qquad\qquad Q^{ij}=-Q^{ji}=-\lambda\varepsilon^{ij}\left(\frac{u^2}{2}+gh\right). \tag{1.27}$$

Integrating the current on a surface Σ at fixed x^μ, we find

$$\mathcal{Q}=\int_\Sigma(\mathrm{d}x)_\mu N^\mu\mathrel{\hat{=}}\int_\Sigma(\mathrm{d}x)_\mu\partial_\nu Q^{\mu\nu}=\oint_{\partial\Sigma}(\mathrm{d}x)_{\mu\nu}Q^{\mu\nu}. \tag{1.28}$$

In particular, for a surface at fixed time we find

$$\mathcal{Q}=\Gamma_\lambda=\oint_{\partial\Sigma}(\mathrm{d}x)_i\,\lambda\varepsilon^{ij}u_j, \tag{1.29}$$

which reduces to (1.4) when λ is constant.

2 Linearized shallow water

Let us now consider the linearized shallow water system. For this, we linearize around a fixed fluid height H by writing

$$h(t,\boldsymbol{x}) = H + \eta(t,\boldsymbol{x}). \tag{2.1}$$

Keeping only terms linear in η and $\boldsymbol{u}$, the covariant derivative D_t will always reduce to ∂_t. The fluid equations (1.2) become

$$\partial_t\eta + H(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = 0, \qquad \partial_t\eta + H\partial_i u^i = 0, \tag{2.2a}$$

$$\partial_t\boldsymbol{u} + g\boldsymbol{\nabla}\eta - f{*}\boldsymbol{u} = 0, \qquad \partial_t u^i + g\partial^i\eta - f\varepsilon^{ij}u_j = 0. \tag{2.2b}$$

The conservation equations (1.3) now become

$$\partial_t\eta + H(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = 0, \qquad \partial_t\eta + H\partial_i u^i = 0, \tag{2.3a}$$

$$\partial_t\zeta + f(\boldsymbol{\nabla}\cdot\boldsymbol{u}) = 0, \qquad \partial_t\zeta + f\partial_i u^i = 0, \tag{2.3b}$$

which can be used to obtain

$$\partial_t q = 0, \qquad q(\boldsymbol{x}) \coloneqq H\zeta - f\eta. \tag{2.4}$$

This potential vorticity is related to the non-linear one (1.5) by

$$q_{\text{SW}} = \frac{\xi}{h} = \frac{1}{H}(\zeta+f)\left(1+\frac{\eta}{H}\right)^{-1} \simeq \frac{1}{H}(\zeta+f)\left(1-\frac{\eta}{H}\right) = \frac{q}{H^2} + \frac{f}{H}. \tag{2.5}$$

The electric and magnetic fields (1.6) become

$$B = H + \eta = \frac{1}{f}(H\xi - q), \qquad \boldsymbol{E} = H{*}\boldsymbol{u}, \tag{2.6a}$$

$$\tilde{B} = \xi, \qquad \tilde{\boldsymbol{E}} = f{*}\boldsymbol{u}, \tag{2.6b}$$

One can see in particular that

$$\tilde{B} = \frac{1}{H}(fB + q), \qquad \tilde{\boldsymbol{E}} = \frac{f}{H}\boldsymbol{E}, \tag{2.7}$$

which means, using the definitions (1.8) of the electric and magnetic fields, that up to a gauge transformation we can write

$$\tilde{A}_i = \frac{1}{H}\left(fA_i - \frac{q}{2}\varepsilon_{ij}x^j\right), \qquad \tilde{A}_0 = \frac{f}{H}A_0. \tag{2.8}$$

Finally, let us note that in term of B and $\boldsymbol{E}$ the linearized shallow water equations (2.2) take the form

$$\partial_t B - \boldsymbol{\nabla}\cdot{*}\boldsymbol{E} = 0, \tag{2.9a}$$

$$\partial_t{*}\boldsymbol{E} - c^2\boldsymbol{\nabla}B + f\boldsymbol{E} = 0, \tag{2.9b}$$

where $c = \sqrt{gH}$.

2.1 Lagrangian and gauge symmetries

The first equation in (2.9) is the Bianchi identity, and is therefore automatically satisfied one we work with the potentials A_μ. The second equation needs to be derived from a variational principle. For this, we consider the Lagrangian

$$L[A]=\frac{1}{2}\Big(E^2-c^2B^2-f\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho+2(fH-q)A_0\Big). \tag{2.10}$$

This is the Lagrangian for Maxwell–Chern–Simons theory with a non-trivial $(2+1)$-dimensional metric and a background electric charge introduced with a source in A_0. Its variation is

$$\begin{aligned}\delta L&=\Big(\partial_iE^i-fB+fH-q\Big)\delta A_0-\Big(\partial_tE^i+c^2\varepsilon^{ij}\partial_jB-f\varepsilon^{ij}E_j\Big)\delta A_i\\&\quad+\partial_t(E^i\delta A_i)-\partial_i\Big(E^i\delta A_0+c^2B\varepsilon^{ij}\delta A_j\Big)+\frac{1}{2}\partial_\mu\big(f\varepsilon^{\mu\nu\rho}A_\nu\delta A_\rho\big),\end{aligned} \tag{2.11}$$

and the EOMs are

$$\delta A_0\qquad\Rightarrow\qquad\partial_iE^i-fB+fH-q=0, \tag{2.12a}$$

$$\delta A_i\qquad\Rightarrow\qquad\partial_tE^i+c^2\varepsilon^{ij}\partial_jB-f\varepsilon^{ij}E_j=0. \tag{2.12b}$$

Remembering that (2.6) implies $\partial_iE^i=H\zeta$, the first EOM (which is the Gauss law) tells us that $q=H\zeta-f\eta$. Taking the dual of the second EOM and recalling that $*^2=-1$, we recover (2.9b). Let us now consider the gauge transformations $\delta_\lambda A_\mu=\partial_\mu\lambda$. They act on the Lagrangian as

$$\delta_\lambda L=-\frac{1}{2}\partial_\mu\big(f\varepsilon^{\mu\nu\rho}\lambda\partial_\nu A_\rho\big)+(fH-q)\partial_t\lambda=\partial_\mu b^\mu. \tag{2.13}$$

We also have

$$\begin{aligned}\text{EOM}\delta_\lambda\Phi&=\Big(\partial_iE^i-fB+fH-q\Big)\partial_t\lambda-\Big(\partial_tE^i+c^2\varepsilon^{ij}\partial_jB-f\varepsilon^{ij}E_j\Big)\partial_i\lambda\\&=\Big(f\partial_tB-f\varepsilon^{ij}\partial_iE_j+\partial_tq\Big)\lambda\\&\quad+\partial_t\Big[\Big(\partial_iE^i-fB+fH-q\Big)\lambda\Big]-\partial_i\Big[\Big(\partial_tE^i+c^2\varepsilon^{ij}\partial_jB-f\varepsilon^{ij}E_j\Big)\lambda\Big].\end{aligned} \tag{2.14}$$

For the Noether current $N^\mu\coloneqq\theta^\mu[\delta_\lambda]-b^\mu$ we find

$$N^0=-\Big(\partial_iE^i-fB+fH-q\Big)\lambda+\partial_i\left[\left(E^i-\frac{1}{2}f\varepsilon^{ij}A_j\right)\lambda\right], \tag{2.15a}$$

$$\begin{aligned}N^i&=\Big(\partial_tE^i+c^2\varepsilon^{ij}\partial_jB-f\varepsilon^{ij}E_j\Big)\lambda\\&\quad-\partial_t\left[\left(E^i-\frac{1}{2}f\varepsilon^{ij}A_j\right)\lambda\right]-\varepsilon^{ij}\partial_j\left[\left(c^2B+\frac{1}{2}fA_0\right)\lambda\right],\end{aligned} \tag{2.15b}$$

which is as expected of the form $N^\mu=C^\mu+\partial_\nu Q^{\mu\nu}$ with

$$Q^{0i}=-Q^{i0}=\lambda\left(E^i-\frac{1}{2}f\varepsilon^{ij}A_j\right),\qquad\qquad Q^{ij}=-Q^{ji}=-\lambda\varepsilon^{ij}\left(c^2B+\frac{1}{2}fA_0\right). \tag{2.16}$$

In particular, for a surface at fixed time we find

$$\mathcal{Q}=\Gamma_\lambda=\oint_{\partial\Sigma}(\mathrm{d}x)_i\,\lambda\left(E^i-\frac{1}{2}f\varepsilon^{ij}A_j\right)=\oint_{\partial\Sigma}(\mathrm{d}x)_i\,\lambda H\varepsilon^{ij}\left(u_j-\frac{1}{2H}fA_j\right). \tag{2.17}$$

Up to a rescaling, the global charge obtained for $\lambda=1$ is the circulation charge (1.4) if we identify

$$A_i=H\varepsilon_{ij}x^j. \tag{2.18}$$

In the temporal gauge where $A_0=0$ and $E_i=\partial_tA_i$, this is exactly what we obtain when using the relation $E_i=H\varepsilon_{ij}u^j=H\varepsilon_{ij}\partial_tu^j$.

2.2 Flat band solution and Poincaré waves

Let us consider the ansatz

$$\boldsymbol{u}=\hat{\boldsymbol{u}}e^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})},\qquad\qquad \eta=\hat{\eta}e^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}. \tag{2.19}$$

Plugging this into the linearized shallow water equations (2.2) we obtain the eigenvalue equation

$$\begin{pmatrix}0 & Hk_1 & Hk_2\\ gk_1 & 0 & -if\\ gk_2 & if & 0\end{pmatrix}\begin{pmatrix}\hat{\eta}\\ \hat{u}_1\\ \hat{u}_2\end{pmatrix}=\omega\begin{pmatrix}\hat{\eta}\\ \hat{u}_1\\ \hat{u}_2\end{pmatrix}. \tag{2.20}$$

There are two types of solutions.

Flat band solution. This solution is given by

$$\omega=0,\qquad\qquad \begin{pmatrix}\hat{\eta}\\ \hat{u}_1\\ \hat{u}_2\end{pmatrix}=\frac{1}{f}\begin{pmatrix}f\\ igk_2\\ -igk_1\end{pmatrix},\qquad\qquad q\sim c^2k^2+f^2. \tag{2.21}$$

Poincaré waves. These solutions are given by

$$\omega^2=c^2k^2+f^2,\qquad\qquad \begin{pmatrix}\hat{\eta}\\ \hat{u}_1\\ \hat{u}_2\end{pmatrix}=\begin{pmatrix}Hk^2\\ k_1\omega-ifk_2\\ k_2\omega+ifk_1\end{pmatrix},\qquad\qquad q=0. \tag{2.22}$$

Small wavelength modes propagate at c and the Coriolis parameter f appears as an effective mass (like when we introduce a photon mass by adding a Chern–Simons term to 3d Maxwell).

2.3 Memory

Let us now consider for simplicity the case $q=0$ and work in the temporal gauge $A_0=0$. Let us consider the circulation charge aspect in (2.17) and denote it by

$$\gamma_i\coloneqq u_i-\frac{1}{2H}fA_i. \tag{2.23}$$

From the temporal gauge we get

$$u_i = -\frac{1}{H}\varepsilon_{ij}E^j = -\frac{1}{H}\varepsilon_{ij}\partial_t A^j \quad \Rightarrow \quad \varepsilon_{ij}u^j = \frac{1}{H}\partial_t A_i \quad \Rightarrow \quad \zeta = \varepsilon^{ij}\partial_i u_j = \frac{1}{H}\partial_t\partial_i A^i, \tag{2.24}$$

and from the vanishing potential vorticity $q = 0$ we get $\eta = H\zeta/f$. The EOM (2.2b) can then be rewritten as

$$\partial_t u_i = -\frac{g}{f}\partial_t\partial_i\partial_j A^j + \frac{f}{H}\partial_t A_i, \tag{2.25}$$

and the time evolution of the charge aspect becomes

$$\partial_t \gamma_i = \frac{f}{2H}\partial_t A_i - \frac{g}{f}\partial_t\partial_i\partial_j A^j. \tag{2.26}$$

For time-independent λ, we can therefore get the memory equation for the variation of the charge $\Delta\Gamma_\lambda = \Gamma_\lambda(t=+\infty) - \Gamma_\lambda(t=-\infty)$.

2.4 Edge modes

Let us consider coordinates $(t, x^i) = (t, x, y)$ and a boundary at fixed x. The variation of the Chern–Simons action is

$$\begin{aligned}
\delta S &= \delta \int_M \varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \\
&= 2\int_M \varepsilon^{\mu\nu\rho}\delta A_\mu\partial_\nu A_\rho - \int_{\partial M}\varepsilon^{\mu\nu\rho}n_\mu A_\nu\delta A_\rho \\
&= 2\int_M \varepsilon^{\mu\nu\rho}\delta A_\mu\partial_\nu A_\rho + \int_{\partial M}(A_t\delta A_y - A_y\delta A_t).
\end{aligned} \tag{2.27}$$

as boundary conditions we could impose for example $A_y|_{\partial M} = cA_t$. Imposing the constraint $F_{ij} = 0$, which is solved by $A_i = \partial_i a$, the bare action becomes a surface integral

$$S = \int_{\partial M}(\partial_t a - A_t)\partial_y a = \int_{\partial M}(\partial_t a - c\partial_y a)\partial_y a, \tag{2.28}$$

where we have used the boundary condition. The equation of motion is $\partial_t\partial_y a - c\partial_y^2 a = 0$, or in terms of $\chi = \partial_y a$ a chiral equation $\partial_t\chi - c\partial_y\chi = 0$ with solution $\chi(y + ct)$.

For linearized shallow water, using (2.11) we find that the on-shell variation of the action with boundary at fixed x is

$$\delta S = \frac{1}{2}\int_{\partial M}\Big(f(A_y\delta A_t - A_t\delta A_y) - 2E_x\delta A_t - 2c^2 B\delta A_y\Big). \tag{2.29}$$

Let us consider the boundary conditions $A_t|_{\partial M} = \mathrm{C}^{\mathrm{ste}}$ and $A_y|_{\partial M} = \mathrm{C}^{\mathrm{ste}}$. This implies that $E_y|_{\partial M} = 0 = u_x|_{\partial M}$, i.e. that there is no flow through the boundary. We can look for solutions which extend the boundary conditions throughout the bulk, i.e. have $A_t = 0 = A_y$ everywhere, which also implies u_x everywhere. Let us then consider the two EOMs

$$\partial_i E^i - fB = 0, \qquad \partial_t E^i + c^2\varepsilon^{ij}\partial_j B - f\varepsilon^{ij}E_j = 0. \tag{2.30}$$

Using the ansatz

$$A_x(t, \boldsymbol{x}) = A(x)e^{i(\omega t - ky)}, \tag{2.31}$$

the second EOM tells us that

$$\partial_t E_x = -c^2 \partial_y B \qquad \Rightarrow \qquad \omega^2 = c^2 k^2, \tag{2.32}$$

and the first one then gives

$$\partial_x E_x = fB \qquad \Rightarrow \qquad \omega \partial_x A = kfA \qquad \Rightarrow \qquad \partial_x A = \pm \frac{f}{c} A. \tag{2.33}$$

In the northern hemisphere we have $f > 0$, and the normalizable solution is given by

$$A(x) \propto e^{-x/R}, \tag{2.34}$$

where $R = c/f$ is the Rossby radius of deformation. These solutions, travel in the direction of decreasing y, meaning towards the south. They are concentrated around $x = 0$, and are therefore "edge modes".

3 Incompressible Euler

Assuming that the density is $\rho = 1$, the incompressible Euler equations are

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \qquad \partial_i u^i = 0, \tag{3.1a}$$

$$D_t \boldsymbol{u} + \boldsymbol{\nabla} p = 0, \qquad D_t u^i + \partial^i p = 0, \tag{3.1b}$$

where p is the pressure. From (3.1a), the velocity can be written as the gradient of a stream function (or velocity potential) as

$$\boldsymbol{u} = *\boldsymbol{\nabla}\psi, \qquad u^i = \varepsilon^{ij}\partial_j \psi, \tag{3.2}$$

and when computing the vorticity we therefore obtain

$$\zeta = \boldsymbol{\nabla} \cdot *\boldsymbol{u} = \varepsilon^{ij}\partial_i u_j = -\Box\psi. \tag{3.3}$$

Acting with $\boldsymbol{\nabla} \cdot *$ on (3.1b) and using (3.1a), we obtain the ideal vorticity equation

$$\partial_t \zeta + \boldsymbol{\nabla} \cdot (\zeta \boldsymbol{u}) = \partial_t \zeta + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\zeta = D_t \zeta = 0. \tag{3.4}$$

This is a conservation equation $\partial_\mu J^\mu = \partial_t J^0 + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ for a current $J^\mu = (J^0, \boldsymbol{J}) = (\zeta, \zeta \boldsymbol{u})$. Let us now introduce the electric and magnetic fields

$$B := J^0 = \zeta, \qquad \boldsymbol{E} := *\boldsymbol{J} = \zeta * \boldsymbol{u} = -\zeta \boldsymbol{\nabla}\psi, \tag{3.5}$$

and also write them as (1.8a). Following [4], let us now consider the Lagrangian

$$L[A] = \frac{E^2}{2B} - pB - \frac{1}{2}\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{3.6}$$

Its variation is

$$\begin{aligned}\delta L &= \frac{E^i}{B}\delta E_i - \left(\frac{E^2}{2B^2}+p\right)\delta B - \frac{1}{2}\varepsilon^{\mu\nu\rho}\big(\delta A_\mu \partial_\nu A_\rho + A_\mu \partial_\nu \delta A_\rho\big) \\ &= \left[\partial_i\left(\frac{E^i}{B}\right) - B\right]\delta A_0 - \left[\partial_t\left(\frac{E^i}{B}\right) + \varepsilon^{ij}\partial_j\left(\frac{E^2}{2B^2}+p\right) - \varepsilon^{ij}E_j\right]\delta A_i \\ &\quad + \partial_t\left(\frac{E^i}{B}\delta A_i\right) - \partial_i\left(\frac{E^i}{B}\delta A_0 + \left(\frac{E^2}{2B^2}+p\right)\varepsilon^{ij}\delta A_j\right) + \frac{1}{2}\partial_\mu\big(\varepsilon^{\mu\nu\rho}A_\nu \delta A_\rho\big).\end{aligned} \tag{3.7}$$

The EOMs are

$$\delta A_0 \qquad \Rightarrow \qquad \partial_i\left(\frac{E^i}{B}\right) - B = 0, \tag{3.8a}$$

$$\delta A_i \qquad \Rightarrow \qquad \partial_t\left(\frac{E^i}{B}\right) + \varepsilon^{ij}\partial_j\left(\frac{E^2}{2B^2}+p\right) - \varepsilon^{ij}E_j = 0. \tag{3.8b}$$

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