Stability of small periodic waves for the nonlinear Schrödinger equation

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Abstract

The nonlinear Schrödinger equation possesses three distinct six-parameter families of complex-valued quasi-periodic travelling waves, one in the defocusing case and two in the focusing case. All these solutions have the property that their modulus is a periodic function of $x - ct$ for some $c \in \mathbb{R}$. In this paper we investigate the stability of the small amplitude travelling waves, both in the defocusing and the focusing case. Our first result shows that these waves are orbitally stable within the class of solutions which have the same period and the same Floquet exponent as the original wave. Next, we consider general bounded perturbations and focus on spectral stability. We show that the small amplitude travelling waves are stable in the defocusing case, but unstable in the focusing case. The instability is of side-band type, and therefore cannot be detected in the periodic set-up used for the analysis of orbital stability.

1 Introduction

We consider the one-dimensional cubic nonlinear Schrödinger equation (NLS)

$$\mathrm{i}U_t(x,t) + U_{xx}(x,t) \pm |U(x,t)|^2 U(x,t) \;=\; 0 \;,$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, $U(x,t) \in \mathbb{C}$, and the signs $+$ and $-$ in the nonlinear term correspond to the focusing and the defocusing case, respectively. In both cases the NLS equation possesses *quasi-periodic* solutions of the general form

$$U(x,t) \;=\; \mathrm{e}^{\mathrm{i}(px - \omega t)}\, V(x - ct)\;, \quad x \in \mathbb{R}\;, \quad t \in \mathbb{R}\;, \tag{1.1}$$

where p, ω, c are real parameters and the wave profile V is a *complex-valued periodic* function of its argument. The aim of the present paper is to investigate the stability properties of these particular solutions, at least when the wave profile V is small. It turns out that the discussion is very similar in both cases, so for simplicity we restrict our presentation to the defocusing equation

$$\mathrm{i}U_t(x,t) + U_{xx}(x,t) - |U(x,t)|^2 U(x,t) \;=\; 0 \;, \tag{1.2}$$

and only discuss the differences which occur in the focusing case at the end of the paper.

A crucial role in the stability analysis is played by the various symmetries of the NLS equation. The most important ones for our purposes are the four continuous symmetries:

- *phase invariance:* $U(x,t) \mapsto U(x,t)\,\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in \mathbb{R}$;

- *translation invariance:* $U(x,t) \mapsto U(x + \xi, t)$, $\xi \in \mathbb{R}$;

- *Galilean invariance:* $U(x,t) \mapsto \mathrm{e}^{-\mathrm{i}\left(\frac{v}{2}x + \frac{v^2}{4}t\right)}U(x + vt, t)$, $v \in \mathbb{R}$;

- *dilation invariance:* $U(x,t) \mapsto \lambda U(\lambda x, \lambda^2 t)$, $\lambda > 0$;

and the two discrete symmetries:

- *reflection symmetry:* $U(x,t) \mapsto U(-x,t)$,

- *conjugation symmetry:* $U(x,t) \mapsto \overline{U}(x,-t)$.

As is well-known, the Cauchy problem for equation (1.2) is globally well-posed on the whole real line in the Sobolev space $H^1(\mathbb{R}, \mathbb{C})$, see e.g. [9, 13, 14, 21]. Alternatively, one can solve the NLS equation on a bounded interval $[0, L]$ with periodic boundary conditions, in which case an appropriate function space is $H^1_{\mathrm{per}}([0, L], \mathbb{C})$. In both situations, we have the following conserved quantities:

$$E_1(U) \;=\; \frac{1}{2}\int_{\mathcal{I}} |U(x,t)|^2\,\mathrm{d}x \;,$$

$$E_2(U) \;=\; \frac{\mathrm{i}}{2}\int_{\mathcal{I}} \overline{U}(x,t)U_x(x,t)\,\mathrm{d}x \;,$$

$$E_3(U) \;=\; \int_{\mathcal{I}} \left(\frac{1}{2}|U_x(x,t)|^2 + \frac{1}{4}|U(x,t)|^4\right)\mathrm{d}x \;,$$

where $\mathcal{I}$ denotes either the whole real line or the bounded interval $[0, L]$. The quantities E_1 and E_2 are conserved due to the phase invariance and the translation invariance, respectively, whereas the conservation of E_3 originates in the fact that equation (1.2) is autonomous.

The symmetries listed above are also useful to understand the structure of the set of all quasi-periodic solutions of (1.2). Assume that $U(x,t)$ is a solution of (1.2) of the form (1.1), where $V : \mathbb{R} \to \mathbb{C}$ is a bounded function. Since $|U(x,t)| = |V(x - ct)|$, the translation speed $c \in \mathbb{R}$ is uniquely determined by U, except if the modulus $|V|$ is constant. In any case, using the Galilean invariance, we can transform $U(x,t)$ into another solution of the form (1.1) with $c = 0$. Once this is done, the temporal frequency ω is in turn uniquely determined by $U(x,t)$, except in the trivial case where V is identically zero. In view of the dilation invariance, only the sign of ω is important, so we can assume without loss of generality that $\omega \in \{-1; 0; 1\}$. Setting $U(x,t) = \mathrm{e}^{-\mathrm{i}\omega t} W(x)$, we see that $W(x) = \mathrm{e}^{\mathrm{i}px} V(x)$ is a bounded solution of the ordinary differential equation

$$W_{xx}(x) + \omega W(x) - |W(x)|^2 W(x) = 0 \ , \quad x \in \mathbb{R} \ . \tag{1.3}$$

If $\omega = 0$ or $\omega = -1$, is it straightforward to verify that $W \equiv 0$ is the only bounded solution of (1.3), thus we assume from now on that $\omega = 1$. Equation (1.3) is actually the stationary Ginzburg-Landau equation and the set of its bounded solutions is well-known [6, 10, 11, 12]. There are two kinds of solutions of (1.3) which lead to quasi-periodic solutions of the NLS equation of the form (1.1):

- A family of *periodic solutions* with constant modulus $W(x) = (1 - p^2)^{1/2}\, \mathrm{e}^{\mathrm{i}(px + \varphi)}$, where $p \in [-1, 1]$ and $\varphi \in [0, 2\pi]$. The corresponding solutions of (1.2) are called *plane waves*. The general form of these waves is
$$U(x,t) = \mathrm{e}^{\mathrm{i}(px - \omega t)}\, V \ ,$$
where $p \in \mathbb{R}$, $\omega \in \mathbb{R}$, and $V \in \mathbb{C}$ satisfy the dispersion relation $\omega = p^2 + |V|^2$.

- A family of *quasi-periodic solutions* of the form $W(x) = r(x)\, \mathrm{e}^{\mathrm{i}\varphi(x)}$, where the modulus $r(x)$ and the derivative of the phase $\varphi(x)$ are periodic with the same period. Any such solution can be written in the equivalent form $W(x) = \mathrm{e}^{\mathrm{i}px}\, Q(2kx)$, where $p \in \mathbb{R}$, $k > 0$, and $Q : \mathbb{R} \to \mathbb{C}$ is 2π-periodic. In particular,
$$U(x,t) = \mathrm{e}^{-\mathrm{i}t} W(x) = \mathrm{e}^{\mathrm{i}(px - t)} Q(2kx) \tag{1.4}$$
is a quasi-periodic solution of (1.2) of the form (1.1) (with $c = 0$ and $\omega = 1$). We shall refer to such a solution as a *periodic wave*, because its profile $|U(x,t)|$ is a (non-trivial) periodic function of the space variable x. Important quantities related to the periodic wave (1.4) are the period of the modulus $T = \pi/k$, and the Floquet multiplier $\mathrm{e}^{\mathrm{i}pT}$. For small amplitude solutions ($|Q| \ll 1$) the minimal period T is close to π, hence $k \approx 1$, and the Floquet multiplier is close to -1, so that we can choose $p \approx 1$.

While the plane waves form a three-parameter family, we will see in Section 2 that the periodic waves form a six-parameter family of solutions of (1.2). However, the number of independent parameters can be substantially reduced if we use the four continuous symmetries listed above. Indeed it is easy to verify that any plane wave is equivalent either to $U_1(x,t) = 0$ or to $U_2(x,t) = \mathrm{e}^{-\mathrm{i}t}$. In a similar way, the set of all periodic waves reduces to a two-parameter family.

As far as the stability of the *plane waves* is concerned, the conserved quantities E_1 and E_3 immediately show that the trivial solution $U_1 = 0$ is stable (in the sense of Lyapunov) with respect to perturbations in $H^1(\mathbb{R})$ or $H^1_{\mathrm{per}}([0, L])$, for any $L > 0$. The same conservation laws also imply that the plane wave $U_2 = \mathrm{e}^{-\mathrm{i}t}$ is *orbitally stable* in the following sense. Assume that $\mathcal{I} = [0, L]$ is a bounded interval, and let $U(x, t)$ be the solution of (1.2) with initial data $U(x, 0) = 1 + V_0(x)$, where $V_0 \in H^1_{\mathrm{per}}(\mathcal{I})$ and $\|V_0\|_{H^1(\mathcal{I})} \le \epsilon$. If $\epsilon > 0$ is small enough, then

$$\inf_{\varphi \in [0, 2\pi]} \|U(\cdot, t) - \mathrm{e}^{\mathrm{i}\varphi}\|_{H^1(\mathcal{I})} \le C(\mathcal{I})\epsilon \,, \quad \text{for all } t \in \mathbb{R} \,, \tag{1.5}$$

where the constant $C(\mathcal{I})$ depends only on the length of the interval $\mathcal{I}$. This stability property is easily established using the conserved quantity

$$E(U) = \int_{\mathcal{I}} \left(\frac{1}{2}|U_x(x, t)|^2 + \frac{1}{4}(|U(x, t)|^2 - 1)^2 \right) \mathrm{d}x = E_3(U) - E_1(U) + \frac{1}{4}|\mathcal{I}| \,.$$

A similar result holds for small perturbations of U_2 in $H^1(\mathbb{R})$. In that case, the bound (1.5) holds for any bounded interval $\mathcal{I} \subset \mathbb{R}$, but the conservation of $E(U)$ does not prevent the norm $\|U(\cdot, t) - \mathrm{e}^{-\mathrm{i}t}\|_{H^1(\mathbb{R})}$ from growing as $|t| \to \infty$. We refer to [30, Section 3.3] for a detailed analysis of the stability of plane waves.

The stability question is much more difficult for *periodic waves*. In contrast to dissipative systems for which nonlinear stability of periodic patterns has been established for rather general classes of perturbations, including localized ones (see e.g. [28]), no such result is available so far for dispersive equations. In the particular case of NLS, the stability of the ground state solitary waves has been intensively studied (see e.g. [8, 29]), but relatively little seems to be known about the corresponding question for periodic waves. A partial spectral analysis is carried out by Rowlands [25], who shows that the periodic waves are unstable in the focusing case and stable in the defocusing case, provided disturbances lie in the long-wave regime (see also [3, 19] and the references therein). Spectral stability has also been addressed for certain NLS-type equations with periodic potentials [7, 23]. Very recently, Angulo [1] has shown that the family of "dnoidal waves" of the focusing NLS equation is orbitally stable with respect to perturbations which have the same period as the wave itself. In most of these previous works, the wave profile V is assumed to be real-valued. Here we restrict ourselves to small amplitude solutions, but allow for general complex-valued wave profiles. While the nonlinear stability of these waves with respect to bounded or localized perturbations remains an open problem, we treat here two particular questions: orbital stability with respect to periodic perturbations, and spectral stability with respect to bounded or localized perturbations.

Our first result shows that the periodic waves of (1.2) are *orbitally stable* within the class of solutions which have the same period and the same Floquet multiplier as the original wave:

Theorem 1 (Orbital stability)
Let $X = H^1_{\mathrm{per}}([0, 2\pi], \mathbb{C})$. There exist positive constants C_0, ϵ_0, and δ_0 such that the following holds. Assume that $W(x) = \mathrm{e}^{\mathrm{i}px} Q_{\mathrm{per}}(2kx)$ is a solution of (1.3) with $Q_{\mathrm{per}} \in X$, $\|Q_{\mathrm{per}}\|_X \le \delta_0$, and $p, k \approx 1$. For all $R \in X$ such that $\|R\|_X \le \epsilon_0$, the solution $U(x, t) = \mathrm{e}^{\mathrm{i}(px-t)}Q(2kx, t)$ of (1.2) with initial data $U(x, 0) = \mathrm{e}^{\mathrm{i}px}(Q_{\mathrm{per}}(2kx) + R(2kx))$ satisfies, for all $t \in \mathbb{R}$,

$$\inf_{\varphi, \xi \in [0, 2\pi]} \|Q(\cdot, t) - \mathrm{e}^{\mathrm{i}\varphi}Q_{\mathrm{per}}(\cdot - \xi)\|_X \le C_0\|R\|_X \,. \tag{1.6}$$

4

Remarks

1. We point out that Theorem 1 holds uniformly for all quasi-periodic solutions of (1.2) with small amplitude. In particular the unperturbed solution $\mathrm{e}^{\mathrm{i}(px-t)}\,Q_{\mathrm{per}}(2kx)$ can be either a periodic wave, or a plane wave, or even the zero solution.

2. The proof of Theorem 1 relies on the classical approach to orbital stability which goes back to Benjamin [4] (see also [2, 5, 29]). While for solitary waves this method gives a rather complete answer to the stability question, in the case of periodic waves it allows to prove orbital stability only if we restrict ourselves to solutions which have the same periodicity properties as the original wave (see however Remark 3.11 below for a discussion of this limitation). In this paper we shall use the general framework developed by Grillakis, Shatah, and Strauss [15, 16], with appropriate modifications to obtain a uniform stability result for small waves. Note that a direct application of the stability theorem in [16] would give the same conclusion as in Theorem 1, but with stability constants C_0 and ϵ_0 depending on the wave profile Q_{per}.

3. Following the approach of [16] it is shown in [12] that all periodic waves of (1.2) are orbitally stable in the sense of (1.6), without any restriction on the amplitude of the wave profile Q_{per}. The argument in [12] relies in part on the results obtained in the present paper, and uses a global parametrization of the set of quasi-periodic solutions of (1.3) which is very different from the explicit series expansions that we use here to describe the small amplitude solutions.

4. It is worth considering what Theorem 1 exactly means in the particular case where W is a *real-valued* periodic solution of (1.3) (such a solution is often referred to as a "cnoidal wave" in the literature). In that case we have $W(x) = \mathrm{e}^{\mathrm{i}px}\,Q_{\mathrm{per}}(2kx)$ where $p = k = \pi/T$ and $T > \pi$ is the minimal period of $|W|$. The Floquet multiplier $\mathrm{e}^{\mathrm{i}pT}$ is therefore exactly equal to -1, so that $W(x+T) = -W(x)$ for all $x \in \mathbb{R}$. In particular, we see that W is periodic with (minimal) period $L = 2T$. Thus Theorem 1 shows that the L-periodic cnoidal wave $U(x,t) = \mathrm{e}^{-\mathrm{i}t}W(x)$ is orbitally stable with respect to L-periodic perturbations $\widetilde{W}$ *provided* that $\widetilde{W}(x + L/2) = -\widetilde{W}(x)$ for all $x \in \mathbb{R}$. As is explained in [1], without this additional assumption the classical approach does not allow to prove the stability of cnoidal waves with respect to periodic perturbations.

Next, we investigate the spectral stability of the periodic waves with respect to bounded, or localized, perturbations. Although spectral stability is weaker than nonlinear stability, it provides valuable information about the the linearization of the system at the periodic wave. Our second result is:

Theorem 2 (Spectral stability)
Let $Y = L^2(\mathbb{R}, \mathbb{C})$ or $Y = C_b(\mathbb{R}, \mathbb{C})$. There exists $\delta_1 > 0$ such that the following holds. Assume that $W(x) = \mathrm{e}^{\mathrm{i}px}\,Q_{\mathrm{per}}(2kx)$ is a solution of (1.3) with $Q_{\mathrm{per}} \in X$, $\|Q_{\mathrm{per}}\|_X \leq \delta_1$, and $p, k \approx 1$, just as in Theorem 1. Then the spectrum of the linearization of (1.2) about the periodic wave $\mathrm{e}^{-\mathrm{i}t}W(x)$ in the space Y entirely lies on the imaginary axis. Consequently, this wave is spectrally stable in Y.

The proof of Theorem 2 is based on the so-called Bloch-wave decomposition, which reduces the spectral study of the linearized operator in the space Y to the study of the spectra of a family of linear operators in a space of periodic functions. Bloch waves are well-known for Schrödinger operators with periodic potentials [24] and have been extensively used in dissipative problems [22, 26, 27, 28], but also in a number of dispersive problems [7, 17, 23]. The advantage of such

a decomposition is that the resulting operators have compact resolvent, and therefore only point spectra. The main step in the analysis consists in locating these point spectra. For our problem, we rely on perturbation arguments for linear operators in which we regard the operators resulting from the Bloch-wave decomposition as small perturbations of operators with constant coefficients. The latter ones are actually obtained from the linearization of (1.2) about zero, and Fourier analysis allows to compute their spectra explicitly. The restriction to small amplitudes is essential in this perturbation argument, and we do not know whether spectral stability holds for large waves.

The rest of the paper is organized as follows. In Section 2, we briefly describe the set of all bounded solutions of (1.3), and we introduce an analytic parametrization of the small amplitude solutions which will be used throughout the paper. In Section 3 we recall the main ideas of the orbital stability method, and we apply it with appropriate modifications to prove Theorem 1. Spectral stability is established in Section 4, using Bloch-wave decomposition and the pertubation argument mentioned above.

Finally, in Section 5 we discuss the stability of the small periodic waves of the focusing NLS equation. In contrast to the defocusing case, the focusing NLS equation possesses two different families of quasi-periodic solutions of the form (1.1), one for $\omega > 0$ and the other for $\omega < 0$ [12]. Small solutions exist only within the first family, and their stability properties can be analyzed as in the defocusing case. However, while for periodic perturbations we obtain the same orbital stability result as in Theorem 1, it turns out that the small periodic waves are spectrally *unstable* in the focusing case. Unstable spectrum is detected for perturbations with wave-numbers which are close to that of the original wave (side-band instability). As for the second family, which contains only large waves, we refer to [1, 12] for a proof of orbital stability and to [23] for a discussion of spectral stability.

2   Parametrization of small periodic waves

In this section, we briefly review the bounded solutions of equation (1.3) with $\omega = 1$:

$$W_{xx}(x) + W(x) - |W(x)|^2 W(x) \,=\, 0 \;, \quad W : \mathbb{R} \to \mathbb{C} \;, \tag{2.1}$$

and we give a convenient parametrization of all small solutions. If we interpret the spatial variable $x \in \mathbb{R}$ as a "time", equation (2.1) becomes an integrable Hamiltonian dynamical system with two degrees of freedom. The conserved quantities are the "angular momentum" J and the "energy" E:

$$J \,=\, \mathrm{Im}(\overline{W} W_x) \;, \qquad E \,=\, \frac{1}{2}|W_x|^2 + \frac{1}{2}|W|^2 - \frac{1}{4}|W|^4 \;. \tag{2.2}$$

If W is a solution of (2.1) with $J \neq 0$, then $W(x) \neq 0$ for all $x \in \mathbb{R}$, so that we can introduce the polar coordinates $W(x) = r(x)\,\mathrm{e}^{\mathrm{i}\varphi(x)}$. The invariants then become

$$J \,=\, r^2 \varphi_x \;, \qquad E \,=\, \frac{r_x^2}{2} + \frac{J^2}{2r^2} + \frac{r^2}{2} - \frac{r^4}{4} \;.$$

The set of bounded solutions of (2.1) can be entirely described in terms of these two invariants [6, 10, 11, 12]. In the parameter space (J, E) there is an open set

$$D = \left\{ (J, E) \in \mathbb{R}^2 \,\middle|\, J^2 < 4/27, \ E_-(J) < E < E_+(J) \right\}, \tag{2.3}$$

where E_-, E_+ are explicit functions of J, such that the closure $\overline{D}$ consists of all values of (J, E) which give rise to bounded solutions W of (2.1) (see Fig. 1). Furthermore, we have the following classification for (J, E) in $\overline{D}$:

(i) If $E = E_-(J)$, then W is a periodic solution with constant modulus and linear phase, i.e. $W(x) = W_{p,\varphi}(x) = (1 - p^2)^{1/2} \, \mathrm{e}^{\mathrm{i}(px + \varphi)}$ with $1/3 \le p^2 \le 1$ and $\varphi \in [0, 2\pi]$.

(ii) If $E = E_+(J)$, then either $W = W_{p,\varphi}$ for some $p^2 \le 1/3$ and some $\varphi \in [0, 2\pi]$, or W is a homoclinic orbit connecting W_{p,φ_-} at $x = -\infty$ to W_{p,φ_+} at $x = +\infty$ for some $\varphi_-, \varphi_+ \in [0, 2\pi]$.

(iii) If $E_-(J) < E < E_+(J)$ and $J \ne 0$, then the modulus and the phase derivative of W are both periodic with the same period $T(J, E) > \pi$. Let $\Phi(J, E)$ be the increment of the phase over a period of the modulus, so that $W(x + T) = \mathrm{e}^{\mathrm{i}\Phi} W(x)$ for all $x \in \mathbb{R}$. In general Φ is not a rational multiple of π, hence the solution W of (2.1) is typically not periodic, but only quasi-periodic. In the particular case where $J = 0$, then $\mathrm{e}^{\mathrm{i}\Phi} = -1$ and W is periodic with period $2T(0, E)$.
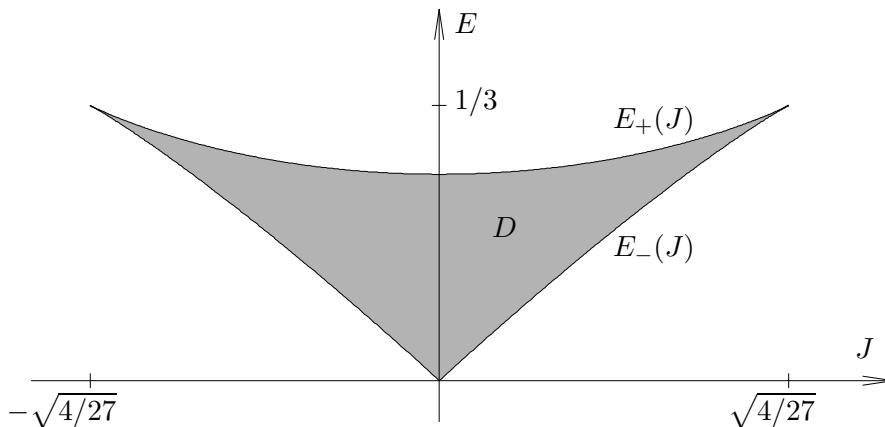

Fig. 1: The region D in the parameter space (J, E) for which (2.1) has bounded solutions.

For a fixed pair $(J, E) \in \overline{D}$, the bounded solution W of (2.1) satisfying (2.2) is unique up to a translation and a phase factor. In case (iii), the period T and the phase increment Φ (or the Floquet multiplier $\mathrm{e}^{\mathrm{i}\Phi}$) are important quantities which play a crucial role in the stability analysis of the quasi-periodic solutions of (2.1), both for the Schrödinger and the Ginzburg-Landau dynamics. A number of properties of T and Φ are collected in [12]. In particular, if we define the renormalized phase

$$\Psi(J, E) = \begin{cases} \Phi(J, E) - \pi \operatorname{sign}(J) & \text{if } J \ne 0, \\ 0 & \text{if } J = 0, \end{cases} \tag{2.4}$$

then $T : D \to \mathbb{R}$ and $\Psi : D \to \mathbb{R}$ are smooth functions of $(J, E) \in D$, in contrast to $\Phi(J, E)$ which is discontinuous at $J = 0$. In addition, $T \approx \pi$ and $\Psi \approx 0$ for small solutions $W \approx 0$.

The periodic solutions $W_{p,\varphi}$ of (2.1) correspond to plane waves of the NLS equation. We are mainly interested here in the quasi-periodic solutions described in (iii) above, which correspond to periodic waves of the form (1.1). To see this, fix $(J, E) \in D$ and let $W : \mathbb{R} \to \mathbb{C}$ be a bounded solution of (2.1) satisfying (2.2). We set

$$W(x) = \mathrm{e}^{\mathrm{i}\ell x} P(kx) , \quad x \in \mathbb{R} , \tag{2.5}$$

in which k and ℓ are related to the period $T(J, E)$ and the renormalized phase $\Psi(J, E)$ through

$$k = \frac{\pi}{T(J, E)} , \quad \text{and} \quad \ell = \frac{\Psi(J, E)}{T(J, E)} . \tag{2.6}$$

As $|W(x)| = |P(kx)|$ is T-periodic (in x) by the definition of $T(J, E)$, it is clear that $|P(y)|$ is π-periodic (in y). Moreover, since $W(x+T) = \mathrm{e}^{\mathrm{i}\Phi}W(x) = -\mathrm{e}^{\mathrm{i}\Psi}W(x)$, we also have $P(y+\pi) = -P(y)$ for all $y \in \mathbb{R}$, hence P is 2π-periodic. Thus $U(x,t) = \mathrm{e}^{-\mathrm{i}t}W(x) = \mathrm{e}^{\mathrm{i}\ell x}\mathrm{e}^{-\mathrm{i}t}P(kx)$ is a quasi-periodic solution of (1.2) of the form (1.1), with $\omega = 1$ and $c = 0$.

Remark 2.1 *Using the phase and the translation invariance of the NLS equation, from this family of quasi-periodic solutions we obtain a four-parameter family of periodic waves of (1.2) of the form (1.1) with $\omega = 1$ and $c = 0$. Taking into account the Galilean and the dilation invariance, we obtain altogether a six-parameter family of periodic waves,*

$$U_{v,\lambda,\varphi,\xi}(x,t) = \lambda \, \mathrm{e}^{\mathrm{i}p_{v,\lambda}x} \, \mathrm{e}^{-\mathrm{i}\omega_{v,\lambda}t} \, \mathrm{e}^{\mathrm{i}\varphi} P(k\lambda(x+vt)+\xi) ,$$

where k and ℓ are given by (2.6), $p_{v,\lambda} = \lambda\ell - v/2$, $\omega_{v,\lambda} = \lambda^2 - v\lambda\ell + v^2/4$, and $v \in \mathbb{R}$, $\lambda > 0$, $\varphi, \xi \in [0, 2\pi]$. Similarly, in the case of the periodic solutions with constant modulus and linear phase $W_{p,\varphi}$ for $(J, E) \in \overline{D} \setminus D$, we find a three-parameter family of plane waves of (1.2) (for such solutions, a phase rotation is just a translation, and a Galilean transformation amounts to a dilation and a shift in the parameter p).

Alternatively, we can write the solution (2.5) of (2.1) in the form

$$W(x) = \mathrm{e}^{\mathrm{i}(\ell+k)x} Q^+(2kx) = \mathrm{e}^{\mathrm{i}(\ell-k)x} Q^-(2kx) , \quad x \in \mathbb{R} , \tag{2.7}$$

where $Q^{\pm}(z) = \mathrm{e}^{\mp\mathrm{i}z/2}P(z/2)$. By construction, $Q^{\pm}$ and $|Q^{\pm}|$ are now periodic functions with the *same* minimal period 2π. The representation (2.7) turns out to be more convenient than (2.5) to study the orbital stability of the periodic waves in the next section.

The global parametrization of the quasiperiodic solutions of (2.1) in terms of the invariants (J, E) is natural, but it is not very convenient as far as small solutions are concerned because the admissible domain D is not smooth near the origin (see Fig. 1). For this reason, we now introduce an analytic parametrization of the *small* solutions of (2.1). We start from the representation (2.5), and we choose as parameters the first nonzero Fourier coefficients of the 2π-periodic function P:

$$a = \frac{1}{2\pi} \int_0^{2\pi} P(y) \, \mathrm{e}^{\mathrm{i}y} \, \mathrm{d}y , \quad b = \frac{1}{2\pi} \int_0^{2\pi} P(y) \, \mathrm{e}^{-\mathrm{i}y} \, \mathrm{d}y .$$

(Remark that P has zero mean over a period.) Replacing $P(y)$ with $e^{-i\varphi}P(y+\xi)$ if needed, we can assume that both a and b are real. If P (hence also W) is small, we have $T \approx \pi$ and $\Psi \approx 0$, hence $k \approx 1$ and $\ell \approx 0$. This determines uniquely the expansion of P, k, ℓ in powers of a and b. Setting

$$W_{a,b}(x) = e^{i\ell_{a,b}x}P_{a,b}(k_{a,b}x) , \quad x \in \mathbb{R} , \tag{2.8}$$

we obtain after straightforward calculations:

$$\begin{aligned} \ell_{a,b} &= \frac{1}{4}(b^2 - a^2) + \mathcal{O}(a^4 + b^4) , \\[2mm] k_{a,b} &= 1 - \frac{3}{4}(a^2 + b^2) + \mathcal{O}(a^4 + b^4) , \\[2mm] P_{a,b}(y) &= a\,e^{-iy} + b\,e^{iy} - \frac{a^2b}{8}\,e^{-3iy} - \frac{ab^2}{8}\,e^{3iy} + \mathcal{O}(|ab|(|a|^3 + |b|^3)) , \end{aligned} \tag{2.9}$$

as $(a, b) \to (0, 0)$. Notice also that the invariants J, E have the following expressions:

$$\begin{aligned} J &= b^2 - a^2 + \frac{1}{2}(a^4 - b^4) + \mathcal{O}(a^6 + b^6) , \\[2mm] E &= a^2 + b^2 - 3a^2b^2 - \frac{3}{4}(a^4 + b^4) + \mathcal{O}(a^6 + b^6) . \end{aligned}$$

With this parametrization, replacing a with $-a$ or b with $-b$ gives the same function P up to a translation and a phase factor:

$$P_{-a,b}(y) = -iP_{a,b}(y + \pi/2) , \quad P_{-a,-b}(y) = -P_{a,b}(y) = P_{a,b}(y + \pi) , \quad y \in \mathbb{R} .$$

It follows that J, E, hence also k, ℓ, are even functions of a and b. Similarly, $P_{b,a}(y) = \overline{P_{a,b}(y)}$. This conjugation leaves E unchanged but reverses the sign of J, hence $k_{a,b} = k_{b,a}$ and $\ell_{a,b} = -\ell_{b,a}$. Therefore, using the symmetries of (2.1), we can restrict ourselves to the parameter region $\{b \geq a \geq 0\}$ without loss of generality.

Two particular cases will play a special role in what follows.

(i) **(Cnoidal waves)** If $a = b$, then $\ell_{a,a} = 0$ and we obtain a family of real-valued periodic solutions $W_{a,a}(x) = P_{a,a}(k_{a,a}x)$, where

$$k_{a,a} = 1 - \frac{3}{2}a^2 + \mathcal{O}(a^4) , \quad P_{a,a}(y) = 2a\cos y - \frac{a^3}{4}\cos(3y) + \mathcal{O}(|a|^5) .$$

Observe that $J = 0$ in that case.

(ii) **(Plane waves)** If $a = 0$, then $P_{0,b}(y) = b\,e^{iy}$, hence $W_{0,b}$ has constant modulus. It follows that

$$W_{0,b}(x) = b\,e^{i\sqrt{1-b^2}x} , \quad \text{and} \quad k_{0,b} + \ell_{0,b} = \sqrt{1-b^2} .$$

In addition, one also finds $k_{0,b} = \sqrt{1 - 3b^2/2}$. It is advantageous here to use the representation (2.7), namely $W_{0,b}(y) = e^{ip_{0,b}^+ y}Q_{0,b}^+(y)$ with $p_{0,b}^+ = \ell_{0,b} + k_{0,b} = (1 - b^2)^{1/2}$ and $Q_{0,b}^+ \equiv b$. Similarly, if $b = 0$ we have $P_{a,0}(y) = a\,e^{-iy}$ and thus

$$W_{a,0}(x) = a\,e^{-i\sqrt{1-a^2}x} , \quad \text{and} \quad k_{a,0} - \ell_{a,0} = \sqrt{1-a^2} , \quad k_{a,0} = \sqrt{1 - 3a^2/2} .$$

Alternatively, $W_{a,0}(y) = e^{ip_{a,0}^- y}Q_{a,0}^-(y)$ with $p_{a,0}^- = \ell_{a,0} - k_{a,0} = -(1 - a^2)^{1/2}$ and $Q_{a,0}^- \equiv a$.

3 Orbital stability

In this section we prove the orbital stability result in Theorem 1. Since we restrict ourselves to periodic waves with small amplitude, we shall use the local parametrization (2.8), (2.9) of the small solutions of (2.1). Given $(a, b) \in \mathbb{R}^2$ with $\|(a, b)\|$ sufficiently small, we consider the periodic wave $U_{a,b}(x, t) = \mathrm{e}^{-\mathrm{i}t} W_{a,b}(x)$, where

$$W_{a,b}(x) \;=\; \mathrm{e}^{\mathrm{i}\ell_{a,b}x} P_{a,b}(k_{a,b}x) \;=\; \mathrm{e}^{\mathrm{i}p_{a,b}x} Q_{a,b}(2k_{a,b}x) \;, \quad x \in \mathbb{R} \;.$$

Here $\ell_{a,b}, k_{a,b}, P_{a,b}$ are defined in (2.9), and the last expression in the right-hand side corresponds to the first choice in (2.7), namely

$$p_{a,b} \;=\; \ell_{a,b} + k_{a,b} \;, \quad Q_{a,b}(z) \;=\; \mathrm{e}^{-\mathrm{i}z/2} P_{a,b}(z/2) \;. \tag{3.1}$$

¿From the properties of $P_{a,b}$ we deduce

$$Q_{-a,b}(z) = Q_{a,b}(z + \pi), \quad Q_{-a,-b}(z) = -Q_{a,b}(z), \quad Q_{b,a}(z) = \mathrm{e}^{-\mathrm{i}z}\overline{Q_{a,b}(z)} \;, \tag{3.2}$$

and that the real and imaginary parts of $Q_{a,b}$ are even and odd functions of z, respectively.

Remark 3.1 *Without loss of generality, we shall assume henceforth that $b \geq a \geq 0$. Note that the second choice in (2.7) would be preferable when $a^2 \geq b^2$.*

3.1 Main result and strategy of proof

To study the stability of $U_{a,b}(x, t)$ we consider solutions of (1.2) of the form

$$U(x, t) \;=\; \mathrm{e}^{\mathrm{i}(p_{a,b}x - t)} Q(2k_{a,b}x, t) \;, \tag{3.3}$$

where $Q(z, t)$ is a 2π-periodic function of z which satisfies the evolution equation

$$\mathrm{i}Q_t + 4\mathrm{i}p_{a,b}k_{a,b}Q_z + 4k_{a,b}^2 Q_{zz} + (1 - p_{a,b}^2)Q - |Q|^2 Q \;=\; 0 \;. \tag{3.4}$$

By construction, $Q_{a,b}(z)$ is now a stationary solution of (3.4) and our goal is to show that this equilibrium is stable with respect to 2π-periodic perturbations. We thus introduce the function space

$$X \;=\; H_{\mathrm{per}}^1([0, 2\pi], \mathbb{C}) \;=\; \left\{ u \in H_{\mathrm{loc}}^1(\mathbb{R}, \mathbb{C}) \,\middle|\, u(z+2\pi) = u(z), \; \forall z \in \mathbb{R} \right\} \;,$$

which is viewed as a *real* Hilbert space equipped with the scalar product

$$(u, v)_X \;=\; \mathrm{Re} \int_0^{2\pi} (u(z)\overline{v}(z) + u_z(z)\overline{v}_z(z)) \,\mathrm{d}z \;, \quad u, v \in X \;.$$

As usual, the dual space X^* will be identified with $H_{\mathrm{per}}^{-1}([0, 2\pi], \mathbb{C})$ through the pairing

$$\langle u, v \rangle \;=\; \mathrm{Re} \int_0^{2\pi} u(z)\overline{v}(z) \,\mathrm{d}z \;, \quad u \in X^* \;, \quad v \in X \;.$$

It is well-known that the Cauchy problem for (3.4) is globally well-posed in the space X. Moreover, the evolution defined by (3.4) on X is invariant under a two-parameter group of isometries.

The symmetry group is the two-dimensional torus $G = \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$, a compact abelian Lie group which acts on X through the unitary representation $\mathcal{R}$ defined by

$$(\mathcal{R}_{(\varphi,\xi)}u)(z) = \mathrm{e}^{-\mathrm{i}\varphi}u(z+\xi) \ , \quad u \in X \ , \quad (\varphi,\xi) \in G \ .$$

Due to these symmetries, it is useful to introduce the semi-distance ρ on X defined by

$$\rho(u,v) = \inf_{(\varphi,\xi)\in G} \|u - \mathcal{R}_{(\varphi,\xi)}v\|_X \ , \quad u,v \in X \ . \tag{3.5}$$

In words, $\rho(u,v)$ is small if u is close to v (in the topology of X) up to a translation and a phase rotation. Our stability result in Theorem 1 can now be formulated as follows:

Proposition 3.2 *There exist $C_0 > 0$, $\epsilon_0 > 0$, and $\delta_0 > 0$ such that, for all $(a,b) \in \mathbb{R}^2$ with $\|(a,b)\| \le \delta_0$, the following holds. If $Q_0 \in X$ satisfies $\rho(Q_0, Q_{a,b}) \le \epsilon$ for some $\epsilon \le \epsilon_0$, then the solution $Q(z,t)$ of (3.4) with initial data Q_0 satisfies $\rho(Q(\cdot,t), Q_{a,b}) \le C_0\epsilon$ for all $t \in \mathbb{R}$.*

For each fixed value of (a,b), the stability of the periodic (or plane) wave $Q_{a,b}$ can be proved using the abstract results of Grillakis, Shatah and Strauss [15, 16]. However, this approach would not give a stability theorem that holds uniformly in a neighborhood of the origin, as it is the case in Proposition 3.2. A difficulty in proving such a uniform result is that we have to deal simultaneously with three sorts of solutions: the zero solution ($a = b = 0$), plane waves ($ab = 0$) and periodic waves ($ab \ne 0$). These equilibria are genuinely different from the point of view of orbital stability theory, because their orbits under the action of the symmetry group G have different dimensions (0, 1, and 2, respectively). In what follows, we shall concentrate on the periodic waves, and at the end we shall indicate how the other cases can be incorporated to obtain a uniform result. Whenever possible, we shall adopt similar notations as in [16] to facilitate comparison.

Due to its symmetries, equation (3.4) has the same conserved quantities as the original NLS equation, namely

$$N(Q) = \frac{1}{2}\int_0^{2\pi} |Q(z)|^2 \, \mathrm{d}z \ ,$$

$$M(Q) = \frac{\mathrm{i}}{2}\int_0^{2\pi} \overline{Q}(z)Q_z(z) \, \mathrm{d}z \ ,$$

$$\mathcal{E}(Q) = \int_0^{2\pi} \left(2k_{a,b}^2|Q_z(z)|^2 + \frac{1}{4}|Q(z)|^4 \right) \mathrm{d}z \ .$$

The charge N, the momentum M and the energy $\mathcal{E}$ are smooth real-valued functions on X. Their first order derivatives are therefore smooth maps from X into X^*:

$$N'(Q) = Q \ , \quad M'(Q) = \mathrm{i}Q_z \ , \quad \mathcal{E}'(Q) = -4k_{a,b}^2 Q_{zz} + |Q|^2 Q \ .$$

Similarly, the second order derivatives are smooth maps from X into $\mathcal{L}(X, X^*)$, the space of all bounded linear operators from X into X^*:

$$N''(Q) = \mathbf{1} \ , \quad M''(Q) = \mathrm{i}\partial_z \ , \quad \mathcal{E}''(Q) = -4k_{a,b}^2\partial_{zz} + |Q|^2 + 2Q \otimes Q \ ,$$

where

$$\langle (Q \otimes Q)u, v \rangle = \int_0^{2\pi} \operatorname{Re}(Q\overline{u}) \operatorname{Re}(Q\overline{v}) \, dz, \quad \forall \, u, v \in X.$$

¿From now on, we fix $(a, b) \in \mathbb{R}^2$ with $\|(a, b)\|$ sufficiently small. As is explained above, we assume for the moment that $ab \neq 0$, in which case the function $Q_{a,b} \in X$ defined by (3.1) is a stationary solution of (3.4) corresponding to a periodic wave of the original NLS equation, i.e. $|Q_{a,b}|$ is not constant. By construction, $Q_{a,b}$ is a critical point of the modified energy

$$\mathcal{E}_{a,b}(Q) = \mathcal{E}(Q) - (1 - p_{a,b}^2)N(Q) - 4p_{a,b}k_{a,b}M(Q) \,, \tag{3.6}$$

namely $\mathcal{E}'_{a,b}(Q_{a,b}) = 0$. The orbital stability argument is based on two essential ingredients:

Claim 1: The equilibrium $Q_{a,b}$ is a *local minimum* of the function $\mathcal{E}_{a,b}$ restricted to the codimension two submanifold

$$\Sigma_{a,b}(Q) = \left\{ Q \in X \,\middle|\, N(Q) = N(Q_{a,b}), \; M(Q) = M(Q_{a,b}) \right\} \,. \tag{3.7}$$

Note that this manifold contains the entire orbit of $Q_{a,b}$ under the action of G.

Claim 2: The equilibrium $Q_{a,b}$ is a member of a *two-parameter family* of travelling and rotating waves of the form

$$Q(z,t) = e^{-i\omega t} Q_{a,b}^{\omega,c}(z + ct) \,, \quad z \in \mathbb{R} \,, \quad t \in \mathbb{R} \,, \tag{3.8}$$

where (ω, c) lies in a neighborhood of the origin in $\mathbb{R}^2$ (the Lie algebra of G) and $Q_{a,b}^{\omega,c} \in X$ is a smooth function of (ω, c) with $Q_{a,b}^{0,0} = Q_{a,b}$. Moreover the map $(\omega, c) \mapsto (N(Q_{a,b}^{\omega,c}), M(Q_{a,b}^{\omega,c}))$ is a *local diffeomorphism* near $(\omega, c) = (0, 0)$.

3.2 Proof of Claim 2

The second claim is easily justified using the continuous symmetries of the NLS equation. Indeed, let $(a', b') \in \mathbb{R}^2$ be close to (a, b). Then

$$U(x, t) = e^{i(p_{a',b'}x - t)} Q_{a',b'}(2k_{a',b'}x) \,, \quad x \in \mathbb{R} \,, \quad t \in \mathbb{R} \,,$$

is a solution of the NLS equation, but it is not of the form (3.3) because $p_{a',b'} \neq p_{a,b}$ and $k_{a',b'} \neq k_{a,b}$ in general. However we can transform $U(x, t)$ into a solution of (1.2) of the form (3.3), (3.8) by applying successively a dilation of factor λ and a Galilean transformation of speed v, where

$$\lambda = \lambda_{a,b}^{a',b'} = \frac{k_{a,b}}{k_{a',b'}} \,, \quad v = v_{a,b}^{a',b'} = 2(\lambda_{a,b}^{a',b'} p_{a',b'} - p_{a,b}) \,. \tag{3.9}$$

After some elementary algebra, we obtain $Q_{a,b}^{\omega,c}(z) = \lambda_{a,b}^{a',b'} Q_{a',b'}(z)$ with

$$\omega = (\lambda_{a,b}^{a',b'})^2 (1 - p_{a',b'}^2) - (1 - p_{a,b}^2) \,, \quad c = 4(\lambda_{a,b}^{a',b'})^2 k_{a',b'} p_{a',b'} - 4k_{a,b}p_{a,b} \,. \tag{3.10}$$

Using the expansions (2.9), it is straightforward to verify that

$$\mathcal{M}_{a,b} \overset{\text{def}}{=} \left. \begin{pmatrix} \frac{\partial \omega}{\partial a'} & \frac{\partial c}{\partial a'} \\ \frac{\partial \omega}{\partial b'} & \frac{\partial c}{\partial b'} \end{pmatrix} \right|_{(a',b')=(a,b)} = \begin{pmatrix} 4a & -2a \\ 2b & 2b \end{pmatrix} (\mathbf{1} + \mathcal{O}(a^2+b^2)) \,.$$

Since we assumed that $ab \neq 0$, the matrix $\mathcal{M}_{a,b}$ is invertible for a, b sufficiently small, hence the mapping $(a', b') \mapsto (\omega, c)$ defined by (3.10) is a diffeomorphism from a neighborhood of (a, b) onto a neighborhood of $(0, 0)$. This proves the existence of the travelling and rotating wave (3.8) for $(\omega, c) \in \mathbb{R}^2$ sufficiently small. Remark that the profile $Q_{a,b}^{\omega;c}$ is a critical point of the functional

$$\mathcal{E}_{a,b}^{\omega;c}(Q) = \mathcal{E}_{a,b}(Q) - \omega N(Q) - cM(Q) , \quad Q \in X .$$

Following [16], we define $d_{a,b}(\omega, c) = \mathcal{E}_{a,b}^{\omega;c}(Q_{a,b}^{\omega;c})$. The properties of the function $d_{a,b}$ will play an important role in the orbital stability argument.

Lemma 3.3 *The Hessian matrix of the function $d_{a,b}$ satisfies:*

$$\mathcal{H}_{a,b} \overset{def}{=} \begin{pmatrix} \frac{\partial^2 d_{a,b}}{\partial \omega^2} & \frac{\partial^2 d_{a,b}}{\partial \omega \, \partial c} \\ \frac{\partial^2 d_{a,b}}{\partial c \, \partial \omega} & \frac{\partial^2 d_{a,b}}{\partial c^2} \end{pmatrix} \Bigg|_{(\omega,c)=(0,0)} = \frac{\pi}{3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} (\mathbf{1} + \mathcal{O}(a^2+b^2)) .$$

Proof. Since $Q_{a,b}^{\omega;c}$ is a critical point of $\mathcal{E}_{a,b}^{\omega;c}$, we have

$$\frac{\partial}{\partial \omega} d_{a,b}(\omega, c) = -N(Q_{a,b}^{\omega;c}) , \quad \frac{\partial}{\partial c} d_{a,b}(\omega, c) = -M(Q_{a,b}^{\omega;c}) . \tag{3.11}$$

To compute the second-order derivatives, we parametrize (ω, c) by (a', b') as above. Using (3.10) we find $\mathcal{H}_{a,b} = -(\mathcal{M}_{a,b})^{-1} \mathcal{K}_{a,b}$, where

$$\mathcal{K}_{a,b} = \begin{pmatrix} \frac{\partial}{\partial a'} N(Q_{a,b}^{\omega;c}) & \frac{\partial}{\partial a'} M(Q_{a,b}^{\omega;c}) \\ \frac{\partial}{\partial b'} N(Q_{a,b}^{\omega;c}) & \frac{\partial}{\partial b'} M(Q_{a,b}^{\omega;c}) \end{pmatrix} \Bigg|_{(a',b')=(a,b)} .$$

As $Q_{a,b}^{\omega;c} = \lambda_{a,b}^{a',b'} Q_{a',b'}(z)$, we have

$$N(Q_{a,b}^{\omega;c}) = (\lambda_{a,b}^{a',b'})^2 N(Q_{a',b'}) , \quad M(Q_{a,b}^{\omega;c}) = (\lambda_{a,b}^{a',b'})^2 M(Q_{a',b'}) .$$

On the other hand, using the expansion

$$Q_{a,b}(z) = a\,e^{-iz} + b - \frac{a^2 b}{8} e^{-2iz} - \frac{ab^2}{8} e^{iz} + \mathcal{O}(|ab|(|a|^3 + |b|^3)) , \tag{3.12}$$

which follows from (2.9), (3.1), we easily find

$$N(Q_{a,b}) = \pi(a^2 + b^2) + \mathcal{O}(a^2 b^2(a^2 + b^2)) , \quad M(Q_{a,b}) = \pi a^2 + \mathcal{O}(a^2 b^2(a^2 + b^2)) .$$

Combining these results, we obtain

$$(\mathcal{M}_{a,b})^{-1} = \frac{1}{6ab} \begin{pmatrix} b & a \\ -b & 2a \end{pmatrix} (\mathbf{1} + \mathcal{O}(a^2+b^2)) , \quad \mathcal{K}_{a,b} = 2\pi \begin{pmatrix} a & a \\ b & 0 \end{pmatrix} (\mathbf{1} + \mathcal{O}(a^2+b^2)) ,$$

and the conclusion follows. $\qquad \square$

Lemma 3.3 implies that the Hessian matrix $\mathcal{H}_{a,b}$ is nondegenerate for $\|(a, b)\|$ sufficiently small (in fact, $\mathcal{H}_{a,b}$ has one positive and one negative eigenvalue.) It follows that the map $(\omega, c) \mapsto (N(Q_{a,b}^{\omega;c}), M(Q_{a,b}^{\omega;c}))$ is a local diffeomorphism near $(\omega, c) = (0, 0)$, because by (3.11) the Jacobian matrix of this map at the origin is just $-\mathcal{H}_{a,b}$. Thus Claim 2 above is completely justified.

Remark 3.4 *At this point we could apply the general result of [16], but as already mentioned this would not give the uniform result in Theorem 1. According to the Stability Theorem in [16], in order to establish the orbital stability of a single wave $Q_{a,b}$ with $ab \neq 0$ it suffices to show that the linear operator*

$$H_{a,b} = \mathcal{E}''_{a,b}(Q_{a,b}) = -4k^2_{a,b}\partial_{zz} - 4ip_{a,b}k_{a,b}\partial_z - (1-p^2_{a,b}) + |Q_{a,b}|^2 + 2Q_{a,b} \otimes Q_{a,b} , \qquad (3.13)$$

has precisely one simple negative eigenvalue, a two-dimensional kernel spanned by

$$\frac{\partial}{\partial\varphi}\mathcal{R}_{(\varphi,\xi)}Q_{a,b}\Big|_{(\varphi,\xi)=(0,0)} = -iQ_{a,b} , \quad \frac{\partial}{\partial\xi}\mathcal{R}_{(\varphi,\xi)}Q_{a,b}\Big|_{(\varphi,\xi)=(0,0)} = \partial_z Q_{a,b} , \qquad (3.14)$$

and that the rest of its spectrum is strictly positive. Observe that $H_{a,b}$ is self-adjoint in the real Hilbert space $L^2_{\mathrm{per}}([0,2\pi], \mathbb{C})$ equipped with the scalar product $\langle\cdot,\cdot\rangle$. Clearly, the vectors (3.14) always belong to the kernel of $H_{a,b}$. In fact, for small (a,b) we can determine the spectrum of $H_{a,b}$ by a perturbation argument similar to the one used for the spectral analysis of the operators $\mathcal{A}_{a,b,\gamma}$ in Section 4. We find that $H_{a,b}$ has exactly four eigenvalues in a neighborhood of the origin, the rest of the spectrum being positive and bounded away from zero. Among these four eigenvalues, two are always zero, and the other two have negative product $-12a^2b^2(1+\mathcal{O}(a^2+b^2))$. This implies that $H_{a,b}$ has the required properties, so that the wave profile $Q_{a,b}$ is orbitally stable if $ab \neq 0$. This information on the spectrum of $H_{a,b}$ will not be used in the remainder of this section. However, since it provides the starting point for the stability analysis of large waves in [12], we give a brief proof in the Appendix.

3.3 Proof of Claim 1

We now turn back to Claim 1 and study the behavior of the energy $\mathcal{E}_{a,b}$ on the manifold $\Sigma_{a,b}$ defined by (3.7). In the arguments below, we assume $b \geq a > 0$, so exclude for the moment the plane wave corresponding to $a = 0$. Let $\mathcal{T}_{a,b}$ be the tangent space to $\Sigma_{a,b}$ at the point $Q_{a,b}$:

$$\mathcal{T}_{a,b} = \left\{ Q \in X \,\middle|\, \langle N'(Q_{a,b}), Q\rangle = \langle M'(Q_{a,b}), Q\rangle = 0 \right\} .$$

Then $X = \mathcal{T}_{a,b} \oplus \mathcal{N}_{a,b}$, where $\mathcal{N}_{a,b}$ (the "normal" space) is the two-dimensional subspace of X spanned by $N'(Q_{a,b}) = Q_{a,b}$ and $M'(Q_{a,b}) = i\partial_z Q_{a,b}$. When (a,b) is small, a more convenient basis of $\mathcal{N}_{a,b}$ is $\{\xi_{a,b}, \eta_{a,b}\}$, where

$$\xi_{a,b} = \frac{i}{a}\partial_z Q_{a,b} = e^{-iz} + \mathcal{O}(|ab| + b^2) , \quad \eta_{a,b} = \frac{1}{b}(Q_{a,b} - i\partial_z Q_{a,b}) = 1 + \mathcal{O}(a^2 + |ab|) . \quad (3.15)$$

The tangent space is further decomposed as $\mathcal{T}_{a,b} = Y_{a,b} \oplus Z_{a,b}$, where

$$Y_{a,b} = \left\{ Q \in \mathcal{T}_{a,b} \,\middle|\, \langle iQ_{a,b}, Q\rangle = \langle\partial_z Q_{a,b}, Q\rangle = 0 \right\} ,$$

and $Z_{a,b}$ is the two-dimensional space spanned by $iQ_{a,b}$ and $\partial_z Q_{a,b}$. In view of (3.14), $Z_{a,b}$ is just the tangent space to the orbit of $Q_{a,b}$ under the action of G. Again, a convenient basis of $Z_{a,b}$ is $\{i\xi_{a,b}, i\eta_{a,b}\}$.

As in [16], we introduce an appropriate coordinate system in a neighborhood of the orbit of $Q_{a,b}$ under the action of G:

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Lemma 3.5 *Assume that $\|(a,b)\|$ is sufficiently small and $b \geq a > 0$. There exist $\kappa > 0$, $C_1 > 0$, and $C_2 > 0$ such that any $Q \in X$ with $\rho(Q, Q_{a,b}) \leq \kappa a$ can be represented as*

$$Q = \mathcal{R}_{(\varphi,\xi)}(Q_{a,b} + \nu + y) , \qquad (3.16)$$

where $(\varphi, \xi) \in G$, $\nu \in \mathcal{N}_{a,b}$, $y \in Y_{a,b}$, and $\|\nu\|_X + \|y\|_X \leq C_1 \rho(Q, Q_{a,b})$. Moreover, if $Q \in \Sigma_{a,b}$, then $\|\nu\|_X \leq (C_2/a)\|y\|_X^2$.

Remark 3.6 *Here and in the sequel, all constants $C_1, C_2, \ldots$ are independent of (a, b) provided $\|(a, b)\|$ is sufficiently small.*

Proof. It is clearly sufficient to prove the result for all $Q \in X$ with $\|Q - Q_{a,b}\|_X \leq \kappa a$, where $\kappa > 0$ is a (small) constant that will be fixed below. Since $X = \mathcal{N}_{a,b} \oplus Y_{a,b} \oplus Z_{a,b}$, any such Q can be decomposed as

$$Q = Q_{a,b} - x_1 \mathrm{i} Q_{a,b} + x_2 \partial_z Q_{a,b} + \nu_1 + y_1 ,$$

where $\nu_1 \in \mathcal{N}_{a,b}$, $y_1 \in Y_{a,b}$, and $(x_1, x_2) \in \mathbb{R}^2$ is the solution of the linear system

$$\begin{pmatrix} \langle \mathrm{i} Q_{a,b}, \mathrm{i} Q_{a,b} \rangle & -\langle \mathrm{i} Q_{a,b}, \partial_z Q_{a,b} \rangle \\ -\langle \partial_z Q_{a,b}, \mathrm{i} Q_{a,b} \rangle & \langle \partial_z Q_{a,b}, \partial_z Q_{a,b} \rangle \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\langle \mathrm{i} Q_{a,b}, Q - Q_{a,b} \rangle \\ \langle \partial_z Q_{a,b}, Q - Q_{a,b} \rangle \end{pmatrix} . \qquad (3.17)$$

The matrix $\hat{\mathcal{M}}_{a,b}$ in the left-hand side of (3.17) is invertible, and using the expansions (3.12) we find

$$(\hat{\mathcal{M}}_{a,b})^{-1} = \frac{1}{2\pi} \begin{pmatrix} b^{-2} & -b^{-2} \\ -b^{-2} & a^{-2} + b^{-2} \end{pmatrix} (\mathbf{1} + \mathcal{O}(a^2 + b^2)) .$$

Since $b \geq a$, it follows that $|x_1| + |x_2| \leq (C/a)\|Q - Q_{a,b}\|_X \leq C\kappa$ for some $C > 0$ (independent of a, b). Now, since

$$\mathcal{R}_{(\varphi,\xi)} Q_{a,b} = Q_{a,b} - \varphi \mathrm{i} Q_{a,b} + \xi \partial_z Q_{a,b} + \mathcal{O}(\varphi^2 + \xi^2) ,$$

the Implicit Function Theorem implies that, if (x_1, x_2) is sufficiently small, there exists a unique pair $(\varphi, \xi) \in \mathbb{R}^2$ with $(\varphi, \xi) = (x_1, x_2) + \mathcal{O}(x_1^2 + x_2^2)$ such that $\mathcal{R}_{(\varphi,\xi)}^{-1} Q - Q_{a,b} \in \mathcal{N}_{a,b} \oplus Y_{a,b}$ (see Lemma 4.2 in [16] for a similar argument). Setting $\mathcal{R}_{(\varphi,\xi)}^{-1} Q - Q_{a,b} = \nu + y$, we obtain the desired decomposition (assuming that $\kappa > 0$ is small enough so that we can apply the Implicit Function Theorem). This choice of (φ, ξ) does not minimize the distance in X between Q and $\mathcal{R}_{(\varphi,\xi)} Q_{a,b}$, because the subspaces $\mathcal{N}_{a,b}$, $Z_{a,b}$, and $Y_{a,b}$ are not mutually orthogonal for the scalar product of X. However, since the *minimum gap* between these spaces is strictly positive (uniformly in a, b), we still have $\|\nu\|_X + \|y\|_X \leq C\|\nu + y\|_X \leq C_1 \rho(Q, Q_{a,b})$. (We refer to [20] for the definition and the properties of the minimum gap between closed subspaces of a Banach space.)

Now, we assume in addition that $Q \in \Sigma_{a,b}$, i.e. $N(Q) = N(Q_{a,b})$ and $M(Q) = M(Q_{a,b})$. In view of (3.16), we have

$$N(Q) = N(Q_{a,b} + \nu + y) \equiv N(Q_{a,b}) + \langle Q_{a,b}, \nu + y \rangle + N(\nu + y) ,$$

and using the fact that $y \in Y_{a,b} \subset \mathcal{T}_{a,b}$ we obtain $\langle Q_{a,b}, \nu \rangle + N(\nu + y) = 0$. A similar argument shows that $\langle \mathrm{i} \partial_z Q_{a,b}, \nu \rangle + M(\nu + y) = 0$. Thus $\nu = \nu_1 Q_{a,b} + \nu_2 \mathrm{i} \partial_z Q_{a,b}$, where

$$\begin{pmatrix} \langle Q_{a,b}, Q_{a,b} \rangle & \langle Q_{a,b}, \mathrm{i} \partial_z Q_{a,b} \rangle \\ \langle \mathrm{i} \partial_z Q_{a,b}, Q_{a,b} \rangle & \langle \mathrm{i} \partial_z Q_{a,b}, \mathrm{i} \partial_z Q_{a,b} \rangle \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = - \begin{pmatrix} N(\nu + y) \\ M(\nu + y) \end{pmatrix} .$$

Observe that the matrix of this system is exactly the same one as in (3.17). Thus, proceeding as above, we obtain $\|\nu\|_X \le (C/a)\|\nu + y\|_X^2$ for some $C > 0$ independent of a, b. Since we already know that $\|\nu\|_X \le C_1 \kappa a$, it follows that $\|\nu\|_X \le (C_2/a)\|y\|_X^2$ provided $\kappa > 0$ is sufficiently small. $\square$

To show that the energy $\mathcal{E}_{a,b}$ has a local minimum on $\Sigma_{a,b}$ at $Q_{a,b}$, we consider the second variation of $\mathcal{E}_{a,b}$ at $Q_{a,b}$, i.e. the linear operator $H_{a,b}$ defined in (3.13).

Lemma 3.7 *If $\|(a,b)\|$ is sufficiently small and $ab \ne 0$, then*

$$\langle H_{a,b}\, y, y\rangle \ge 6\|y\|_X^2 \;, \quad \text{for all } y \in Y_{a,b} \;. \tag{3.18}$$

Proof. We use a perturbation argument. When $(a,b) = (0,0)$, the operator $H_{a,b}$ reduces to a differential operator with constant coefficients: $H_0 = -4\partial_{zz} - 4\mathrm{i}\partial_z$. This operator is self-adjoint in the real Hilbert space $L^2_{\mathrm{per}}([0,2\pi],\mathbb{C})$ equipped with the scalar product $\langle \cdot, \cdot \rangle$, and its spectrum is $\sigma(H_0) = \{4n(n \pm 1)\,|\,n \in \mathbb{Z}\}$. The null space of H_0 is spanned by the four vectors $\xi_0, \mathrm{i}\xi_0, \eta_0, \mathrm{i}\eta_0$, where $\xi_0 = \mathrm{e}^{-\mathrm{i}z}$ and $\eta_0 = 1$ (see (3.15)). The other eigenvalues of H_0 are positive and greater or equal to 8, hence the quadratic form $h_0 : X \to \mathbb{R}$ associated to H_0 satisfies

$$h_0(Q) \overset{\mathrm{def}}{=} \langle H_0 Q, Q\rangle \ge 8\|Q\|_X^2 \;, \quad \text{for all } Q \in Y_0 \;,$$

where

$$Y_0 = \left\{ Q \in X \,|\, \langle \xi_0, Q\rangle = \langle \mathrm{i}\xi_0, Q\rangle = \langle \eta_0, Q\rangle = \langle \mathrm{i}\eta_0, Q\rangle = 0 \right\} \;.$$

We now consider the quadratic form $h_{a,b} : X \to \mathbb{R}$ defined by $h_{a,b}(Q) = \langle H_{a,b} Q, Q\rangle$. This form is uniformly bounded for (a,b) in a neighborhood of zero, i.e. there exists $C_3 > 0$ such that $h_{a,b}(Q) \le C_3\|Q\|_X^2$ for all $Q \in X$. Moreover, $h_{a,b}$ converges to h_0 as $(a,b) \to (0,0)$ in the following sense:

$$\sup\left\{ |h_{a,b}(Q) - h_0(Q)| \,\Big|\, Q \in X\,, \ \|Q\|_X = 1 \right\} = \mathcal{O}(a^2 + b^2) \;.$$

In particular, we have $h_{a,b}(Q) \ge 7\|Q\|_X^2$ for all $Q \in Y_0$ if $\|(a,b)\|$ is sufficiently small.

On the other hand, since $\|\xi_{a,b} - \xi_0\|_X + \|\eta_{a,b} - \eta_0\|_X = \mathcal{O}(a^2 + b^2)$, it is straightforward to verify that the subspace

$$Y_{a,b} = \left\{ Q \in X \,|\, \langle \xi_{a,b}, Q\rangle = \langle \mathrm{i}\xi_{a,b}, Q\rangle = \langle \eta_{a,b}, Q\rangle = \langle \mathrm{i}\eta_{a,b}, Q\rangle = 0 \right\}$$

converges to Y_0 as $(a,b) \to (0,0)$ in the following sense:

$$\delta(Y_{a,b}, Y_0) \overset{\mathrm{def}}{=} \sup\left\{ \mathrm{dist}_X(Q, Y_0) \,\Big|\, Q \in Y_{a,b}\,, \ \|Q\|_X = 1 \right\} = \mathcal{O}(a^2 + b^2) \;.$$

In particular, if $Q \in Y_{a,b}$ satisfies $\|Q\|_X = 1$, we can find $\tilde{Q} \in Y_0$ with $\|\tilde{Q}\|_X = 1$ and $\|Q - \tilde{Q}\|_X$ as small as we want, provided (a,b) is close to zero. Since $h_{a,b}(\tilde{Q}) \ge 7$ and

$$|h_{a,b}(Q) - h_{a,b}(\tilde{Q})| \le \|h_{a,b}\|(\|Q\|_X + \|\tilde{Q}\|_X)\|Q - \tilde{Q}\|_X \le 2C_3\|Q - \tilde{Q}\|_X \;,$$

we conclude that $h_{a,b}(Q) \ge 6$ if $\|(a,b)\|$ is sufficiently small. This proves (3.18). $\square$

Using Lemmas 3.5 and 3.7, we are able to give a more precise version of Claim 1 above:

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Lemma 3.8 *There exists $\kappa_1 > 0$ such that, if $\|(a,b)\|$ is sufficiently small and $b \geq a > 0$, then for all $Q \in \Sigma_{a,b}$ satisfying $\rho(Q, Q_{a,b}) \leq \kappa_1 a$ one has the inequality*

$$\mathcal{E}_{a,b}(Q) - \mathcal{E}_{a,b}(Q_{a,b}) \geq \rho(Q, Q_{a,b})^2 \ . \tag{3.19}$$

Proof. If $Q \in \Sigma_{a,b}$ satisfies $\rho(Q, Q_{a,b}) \leq \kappa_1 a$ for some $\kappa_1 > 0$ sufficiently small, we know from Lemma 3.5 that $Q = \mathcal{R}_{(\varphi,\xi)}(Q_{a,b}+\nu+y)$, where $(\varphi,\xi) \in G$, $\nu \in \mathcal{N}_{a,b}$, $y \in Y_{a,b}$, $\|y\|_X \leq C_1 \rho(Q, Q_{a,b})$, and $\|\nu\|_X \leq (C_2/a)\|y\|_X^2$. In particular, $\|\nu\|_X \leq \kappa_1 C_1 C_2 \|y\|_X$. Since the energy $\mathcal{E}_{a,b}$ is invariant under the action of G, we have $\mathcal{E}_{a,b}(Q) = \mathcal{E}_{a,b}(Q_{a,b}+\nu+y)$. As $\mathcal{E}'_{a,b}(Q_{a,b}) = 0$ and $\mathcal{E}''_{a,b}(Q_{a,b}) = H_{a,b}$, we obtain using Taylor's formula:

$$\mathcal{E}_{a,b}(Q) - \mathcal{E}_{a,b}(Q_{a,b}) = \frac{1}{2}\langle H_{a,b}(y+\nu), (y+\nu)\rangle + \mathcal{O}(\|y\|_X^3) \ .$$

But $\langle H_{a,b}\, y, y \rangle \geq 6\|y\|_X^2$ by Lemma 3.7, hence

$$
\begin{aligned}
\frac{1}{2}\langle H_{a,b}(y+\nu), (y+\nu)\rangle &\geq\; 3\|y\|_X^2 - C_3\|y\|_X\|\nu\|_X - \tfrac{1}{2}C_3\|\nu\|_X^2 \\
&\geq\; \|y\|_X^2\bigl(3 - \kappa_1 C_1 C_2 C_3 - \tfrac{1}{2}(\kappa_1 C_1 C_2)^2 C_3\bigr) \ ,
\end{aligned}
$$

where C_3 is the constant in the proof of Lemma 3.7. Thus, if κ_1 is sufficiently small, we obtain $\mathcal{E}_{a,b}(Q) - \mathcal{E}_{a,b}(Q_{a,b}) \geq 2\|y\|_X^2$. Under the same assumption, we also have $\rho(Q, Q_{a,b}) \leq \|y+\nu\|_X \leq \|y\|_X(1 + \kappa_1 C_1 C_2) \leq \sqrt{2}\|y\|_X$, and (3.19) follows. $\qquad\square$

3.4 Proof of Proposition 3.2

The proof of Proposition 3.2 consists of three steps in which we treat successively the three types of waves: the zero solution ($a = b = 0$), the plane waves ($ab = 0$), and the periodic waves ($ab \neq 0$). In each case, the arguments rely upon energy estimates as the one given in Lemma 3.8 for the periodic waves ($ab \neq 0$). In the case of the plane wave $Q_{0,b} \equiv b$ the stability is proved using the functional

$$\mathcal{E}_b(Q) = \mathcal{E}(Q) - b^2 N(Q) = \int_0^{2\pi}\left((2-3b^2)|Q_z|^2 + \frac{1}{4}(|Q|^2 - b^2)^2\right)\mathrm{d}z - \frac{\pi b^4}{2} \ ,$$

for which we have the analog of Lemma 3.8:

Lemma 3.9 *There exists $\kappa_2 > 0$ such that, if $b > 0$ is sufficiently small, then for all $Q \in X$ satisfying $\rho(Q, Q_{0,b}) \leq \kappa_2 b$ and $N(Q) = N(Q_{0,b})$ one has the inequality*

$$\mathcal{E}_b(Q) - \mathcal{E}_b(Q_{0,b}) \geq \frac{1}{6}\,\rho(Q, Q_{0,b})^2 \ . \tag{3.20}$$

Proof. Without loss of generality, we assume that $\|Q - Q_{0,b}\|_X \leq \kappa_2 b$. Since

$$|Q(z) - b| \leq \|Q - Q_{0,b}\|_X \leq \kappa_2 b < b \ , \quad \text{for all } z \in \mathbb{R} \ ,$$

we can write $Q = (b+r)\mathrm{e}^{\mathrm{i}\varphi}$, where $r, \varphi \in X$ are real functions satisfying

$$|r(z)| \leq \kappa_2 b \ , \quad |\mathrm{e}^{\mathrm{i}\varphi(z)} - 1| \leq 2\kappa_2 \ , \quad \text{for all } z \in \mathbb{R} \ .$$

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These inequalities imply $\|r\|_X + b\|\varphi\|_X \leq C\kappa_2 b$, for some $C > 0$ independent of a, b. Thus

$$\mathcal{E}_b(Q) - \mathcal{E}_b(Q_{0,b}) = \int_0^{2\pi} \left\{ (2-3b^2)\left(r_z^2 + (b+r)^2\varphi_z^2\right) + \frac{1}{4}(2br + r^2)^2 \right\} \mathrm{d}z$$

$$\geq \int_0^{2\pi} (r_z^2 + b^2\varphi_z^2)\,\mathrm{d}z \ . \tag{3.21}$$

On the other hand, since $N(Q) = N(Q_{0,b})$, we have $\int_0^{2\pi}(2br + r^2)\,\mathrm{d}z = 0$, and Poincaré's inequality implies

$$\int_0^{2\pi} r^2\left(1 + \frac{r}{2b}\right)^2 \mathrm{d}z \leq \int_0^{2\pi} r_z^2\left(1 + \frac{r}{b}\right)^2 \mathrm{d}z \ , \quad \text{hence} \quad \int_0^{2\pi} r^2\,\mathrm{d}z \leq 2\int_0^{2\pi} r_z^2\,\mathrm{d}z \ .$$

Finally, if $\bar\varphi = (2\pi)^{-1}\int_0^{2\pi}\varphi(z)\,\mathrm{d}z$, we have

$$\rho(Q,Q_{0,b})^2 \leq \|Q - b\,\mathrm{e}^{\mathrm{i}\bar\varphi}\|_X^2 \leq 2\|r\mathrm{e}^{\mathrm{i}\varphi}\|_X^2 + 2b^2\|\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{\mathrm{i}\bar\varphi}\|_X^2$$

$$\leq 2\|r\|_X^2 + 2\int_0^{2\pi} r^2\varphi_z^2\,\mathrm{d}z + 2b^2\|\varphi - \bar\varphi\|_X^2 \leq 6\int_0^{2\pi} (r_z^2 + b^2\varphi_z^2)\,\mathrm{d}z \ , \tag{3.22}$$

again by Poincaré's inequality. Combining (3.21) and (3.22), we obtain (3.20) . $\qquad\square$

We are now in position to prove Proposition 3.2.

Proof of Proposition 3.2. Throughout the proof, we assume that $\|(a,b)\|$ is sufficiently small and that $b \geq a \geq 0$. Given $Q_0 \in X$ with $\rho(Q_0, Q_{a,b}) \leq \epsilon$, we consider the solution $Q(z,t)$ of (3.4) with initial data Q_0. Replacing Q_0 with $\mathcal{R}_{(\varphi,\xi)}Q_0$ if needed, we can assume that $\|Q_0 - Q_{a,b}\|_X \leq \epsilon$. We distinguish three cases:

Case 1: $a = b = 0$, i.e. $Q_{a,b} = 0$. In this case, if $\epsilon > 0$ is small enough, the solution $Q(\cdot, t)$ of (3.4) satisfies $\|Q(\cdot, t)\|_X \leq 2\epsilon$ for all $t \in \mathbb{R}$. This is obvious because the quantity

$$\mathcal{E}(Q) + 4N(Q) = \int_0^{2\pi} \left(2|Q_z|^2 + \frac{1}{4}|Q|^4 + 2|Q|^2 \right) \mathrm{d}z$$

is invariant under the evolution of (3.4), and satisfies $2\|Q\|_X^2 \leq \mathcal{E}(Q) + 4N(Q) \leq 4\|Q\|_X^2$ if $\|Q\|_X$ is small.

Remark: As a consequence of this preliminary case, we assume from now on that $\epsilon \leq \kappa_3(a^2 + b^2)^{1/2}$ for some small $\kappa_3 > 0$. Indeed, if $\epsilon \geq \kappa_3(a^2 + b^2)^{1/2}$, we can use the trivial estimate

$$\|Q(\cdot, t) - Q_{a,b}\|_X \leq \|Q(\cdot, t)\|_X + \|Q_{a,b}\|_X \leq 2\|Q_0\|_X + \|Q_{a,b}\|_X \leq 2\epsilon + 3\|Q_{a,b}\|_X \ ,$$

which gives the desired result since $\|Q_{a,b}\|_X \leq C(a^2 + b^2)^{1/2} \leq (C/\kappa_3)\epsilon$.

Case 2: $b > a = 0$, i.e. $Q_{a,b} = b$ is a plane wave. We consider initial data $Q_0 \in X$ such that $\|Q_0 - Q_{0,b}\|_X \leq \epsilon \leq \kappa_3 b$. If $N(Q_0) = N(Q_{0,b})$, then $N(Q(\cdot, t)) = N(Q_{0,b})$ for all $t \in \mathbb{R}$, and Lemma 3.9 implies

$$\rho(Q(\cdot, t), Q_{0,b})^2 \leq 6(\mathcal{E}_b(Q(\cdot, t)) - \mathcal{E}_b(Q_{0,b})) = 6(\mathcal{E}_b(Q_0) - \mathcal{E}_b(Q_{0,b})) \leq C\epsilon^2 \ ,$$

provided that $C\epsilon^2 \leq \kappa_2^2 b^2$, which is the case if $C\kappa_3^2 \leq \kappa_2^2$. If $N(Q_0) \neq N(Q_{0,b})$, we define $\omega = \pi^{-1}(N(Q_0) - N(Q_{0,b}))$, so that $N(Q_0) = N(Q_{0,b}^\omega)$, where $Q_{0,b}^\omega = (b^2+\omega)^{1/2}$. So we are led to study the stability of the rotating wave $Q_{0,b}^\omega\, \mathrm{e}^{-\mathrm{i}\omega t}$ of (3.4) with respect to perturbations preserving the charge N. This can be proved exactly as above, and we obtain $\rho(Q(\cdot,t), Q_{0,b}^\omega) \leq C\epsilon$ for all $t \in \mathbb{R}$. Since $\|Q_{0,b}^\omega - Q_{0,b}\|_X \leq C|\omega|/b \leq C\epsilon$, we have the desired result.

Remark: As a consequence, we can assume from now on that $\epsilon \leq \kappa_4 a$ for some small $\kappa_4 > 0$. Indeed, if $\epsilon \geq \kappa_4 a$, we can use the easy estimate

$$\rho(Q(\cdot,t), Q_{a,b}) \ \leq\ \rho(Q(\cdot,t), Q_{0,b}) + \|Q_{0,b} - Q_{a,b}\|_X \ . \tag{3.23}$$

Observe that $\|Q_{0,b} - Q_{a,b}\|_X \leq Ca$ for some $C > 0$ independent of a, b. In particular

$$\|Q_0 - Q_{0,b}\|_X \ \leq\ \|Q_0 - Q_{a,b}\|_X + \|Q_{a,b} - Q_{0,b}\|_X \ \leq\ \epsilon + Ca \ \leq\ (1 + C/\kappa_4)\epsilon \ ,$$

hence for $\epsilon > 0$ small enough we have $\rho(Q(\cdot,t), Q_{0,b}) \leq C'\epsilon$ for all $t \in \mathbb{R}$. It then follows from (3.23) that $\rho(Q(\cdot,t), Q_{a,b}) \leq C''\epsilon$ for all $t \in \mathbb{R}$, which is the desired result.

Case 3: $b \geq a > 0$, i.e. $Q_{a,b}$ is a nontrivial periodic equilibrium of (3.4) corresponding to a periodic wave of (1.2). Assume that $Q_0 \in X$ satisfies $\|Q_0 - Q_{a,b}\|_X \leq \epsilon \leq \kappa_4 a$. If $Q_0 \in \Sigma_{a,b}$, then $Q(\cdot,t) \in \Sigma_{a,b}$ for all $t \in \mathbb{R}$ and Lemma 3.8 implies that

$$\rho(Q(\cdot,t), Q_{a,b})^2 \ \leq\ \mathcal{E}_{a,b}(Q(\cdot,t)) - \mathcal{E}_{a,b}(Q_{a,b}) \ =\ \mathcal{E}_{a,b}(Q_0) - \mathcal{E}_{a,b}(Q_{a,b}) \ \leq\ C\epsilon^2 \ ,$$

provided $C\epsilon^2 \leq \kappa_1^2 a^2$, which is the case if $C\kappa_4^2 \leq \kappa_1^2$. If $Q_0 \notin \Sigma_{a,b}$, then by Claim 2 above there exists $(\omega, c) \in \mathbb{R}^2$ with $|\omega| + |c| \leq Cb\epsilon$ such that $N(Q_0) = N(Q_{a,b}^{\omega,c})$ and $M(Q_0) = M(Q_{a,b}^{\omega,c})$. So we are led to study the stability of the periodic wave $u(x,t) = \mathrm{e}^{\mathrm{i}(p_{a,b}x - (1+\omega)t)}\, Q_{a,b}^{\omega,c}(2k_{a,b}x + ct)$ of (1.2) among solutions of the form $\mathrm{e}^{\mathrm{i}(p_{a,b}x - t)}\, Q(2k_{a,b}x, t)$ for which the charge N and the momentum M have the same values as for the periodic wave. But if we apply a dilation of factor λ and a Galilean transformation of speed v, where λ, v are given by (3.9), the periodic wave becomes $u(x,t) = \mathrm{e}^{\mathrm{i}(p_{a',b'}x - t)}\, Q_{a',b'}(2k_{a',b'}x)$ for some (a', b') close to (a, b), and we are back to the previous case. As $\|Q_0 - Q_{a,b}^{\omega,c}\|_X \leq \|Q_0 - Q_{a,b}\|_X + \|Q_{a,b} - Q_{a,b}^{\omega,c}\|_X \leq C\epsilon$, this shows that $\rho(Q(\cdot,t), Q_{a,b}^{\omega,c}) \leq C\epsilon$ for all $t \in \mathbb{R}$, and the result follows. This concludes the proof of Proposition 3.2. $\qquad\square$

Remark 3.10 *In [16] the authors use the decomposition $X = \mathcal{T}_{a,b} \oplus \tilde{\mathcal{N}}_{a,b}$, where $\tilde{\mathcal{N}}_{a,b}$ is the two-dimensional space spanned by*

$$\partial_\omega Q_{a,b} = \frac{\partial}{\partial\omega} Q_{a,b}^{\omega,c}\Big|_{(\omega,c)=(0,0)} \ , \quad \partial_c Q_{a,b} = \frac{\partial}{\partial c} Q_{a,b}^{\omega,c}\Big|_{(\omega,c)=(0,0)} \ .$$

This alternative decomposition has the advantage that $\langle H_{a,b}u, v\rangle = 0$ for all $u \in \tilde{\mathcal{N}}_{a,b}$ and all $v \in \mathcal{T}_{a,b}$, because

$$H_{a,b}(\partial_\omega Q_{a,b}) = N'(Q_{a,b}) = Q_{a,b} \ , \quad H_{a,b}(\partial_c Q_{a,b}) = M'(Q_{a,b}) = \mathrm{i}\partial_z Q_{a,b} \ .$$

Using in addition (3.11) we also find

$$\mathcal{H}_{a,b} = -\begin{pmatrix} \langle H_{a,b}(\partial_\omega Q_{a,b}), \partial_\omega Q_{a,b}\rangle & \langle H_{a,b}(\partial_\omega Q_{a,b}), \partial_c Q_{a,b}\rangle \\ \langle H_{a,b}(\partial_c Q_{a,b}), \partial_\omega Q_{a,b}\rangle & \langle H_{a,b}(\partial_c Q_{a,b}), \partial_c Q_{a,b}\rangle \end{pmatrix} \ .$$

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Remark that the spaces $\mathcal{N}_{a,b}$ and $\tilde{\mathcal{N}}_{a,b}$ are very close when (a,b) is small, since

$$\partial_\omega Q_{a,b} = \frac{1}{6ab}(a + be^{-iz})(1 + \mathcal{O}(a^2 + b^2)) , \quad \partial_c Q_{a,b} = \frac{1}{6ab}(2a - be^{-iz})(1 + \mathcal{O}(a^2 + b^2)) .$$

Remark 3.11 ($2n\pi$–periodic perturbations) *As an intermediate step between the periodic set-up considered in Theorem 1 and the case of arbitrary bounded perturbations for which no result is available so far, one can try to study the orbital stability of the travelling waves of (1.2) with respect to perturbations whose periods are integer multiples of the period of the original wave. This amounts to replacing the space X in Proposition 3.2 by $H^1_{\mathrm{per}}([0, 2n\pi], \mathbb{C})$ for some $n \geq 2$. In that case most of the results above remain valid, but the linear operator $H_{a,b}$ has now $2n - 1$ negative eigenvalues. Thus the number of negative eigenvalues of $H_{a,b}$ minus the number of positive eigenvalues of the Hessian matrix $\mathcal{H}_{a,b}$ is equal to $2n - 2$, a nonzero even integer. This means that neither the Stability Theorem nor the Instability Theorem in [16] applies if $n \geq 2$. The only way out of this difficulty would be to replace the manifolds $\Sigma_{a,b}$ defined in (3.7) by invariant manifolds higher codimension, which amounts to use additional conserved quantities of (1.2) instead of N and M only.*

4 Spectral stability

In this section we prove the spectral stability result in Theorem 2. We start with the evolution equation (3.4), which we linearize about the stationary solution $Q_{a,b}(z)$ corresponding to the periodic wave $U_{a,b}(x,t) = e^{i(p_{a,b}x - t)}Q_{a,b}(2k_{a,b}x)$ of the NLS equation. We find the linear operator

$$\mathcal{A}_{a,b}Q = 4ik_{a,b}^2 Q_{zz} - 4p_{a,b}k_{a,b}Q_z + i(1 - p_{a,b}^2)Q - 2i|Q_{a,b}|^2 Q - iQ_{a,b}^2\overline{Q} , \qquad (4.1)$$

which we consider in either the real Hilbert space $Y = L^2(\mathbb{R}, \mathbb{C})$ (localized perturbations) or the real Banach space $Y = C_b(\mathbb{R}, \mathbb{C})$ (bounded pertubations). To study the spectrum $\mathcal{A}_{a,b}$, it is convenient to decompose the elements of Y into real and imaginary parts, in which case we obtain the matrix operator

$$\mathcal{A}_{a,b} = \begin{pmatrix} -4p_{a,b}k_{a,b}\partial_z + 2R_{a,b}I_{a,b} & -4k_{a,b}^2\partial_{zz} + (p_{a,b}^2 - 1) + R_{a,b}^2 + 3I_{a,b}^2 \\ 4k_{a,b}^2\partial_{zz} - (p_{a,b}^2 - 1) - 3R_{a,b}^2 - I_{a,b}^2 & -4p_{a,b}k_{a,b}\partial_z - 2R_{a,b}I_{a,b} \end{pmatrix} , \quad (4.2)$$

where $Q_{a,b} = R_{a,b} + iI_{a,b}$. We are now interested in the spectrum of this matrix operator in the (complexified) spaces $L^2(\mathbb{R}, \mathbb{C}^2)$ and $C_b(\mathbb{R}, \mathbb{C}^2)$. We prove that the spectrum of $\mathcal{A}_{a,b}$ in both spaces lies entirely on the imaginary axis, if $\|(a,b)\|$ is sufficiently small. This means that the periodic wave $U_{a,b}$ is spectrally stable in Y.

4.1 Bloch-wave decomposition and symmetries

The spectral analysis of $\mathcal{A}_{a,b}$ relies upon the so-called Bloch-wave decomposition for differential operators with periodic coefficients. This method allows to show that the spectrum of $\mathcal{A}_{a,b}$ is exactly the same in both spaces $L^2(\mathbb{R}, \mathbb{C}^2)$ and $C_b(\mathbb{R}, \mathbb{C}^2)$, and can be described as the union of the point spectra of a family of operators with compact resolvent (see e.g. [22, 24]). In our case,

the operator $\mathcal{A}_{a,b}$ has 2π-periodic coefficients and its spectrum in both $L^2(\mathbb{R}, \mathbb{C}^2)$ and $C_b(\mathbb{R}, \mathbb{C}^2)$ is given by

$$\sigma(\mathcal{A}_{a,b}) = \bigcup_{\gamma \in (-\frac{1}{2}, \frac{1}{2}]} \sigma(\mathcal{A}_{a,b,\gamma}) \;, \tag{4.3}$$

where the Bloch operators

$$\mathcal{A}_{a,b,\gamma} = \begin{pmatrix} -4p_{a,b}k_{a,b}(\partial_z + \mathrm{i}\gamma) + 2R_{a,b}I_{a,b} & -4k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 + (p_{a,b}^2 - 1) + R_{a,b}^2 + 3I_{a,b}^2 \\ 4k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 - (p_{a,b}^2 - 1) - 3R_{a,b}^2 - I_{a,b}^2 & -4p_{a,b}k_{a,b}(\partial_z + \mathrm{i}\gamma) - 2R_{a,b}I_{a,b} \end{pmatrix}$$

are linear operators in the Hilbert space of 2π-periodic functions $L^2_{\text{per}}([0, 2\pi], \mathbb{C}^2)$. We can now reformulate the spectral result of Theorem 2 as follows:

Proposition 4.1 *There exists $\delta_1 > 0$ such that, for any $\gamma \in (-\frac{1}{2}, \frac{1}{2}]$ and any $(a, b) \in \mathbb{R}^2$ with $\|(a, b)\| \le \delta_1$, the spectrum of the operator $\mathcal{A}_{a,b,\gamma}$ in $L^2_{\text{per}}([0, 2\pi], \mathbb{C}^2)$ satisfies $\sigma(\mathcal{A}_{a,b,\gamma}) \subset \mathrm{i}\mathbb{R}$.*

We equip the Hilbert space $L^2_{\text{per}}([0, 2\pi], \mathbb{C}^2)$ with the usual scalar product defined through

$$\langle (Q_1, Q_2)^t, (R_1, R_2)^t \rangle = \int_0^{2\pi} \left(Q_1(z)\overline{R}_1(z) + Q_2(z)\overline{R}_2(z) \right) \mathrm{d}z \;.$$

The operators $\mathcal{A}_{a,b,\gamma}$ are closed in this space with compactly embedded domain $H^2_{\text{per}}([0, 2\pi], \mathbb{C}^2)$. An immediate consequence of the latter property is that these operators have compact resolvent, so that their spectra are purely point spectra consisting of isolated eigenvalues with finite algebraic multiplicities. Our problem consists in locating these eigenvalues.

The spectra of the operators $\mathcal{A}_{a,b}$ and $\mathcal{A}_{a,b,\gamma}$ possess several symmetries originating from the discrete symmetries of (1.2) and the symmetries of the wave profile $Q_{a,b}$. First, since the operator $\mathcal{A}_{a,b}$ has real coefficients, its spectrum is symmetric with respect to the real axis: $\sigma(\mathcal{A}_{a,b}) = \overline{\sigma(\mathcal{A}_{a,b})}$. For the Bloch operator $\mathcal{A}_{a,b,\gamma}$, the corresponding property is $\sigma(\mathcal{A}_{a,b,\gamma}) = \overline{\sigma(\mathcal{A}_{a,b,-\gamma})}$. Next, it is straightforward to check that $\mathcal{A}_{a,b}$ has a *reversibility symmetry*, i.e. it anticommutes with the isometry $\mathcal{S}$ defined by

$$\mathcal{S} \begin{pmatrix} Q_1(z) \\ Q_2(z) \end{pmatrix} = \begin{pmatrix} Q_1(-z) \\ -Q_2(-z) \end{pmatrix} \;. \tag{4.4}$$

Thus $\mathcal{S}\mathcal{A}_{a,b} = -\mathcal{A}_{a,b}\mathcal{S}$, which implies that the spectrum of $\mathcal{A}_{a,b}$ is symmetric with respect to the origin in the complex plane: $\sigma(\mathcal{A}_{a,b}) = -\sigma(\mathcal{A}_{a,b})$. The corresponding property for the Bloch operators is $\mathcal{S}\mathcal{A}_{a,b,\gamma} = -\mathcal{A}_{a,b,-\gamma}\mathcal{S}$, which implies that $\sigma(\mathcal{A}_{a,b,\gamma}) = -\sigma(\mathcal{A}_{a,b,-\gamma})$. Summarizing, the spectrum of $\mathcal{A}_{a,b}$ is symmetric with respect to both the real and the imaginary axis, and the spectra of the Bloch operators $\mathcal{A}_{a,b,\gamma}$ satisfy

$$\sigma(\mathcal{A}_{a,b,\gamma}) = \overline{\sigma(\mathcal{A}_{a,b,-\gamma})} = -\sigma(\mathcal{A}_{a,b,-\gamma}) = -\overline{\sigma(\mathcal{A}_{a,b,\gamma})} \;. \tag{4.5}$$

In particular, the spectrum of $\mathcal{A}_{a,b,\gamma}$ is symmetric with respect to the imaginary axis and we can restrict ourselves to positive values $\gamma \in [0, \frac{1}{2}]$ without loss of generality.

Using now the relations (3.2) for the wave profile $Q_{a,b}$, we see that the spectra of $\mathcal{A}_{a,b}$ satisfy $\sigma(\mathcal{A}_{a,b}) = \sigma(\mathcal{A}_{-a,b}) = \sigma(\mathcal{A}_{-a,-b})$ and $\sigma(\mathcal{A}_{b,a}) = -\overline{\sigma(\mathcal{A}_{a,b})}$. (Actually, the last equality is easier to

establish if we use the complex form (4.1) of the operator $\mathcal{A}_{a,b}$, for which we have $\mathcal{A}_{b,a}(\mathrm{e}^{-\mathrm{i}z}Q) = -\mathrm{e}^{-\mathrm{i}z}\overline{\mathcal{A}_{a,b}}Q$.) Similarly, we find for the Bloch operators

$$\sigma(\mathcal{A}_{a,b,\gamma}) = \sigma(\mathcal{A}_{-a,b,\gamma}) = \sigma(\mathcal{A}_{-a,-b,\gamma}) , \quad \text{and} \quad \sigma(\mathcal{A}_{b,a,\gamma}) = -\overline{\sigma(\mathcal{A}_{a,b,\gamma})} . \qquad (4.6)$$

Finally, we note the formal relation $\mathcal{A}_{a,b} = -\mathrm{i}H_{a,b}$ between the linearized operator (4.1) and the second variation of the energy defined in (3.13). When written for the matrix operators this relation becomes

$$\mathcal{A}_{a,b} = JH_{a,b} , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

so that for the Bloch operators we have $\mathcal{A}_{a,b,\gamma} = JH_{a,b,\gamma}$ with

$$H_{a,b,\gamma} = \begin{pmatrix} -4k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 + (p_{a,b}^2 - 1) + 3R_{a,b}^2 + I_{a,b}^2 & 4p_{a,b}k_{a,b}(\partial_z + \mathrm{i}\gamma) + 2R_{a,b}I_{a,b} \\ -4p_{a,b}k_{a,b}(\partial_z + \mathrm{i}\gamma) + 2R_{a,b}I_{a,b} & -4k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 + (p_{a,b}^2 - 1) + R_{a,b}^2 + 3I_{a,b}^2 \end{pmatrix}.$$

Actually, this property is a consequence of the Hamiltonian structure of the NLS equation. Though some properties induced by this structure are exploited, we shall not make an explicit use of the Hamiltonian structure itself in the proof of spectral stability.

4.2   First perturbation argument and properties of the unperturbed operators

Our spectral analysis for the operators $\mathcal{A}_{a,b,\gamma}$ relies upon perturbation arguments in which we regard $\mathcal{A}_{a,b,\gamma}$ as small bounded perturbations of the operators with constant coefficients

$$\mathcal{A}_{a,b,\gamma}^0 = \begin{pmatrix} -4p_{a,b}k_{a,b}(\partial_z + \mathrm{i}\gamma) & -4k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 + (p_{a,b}^2 - 1) \\ 4k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 - (p_{a,b}^2 - 1) & -4p_{a,b}k_{a,b}(\partial_z + \mathrm{i}\gamma) \end{pmatrix} .$$

The difference $\mathcal{A}_{a,b}^1 := \mathcal{A}_{a,b,\gamma} - \mathcal{A}_{a,b,\gamma}^0$ is a bounded operator with norm $\|\mathcal{A}_{a,b}^1\| = \mathcal{O}(a^2 + b^2)$, as $(a,b) \to (0,0)$.

A straightforward Fourier analysis allows to compute the spectra of the operators $A_{a,b,\gamma}^0$:

$$\sigma(\mathcal{A}_{a,b,\gamma}^0) = \{\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}, \ \omega_{a,b,\gamma}^{\pm,n} = -4p_{a,b}k_{a,b}(n+\gamma) \pm (4k_{a,b}^2(n+\gamma)^2 + p_{a,b}^2 - 1), \ n \in \mathbb{Z}\} \subset \mathrm{i}\mathbb{R} , \quad (4.7)$$

in which the eigenvalues are all semi-simple with eigenfunctions

$$e^{\pm,n} = \mathrm{e}^{\mathrm{i}nz} \begin{pmatrix} 1 \\ \pm\mathrm{i} \end{pmatrix} , \quad \mathcal{A}_{a,b,\gamma}^0 e^{\pm,n} = \mathrm{i}\omega_{a,b,\gamma}^{\pm,n} e^{\pm,n} .$$

Furthermore, the resolvent operators $\mathcal{R}_{a,b,\gamma}^0(\lambda) = (\lambda\mathbf{1} - \mathcal{A}_{a,b,\gamma}^0)^{-1}$ have norms

$$\|\mathcal{R}_{a,b,\gamma}^0(\lambda)\| = \frac{1}{\mathrm{dist}(\lambda, \sigma(\mathcal{A}_{a,b,\gamma}^0))} , \quad \lambda \notin \sigma(\mathcal{A}_{a,b,\gamma}^0) .$$

A simple perturbation argument shows now that the spectrum of $\mathcal{A}_{a,b,\gamma}$ stays close to $\sigma(\mathcal{A}_{a,b,\gamma}^0)$ provided $\|(a,b)\|$ is sufficiently small. More precisely, we have the following result.

Lemma 4.2 *For any $c > 0$ there exists $\delta > 0$ such that for any $\gamma \in [0, \frac{1}{2}]$ and any $(a, b) \in \mathbb{R}^2$ with $\|(a, b)\| \leq \delta$ the spectrum of $\mathcal{A}_{a,b,\gamma}$ satisfies*

$$\sigma(\mathcal{A}_{a,b,\gamma}) \subset \bigcup_{n \in \mathbb{Z}} B(\mathrm{i}\omega_{a,b,\gamma}^{-,n}; c) \cup \bigcup_{n \in \mathbb{Z}} B(\mathrm{i}\omega_{a,b,\gamma}^{+,n}; c),$$

in which $B(\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}; c)$ represents the open ball centered at $\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}$ with radius c.

Proof. For any $\lambda \notin \bigcup_{n \in \mathbb{Z}} B(\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}; c)$, we write

$$\lambda \mathbf{1} - \mathcal{A}_{a,b,\gamma} = \lambda \mathbf{1} - \mathcal{A}_{a,b,\gamma}^0 - \mathcal{A}_{a,b}^1 = (\lambda \mathbf{1} - \mathcal{A}_{a,b,\gamma}^0)\left(\mathbf{1} - \mathcal{R}_{a,b,\gamma}^0(\lambda)\mathcal{A}_{a,b}^1\right) \ .$$

Since

$$\|\mathcal{R}_{a,b,\gamma}^0(\lambda)\mathcal{A}_{a,b}^1\| \leq \frac{1}{c}\|\mathcal{A}_{a,b}^1\| \ , \quad \|\mathcal{A}_{a,b}^1\| = \mathcal{O}(a^2 + b^2) \ ,$$

upon choosing δ sufficiently small, we have that $\mathbf{1} - \mathcal{R}_{a,b,\gamma}^0(\lambda)\mathcal{A}_{a,b}^1$ is invertible, so that $\lambda \mathbf{1} - \mathcal{A}_{a,b,\gamma}$ is invertible, as well. This proves that λ does not belong to $\sigma(\mathcal{A}_{a,b,\gamma})$. $\square$

In order to locate the spectra of $\mathcal{A}_{a,b,\gamma}$, we need a more precise description of the spectra of $\mathcal{A}_{a,b,\gamma}^0$. Looking at $a = b = 0$ we find that:

- if $\gamma = 0$, all nonzero eigenvalues of $\mathcal{A}_{0,0,0}^0$ are double,

$$\omega_{0,0,0}^{+,n} = \omega_{0,0,0}^{+,1-n} \ , \quad \omega_{0,0,0}^{-,n} = \omega_{0,0,0}^{-,-1-n} \ ,$$

  and zero is an eigenvalue of multiplicity 4,

$$\omega_{0,0,0}^{\pm,0} = \omega_{0,0,0}^{+,1} = \omega_{0,0,0}^{-,-1} = 0 \ ;$$

- if $0 < \gamma < \frac{1}{2}$ all eigenvalues are simple;

- if $\gamma = \frac{1}{2}$, there is a pair of simple eigenvalues $\pm \mathrm{i}$,

$$\omega_{0,0,\frac{1}{2}}^{-,-1} = -\omega_{0,0,\frac{1}{2}}^{+,0} = 1 \ ,$$

  and the other eigenvalues are all double,

$$\omega_{0,0,\frac{1}{2}}^{+,n} = \omega_{0,0,\frac{1}{2}}^{+,-n} \ , \quad \omega_{0,0,\frac{1}{2}}^{-,n} = \omega_{0,0,\frac{1}{2}}^{-,-2-n} \ .$$

We therefore distinguish three cases: $\gamma \approx 0$, $\gamma \approx \frac{1}{2}$, and $\gamma \in [\gamma_*, \frac{1}{2} - \gamma_*]$ for some $\gamma_* \in (0, \frac{1}{4})$, which we treat separately in the next paragraphs. In each case, the starting point is an estimate of the distance between any pair of eigenvalues of $\mathcal{A}_{a,b,\gamma}^0$, which is directly obtained from the explicit formulas (4.7). We use this estimate to construct an infinite family of mutually disjoint sets (balls or finite unions of balls) with the property that the spectrum of $\mathcal{A}_{a,b,\gamma}$ is contained in their union. Inside each set $\mathcal{A}_{a,b,\gamma}$ will have a finite number of eigenvalues (one, two or four) so that the problem reduces to showing that these eigenvalues are purely imaginary. In Propositions 4.4, 4.6, 4.8, and 4.10 below, we show that in all three cases the spectrum of $\mathcal{A}_{a,b,\gamma}$ is purely imaginary, provided $\|(a, b)\|$ is sufficiently small. This proves Proposition 4.1.

4.3 Spectrum for γ away from 0 and $\frac{1}{2}$

We start with the case $\gamma \in [\gamma_*, \frac{1}{2} - \gamma_*]$, when the operators with constant coefficients $\mathcal{A}^0_{a,b,\gamma}$ have only simple eigenvalues:

Lemma 4.3 *For any $\gamma_* \in (0, \frac{1}{4})$, there exist positive constants c_* and δ_* such that for any $\gamma \in [\gamma_*, \frac{1}{2} - \gamma_*]$ and any (a,b) with $\|(a,b)\| \leq \delta_*$, we have*

$$|\mathrm{i}\omega^{\sigma,n}_{a,b,\gamma} - \mathrm{i}\omega^{\tau,p}_{a,b,\gamma}| \geq c_* , \quad \text{for all } p, n \in \mathbb{Z} \text{ and all } \sigma, \tau \in \{-, +\} \text{ with } (\sigma, n) \neq (\tau, p) .$$

This lemma shows that the distance between any pair of eigenvalues of $\mathcal{A}^0_{a,b,\gamma}$ is strictly positive, uniformly for small $\|(a,b)\|$ and $\gamma \in [\gamma_*, \frac{1}{2} - \gamma_*]$. This allows us to find an infinite sequence of mutually disjoint balls with the property that the spectrum of $\mathcal{A}_{a,b,\gamma}$ is contained in their union, and that inside each ball both operators have precisely one simple eigenvalue. The symmetry of the spectrum of $\mathcal{A}_{a,b,\gamma}$ with respect to the imaginary axis then implies that this simple eigenvalue is purely imaginary. We point out that classical perturbation results for families of simple eigenvalues [20] do not directly apply, since here we have infinitely many eigenvalues.

Proposition 4.4 *Fix $\gamma_* \in (0, \frac{1}{4})$. Then there exist positive constants c and δ such that for any $\gamma \in [\gamma_*, \frac{1}{2} - \gamma_*]$ and any (a,b) with $\|(a,b)\| \leq \delta$, the following properties hold:*

(i) The spectrum of $\mathcal{A}_{a,b,\gamma}$ satisfies

$$\sigma(\mathcal{A}_{a,b,\gamma}) \subset \bigcup_{n\in\mathbb{Z}} B(\mathrm{i}\omega^{-,n}_{a,b,\gamma}; c) \cup \bigcup_{n\in\mathbb{Z}} B(\mathrm{i}\omega^{+,n}_{a,b,\gamma}; c) ,$$

and the closed balls $\overline{B(\mathrm{i}\omega^{\pm,n}_{a,b,\gamma}; c)}$ are mutually disjoints.

(ii) Inside each ball $B(\mathrm{i}\omega^{\pm,n}_{a,b,\gamma}; c)$ the operator $\mathcal{A}_{a,b,\gamma}$ has precisely one eigenvalue, which is purely imaginary.[1]

Proof. (i) We may choose any $c \leq c_*/4$ with c_* the constant in Lemma 4.3, so that the balls $\overline{B(\mathrm{i}\omega^{\pm,n}_{a,b,\gamma}; c)}$ are mutually disjoints, and then apply Lemma 4.2.

(ii) Consider a ball $B(\mathrm{i}\omega^{\pm,n}_{a,b,\gamma}; c)$. Inside this ball $\mathcal{A}^0_{a,b,\gamma}$ has precisely one eigenvalue $\mathrm{i}\omega^{\pm,n}_{a,b,\gamma}$ with associated spectral projection $\Pi^{0,n}_{a,b,\gamma}$ satisfying $\|\Pi^{0,n}_{a,b,\gamma}\| = 1$. The result (i) provides us with a spectral decomposition for the operator $\mathcal{A}_{a,b,\gamma}$, and we can compute the spectral projection $\Pi^n_{a,b,\gamma}$ associated to $B(\mathrm{i}\omega^{\pm,n}_{a,b,\gamma}; c)$ via the Dunford integral formula

$$\Pi^n_{a,b,\gamma} = \frac{1}{2\pi\mathrm{i}} \oint_{C_n} \mathcal{R}_{a,b,\gamma}(\lambda)\mathrm{d}\lambda , \tag{4.8}$$

in which C_n is the boundary of $B(\mathrm{i}\omega^{\pm,n}_{a,b,\gamma}; c)$ and $\mathcal{R}_{a,b,\gamma}(\lambda) = (\lambda\mathbf{1} - \mathcal{A}_{a,b,\gamma})^{-1}$. Using the formula for the resolvent

$$\mathcal{R}_{a,b,\gamma}(\lambda) = \mathcal{R}^0_{a,b,\gamma}(\lambda) \left(\mathbf{1} - \mathcal{A}^1_{a,b}\mathcal{R}^0_{a,b,\gamma}(\lambda)\right)^{-1} ,$$

---

[1]Here and in the rest of the paper we say that "$\mathcal{A}_{a,b,\gamma}$ has n eigenvalues inside the set B" when the sum of the algebraic multiplicities of the eigenvalues of $\mathcal{A}_{a,b,\gamma}$ inside B is equal to n.

which holds for sufficiently small δ since $\|\mathcal{A}_{a,b}^1\| = \mathcal{O}(a^2 + b^2)$, we compute the difference

$$\Pi_{a,b,\gamma}^n - \Pi_{a,b,\gamma}^{0,n} = \frac{1}{2\pi\mathrm{i}} \oint_{C_n} \mathcal{R}_{a,b,\gamma}^0(\lambda) \sum_{k \geq 1} \left(\mathcal{A}_{a,b}^1 \mathcal{R}_{a,b,\gamma}^0(\lambda)\right)^k \mathrm{d}\lambda \ .$$

Since $\|\mathcal{R}_{a,b,\gamma}^0(\lambda)\| = 1/c$, for $\lambda \in C_n$, we conclude that

$$\|\Pi_{a,b,\gamma}^n - \Pi_{a,b,\gamma}^{0,n}\| \leq \sum_{k \geq 1} \left(\frac{1}{c} \|\mathcal{A}_{a,b}^1\|\right)^k = \frac{\|\mathcal{A}_{a,b}^1\|}{c - \|\mathcal{A}_{a,b}^1\|} \ .$$

Upon choosing δ small enough we achieve

$$\begin{aligned}
\|\Pi_{a,b,\gamma}^n - \Pi_{a,b,\gamma}^{0,n}\| &< \frac{1}{1 + \|\Pi_{a,b,\gamma}^n - \Pi_{a,b,\gamma}^{0,n}\|} = \frac{1}{\|\Pi_{a,b,\gamma}^{0,n}\| + \|\Pi_{a,b,\gamma}^n - \Pi_{a,b,\gamma}^{0,n}\|} \\
&\leq \min\left(\frac{1}{\|\Pi_{a,b,\gamma}^{0,n}\|}, \frac{1}{\|\Pi_{a,b,\gamma}^n\|}\right) \ ,
\end{aligned}$$

so that the projections $\Pi_{a,b,\gamma}^n$ and $\Pi_{a,b,\gamma}^{0,n}$ realize isomorphisms between the associated spectral subspaces of $\mathcal{A}_{a,b,\gamma}^0$ and $\mathcal{A}_{a,b,\gamma}$ ([17, Lemma B.1]; see also [20, Chapter I §6.8]). In particular, they have the same finite rank, so that $\mathcal{A}_{a,b,\gamma}$ has precisely one simple eigenvalue inside the ball $B(\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}; c)$. Finally, since the spectrum is symmetric with respect to the imaginary axis (4.5) this simple eigenvalue is necessarily purely imaginary, which concludes the proof. $\qquad\square$

4.4 Spectrum for small γ

We start again by analyzing the distance between the eigenvalues of $\mathcal{A}_{a,b,\gamma}^0$, now for small values of γ. Since at $a = b = \gamma = 0$ the spectrum of $\mathcal{A}_{a,b,\gamma}^0$ consists of double nonzero eigenvalues and a quadruple eigenvalue at zero, for small a, b, and γ we expect pairs of arbitrarily close eigenvalues together with four eigenvalues close to the origin. A precise description of the location of these eigenvalues is given in the following lemma.

Lemma 4.5 *There exist positive constants γ_0, c, and δ, such that the following holds, for any $\gamma \in [0, \gamma_0]$, and (a, b) with $\|(a, b)\| \leq \delta$:*

(i) $\mathrm{i}\omega_{a,b,\gamma}^{\pm,0}$, $\mathrm{i}\omega_{a,b,\gamma}^{+,1}$, $\mathrm{i}\omega_{a,b,\gamma}^{-,-1} \in B(0; 1)$;

(ii) $\mathrm{i}\omega_{a,b,\gamma}^{-,1}$, $\mathrm{i}\omega_{a,b,\gamma}^{+,-1}$, $\mathrm{i}\omega_{a,b,\gamma}^{\pm,n} \notin B(0; 4)$, $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$;

(iii) $|\mathrm{i}\omega_{a,b,\gamma}^{+,n} - \mathrm{i}\omega_{a,b,\gamma}^{+,p}| \geq c$, $n, p \in \mathbb{Z} \setminus \{0, 1\}$, $p \neq n$, $p \neq 1 - n$;

(iv) $|\mathrm{i}\omega_{a,b,\gamma}^{-,n} - \mathrm{i}\omega_{a,b,\gamma}^{-,p}| \geq c$, $n, p \in \mathbb{Z} \setminus \{-1, 0\}$, $p \neq n$, $p \neq -1 - n$;

(v) $|\mathrm{i}\omega_{a,b,\gamma}^{+,n} - \mathrm{i}\omega_{a,b,\gamma}^{-,p}| \geq c$, $n \in \mathbb{Z} \setminus \{0, 1\}$, $p \in \mathbb{Z} \setminus \{-1, 0\}$.

The first two properties (i)-(ii) in this lemma together with the perturbation result in Lemma 4.2 provides us with a spectral splitting for $\mathcal{A}_{a,b,\gamma}$:

$$\sigma(\mathcal{A}_{a,b,\gamma}) = \sigma_1(\mathcal{A}_{a,b,\gamma}) \cup \sigma_2(\mathcal{A}_{a,b,\gamma}) \, ,$$

with

$$\sigma_1(\mathcal{A}_{a,b,\gamma}) \subset B(0;2) \, , \quad \sigma_2(\mathcal{A}_{a,b,\gamma}) \cap B(0;3) = \emptyset \, .$$

Inside the ball $B(0;2)$ we find the part of the spectrum of $\mathcal{A}_{a,b,\gamma}$ which is close to the quadruple zero eigenvalue of $\mathcal{A}_{0,0,0}^0$, whereas the rest of the spectrum lies outside the ball $B(0;3)$. The last properties (iii)-(v) show that the eigenvalues outside $B(0;3)$ are well separated except for the pairs $(\mathrm{i}\omega_{a,b,\gamma}^{+,n}, \mathrm{i}\omega_{a,b,\gamma}^{+,1-n})$, $n \in \mathbb{Z} \setminus \{0,1\}$, and $(\mathrm{i}\omega_{a,b,\gamma}^{-,n}, \mathrm{i}\omega_{a,b,\gamma}^{-,-1-n})$, $n \in \mathbb{Z} \setminus \{-1,0\}$, which may be arbitrarily close. At $a = b = \gamma = 0$, these are precisely the double eigenvalues of $\mathcal{A}_{0,0,0}^0$. Notice however that, for fixed a, b, γ, the distances $|\mathrm{i}\omega_{a,b,\gamma}^{+,n} - \mathrm{i}\omega_{a,b,\gamma}^{+,1-n}|$ and $|\mathrm{i}\omega_{a,b,\gamma}^{-,n} - \mathrm{i}\omega_{a,b,\gamma}^{-,-1-n}|$ typically grow like $\mathcal{O}(n^2)$ as $|n| \to \infty$, see (4.7).

We analyze these two parts $\sigma_1(\mathcal{A}_{a,b,\gamma})$ and $\sigma_2(\mathcal{A}_{a,b,\gamma})$ of the spectrum of $\mathcal{A}_{a,b,\gamma}$ separately in the Propositions 4.6 and 4.8 below.

Proposition 4.6 *There exist positive constants γ_0, c, and δ, such that for any $\gamma \in [0, \gamma_0]$, and (a,b) with $\|(a,b)\| \leq \delta$, the following holds.*

(i) The spectrum $\sigma_2(\mathcal{A}_{a,b,\gamma})$ satisfies

$$\sigma_2(\mathcal{A}_{a,b,\gamma}) \subset B(\mathrm{i}\omega_{a,b,\gamma}^{-,1}; c) \cup B(\mathrm{i}\omega_{a,b,\gamma}^{+,-1}; c) \cup \bigcup_{n \neq \pm 1, 0} B(\mathrm{i}\omega_{a,b,\gamma}^{-,n}; c) \cup \bigcup_{n \neq \pm 1, 0} B(\mathrm{i}\omega_{a,b,\gamma}^{+,n}; c) \, ,$$

in which the balls $B(\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}; c)$ are mutually disjoints, except for some pairs $(\mathrm{i}\omega_{a,b,\gamma}^{+,n}, \mathrm{i}\omega_{a,b,\gamma}^{+,1-n})$, $n \in \mathbb{Z} \setminus \{0,1\}$, or $(\mathrm{i}\omega_{a,b,\gamma}^{-,n}, \mathrm{i}\omega_{a,b,\gamma}^{-,-1-n})$, $n \in \mathbb{Z} \setminus \{-1,0\}$.

(ii) Inside each ball $B(\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}; c)$ the operator $\mathcal{A}_{a,b,\gamma}$ has either one or two eigenvalues, which are purely imaginary.

Proof. The result (i) is obtained from Lemma 4.5 (iii)-(v) and Lemma 4.2, just as the first part of Proposition 4.4. The only difference is that here we have pairs of balls which are not disjoint. As was noticed above, the distances $|\mathrm{i}\omega_{a,b,\gamma}^{+,n} - \mathrm{i}\omega_{a,b,\gamma}^{+,1-n}|$ and $|\mathrm{i}\omega_{a,b,\gamma}^{-,n} - \mathrm{i}\omega_{a,b,\gamma}^{-,-1-n}|$ grow like $\mathcal{O}(n^2)$ as $|n| \to \infty$, so that we have in general a finite number of such pairs for a given value of a, b, γ.

(ii) For the balls $B(\mathrm{i}\omega_{a,b,\gamma}^{\pm,n}; c)$ which are disjoint from all the others we can argue and conclude as in the proof of Proposition 4.4. It remains to consider the case of two balls which are not disjoint. Choose a pair of eigenvalues $(\mathrm{i}\omega_{a,b,\gamma}^{+,n}, \mathrm{i}\omega_{a,b,\gamma}^{+,1-n})$ such that $B(\mathrm{i}\omega_{a,b,\gamma}^{+,n}; c) \cap B(\mathrm{i}\omega_{a,b,\gamma}^{+,1-n}; c) \neq \emptyset$ (the argument is similar for a pair $(\mathrm{i}\omega_{a,b,\gamma}^{-,n}, \mathrm{i}\omega_{a,b,\gamma}^{-,-1-n})$). We construct the spectral projection $\Pi_{a,b,\gamma}^{n,1-n}$ for $\mathcal{A}_{a,b,\gamma}$ corresponding to the union of these balls with the help of the Dunford integral formula (4.8), in which the circle C_n is replaced by the smallest circle C_r with radius $c < r < 2c$, centered on the imaginary axis, which contains both balls. The spectral projection $\Pi_{a,b,\gamma}^{0,n,1-n}$ for $\mathcal{A}_{a,b,\gamma}^0$ has unit norm again, and since $\|\mathcal{R}_{a,b,\gamma}^0(\lambda)\| \leq 1/c$ for $\lambda \in C_r$, we easily find

$$\|\Pi_{a,b,\gamma}^{n,1-n} - \Pi_{a,b,\gamma}^{0,n,1-n}\| \leq \frac{r}{c} \frac{\|\mathcal{A}_{a,b}^1\|}{c - \|\mathcal{A}_{a,b}^1\|} \leq \frac{2\|\mathcal{A}_{a,b}^1\|}{c - \|\mathcal{A}_{a,b}^1\|} = \mathcal{O}(a^2 + b^2) \, . \tag{4.9}$$

As in the proof of Proposition 4.4 we now choose δ sufficiently small such that these projections realize isomorphisms between the associated spectral subspaces of $\mathcal{A}^0_{a,b,\gamma}$ and $\mathcal{A}_{a,b,\gamma}$. In particular, these subspaces have the same finite rank equal to 2, which proves that $\mathcal{A}_{a,b,\gamma}$ has precisely two eigenvalues in $B(\mathrm{i}\omega^{+;n}_{a,b,\gamma};c) \cup B(\mathrm{i}\omega^{+;1-n}_{a,b,\gamma};c)$.

In order to show that these two eigenvalues do not move off the imaginary axis we choose an appropriate basis of the associated two-dimensional eigenspace and compute the 2×2 matrix representing the action of $\mathcal{A}_{a,b,\gamma}$ on this space. Then it suffices to show that this matrix has purely imaginary eigenvalues. We start with the basis

$$\xi_{00} \;=\; \frac{1}{2\sqrt{\pi}}\, \mathrm{e}^{\mathrm{i}nz} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \;, \quad \xi_{01} \;=\; \frac{1}{2\sqrt{\pi}}\, \mathrm{e}^{\mathrm{i}(1-n)z} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \;,$$

of the two-dimensional eigenspace of $\mathcal{A}^0_{a,b,\gamma}$, which satisfies $\langle -\mathrm{i}J\xi_{0k}, \xi_{0\ell} \rangle = \delta_{k\ell}$. We claim that for $\mathcal{A}_{a,b,\gamma}$ we can find a basis with the same property. Indeed, consider the vectors

$$\widetilde{\xi}_0 \;=\; \Pi^{n,1-n}_{a,b,\gamma} \xi_{00} \;, \quad \widetilde{\xi}_1 \;=\; \Pi^{n,1-n}_{a,b,\gamma} \xi_{01} \;,$$

which form a basis of the two-dimensional eigenspace of $\mathcal{A}_{a,b,\gamma}$. From (4.9) we obtain $\langle -\mathrm{i}J\widetilde{\xi}_0, \widetilde{\xi}_0 \rangle = 1 + \mathcal{O}(a^2 + b^2) > 0$, so that the vector ξ_0 defined by

$$\xi_0 \;=\; \frac{1}{\langle -\mathrm{i}J\widetilde{\xi}_0, \widetilde{\xi}_0 \rangle^{1/2}}\, \widetilde{\xi}_0 \;,$$

satisfies $\langle -\mathrm{i}J\xi_0, \xi_0 \rangle = 1$. Then we define successively

$$\widehat{\xi}_1 \;=\; \widetilde{\xi}_1 - \overline{\langle -\mathrm{i}J\xi_0, \widetilde{\xi}_1 \rangle}\, \xi_0 \;, \quad \xi_1 \;=\; \frac{1}{\langle -\mathrm{i}J\widehat{\xi}_1, \widehat{\xi}_1 \rangle^{1/2}}\, \widehat{\xi}_1 \;,$$

and find $\langle -\mathrm{i}J\xi_0, \xi_1 \rangle = 0$ and $\langle -\mathrm{i}J\xi_1, \xi_1 \rangle = 1$, which proves the claim.

The property $\langle -\mathrm{i}J\xi_k, \xi_\ell \rangle = \delta_{k\ell}$ implies that the action of $\mathcal{A}_{a,b,\gamma}$ on the two-dimensional space spanned by $\{\xi_0, \xi_1\}$ is given by the matrix

$$\mathcal{M}_{a,b,\gamma} \;=\; \begin{pmatrix} \langle \mathcal{A}_{a,b,\gamma}\xi_0, -\mathrm{i}J\xi_0 \rangle & \langle \mathcal{A}_{a,b,\gamma}\xi_1, -\mathrm{i}J\xi_0 \rangle \\ \langle \mathcal{A}_{a,b,\gamma}\xi_0, -\mathrm{i}J\xi_1 \rangle & \langle \mathcal{A}_{a,b,\gamma}\xi_1, -\mathrm{i}J\xi_1 \rangle \end{pmatrix} \;.$$

Using the decomposition $\mathcal{A}_{a,b,\gamma} = JH_{a,b,\gamma}$ we find

$$\langle \mathcal{A}_{a,b,\gamma}\xi_k, -\mathrm{i}J\xi_l \rangle \;=\; \langle JH_{a,b,\gamma}\xi_k, -\mathrm{i}J\xi_l \rangle \;=\; \langle H_{a,b,\gamma}\xi_k, -\mathrm{i}J^{-1}J\xi_l \rangle \;=\; \mathrm{i}\langle H_{a,b,\gamma}\xi_k, \xi_l \rangle \;,$$

so that

$$\mathcal{M}_{a,b,\gamma} \;=\; \mathrm{i} \begin{pmatrix} \langle H_{a,b,\gamma}\xi_0, \xi_0 \rangle & \langle H_{a,b,\gamma}\xi_1, \xi_0 \rangle \\ \langle H_{a,b,\gamma}\xi_0, \xi_1 \rangle & \langle H_{a,b,\gamma}\xi_1, \xi_1 \rangle \end{pmatrix} \;.$$

Since $\langle H_{a,b,\gamma}Q, R \rangle = \overline{\langle H_{a,b,\gamma}R, Q \rangle}$, we conclude that this matrix always has purely imaginary eigenvalues. This completes the proof. $\square$

Remark 4.7 *The last part of this proof is a simple version of the well-known result for general Hamiltonian systems which asserts that colliding purely imaginary eigenvalues do not leave the imaginary axis when they have the same Krein signature (see e.g. [18]). In the case of the four eigenvalues close to the origin, which we treat in the next proposition, the same argument does not work anymore (these eigenvalues have opposite Krein signatures). Instead, we compute an explicit expansion of the restriction of $\mathcal{A}_{a,b,\gamma}$ to the associated eigenspace which allows to show that these four eigenvalues are purely imaginary.*

Proposition 4.8 *There exist positive constants γ_0, c, and δ, such that for any $\gamma \in [0, \gamma_0]$, and (a, b) with $\|(a, b)\| \le \delta$, the set $\sigma_1(\mathcal{A}_{a,b,\gamma})$ consists of four purely imaginary eigenvalues.*

Proof.   As in the previous cases, upon choosing δ sufficiently small, we obtain that $\mathcal{A}_{a,b,\gamma}$ has precisely four eigenvalues inside the ball $B(0; 2)$. In order to locate these four eigenvalues we construct a suitable basis for the associated eigenspace and compute the 4×4 matrix $\mathcal{M}_{a,b,\gamma}$ representing the action of $\mathcal{A}_{a,b,\gamma}$ on this space. Then we show that this matrix has purely imaginary eigenvalues.

We start with the particular cases $a = b = 0$, and $\gamma = 0$. In the first case, the operator $\mathcal{A}_{0,0,\gamma} = \mathcal{A}_{0,0,\gamma}^0$ has constant coefficients, so that we can explicitly compute the basis and the matrix. We choose the real basis

$$\xi_{0,0,\gamma}^{(0)} = \begin{pmatrix} \sin z \\ \cos z \end{pmatrix} , \quad \xi_{0,0,\gamma}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \xi_{0,0,\gamma}^{(2)} = \begin{pmatrix} \cos z \\ -\sin z \end{pmatrix} , \quad \xi_{0,0,\gamma}^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

in which we find

$$\mathcal{M}_{0,0,\gamma} = \begin{pmatrix} 4\mathrm{i}\gamma \, \mathbf{D}_2 & -4\gamma^2 \mathbf{1}_2 \\ 4\gamma^2 \mathbf{1}_2 & 4\mathrm{i}\gamma \, \mathbf{D}_2 \end{pmatrix} , \quad \mathbf{D}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Next, we consider the operator $\mathcal{A}_{a,b,0}$. As for the operator $H_{a,b}$ in Section 3, we have that

$$\left. \frac{\partial}{\partial \varphi} \mathcal{R}_{(\varphi, \xi)} Q_{a,b} \right|_{(\varphi, \xi) = (0,0)} = -\mathrm{i} Q_{a,b} , \quad \left. \frac{\partial}{\partial \xi} \mathcal{R}_{(\varphi, \xi)} Q_{a,b} \right|_{(\varphi, \xi) = (0,0)} = \partial_z Q_{a,b} ,$$

belong to the kernel of $\mathcal{A}_{a,b,0}$. In addition, since

$$H_{a,b}(\partial_\omega Q_{a,b}) = N'(Q_{a,b}) = Q_{a,b} , \quad H_{a,b}(\partial_c Q_{a,b}) = M'(Q_{a,b}) = \mathrm{i}\partial_z Q_{a,b} ,$$

(see Remark 3.10) and $\mathcal{A}_{a,b,0} = -\mathrm{i}H_{a,b}$, we have

$$\mathcal{A}_{a,b,0}(\partial_\omega Q_{a,b}) = -\mathrm{i} Q_{a,b} , \quad \mathcal{A}_{a,b,0}(\partial_c Q_{a,b}) = \partial_z Q_{a,b} ,$$

which provides us with two principal vectors in the generalized kernel of $\mathcal{A}_{a,b,0}$. Together with the two vectors in the kernel of $\mathcal{A}_{a,b,0}$ these give us a basis for the four-dimensional eigenspace of $\mathcal{A}_{a,b,0}$. At $a = b = 0$ we must find the basis above so that we set

$$\xi_{a,b,0}^{(0)} = -\frac{1}{a} \partial_z Q_{a,b} = \begin{pmatrix} \sin z \\ \cos z \end{pmatrix} + \mathcal{O}(|b|(|a| + |b|)) ,$$

$$\xi_{a,b,0}^{(1)} = \frac{1}{b}(\mathrm{i} Q_{a,b} + \partial_z Q_{a,b}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}(|a|(|a| + |b|)) ,$$

$$\xi_{a,b,0}^{(2)} = 2a(2\partial_\omega Q_{a,b} - \partial_c Q_{a,b}) = \begin{pmatrix} \cos z \\ -\sin z \end{pmatrix} + \mathcal{O}(a^2 + b^2) ,$$

$$\xi_{a,b,0}^{(3)} = 2b(\partial_\omega Q_{a,b} + \partial_c Q_{a,b}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(a^2 + b^2) ,$$

and a straightforward calculation gives the matrix

$$\mathcal{M}_{a,b,0} = \begin{pmatrix} \mathbf{0}_2 & \mathbf{M}_2(a, b) \\ \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} , \quad \mathbf{0}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad \mathbf{M}_2 = \begin{pmatrix} -2a^2 & -4ab \\ -4ab & -2b^2 \end{pmatrix} + \mathcal{O}(a^4 + b^4) .$$

Finally, we consider the full operator $\mathcal{A}_{a,b,\gamma}$ and construct a basis $\{\xi^{(0)}_{a,b,\gamma}, \xi^{(1)}_{a,b,\gamma}, \xi^{(2)}_{a,b,\gamma}, \xi^{(3)}_{a,b,\gamma}\}$ for small a, b, and γ, by extending the bases above. Notice first that the vectors in the basis for $\gamma = 0$ satisfy

$$\mathcal{S}\xi^{(0)}_{a,b,0} = -\xi^{(0)}_{a,b,0}, \quad \mathcal{S}\xi^{(1)}_{a,b,0} = -\xi^{(1)}_{a,b,0}, \quad \mathcal{S}\xi^{(2)}_{a,b,0} = \xi^{(2)}_{a,b,0}, \quad \mathcal{S}\xi^{(3)}_{a,b,0} = \xi^{(3)}_{a,b,0} ,$$

where $\mathcal{S}$ is the reversibility operator (4.4). Since for $\gamma \neq 0$ we have $\mathcal{S}\mathcal{A}_{a,b,\gamma} = -\mathcal{A}_{a,b,-\gamma}\mathcal{S}$, the vectors in the basis can be taken such that

$$\mathcal{S}\xi^{(0)}_{a,b,\gamma} = -\xi^{(0)}_{a,b,-\gamma}, \quad \mathcal{S}\xi^{(1)}_{a,b,\gamma} = -\xi^{(1)}_{a,b,-\gamma}, \quad \mathcal{S}\xi^{(2)}_{a,b,\gamma} = \xi^{(2)}_{a,b,-\gamma}, \quad \mathcal{S}\xi^{(3)}_{a,b,\gamma} = \xi^{(3)}_{a,b,-\gamma} ,$$

and then the matrix $\mathcal{M}_{a,b,\gamma}$ satisfies

$$\widetilde{\mathcal{S}}\mathcal{M}_{a,b,\gamma} = -\mathcal{M}_{a,b,-\gamma}\widetilde{\mathcal{S}} , \quad \text{where} \quad \widetilde{\mathcal{S}} = \begin{pmatrix} -\mathbf{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix} .$$

In addition, since $\mathcal{A}_{a,b} = \mathcal{A}_{-a,-b}$, we also have $\mathcal{M}_{a,b,\gamma} = \mathcal{M}_{-a,-b,\gamma}$. Together with the results for $a = b = 0$ and $\gamma = 0$ we conclude that

$$\mathcal{M}_{a,b,\gamma} = \begin{pmatrix} 4\mathrm{i}\gamma\left(\mathbf{D}_2 + \mathcal{O}(a^2 + b^2)\right) & \mathbf{M}_2(a,b) - 4\gamma^2\left(\mathbf{1}_2 + \mathcal{O}(a^2 + b^2)\right) \\ 4\gamma^2\left(\mathbf{1}_2 + \mathcal{O}(a^2 + b^2)\right) & 4\mathrm{i}\gamma\left(\mathbf{D}_2 + \mathcal{O}(a^2 + b^2)\right) \end{pmatrix} .$$

To end the proof we show that the four eigenvalues of this matrix are purely imaginary. The structure of the matrix $\mathcal{M}_{a,b,\gamma}$ implies that its characteristic polynomial is of the form

$$\lambda^4 + \mathrm{i}\gamma c_3\lambda^3 + \gamma^2 c_2\lambda^2 + \mathrm{i}\gamma^3 c_1\lambda + c_0\gamma^4 ,$$

in which the coefficients c_j depend upon a, b, and γ. The four roots of this polynomial are symmetric with respect to the imaginary axis, because the spectrum of $A_{a,b,\gamma}$ is symmetric with respect to the imaginary axis, so that the coefficients c_j are real functions of a, b, γ. In addition, the spectral equalities (4.5)–(4.6) imply that c_j are even in a, b, and γ, and that when replacing (a,b) by (b,a) the coefficients c_0, c_2 do not change, while c_1, c_3 change sign.

We now set $\lambda = \mathrm{i}\gamma X$, and obtain the polynomial with real coefficients,

$$P(X) = X^4 + c_3 X^3 - c_2 X^2 - c_1 X + c_0 . \tag{4.10}$$

At $a = b = 0$ the four eigenvalues of $\mathcal{M}_{a,b,\gamma}$ are known, which then gives

$$P|_{a=b=0}(X) = X^4 - 32(\gamma^2 + 1)X^2 + 256(1 - 2\gamma^2 + \gamma^4) .$$

In addition, using the explicit formulas for the plane waves we compute the roots of P when $a = \gamma = 0$:

$$X^{(1,2)}_b = -4 \pm 2\sqrt{2}b + 5b^2 + \mathcal{O}(b^3), \quad X^{(3)}_b = X^{(4)}_b = 4 - 7b^2 + \mathcal{O}(b^3) .$$

Similarly, when $b = \gamma = 0$, we find

$$X^{(1,2)}_a = 4 \pm 2\sqrt{2}a - 5a^2 + \mathcal{O}(a^3), \quad X^{(3)}_a = X^{(4)}_a = -4 + 7a^2 + \mathcal{O}(a^3) .$$

Combining these formulas with the parity properties mentioned above, we conclude that

$$c_3 = 4(b^2 - a^2) + \mathcal{O}(a^4 + b^4 + \gamma^4), \quad c_2 = 32 - 88(b^2 + a^2) + 32\gamma^2 + \mathcal{O}(a^4 + b^4 + \gamma^4) ,$$

$$c_1 = \mathcal{O}(a^4 + b^4 + \gamma^4), \quad c_0 = 256 - 1664(b^2 + a^2) - 512\gamma^2 + \mathcal{O}(a^4 + b^4 + \gamma^4) .$$

A direct calculation now gives

$$
\begin{aligned}
P(0) &= 256 + \mathcal{O}(a^2 + b^2 + \gamma^2) > 0 , \\
P(X_b^{(4)}) &= -512a^2 - 1024\gamma^2 + \mathcal{O}((a^2 + \gamma^2)(a^2 + b^2 + \gamma^2)) < 0 , \\
P(X_a^{(4)}) &= -512b^2 - 1024\gamma^2 + \mathcal{O}((b^2 + \gamma^2)(a^2 + b^2 + \gamma^2)) < 0 ,
\end{aligned}
$$

for a, b, and γ sufficiently small. This shows that the polynomial P has four real roots, so that the four eigenvalues of $\mathcal{A}_{a,b,\gamma}$ are purely imaginary. This concludes the proof. $\qquad\square$

4.5 Spectrum for γ close to $\frac{1}{2}$

In this case, the arguments are similar to the ones for $\sigma_2(\mathcal{A}_{a,b,\gamma})$ in Section 4.4, and we shall therefore only state the results and omit the proofs. First, we have the following result on the eigenvalues of $\mathcal{A}_{a,b,\gamma}^0$.

Lemma 4.9 *There exist positive constants γ_1, c, and δ, such that the following hold, for any $\gamma \in [\gamma_1, \frac{1}{2}]$, and (a, b) with $\|(a, b)\| \leq \delta$:*

(i) $i\omega_{a,b,\gamma}^{+,0} \in B(-i; \frac{1}{2})$, $i\omega_{a,b,\gamma}^{-,-1} \in B(i; \frac{1}{2})$;

(ii) $i\omega_{a,b,\gamma}^{-,0}$, $i\omega_{a,b,\gamma}^{+,-1}$, $i\omega_{a,b,\gamma}^{\pm,n} \notin B(0; \frac{5}{2})$, $n \in \mathbb{Z} \setminus \{-1, 0\}$;

(iii) $|i\omega_{a,b,\gamma}^{+,n} - i\omega_{a,b,\gamma}^{+,p}| \geq c$, $n, p \in \mathbb{Z} \setminus \{0\}$, $p \neq n$, $p \neq -n$;

(iv) $|i\omega_{a,b,\gamma}^{-,n} - i\omega_{a,b,\gamma}^{-,p}| \geq c$, $n, p \in \mathbb{Z} \setminus \{-1\}$, $p \neq n$, $p \neq -2 - n$;

(v) $|i\omega_{a,b,\gamma}^{+,n} - i\omega_{a,b,\gamma}^{-,p}| \geq c$, $n \in \mathbb{Z} \setminus \{0\}$, $p \in \mathbb{Z} \setminus \{-1\}$.

Next, we proceed as in the proof of Proposition 4.6 and obtain:

Proposition 4.10 *There exist positive constants γ_1, c, and δ, such that for any $\gamma \in [\gamma_1, \frac{1}{2}]$, and (a, b) with $\|(a, b)\| \leq \delta$, the following holds:*

(i) The spectrum of $\mathcal{A}_{a,b,\gamma}$ satisfies

$$\sigma(\mathcal{A}_{a,b,\gamma}) \subset \bigcup_{n \in \mathbb{Z}} B(i\omega_{a,b,\gamma}^{-,n}; c) \cup \bigcup_{n \in \mathbb{Z}} B(i\omega_{a,b,\gamma}^{+,n}; c) ,$$

in which the balls $B(i\omega_{a,b,\gamma}^{\pm,n}; c)$ are mutually disjoints except for pairs $(i\omega_{a,b,\gamma}^{+,n}, i\omega_{a,b,\gamma}^{+,-n})$, $n \in \mathbb{Z} \setminus \{0\}$, or $(i\omega_{a,b,\gamma}^{-,n}, i\omega_{a,b,\gamma}^{-,-2-n})$, $n \in \mathbb{Z} \setminus \{-1\}$.

(ii) Inside each ball $B(i\omega_{a,b,\gamma}^{\pm,n}; c)$ the operator $\mathcal{A}_{a,b,\gamma}$ has at most two eigenvalues, which are purely imaginary.

5 The focusing NLS equation

We consider in this section the focusing NLS equation

$$iU_t(x,t) + U_{xx}(x,t) + |U(x,t)|^2 U(x,t) = 0 , \qquad (5.1)$$

in which $x \in \mathbb{R}$, $t \in \mathbb{R}$, and $U(x,t) \in \mathbb{C}$. This equation also possesses a family of small periodic waves of the form $U_{a,b}(x) = e^{-it}e^{i\ell_{a,b}x}P_{a,b}(k_{a,b}x)$, but now

$$\ell_{a,b} = \frac{1}{4}(a^2 - b^2) + \mathcal{O}(a^4 + b^4) ,$$

$$k_{a,b} = 1 + \frac{3}{4}(a^2 + b^2) + \mathcal{O}(a^4 + b^4) ,$$

$$P_{a,b}(y) = ae^{-iy} + be^{iy} + \mathcal{O}(|ab|(|a| + |b|)) .$$

Both the equation and the periodic waves have the same symmetry properties as in the defocusing case, so that we can investigate the stability of this family of periodic waves in an analogous way.

As in Section 3, we define $p_{a,b}$ and $Q_{a,b}(z)$ by (3.1), and consider solutions of (5.1) of the form (3.3). The wave profile $Q_{a,b}(z)$ is then an equilibrium of the evolution equation

$$iQ_t + 4ip_{a,b}k_{a,b}Q_z + 4k_{a,b}^2 Q_{zz} + (1 - p_{a,b}^2)Q + |Q|^2 Q = 0 . \qquad (5.2)$$

For the orbital stability, we use the same functional set-up, the same conserved quantities $N(Q)$ and $M(Q)$, and the energy

$$\mathcal{E}(Q) = \int_0^{2\pi} \left(2k_{a,b}^2 |Q_z(z)|^2 - \frac{1}{4}|Q(z)|^4 \right) \mathrm{d}z ,$$

in which only the sign of the last term has been changed. Following the arguments in Section 3 one can show that the result in Theorem 1 holds in this case, as well. We only mention that the Hessian matrix of the function $d_{a,b}$ has now the expression:

$$\mathcal{H}_{a,b} \overset{\text{def}}{=} \left. \begin{pmatrix} \frac{\partial^2 d_{a,b}}{\partial \omega^2} & \frac{\partial^2 d_{a,b}}{\partial \omega \partial c} \\ \frac{\partial^2 d_{a,b}}{\partial c \partial \omega} & \frac{\partial^2 d_{a,b}}{\partial c^2} \end{pmatrix} \right|_{(\omega,c)=(0,0)} = \frac{\pi}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} (\mathbf{1} + \mathcal{O}(a^2 + b^2)) ,$$

so that it has again one positive and one negative eigenvalue.

The analysis is also the same for the spectral stability, when we study the spectrum of the linear operator

$$\mathcal{A}_{a,b}Q = 4ik_{a,b}^2 Q_{zz} - 4p_{a,b}k_{a,b}Q_z + i(1 - p_{a,b}^2)Q + 2i|Q_{a,b}|^2 Q + iQ_{a,b}^2 \overline{Q} .$$

However, in this case the result is different: the periodic waves are spectrally unstable. While we do not attempt a complete description of the spectrum, we focus here on the existence of unstable eigenvalues. It turns out that unstable eigenvalues arise through the unfolding of the quadruple zero eigenvalue of the unperturbed operator at $a = b = \gamma = 0$. These are the eigenvalues of the 4×4 matrix $\mathcal{M}_{a,b,\gamma}$ in Proposition 4.8, which is obtained here in the same way. This matrix has the same structure,

$$\mathcal{M}_{a,b,\gamma} = \begin{pmatrix} 4i\gamma \left(\mathbf{D}_2 + \mathcal{O}(a^2 + b^2) \right) & \mathbf{M}_2(a,b) - 4\gamma^2 \left(\mathbf{1}_2 + \mathcal{O}(a^2 + b^2) \right) \\ 4\gamma^2 \left(\mathbf{1}_2 + \mathcal{O}(a^2 + b^2) \right) & 4i\gamma \left(\mathbf{D}_2 + \mathcal{O}(a^2 + b^2) \right) \end{pmatrix} ,$$

but now

$$\mathbf{M}_2(a,b) = \begin{pmatrix} 2a^2 & 4ab \\ 4ab & 2b^2 \end{pmatrix} + \mathcal{O}(a^4 + b^4) \ .$$

The eigenvalues of $\mathcal{M}_{a,b,\gamma}$ are of the form $\lambda = i\gamma X$, where X is a root of a polynomial of the form (4.10). When $a = b = 0$ we obtain

$$P|_{a=b=0}(X) = X^4 - 32(\gamma^2 + 1)X^2 + 256(1 - 2\gamma^2 + \gamma^4) \ ,$$

and using the plane waves we find the roots of P when $a = \gamma = 0$:

$$X_b^{(1,2)} = -4 \pm i2\sqrt{2}b - 5b^2 + \mathcal{O}(b^3), \quad X_b^{(3)} = X_b^{(4)} = 4 + 7b^2 + \mathcal{O}(b^3) \ ,$$

and when $b = \gamma = 0$:

$$X_a^{(1,2)} = 4 \pm i2\sqrt{2}a + 5a^2 + \mathcal{O}(a^3), \quad X_a^{(3)} = X_a^{(4)} = -4 - 7a^2 + \mathcal{O}(a^3) \ .$$

Then we find the expansions for the coefficients

$$c_3 = -4(b^2 - a^2) + \mathcal{O}(a^4 + b^4 + \gamma^4), \quad c_2 = 32 + 88(b^2 + a^2) + 32\gamma^2 + \mathcal{O}(a^4 + b^4 + \gamma^4),$$

$$c_1 = \mathcal{O}(a^4 + b^4 + \gamma^4), \quad c_0 = 256 + 1664(b^2 + a^2) - 512\gamma^2 + \mathcal{O}(a^4 + b^4 + \gamma^4),$$

which give

$$\begin{aligned} P(0) &= 256 + \mathcal{O}(a^2 + b^2 + \gamma^2) > 0 \ , \\ P(X_b^{(4)}) &= 512a^2 - 1024\gamma^2 + \mathcal{O}((a^2 + \gamma^2)(a^2 + b^2 + \gamma^2)) \ , \\ P(X_a^{(4)}) &= 512b^2 - 1024\gamma^2 + \mathcal{O}((b^2 + \gamma^2)(a^2 + b^2 + \gamma^2)) \ . \end{aligned}$$

This suggests that the polynomial P has complex roots provided γ is small compared to a and b. In order to prove this, we consider the polynomial P when $\gamma = 0$ and show that it has at least two complex roots. We assume that $b \geq a \geq 0$, without loss of generality. Since $P(X) = (X - 4)^2(X + 4)^2 + \mathcal{O}(a^2 + b^2)$, this polynomial is positive outside two $\mathcal{O}(b^{1/2})$-neighborhoods of 4 and -4, when a and b are sufficiently small. Inside each of these neighborhoods, P has at most two real roots. A direct computation gives

$$P(-4+Y) = 512b^2 + (512b^2 + 896a^2)Y + (64 - 40b^2 - 136a^2)Y^2 - (16 - 4a^2 + 4b^2)Y^3 + Y^4 + \mathcal{O}(a^4 + b^4) \ ,$$

from which we conclude that P is positive inside any $\mathcal{O}(b^{1/2})$-neighborhood of -4, for $\|(a,b)\|$ sufficiently small. Summarizing, P has at most two real roots, and we conclude that the operator $\mathcal{A}_{a,b,\gamma}$ has at least one pair of eigenvalues off the imaginary axis, for γ sufficiently small. In view of the symmetry with respect to the imaginary axis of the spectrum of $\mathcal{A}_{a,b,\gamma}$, one of these eigenvalues has positive real part. This proves that the small periodic waves are spectrally unstable in this case.

Appendix: Spectrum of $H_{a,b}$

In this Appendix we discuss the spectrum of the linear self-adjoint operator $H_{a,b}$ defined in (3.13). As in Section 4, we decompose the elements of our function space into real and imaginary parts,

and work with the matrix operator

$$H_{a,b} = \begin{pmatrix} -4k_{a,b}^2\partial_{zz} + (p_{a,b}^2 - 1) + 3R_{a,b}^2 + I_{a,b}^2 & 4p_{a,b}k_{a,b}\partial_z + 2R_{a,b}I_{a,b} \\ -4p_{a,b}k_{a,b}\partial_z + 2R_{a,b}I_{a,b} & -4k_{a,b}^2\partial_{zz} + (p_{a,b}^2 - 1) + R_{a,b}^2 + 3I_{a,b}^2 \end{pmatrix} ,$$

where $Q_{a,b} = R_{a,b} + iI_{a,b}$. We prove the following result:

Proposition A.1 *There exists a positive constant ε_0 such that for all (a,b) with $\|(a,b)\| \leq \varepsilon_0$, the spectrum of the matrix operator $H_{a,b}$ in the Hilbert space of 2π-periodic functions $L^2_{\mathrm{per}}([0,2\pi],\mathbb{C}^2)$ verifies*

$$\sigma(H_{a,b}) = \{0, \lambda_{a,b}^{(2)}, \lambda_{a,b}^{(3)}\} \cup \sigma_1(H_{a,b}), \quad \sigma_1(H_{a,b}) \subset [6, +\infty) ,$$

where 0 is a double eigenvalue and $\lambda_{a,b}^{(j)}$, $j = 2,3$, are simple real eigenvalues with

$$\lambda_{0,0}^{(2)} = \lambda_{0,0}^{(3)} \quad and \quad \lambda_{a,b}^{(2)} < \lambda_{a,b}^{(3)}, \text{ for all } (a,b) \neq 0.$$

Proof.   Notice first that the parity properties with respect to (a,b) of the quantities $k_{a,b}$, $p_{a,b}$, and $Q_{a,b}$ imply that $\sigma(H_{a,b}) = \sigma(H_{-a,b}) = \sigma(H_{a,-b})$, and that $H_{a,b}$ commutes with the symmetry $\mathcal{S}$ introduced in (4.4).

When $a = b = 0$ the operator $H_{a,b}$ reduces to the operator H_0 in the proof of Lemma 3.7 with spectrum $\sigma(H_0) = \{4n(n \pm 1), \ n \in \mathbb{Z}\}$, for which 0 is a quadruple eigenvalue and the other eigenvalues are all positive and greater or equal to 8. Then, a standard perturbation argument shows that the spectrum of $H_{a,b}$ decomposes as

$$\sigma(H_{a,b}) = \{\lambda_{a,b}^{(0)}, \lambda_{a,b}^{(1)}, \lambda_{a,b}^{(2)}, \lambda_{a,b}^{(3)}\} \cup \sigma_1(H_{a,b}), \quad \text{where } \sigma_1(H_{a,b}) \subset [6, +\infty) ,$$

for (a,b) sufficiently small. The four eigenvalues $\lambda_{a,b}^{(j)}$ are the continuation for small (a,b) of the quadruple zero eigenvalue of H_0.

In order to locate these four eigenvalues, we proceed as in the proof of Proposition 4.8: we construct an appropriate basis $\{\xi_{a,b}^{(0)}, \xi_{a,b}^{(1)}, \xi_{a,b}^{(2)}, \xi_{a,b}^{(3)}\}$ for the associated four-dimensional eigenspace, compute the 4×4-matrix $\mathcal{M}_{a,b}$ representing the action of $H_{a,b}$ on this basis, and finally show that the eigenvalues of this matrix have the desired property. When $a = b = 0$ we choose again the basis

$$\xi_{0,0}^{(0)} = \begin{pmatrix} \sin z \\ \cos z \end{pmatrix} , \quad \xi_{0,0}^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \xi_{0,0}^{(2)} = \begin{pmatrix} \cos z \\ -\sin z \end{pmatrix} , \quad \xi_{0,0}^{(3)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

For $(a,b) \neq 0$, the fact that $H_{a,b}$ commutes with the symmetry $\mathcal{S}$ allows us to choose the vectors in the basis such that

$$\mathcal{S}\xi_{a,b}^{(0)} = -\xi_{a,b}^{(0)}, \quad \mathcal{S}\xi_{a,b}^{(1)} = -\xi_{a,b}^{(1)}, \quad \mathcal{S}\xi_{a,b}^{(2)} = \xi_{a,b}^{(2)}, \quad \mathcal{S}\xi_{a,b}^{(3)} = \xi_{a,b}^{(3)} ,$$

and to conclude that the matrix $\mathcal{M}_{a,b}$ is of the form

$$\mathcal{M}_{a,b} = \begin{pmatrix} \mathbf{A}_2(a,b) & 0 \\ 0 & \mathbf{B}_2(a,b) \end{pmatrix} ,$$

where $\mathbf{A}_2(a,b)$ and $\mathbf{B}_2(a,b)$ are 2×2-matrices with coefficients of order $\mathcal{O}(a^2 + b^2)$.

Next, the two vectors $\partial_z Q_{a,b}$ and $\mathrm{i} Q_{a,b}$ in the kernel of $\mathcal{A}_{a,b}$ also belong to the kernel of $H_{a,b}$, so that we can take $\xi_{a,b}^{(j)} = \xi_{a,b,0}^{(j)}$, for $j = 0, 1$, where $\xi_{a,b,0}^{(j)}$ are as in from the proof of Proposition 4.8. Then $\mathbf{A}_2(a,b) = 0$, so that zero is a double eigenvalue: $\lambda_{a,b}^{(0)} = \lambda_{a,b}^{(1)} = 0$. The remaining vectors $\xi_{a,b}^{(j)}$, $j = 2, 3$, and the matrix $\mathbf{B}_2(a,b)$ are computed from the expansions of $k_{a,b}$, $p_{a,b}$, and $Q_{a,b}$. We find $\xi_{a,b}^{(j)} = \xi_{0,0}^{(j)} + \mathcal{O}(a^2 + b^2)$ for $j = 2, 3$, and

$$
\begin{aligned}
\mathbf{B}_2(a,b) &= \frac{1}{2\pi} \begin{pmatrix} \langle H_{a,b}\xi_{0,0}^{(2)}, \xi_{0,0}^{(2)} \rangle & \langle H_{a,b}\xi_{0,0}^{(2)}, \xi_{0,0}^{(3)} \rangle \\ \langle H_{a,b}\xi_{0,0}^{(3)}, \xi_{0,0}^{(2)} \rangle & \langle H_{a,b}\xi_{0,0}^{(3)}, \xi_{0,0}^{(3)} \rangle \end{pmatrix} + \mathcal{O}(|ab|(a^2 + b^2)) \\
&= \begin{pmatrix} 2a^2 & 4ab \\ 4ab & 2b^2 \end{pmatrix} + \mathcal{O}(|ab|(a^2 + b^2)) \ .
\end{aligned}
$$

Since the spectrum of $H_{a,b}$ is the same for all couples $(\pm a, \pm b)$, the determinant of $\mathbf{B}_2(a,b)$ is even in both a and b, which together with the above formula gives

$$
\det\left(\mathbf{B}_2(a,b)\right) = -12a^2 b^2 + \mathcal{O}(a^2 b^2(a^2 + b^2)) < 0 \ .
$$

This shows that $\lambda_{a,b}^{(2)} < 0 < \lambda_{a,b}^{(3)}$, for sufficiently small $\|(a,b)\|$, which concludes the proof. $\qquad\square$

Remark A.2 *We obtain the same result in the focusing case considered in Section 5, when the operator $H_{a,b}$ is given by*

$$
H_{a,b} = \mathcal{E}_{a,b}''(Q_{a,b}) = -4k_{a,b}^2 \partial_{zz} - 4\mathrm{i} p_{a,b} k_{a,b} \partial_z - (1 - p_{a,b}^2) - |Q_{a,b}|^2 - 2Q_{a,b} \otimes Q_{a,b} \ .
$$

The only difference in the proof is the expression of the matrix $\mathbf{B}_2(a,b)$ which is now

$$
\mathbf{B}_2(a,b) = \begin{pmatrix} -2a^2 & -4ab \\ -4ab & -2b^2 \end{pmatrix} + \mathcal{O}(|ab|(a^2 + b^2)) \ ,
$$

but has the same determinant $\det\left(\mathbf{B}_2(a,b)\right) = -12a^2 b^2 + \mathcal{O}(a^2 b^2(a^2 + b^2)) < 0$.

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