SOME MATHEMATICAL ASPECTS OF THE HAWKING EFFECT FOR ROTATING BLACK HOLES

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Abstract. The aim of this work is to give a mathematically rigorous description of the Hawking effect for fermions in the setting of the collapse of a rotating charged star.

1. Introduction

It was in 1975 that S. W. Hawking published his famous paper about the creation of particles by black holes (see [10]). Later this effect was analyzed by other authors in more detail (see e.g. [13]), and we can say that the effect was well-understood from a physical point of view at the end of the 1970s. From a mathematical point of view, however, fundamental questions linked to the Hawking radiation, such as scattering theory for field equations on black hole space-times, were not addressed at that time.

Scattering theory for field equations on the Schwarzschild metric has been studied from a mathematical point of view since the 1980s, see e.g. [7]. In 1999 A. Bachelot [2] gave a mathematically rigorous description of the Hawking effect in the spherically symmetric case. The methods used by Dimock, Kay and Bachelot rely in an essential way on the spherical symmetry of the problem and can't be generalized to the rotating case.

The aim of the present work is to give a mathematically precise description of the Hawking effect for spin-1/2 fields in the setting of the collapse of a rotating charged star, see [9] for a detailed exposition. We show that an observer who is located far away from the black hole and at rest with respect to the Boyer-Lindquist coordinates observes the emergence of a thermal state when his proper time t goes to infinity. In the proof we use the results of [8] as well as their generalizations to the Kerr-Newman case in [4].

Let us give an idea of the theorem describing the effect. Let r_* be the Regge-Wheeler coordinate. We suppose that the boundary of the star is described by $r_* = z(t,\theta)$. The space-time is then given by

$$\mathcal{M}_{col} = \bigcup_t \Sigma_t^{col}, \qquad \Sigma_t^{col} = \big\{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2 \; ; \; r_* \geq z(t,\theta)\big\}.$$

The typical asymptotic behavior of $z(t,\theta)$ is ($A(\theta) > 0$, $\kappa_+ > 0$):

$$z(t,\theta) = -t - A(\theta)e^{-2\kappa_+ t} + B(\theta) + \mathcal{O}(e^{-4\kappa_+ t}), \quad t \to \infty.$$

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Here κ_+ is the surface gravity of the outer horizon. Let

$$\mathcal{H}_t = L^2\big((\Sigma_t^{col}, \mathrm{dVol}); \mathbb{C}^4\big).$$

The Dirac equation can be written as

$$\partial_t \Psi = i \not{D}_t \Psi \quad + \quad \text{boundary condition.} \tag{1.1}$$

We will put an MIT boundary condition on the surface of the star. This condition makes the boundary totally reflecting, we refer to [9, Section 4.5] for details. The evolution of the Dirac field is then described by an isometric propagator $U(s,t) : \mathcal{H}_s \to \mathcal{H}_t$. The Dirac equation on the whole exterior Kerr-Newman space-time $\mathcal{M}_{BH}$ will be written as

$$\partial_t \Psi = i \not{D} \Psi.$$

Here $\not{D}$ is a selfadjoint operator on $\mathcal{H} = L^2((\mathbb{R}_{r_*} \times S^2, dr_* d\omega); \mathbb{C}^4)$. There exists an asymptotic velocity operator $P^\pm$ such that for all continuous functions J with $\lim_{|x|\to\infty} J(x) = 0$ we have

$$J(P^\pm) = \operatorname*{s-lim}_{t\to\pm\infty} e^{-it\not{D}} J\left(\frac{r_*}{t}\right) e^{it\not{D}}.$$

Let $\mathcal{U}_{col}(\mathcal{M}_{col})$ (resp. $\mathcal{U}_{BH}(\mathcal{M}_{BH})$) be the algebras of observables outside the collapsing body (resp. on the space-time describing the eternal black hole) generated by $\Psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)$ (resp. $\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)$). Here $\Psi_{col}(\Phi)$ (resp. $\Psi_{BH}(\Phi)$) are the quantum spin fields on $\mathcal{M}_{col}$ (resp. $\mathcal{M}_{BH}$). Let ω_{col} be a vacuum state on $\mathcal{U}_{col}(\mathcal{M}_{col})$; ω_{vac} a vacuum state on $\mathcal{U}_{BH}(\mathcal{M}_{BH})$ and $\omega^{\eta,\sigma}_{Haw}$ be a KMS-state on $\mathcal{U}_{BH}(\mathcal{M}_{BH})$ with inverse temperature $\sigma > 0$ and chemical potential $\mu = e^{\sigma\eta}$ (see Section 5 for details). For a function $\Phi \in C_0^\infty(\mathcal{M}_{BH})$ we define:

$$\Phi^T(t, r_*, \omega) = \Phi(t - T, r_*, \omega).$$

The theorem about the Hawking effect is the following:

Theorem 1.1 (Hawking effect)**.** *Let*

$$\Phi_j \in \big(C_0^\infty(\mathcal{M}_{col})\big)^4, \quad j = 1, 2.$$

Then we have

(1.2)

$$\begin{aligned}\lim_{T\to\infty} \omega_{col}\big(\Psi^*_{col}(\Phi_1^T)\Psi_{col}(\Phi_2^T)\big) &= \omega^{\eta,\sigma}_{Haw}\big(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_2)\big)\\ &\quad + \omega_{vac}\big(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_2)\big),\end{aligned}$$

$$T_{Haw} = 1/\sigma = \kappa_+/2\pi, \qquad \mu = e^{\sigma\eta}, \qquad \eta = \frac{qQr_+}{r_+^2 + a^2} + \frac{aD_\varphi}{r_+^2 + a^2}.$$

Here q is the charge of the field, Q the charge of the black hole, a is the angular momentum per unit mass of the black hole, $r_+ = M + \sqrt{M^2 - (a^2 + Q^2)}$ defines the outer event horizon, and κ_+ is the surface gravity of this horizon. The interpretation of (1.2) is the following. We start with a vacuum state which we evolve in the proper time of an observer at rest with respect to the Boyer-Lindquist coordinates. The limit as the proper time of this observer goes to infinity is a thermal state coming from the event horizon in formation and a vacuum state coming from infinity as expressed on the R.H.S. of (1.2). The Hawking effect comes from an infinite Doppler effect and the mixing of positive and negative frequencies. To explain

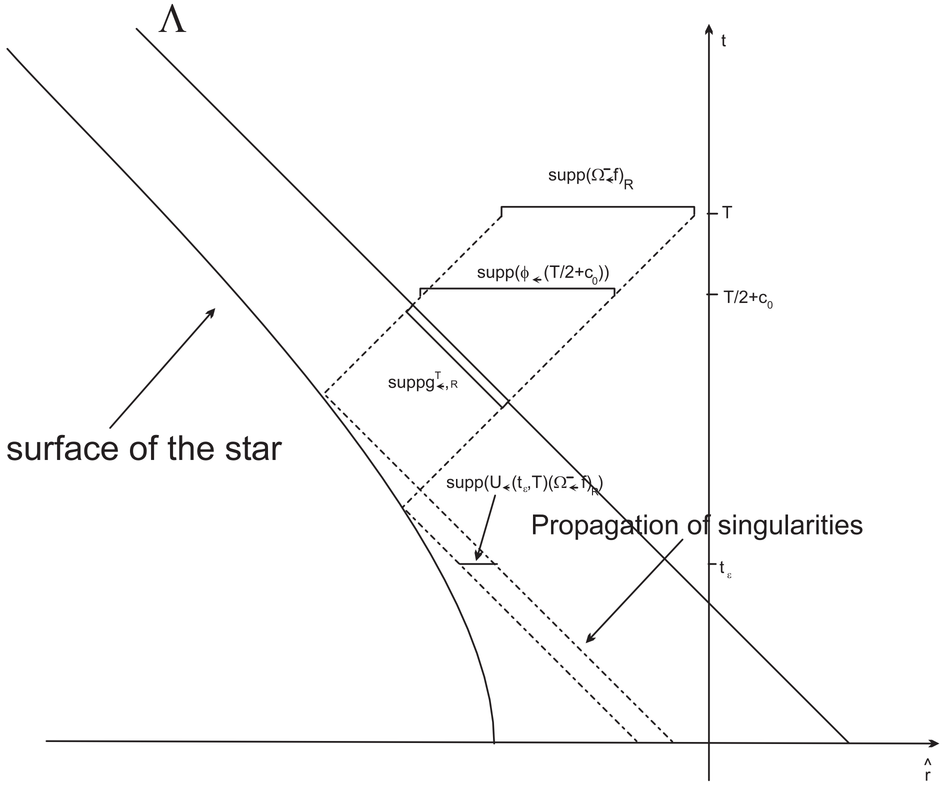

FIGURE 1. The collapse of the star

this a little bit more, we describe the analytic problem behind the effect. Let $f(r_*, \omega) \in C_0^\infty(\mathbb{R} \times S^2)$. The key result about the Hawking effect is:

$$\begin{aligned}\lim_{T\to\infty} \left\|\mathbf{1}_{[0,\infty)}(\not{D}_0)U(0,T)f\right\|_0^2 &= \left\langle \mathbf{1}_{\mathbb{R}^+}(P^-)f,\, \mu e^{\sigma\not{D}}(1+\mu e^{\sigma\not{D}})^{-1}\mathbf{1}_{\mathbb{R}^+}(P^-)f\right\rangle \\ &\quad + \left\|\mathbf{1}_{[0,\infty)}(\not{D})\mathbf{1}_{\mathbb{R}^-}(P^-)f\right\|^2,\end{aligned} \tag{1.3}$$

where μ, η, σ are as in the above theorem. Here $\langle .,. \rangle$ and $\|.\|$ (resp. $\|.\|_0$) are the standard inner product and norm on $\mathcal{H}$ (resp. $\mathcal{H}_0$). Equation (1.3) implies (1.2).

The term on the L.H.S. comes from the vacuum state we consider. We have to project onto the positive frequency solutions (see Section 5 for details). Note that in (1.3) we consider the time-reversed evolution. This comes from the quantization procedure. When time becomes large, the solution hits the surface of the star at a point closer and closer to the future event horizon. Figure 1 shows the situation for an asymptotic comparison dynamics, which satisfies Huygens' principle. For this asymptotic comparison dynamics the support of the solution concentrates more and more when time becomes large, which means that the frequency increases. The consequence of the change in frequency is that the system does not stay in the vacuum state.

2. The analytic problem

Let us consider a model where the eternal black hole is described by a static space-time (although the Kerr-Newman space-time is not even stationary, the problem will be essentially reduced to this kind of situation). Then the problem can be described as follows. Consider a Riemannian manifold Σ_0 with one asymptotically euclidean end and a boundary. The boundary will move when t becomes large asymptotically with the speed of light. The manifold at time t is denoted Σ_t. The "limit" manifold Σ is a manifold with two ends, one asymptotically euclidean and the other asymptotically hyperbolic (see Figure 2). The problem consists in evaluating the limit

$$\lim_{T\to\infty} \left\| \mathbf{1}_{[0,\infty)}(\not{D}_0)U(0,T)f \right\|_0,$$

where $U(0,T)$ is the isometric propagator for the Dirac equation on the manifold with moving boundary and suitable boundary conditions and $\not{D}_0$ is the Dirac Hamiltonian at time $t=0$. It is worth noting that the underlying scattering theory is not the scattering theory for the problem with moving boundary but the scattering theory on the "limit" manifold. It is shown in [9] that the result does not depend on the chiral angle in the MIT boundary condition. Note also that the boundary viewed in $\bigcup_t \{t\} \times \Sigma_t$ is only weakly timelike, a problem that has been rarely considered (but see [1]).

One of the problems for the description of the Hawking effect is to derive a reasonable model for the collapse of the star. We will suppose that the metric outside the collapsing star is always given by the Kerr-Newman metric. Whereas this is a genuine assumption in the rotational case, in the spherically symmetric case Birkhoff's theorem assures that the metric outside the star is the Reissner-Nordström metric. We will suppose that a point on the surface of the star will move along a curve which behaves asymptotically like a timelike geodesic with $L = \mathcal{Q} = \tilde{E} = 0$, where L is the angular momentum, $\tilde{E}$ the rotational energy and $\mathcal{Q}$ the Carter constant. The choice of geodesics is justified by the fact that the collapse creates the space-time, i.e., angular momenta and rotational energy should be zero with respect to the space-time. We will need an additional asymptotic condition on the collapse. It turns out that there is a natural coordinate system $(t, \hat{r}, \omega)$ associated to the collapse. In this coordinate system the surface of the star is described by $\hat{r} = \hat{z}(t,\theta)$. We need to assume the existence of a constant C such that

$$|\hat{z}(t,\theta) + t + C| \to 0, \quad t \to \infty. \tag{2.1}$$

It can be checked that this asymptotic condition is fulfilled if we use the above geodesics for some appropriate initial condition. We think that it is more natural to impose a (symmetric) asymptotic condition than an initial condition. If we would allow in (2.1) a function $C(\theta)$ rather than a constant, the problem would become more difficult. Indeed one of the problems for treating the Hawking radiation in the rotational case is the high frequencies of the solution. In contrast with the spherically symmetric case, the difference between the Dirac operator and an operator with constant coefficients is near the horizon always a differential operator of order one[1]. This explains that in the high-energy regime we are interested

[1] In the spherically symmetric case we can diagonalize the operator. After diagonalization the difference is just a potential.

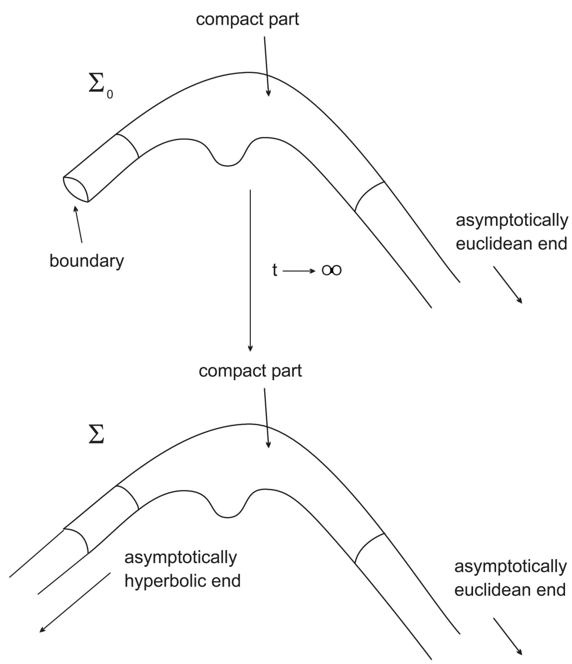

FIGURE 2. The manifold at time $t = 0$ Σ_0 and the limit manifold Σ.

in, the Dirac operator is not close to a constant-coefficient operator. Our method for proving (1.3) is to use scattering arguments to reduce the problem to a problem with a constant-coefficient operator, for which we can compute the radiation explicitly. If we do not impose a condition of type (2.1), then in all coordinate systems the solution has high frequencies, in the radial as well as in the angular directions. With condition (2.1) these high frequencies only occur in the radial direction. Our asymptotic comparison dynamics will differ from the real dynamics only by derivatives in angular directions and by potentials.

Let us now give some ideas of the proof of (1.3). We want to reduce the problem to the evaluation of a limit that can be explicitly computed. To do so, we use the asymptotic completeness results obtained in [8] and [4]. There exists a constant-coefficient operator $\not{D}_{\leftarrow}$ such that the following limits exist:

$$\begin{aligned} W^{\pm}_{\leftarrow} &:= \operatorname*{s-lim}_{t\to\pm\infty} e^{-it\not{D}} e^{it\not{D}_{\leftarrow}} \, \mathbf{1}_{\mathbb{R}^{\mp}}(P^{\pm}_{\leftarrow}), \\ \Omega^{\pm}_{\leftarrow} &:= \operatorname*{s-lim}_{t\to\pm\infty} e^{-it\not{D}_{\leftarrow}} e^{it\not{D}} \, \mathbf{1}_{\mathbb{R}^{\mp}}(P^{\pm}). \end{aligned}$$

Here $P^{\pm}_{\leftarrow}$ is the asymptotic velocity operator associated to the dynamics $e^{it\not{D}_{\leftarrow}}$. Then the R.H.S. of (1.3) equals:

$$\left\|\mathbf{1}_{[0,\infty)}(\not{D})\mathbf{1}_{\mathbb{R}^-}(P^-)f\right\|^2 + \left\langle \Omega^-_{\leftarrow} f,\ \mu e^{\sigma\not{D}_{\leftarrow}}(1+\mu e^{\sigma\not{D}_{\leftarrow}})^{-1}\Omega^-_{\leftarrow} f\right\rangle.$$

The aim is to show that the incoming part is:

$$\lim_{T\to\infty}\left\|\mathbf{1}_{[0,\infty)}(D_{\leftarrow,0})U_{\leftarrow}(0,T)\Omega^-_{\leftarrow} f\right\|_0^2 = \left\langle \Omega^-_{\leftarrow} f,\ \mu e^{\sigma\not{D}_{\leftarrow}}(1+\mu e^{\sigma\not{D}_{\leftarrow}})^{-1}\Omega^-_{\leftarrow} f\right\rangle,$$

where the equality can be shown by explicit calculation. Here $\not{D}_{\leftarrow,t}$ and $U_{\leftarrow}(s,t)$ are the asymptotic operator with boundary condition and the associated propagator. The outgoing part is easy to treat.

Note that (1.3) is of course independent of the choice of the coordinate system and the tetrad, i.e., both sides of (1.3) are independent of these choices.

The proofs of all the results stated in this work can be found in [9]. The work is organized as follows:

- In Section 3 we present the model of the collapsing star. We first analyze geodesics in the Kerr-Newman space-time and explain how the Carter constant can be understood in terms of the Hamiltonian flow. We construct a coordinate system which is well adapted to the collapse.
- In Section 4 we describe classical Dirac fields. The form of the Dirac equation with an adequate choice of the Newman-Penrose tetrad is given. Scattering results are stated.
- Dirac quantum fields are discussed in Section 5. The theorem about the Hawking effect is formulated and discussed in Subsection 5.2.
- In Section 6 we give the main ideas of the proof.

3. The model of the collapsing star

The purpose of this section is to describe the model of the collapsing star. We will suppose that the metric outside the star is given by the Kerr-Newman metric. Geodesics are discussed in Subsection 3.2. We give a description of the Carter constant in terms of the associated Hamiltonian flow. A new position variable is introduced. In Subsection 3.3 we give the asymptotic behavior of the boundary of the star using this new position variable. We require that a point on the surface behaves asymptotically like incoming timelike geodesics with $L = \mathcal{Q} = \tilde{E} = 0$, which are studied in Subsection 3.3.1.

3.1. The Kerr-Newman metric.

We give a brief description of the Kerr-Newman metric, which describes an eternal rotating charged black hole. In Boyer-Lindquist coordinates, a Kerr-Newman black hole is described by a smooth 4-dimensional Lorentzian manifold $\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, whose space-time metric g and electromagnetic vector potential Φ_a are given by:

$$
\begin{aligned}
g &= \left(1 + \frac{Q^2 - 2Mr}{\rho^2}\right) \mathrm{d}t^2 + \frac{2a\sin^2\theta\,(2Mr - Q^2)}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi \\
&\quad - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\,\mathrm{d}\theta^2 - \frac{\sigma^2}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi^2, \\
\rho^2 &= r^2 + a^2\cos^2\theta, \\
\Delta &= r^2 - 2Mr + a^2 + Q^2, \\
\sigma^2 &= (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \\
\Phi_a\,\mathrm{d}x^a &= -\frac{Qr}{\rho^2}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi).
\end{aligned}
\tag{3.1}
$$

Here M is the mass of the black hole, a its angular momentum per unit mass, and Q the charge of the black hole. If $Q = 0$, g reduces to the Kerr metric, and if $Q = a = 0$ we recover the Schwarzschild metric. The expression (3.1) of the

Kerr metric has two types of singularities. While the set of points $\{\rho^2 = 0\}$ (the equatorial ring $\{r = 0,\ \theta = \pi/2\}$ of the $\{r = 0\}$ sphere) is a true curvature singularity, the spheres where Δ vanishes, called horizons, are mere coordinate singularities. We will consider in this work subextremal Kerr-Newman space-times, that is, we suppose $Q^2 + a^2 < M^2$. In this case Δ has two real roots:

$$r_\pm = M \pm \sqrt{M^2 - (a^2 + Q^2)}. \tag{3.2}$$

The spheres $\{r = r_-\}$ and $\{r = r_+\}$ are called event horizons. The two horizons separate $\mathcal{M}_{BH}$ into three connected components called Boyer-Lindquist blocks: B_I, B_{II}, B_{III} $(r_+ < r,\ r_- < r < r_+,\ r < r_-)$. No Boyer-Lindquist block is stationary, that is to say there exists no globally defined timelike Killing vector field on any given block. In the following $\mathcal{M}_{BH}$ will denote only block I of the Kerr-Newman space-time.

3.2. **Some remarks about geodesics in the Kerr-Newman space-time.** It is one of the most remarkable facts about the Kerr-Newman metric that there exist four first integrals for the geodesic equations. If γ is a geodesic in the Kerr-Newman space-time, then $p := \langle \gamma', \gamma' \rangle$ is conserved. The two Killing vector fields ∂_t, ∂_φ give two first integrals, the energy $E := \langle \gamma', \partial_t \rangle$ and the angular momentum $L := -\langle \gamma', \partial_\varphi \rangle$. There exists a fourth constant of motion, the so-called Carter constant $\mathcal{K}$ (see e.g. [3]). We will also use the Carter constant $\mathcal{Q} = \mathcal{K} - (L - aE)^2$, which has a somewhat more geometrical meaning, but gives in general more complicated formulas. Let

$$\mathbf{P} := (r^2 + a^2)E - aL, \qquad \mathbf{D} := L - aE\sin^2\theta. \tag{3.3}$$

Let $\Box_g$ be the d'Alembertian associated to the Kerr-Newman metric. We will consider the Hamiltonian flow of the principal symbol of $\frac{1}{2}\Box_g$ and then use the fact that a geodesic can be understood as the projection of the Hamiltonian flow on $\mathcal{M}_{BH}$. The principal symbol of $\frac{1}{2}\Box_g$ is:

$$P := \frac{1}{2\rho^2}\left(\frac{\sigma^2}{\Delta}\tau^2 - \frac{2a(Q^2 - 2Mr)}{\Delta}q_\varphi\tau - \frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}q_\varphi^2 - \Delta|\xi|^2 - q_\theta^2\right). \tag{3.4}$$

Let

$$\mathcal{C}_p := \left\{(t, r, \theta, \varphi; \tau, \xi, q_\theta, q_\varphi)\ \ ;\ \ P(t, r, \theta, \varphi; \tau, \xi, q_\theta, q_\varphi) = \frac{1}{2}p\right\}.$$

Here $(\tau, \xi, q_\theta, q_\varphi)$ is dual to (t, r, θ, φ). We have the following:

Theorem 3.1. *Let* $x_0 = (t_0, r_0, \varphi_0, \theta_0, \tau_0, \xi_0, q_{\theta_0}, q_{\varphi_0}) \in \mathcal{C}_p$*, and let* $x(s) = \big(t(s), r(s), \theta(s), \varphi(s); \tau(s), \xi(s), q_\theta(s), q_\varphi(s)\big)$ *be the associated Hamiltonian flow line. Then we have the following constants of motion:*

$$\begin{gathered} p = 2P, \qquad\qquad E = \tau, \qquad\qquad L = -q_\varphi, \\ \mathcal{K} = q_\theta^2 + \frac{\mathbf{D}^2}{\sin^2\theta} + pa^2\cos^2\theta = \frac{\mathbf{P}^2}{\Delta} - \Delta|\xi|^2 - pr^2, \end{gathered} \tag{3.5}$$

where $\mathbf{D}$, $\mathbf{P}$ *are defined in* (3.3).

The case $L = \mathcal{Q} = 0$ is of particular interest. Let γ be a null geodesic with energy $E > 0$, Carter constant $\mathcal{Q} = 0$, angular momentum $L = 0$ and given signs of ξ_0 and q_{θ_0}. We can associate a Hamiltonian flow line using (3.5) to define the initial data $\tau_0, \xi_0, q_{\theta_0}, q_{\varphi_0}$ given $t_0, r_0, \theta_0, \varphi_0$. From (3.5) we infer that ξ, q_θ do not change their signs. Note that γ is either in the equatorial plane or it does not cross it. Under the above conditions ξ (resp. q_θ) can be understood as a function of r (resp. θ) alone. In this case let k and l such that

$$\frac{dk(r)}{dr} = \frac{\xi(r)}{E}, \qquad l'(\theta) = \frac{q_\theta(\theta)}{E}, \qquad \hat{r} := k(r) + l(\theta). \tag{3.6}$$

It is easy to check that $(t, \hat{r}, \omega)$ is a coordinate system on block I.

Lemma 3.2. *We have:*

$$\frac{\partial \hat{r}}{\partial t} = -1 \quad \textit{along } \gamma, \tag{3.7}$$

where t is the Boyer-Lindquist time.

We will suppose from now on that our construction is based on incoming null geodesics. From the above lemma follows that for given sign of q_{θ_0} the surfaces $\mathcal{C}^{c,\pm} = \{(t, r_*, \theta, \varphi)\ ;\ \pm t = \hat{r}(r_*, \theta) + c\}$ are characteristic.

Remark 3.3. The variable $\hat{r}$ is a Bondi-Sachs type coordinate. This coordinate system is discussed in some detail in [12]. As in [12], we will call the null geodesics with $L = \mathcal{Q} = 0$ *simple null geodesics (SNGs).*

3.3. **The model of the collapsing star.** Let $\mathcal{S}_0$ be the surface of the star at time $t = 0$. We suppose that elements $x_0 \in \mathcal{S}_0$ will move along curves which behave asymptotically like certain incoming timelike geodesics γ_p. All these geodesics should have the same energy E, angular momentum L, Carter constant $\mathcal{K}$ (resp. $\mathcal{Q} = \mathcal{K} - (L - aE)^2$) and "mass" $p := \langle \gamma_p', \gamma_p' \rangle$. We will suppose:

(A) The angular momentum L vanishes: $L = 0$.
(B) The rotational energy vanishes: $\tilde{E} = a^2(E^2 - p) = 0$.
(C) The total angular momentum about the axis of symmetry vanishes: $\mathcal{Q} = 0$.

The conditions (A)–(C) are imposed by the fact that the collapse itself creates the space-time, so that momenta and rotational energy should be zero with respect to the space-time.

3.3.1. *Timelike geodesics with $L = \mathcal{Q} = \tilde{E} = 0$.* Next, we will study the above family of geodesics. The starting point of the geodesic is denoted $(0, r_0, \theta_0, \varphi_0)$. Given a point in the space-time, the conditions (A)–(C) define a unique cotangent vector provided one adds the condition that the corresponding tangent vector is incoming. The choice of p is irrelevant because it just corresponds to a normalization of the proper time.

Lemma 3.4. *Let γ_p be a geodesic as above. Along this geodesic we have:*

$$\frac{\partial \theta}{\partial t} = 0, \qquad \frac{\partial \varphi}{\partial t} = \frac{a(2Mr - Q^2)}{\sigma^2}, \tag{3.8}$$

where t is the Boyer-Lindquist time.

The function $\frac{\partial \varphi}{\partial t} = \frac{a(2Mr - Q^2)}{\sigma^2}$ is usually called the *local angular velocity* of the space-time. Our next aim is to adapt our coordinate system to the collapse of the star. The most natural way of doing this is to choose an incoming null geodesic γ with $L = Q = 0$ and then use the Bondi-Sachs type coordinate as in the previous subsection. In addition we want that $k(r_*)$ behaves like r_* when $r_* \to -\infty$. We therefore put:

$$k(r_*) = r_* + \int_{-\infty}^{r_*} \left(\sqrt{1 - \frac{a^2\Delta(s)}{(r(s)^2 + a^2)^2}} - 1 \right) \mathrm{d}s, \qquad l(\theta) = a \sin\theta. \tag{3.9}$$

The choice of the sign of l' is not important, the opposite sign would have been possible. Recall that $\cos\theta$ does not change its sign along a null geodesic with $L = \mathcal{Q} = 0$. We put $\hat{r} = k(r_*) + l(\theta)$, and by Lemma 3.2 we have $\frac{\partial \hat{r}}{\partial t} = -1$ along γ.

In order to describe the model of the collapsing star we have to evaluate $\frac{\partial \hat{r}}{\partial t}$ along γ_p. Note that $\theta(t) = \theta_0 = \text{const}$ along γ_p.

Lemma 3.5. *There exist smooth functions $\hat{A}(\theta, r_0) > 0$, $\hat{B}(\theta, r_0)$ such that along γ_p we have uniformly in θ, $r_0 \in [r_1, r_2] \subset (r_+, \infty)$:*

$$\hat{r} = -t - \hat{A}(\theta, r_0)e^{-2\kappa_+ t} + \hat{B}(\theta, r_0) + \mathcal{O}(e^{-4\kappa_+ t}), \ \ t \to \infty, \tag{3.10}$$

where $\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}$ is the surface gravity of the outer horizon.

3.3.2. *Assumptions.* We will suppose that the surface at time $t = 0$ is given in the $(t, \hat{r}, \theta, \varphi)$ coordinate system by $\mathcal{S}_0 = \{(\hat{r}_0(\theta_0), \theta_0, \varphi_0) \ ; \ (\theta_0, \varphi_0) \in S^2\}$, where $\hat{r}_0(\theta_0)$ is a smooth function. As $\hat{r}_0$ does not depend on φ_0, we will suppose that $\hat{z}(t, \theta_0, \varphi_0)$ will be independent of φ_0 : $\hat{z}(t, \theta_0, \varphi_0) = \hat{z}(t, \theta_0) = \hat{z}(t, \theta)$ as this is the case for $\hat{r}(t)$ along timelike geodesics with $L = \mathcal{Q} = 0$. We suppose that $\hat{z}(t, \theta)$ satisfies the asymptotics (3.10) with $\hat{B}(\theta, r_0)$ independent of θ, see [9] for the precise assumptions. Thus the surface of the star is given by:

$$\mathcal{S} = \{(t, \hat{z}(t, \theta), \omega) \ ; \ t \in \mathbb{R}, \ \omega \in S^2\}. \tag{3.11}$$

The space-time of the collapsing star is given by:

$$\mathcal{M}_{col} = \{(t, \hat{r}, \theta, \varphi) \ ; \ \hat{r} \geq \hat{z}(t, \theta)\}.$$

We will also note:

$$\Sigma_t^{col} = \{(\hat{r}, \theta, \varphi) \ ; \ \hat{r} \geq \hat{z}(t, \theta)\}, \quad \text{thus} \ \ \mathcal{M}_{col} = \bigcup_t \Sigma_t^{col}.$$

The asymptotic form (3.10) with $\hat{B}(\theta, r_0)$ can be achieved by incoming timelike geodesics with $L = \mathcal{Q} = \tilde{E} = 0$, see [9, Lemma 3.5].

4. Classical Dirac fields

In this section we will state some results on classical Dirac fields and explain in particular how to overcome the difficulty linked to the high-frequency problem. The key point is the appropriate choice of a Newman-Penrose tetrad. Let $(\mathcal{M}, g)$ be a general globally hyperbolic space-time. Using the Newman-Penrose formalism, the Dirac equation can be expressed as a system of partial differential equations with respect to a coordinate basis. This formalism is based on the choice of a null tetrad, i.e. a set of four vector fields l^a, n^a, m^a and $\bar{m}^a$, the first two being real and future oriented, $\bar{m}^a$ being the complex conjugate of m^a, such that all four vector fields are null and m^a is orthogonal to l^a and n^a, that is to say, $l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a m^a = 0$. The tetrad is said to be normalized if in addition $l_a n^a = 1$, $m_a \bar{m}^a = -1$. The vectors l^a and n^a usually describe "dynamic" or scattering directions, i.e. directions along which light rays may escape towards infinity (or more generally asymptotic regions corresponding to scattering channels). The vector m^a tends to have, at least spatially, bounded integral curves; typically m^a and $\bar{m}^a$ generate rotations. To this Newman-Penrose tetrad is associated a spin frame. The Dirac equation is then a system of partial differential equations for the components of the spinor in this spin frame. For the Weyl equation ($m = 0$) we obtain:

$$\begin{cases} n^a\partial_a\phi_0 - m^a\partial_a\phi_1 + (\mu-\gamma)\phi_0 + (\tau-\beta)\phi_1 = 0, \\ l^a\partial_a\phi_1 - \bar{m}^a\partial_a\phi_1 + (\alpha-\pi)\phi_0 + (\epsilon-\tilde{\rho})\phi_1 = 0. \end{cases}$$

The μ, γ etc. are the so-called spin coefficients, for example

$$\mu = -\bar{m}^a\delta n_a, \qquad \delta = m^a\nabla_a.$$

For the formulas of the spin coefficients and details about the Newman-Penrose formalism see e.g. [11].

Our first result is that there exists a tetrad well-adapted to the high-frequency problem. Let $\mathcal{H} = L^2((\mathbb{R}_{\hat{r}} \times S^2, \mathrm{d}\hat{r}\,\mathrm{d}\omega); \mathbb{C}^4)$, $\Gamma^1 = Diag\,(1,-1,-1,1)$.

Proposition 4.1. *There exists a Newman-Penrose tetrad such that the Dirac equation in the Kerr-Newman space-time can be written as*

$$\partial_t\psi = iH\psi; \qquad H = \Gamma^1 D_{\hat{r}} + P_\omega + W,$$

where W is a real potential and P_ω is a differential operator of order one with derivatives only in the angular directions. The operator H is selfadjoint with domain $D(H) = \{v \in \mathcal{H}\ ;\ Hv \in \mathcal{H}\}$.

Remark 4.2.

(i) l^a, n^a are chosen to be generators of the simple null geodesics.

(ii) Note that the local velocity in $\hat{r}$ direction is ± 1:

$$v = [\hat{r}, H] = \Gamma^1.$$

This comes from the fact that $\frac{\partial \hat{r}}{\partial t} = \pm 1$ along simple null geodesics ($\pm$ depending on whether the geodesic is incoming or outgoing).

(iii) Whereas the above tetrad is well-adapted to the high-frequency analysis, it is not the good choice for the proof of asymptotic completeness results. See [8] for an adequate choice.

(iv) ψ are the components of the spinor which is multiplied by some density.

Let

$$H_{\leftarrow} = \Gamma^1 D_{\hat{r}} - \frac{a}{r_+^2 + a^2} D_\varphi - \frac{qQr_+}{r_+^2 + a^2}\,,$$

$$\mathcal{H}^+ = \big\{v = (v_1, v_2, v_3, v_4) \in \mathcal{H}\ ;\ v_1 = v_4 = 0\big\},$$

$$\mathcal{H}^- = \big\{v = (v_1, v_2, v_3, v_4) \in \mathcal{H}\ ;\ v_2 = v_3 = 0\big\}.$$

The operator $H_{\leftarrow}$ is selfadjoint on $\mathcal{H}$ with domain

$$D(H_{\leftarrow}) = \big\{v \in \mathcal{H}\ ;\ H_{\leftarrow} v \in \mathcal{H}\big\}.$$

Remark 4.3. The above operator is our comparison dynamics. Note that the difference between the full dynamics and the comparison dynamics is a differential operator with derivatives only in the angular directions. The high frequencies will only be present in the $\hat{r}$ directions; this solves the high-frequency problem.

Proposition 4.4. *There exist selfadjoint operators* $P^{\pm}$ *such that for all* $g \in C(\mathbb{R})$ *with* $\lim_{|x|\to\infty} g(x) = 0$*, we have:*

$$g(P^{\pm}) = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{-itH} g\left(\frac{\hat{r}}{t}\right) e^{itH}. \tag{4.1}$$

Let $P_{\mathcal{H}^{\mp}}$ be the projections from $\mathcal{H}$ to $\mathcal{H}^{\mp}$.

Theorem 4.5. *The wave operators*

$$\begin{aligned} W_{\leftarrow}^{\pm} &= \underset{t\to\pm\infty}{\text{s-lim}}\, e^{-itH} e^{itH_{\leftarrow}} P_{\mathcal{H}^{\mp}}, \\ \Omega_{\leftarrow}^{\pm} &= \underset{t\to\pm\infty}{\text{s-lim}}\, e^{-itH_{\leftarrow}} e^{itH}\, \mathbf{1}_{\mathbb{R}^{\mp}}(P^{\pm}) \end{aligned}$$

exist.

Using the above tetrad, the Dirac equation with MIT boundary condition on the surface of the star (chiral angle ν) can be written in the following form:

$$\left.\begin{aligned} \partial_t \Psi &= iH\Psi, \qquad \hat{z}(t,\theta) < \hat{r}, \\ \left(\textstyle\sum_{\hat{\mu}\in\{t,\hat{r},\theta,\varphi\}} \mathcal{N}_{\hat{\mu}} \hat{\gamma}^{\hat{\mu}}\right)\Psi\big(t,\hat{z}(t,\theta),\omega\big) &= -ie^{-i\nu\gamma^5}\Psi\big(t,\hat{z}(t,\theta),\omega\big), \\ \Psi(t=s,.) &= \Psi_s(.). \end{aligned}\right\} \tag{4.2}$$

Here $\mathcal{N}_{\hat{\mu}}$ are the coordinates of the conormal, $\hat{\gamma}^{\hat{\mu}}$ are some appropriate Dirac matrices and $\gamma^5 = Diag(1,1,-1,-1)$. Let

$$\mathcal{H}_t = L^2\Big(\big(\{(\hat{r},\omega)\in\mathbb{R}\times S^2\ ;\ \hat{r}\geq\hat{z}(t,\theta)\},\mathrm{d}\hat{r}\,\mathrm{d}\omega\big);\ \mathbb{C}^4\Big).$$

Proposition 4.6. *The equation* (4.2) *can be solved by a unitary propagator* $U(t,s)$: $\mathcal{H}_s \to \mathcal{H}_t$*.*

5. Dirac quantum fields

We adopt the approach of Dirac quantum fields in the spirit of [5] and [6]. This approach is explained in Section 5.1. In Section 5.2 we present the theorem about the Hawking effect.

5.1. **Quantization in a globally hyperbolic space-time.** Following J. Dimock [6] we construct the local algebra of observables in the space-time outside the collapsing star. This construction does not depend on the choice of the representation of the CARs, or on the spin structure of the Dirac field, or on the choice of the hypersurface. In particular we can consider the Fermi-Dirac-Fock representation and the following foliation of our space-time (see Subsection 3.3):

$$\mathcal{M}_{col} = \bigcup_{t\in\mathbb{R}} \Sigma_t^{col}, \qquad \Sigma_t^{col} = \big\{(t,\hat{r},\theta,\varphi)\ ;\ \hat{r}\geq\hat{z}(t,\theta)\big\}.$$

We construct the Dirac field Ψ_0 and the C^*-algebra $\mathcal{U}(\mathcal{H}_0)$ in the usual way. We define the operator:

$$S_{col}:\ \Phi\in\big(C_0^\infty(\mathcal{M}_{col})\big)^4 \ \mapsto\ S_{col}\Phi := \int_{\mathbb{R}} U(0,t)\Phi(t)\mathrm{d}t \in \mathcal{H}_0, \tag{5.1}$$

where $U(0,t)$ is the propagator defined in Proposition 4.6. The quantum spin field is defined by:

$$\Psi_{col}\ :\Phi\in\big(C_0^\infty(\mathcal{M}_{col})\big)^4 \ \mapsto\ \Psi_{col}(\Phi) := \Psi_0(S_{col}\Phi)\in\mathcal{L}(\mathcal{F}(\mathcal{H}_0)).$$

Here $\mathcal{F}(\mathcal{H}_0)$ is the Dirac-Fermi-Fock space associated to $\mathcal{H}_0$. For an arbitrary set $\mathcal{O} \subset \mathcal{M}_{col}$, we introduce $\mathcal{U}_{col}(\mathcal{O})$, the C^*-algebra generated by $\psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)$, $\operatorname{supp}\Phi_j \subset \mathcal{O},\ \ j=1,2$. Eventually, we have:

$$\mathcal{U}_{col}(\mathcal{M}_{col}) = \overline{\bigcup_{\mathcal{O}\subset\mathcal{M}_{col}} \mathcal{U}_{col}(\mathcal{O})}.$$

Then we define the fundamental state on $\mathcal{U}_{col}(\mathcal{M}_{col})$ as follows:

$$\begin{aligned}\omega_{col}\big(\Psi^*_{col}(\Phi_1)\Psi_{col}(\Phi_2)\big) &:= \omega_{vac}\big(\Psi^*_0(S_{col}\Phi_1)\Psi_0(S_{col}\Phi_2)\big)\\ &= \big\langle \mathbf{1}_{[0,\infty)}(H_0)S_{col}\Phi_1,\, S_{col}\Phi_2\big\rangle.\end{aligned}$$

Let us now consider the future black hole. The algebra $\mathcal{U}_{BH}(\mathcal{M}_{BH})$ and the vacuum state ω_{vac} are constructed as before working now with the group e^{itH} rather than the evolution system $U(t,s)$. We also define the thermal Hawking state (S is analogous to $S_{col},\ \Psi_{BH}$ to $\Psi_{col},$ and Ψ to Ψ_0):

$$\begin{aligned}\omega^{\eta,\sigma}_{Haw}\big(\Psi^*_{BH}(\Phi_1)\Psi_{BH}(\Phi_2)\big) &= \big\langle \mu e^{\sigma H}(1+\mu e^{\sigma H})^{-1}S\Phi_1,\, S\Phi_2\big\rangle_{\mathcal{H}}\\ &=: \omega^{\eta,\sigma}_{KMS}\big(\Psi^*(S\Phi_1)\Psi(S\Phi_2)\big)\end{aligned}$$

with

$$T_{Haw} = \sigma^{-1}, \qquad \mu = e^{\sigma\eta}, \qquad \sigma > 0,$$

where T_{Haw} is the Hawking temperature and μ is the chemical potential.

5.2. **The Hawking effect.** In this subsection we formulate the main result of this work.
Let $\Phi \in (C_0^\infty(\mathcal{M}_{col}))^4$. We put

$$\Phi^T(t,\hat{r},\omega) = \Phi(t-T,\hat{r},\omega). \tag{5.2}$$

Theorem 5.1 (Hawking effect)**.** *Let*

$$\Phi_j \in (C_0^\infty(\mathcal{M}_{col}))^4,\ \ j=1,2.$$

Then we have

(5.3)
$$\begin{aligned}\lim_{T\to\infty}\omega_{col}\big(\Psi^*_{col}(\Phi^T_1)\Psi_{col}(\Phi^T_2)\big) &= \omega^{\eta,\sigma}_{Haw}\big(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^+}(P^-)\Phi_2)\big)\\ &\quad+\omega_{vac}\big(\Psi^*_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_1)\Psi_{BH}(\mathbf{1}_{\mathbb{R}^-}(P^-)\Phi_2)\big),\end{aligned}$$

$$T_{Haw} = 1/\sigma = \kappa_+/2\pi, \qquad \mu = e^{\sigma\eta}, \qquad \eta = \frac{qQr_+}{r_+^2+a^2} + \frac{aD_\varphi}{r_+^2+a^2}.$$

In the above theorem $P^\pm$ is the asymptotic velocity introduced in Section 4. The projections $\mathbf{1}_{\mathbb{R}^\pm}(P^\pm)$ separate outgoing and incoming solutions.

Remark 5.2. The result is independent of the choices of coordinate system, tetrad and chiral angle in the boundary condition.

6. Strategy of the proof

The radiation can be explicitly computed for the asymptotic dynamics near the horizon. For $f = (0, f_2, f_3, 0)$ and T large, the time-reversed solution of the mixed problem for the asymptotic dynamics is well approximated by the so called geometric optics approximation:

$$F^T(\hat{r}, \omega) := \frac{1}{\sqrt{-\kappa_+ \hat{r}}}(f_3, 0, 0, -f_2)\left(T + \frac{1}{\kappa_+}\ln(-\hat{r}) - \frac{1}{\kappa_+}\ln \hat{A}(\theta),\ \omega\right).$$

For this approximation the radiation can be computed explicitly:

Lemma 6.1. *We have:*

$$\lim_{T\to\infty} \left\|\mathbf{1}_{[0,\infty)}(H_{\leftarrow})F^T\right\|^2 = \left\langle f,\ e^{\frac{2\pi}{\kappa_+}H_{\leftarrow}}\left(1 + e^{\frac{2\pi}{\kappa_+}H_{\leftarrow}}\right)^{-1} f\right\rangle.$$

The strategy of the proof is now the following:

i) We decouple the problem at infinity from the problem near the horizon by cut-off functions. The problem at infinity is easy to treat.

ii) We consider $U(t,T)f$ on a characteristic hypersurface Λ. The resulting characteristic data is denoted g^T. We will approximate $\Omega^-_{\leftarrow} f$ by a function $(\Omega^-_{\leftarrow} f)_R$ with compact support and higher regularity in the angular derivatives. Let $U_{\leftarrow}(s,t)$ be the isometric propagator associated to the asymptotic Hamiltonian $H_{\leftarrow}$ with MIT boundary conditions. We also consider $U_{\leftarrow}(t,T)(\Omega^-_{\leftarrow} f)_R$ on Λ. The resulting characteristic data is denoted $g^T_{\leftarrow,R}$. The situation for the asymptotic comparison dynamics is shown in Figure 1.

iii) We solve a characteristic Cauchy problem for the Dirac equation with data $g^T_{\leftarrow,R}$. The solution at time zero can be written in a region near the boundary as

$$G(g^T_{\leftarrow,R}) = U(0, T/2 + c_0)\,\Phi(T/2 + c_0), \tag{6.1}$$

where Φ is the solution of a characteristic Cauchy problem in the whole space (without the star). The solutions of the characteristic problems for the asymptotic Hamiltonian are written in a similar way and denoted $G_{\leftarrow}(g^T_{\leftarrow,R})$ and $\Phi_{\leftarrow}$, respectively.

iv) Using the asymptotic completeness result we show that $g^T - g^T_{\leftarrow,R} \to 0$ when $T, R \to \infty$. By continuous dependence on the characteristic data we see that:

$$G(g^T) - G(g^T_{\leftarrow,R}) \to 0,\ \ T, R \to \infty.$$

v) We write

$$G(g^T_{\leftarrow,R}) - G_{\leftarrow}(g^T_{\leftarrow,R}) = U(0, T/2 + c_0)\big(\Phi(T/2 + c_0) - \Phi_{\leftarrow}(T/2 + c_0)\big) + \big(U(0,T/2+c_0) - U_{\leftarrow}(0,T/2+c_0)\big)\Phi_{\leftarrow}(T/2+c_0).$$

The first term becomes small near the boundary when T becomes large. We then note that for all $\epsilon > 0$ there exists $t_\epsilon > 0$ such that

$$\left\|\big(U(t_\epsilon, T/2 + c_0) - U_{\leftarrow}(t_\epsilon, T/2 + c_0)\big)\Phi_{\leftarrow}(T/2 + c_0)\right\| < \epsilon$$

uniformly in T when T is large. We fix the angular momentum $D_\varphi = n$. The function $U_{\leftarrow}(t_\epsilon, T/2+c_0)\,\Phi_{\leftarrow}(T/2+c_0)$ will be replaced by a geometric optics approximation $F^T_{t_\epsilon}$ which has the following properties:

$$\operatorname{supp} F^T_{t_\epsilon} \subset \big(- t_\epsilon - |\mathcal{O}(e^{-\kappa_+ T})|,\ -t_\epsilon \big), \tag{6.2}$$

$$F^T_{t_\epsilon} \rightharpoonup 0,\ \ T \to \infty, \tag{6.3}$$

$$\forall \lambda > 0 \quad Op\big(\chi(\langle \xi \rangle \le \lambda \langle q \rangle)\big) F^T_{t_\epsilon} \to 0,\ \ T \to \infty. \tag{6.4}$$

Here ξ and q are the dual coordinates to $\hat{r}$ and θ, respectively. $Op(a)$ is the pseudo-differential operator associated to the symbol a (Weyl calculus). The notation $\chi(\langle \xi \rangle \le \lambda \langle q \rangle)$ means that the symbol is supported in $\langle \xi \rangle \le \lambda \langle q \rangle$.

vi) We show that for λ sufficiently large possible singularities of

$$Op\big(\chi(\langle \xi \rangle \ge \lambda \langle q \rangle)\big)\, F^T_{t_\epsilon}$$

are transported by the group $e^{-it_\epsilon H}$ in such a way that they always stay away from the surface of the star.

vii) The points *i) to v) imply:*

$$\lim_{T \to \infty} \big\| \mathbf{1}_{[0,\infty)}(H_0)\, j_- U(0,T) f \big\|_0^2 = \lim_{T \to \infty} \big\| \mathbf{1}_{[0,\infty)}(H_0)\, U(0, t_\epsilon)\, F^T_{t_\epsilon} \big\|_0^2,$$

where j_- is a smooth cut-off which equals 1 near the boundary and 0 at infinity. Let ϕ_δ be a cut-off outside the surface of the star at time 0. If $\phi_\delta = 1$ sufficiently close to the surface of the star at time 0, we see by the previous point that

$$(1 - \phi_\delta) e^{-it_\epsilon H} F^T_{t_\epsilon} \to 0,\ \ T \to \infty. \tag{6.5}$$

Using (6.5) we show that (modulo a small error term):

$$\big(U(0, t_\epsilon) - \phi_\delta e^{-it_\epsilon H}\big) F^T_{t_\epsilon} \to 0,\ \ T \to \infty.$$

Therefore it remains to consider:

$$\lim_{T \to \infty} \big\| \mathbf{1}_{[0,\infty)}(H_0) \phi_\delta\, e^{-it_\epsilon H} F^T_{t_\epsilon} \big\|_0.$$

viii) We show that we can replace $\mathbf{1}_{[0,\infty)}(H_0)$ by $\mathbf{1}_{[0,\infty)}(H)$. This will essentially allow us to commute the energy cut-off and the group. We then show that we can replace the energy cut-off by $\mathbf{1}_{[0,\infty)}(H_\leftarrow)$. We end up with:

$$\lim_{T \to \infty} \big\| \mathbf{1}_{[0,\infty)}(H_\leftarrow)\, e^{-it_\epsilon H_\leftarrow} F^T_{t_\epsilon} \big\|, \tag{6.6}$$

which we have already computed.

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