Three-Dimensional Stability of Burgers Vortices: the Low Reynolds Number Case.

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Abstract

In this paper we establish rigorously that the family of Burgers vortices of the three-dimensional Navier-Stokes equation is stable for small Reynolds numbers. More precisely, we prove that any solution whose initial condition is a small perturbation of a Burgers vortex will converge toward another Burgers vortex as time goes to infinity, and we give an explicit formula for computing the change in the circulation number (which characterizes the limiting vortex completely.) Our result is not restricted to the axisymmetric Burgers vortices, which have a simple analytic expression, but it applies to the whole family of non-axisymmetric vortices which are produced by a general uniaxial strain.

1 Introduction

Numerical simulations of turbulent flows have lead to the general conclusion that vortex tubes serve as important organizing structures for such flows – in the memorable phrase of [11] they form the "sinews of turbulence". After the discovery by Burgers [1] of the explicit vortex solutions of the three-dimensional Navier-Stokes equation which now bear his name, these solutions have been used to model various aspects of turbulent flows [19]. It was also observed in numerical computations of fluid flows that the vortex tubes present in these simulations usually did not exhibit the axial symmetry of the explicit Burgers solution, but rather an elliptical core region. This lead to a search for non-axisymmetric vortices [15], [11], [7]. While no rigorous proof of their existence was available until recently, perturbative calculations and extensive numerical simulations have lead to the expectation that stationary vortical solutions of the three-dimensional Navier-Stokes equation do exist for any Reynolds number and all values of the asymmetry parameter (which we define below) between zero and one.

When addressing the stability of Burgers vortices, it is very important to specify the class of allowed perturbations. If we consider just two-dimensional perturbations (i.e., perturbations which do not depend on the axial variable), then fairly complete answers are known. Robinson and Saffman [15] computed perturbatively the eigenvalues of the linearized operator at the Burgers vortex and proved its stability for sufficiently small Reynolds numbers. Numerical computations of these eigenvalues were performed by Prochazka and Pullin [12], and no instability was found up to $Re = 10^4$. A similar conclusion was drawn for non-symmetric vortices [13]. The first mathematical work is [5], where we proved that the axisymmetric Burgers vortex is *globally stable* with respect to integrable, two-dimensional perturbations, for any value of the Reynolds number. Decay rates in time of spatially localized perturbations were also computed, explaining partially the numerical results of [12]. Building on this work the existence and local stability of slightly asymmetric vortices with respect to two dimensional perturbations was proved in [4] for arbitrary Reynolds numbers.

The stability issue is much more difficult if we allow for perturbations which depend on the axial variable too, and very few results have been obtained so far in this truly three-dimensional case. One early study by Leibovich and Holmes [8] concluded that one could not prove global stability for any Reynolds number solely by means of energy methods. Using a kind of Fourier expansion in the axial variable, Rossi and Le Dizès [16] showed that the point spectrum of the linearized operator is associated with purely two-dimensional perturbations. Crowdy [2] obtained a formal asymptotic expansion of the eigenfunctions in the axial variable. In an important recent work, Schmid and Rossi [18] rewrote the linearized equations in a form which allowed them to compute numerically the evolution of various Fourier modes, from which they concluded that eventually all perturbative modes will be damped out.

In this paper we address rigorously the existence of non-axisymmetric vortices and the stability with respect to three-dimensional perturbations of both the symmetric and non-symmetric vortex solutions. More precisely we first prove that, for all values of the asymmetry parameter between zero and one, non-axisymmetric vortices exist at least for small Reynolds numbers. Then we show that this family of vortex solutions is, in the language of dynamical systems theory, *asymptotically stable with shift*. That is to say, if we take initial conditions that are small perturbations of a vortex solution, the resulting solution of the Navier-Stokes equation will converge toward a vortex solution, but not, in general, the one which we initially perturbed. We also give a formula for computing the limiting vortex toward which the solution converges.

We now state our results more precisely. The three-dimensional Navier-Stokes equation for an incompressible fluid with constant density $\bar{\rho}$ and kinematic viscosity ν is the partial differential equation:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} \,=\, \nu\Delta\mathbf{u} - \frac{1}{\bar{\rho}}\nabla p\;, \quad \nabla\cdot\mathbf{u} \,=\, 0\;. \tag{1}$$

Here $\mathbf{u}(x,t)$ is the velocity of the fluid and $p(x,t)$ its pressure. Equation (1) will be considered in the whole space $\mathbf{R}^3$. Burgers vortices are particular solutions of (1) which are perturbations of the background straining flow

$$\mathbf{u}^s(x) \;=\; \begin{pmatrix} \gamma_1 x_1 \\ \gamma_2 x_2 \\ \gamma_3 x_3 \end{pmatrix} \;, \quad p^s(x) \;=\; -\frac{1}{2}\bar{\rho}(\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + \gamma_3^2 x_3^2) \;, \tag{2}$$

where $\gamma_1, \gamma_2, \gamma_3$ are real constants satisfying $\gamma_1 + \gamma_2 + \gamma_3 = 0$. We restrict ourselves to the case of an *axial strain* aligned with the vertical axis, namely we assume $\gamma_1, \gamma_2 < 0$ and $\gamma_3 > 0$. To be specific, we set

$$\gamma_1 \;=\; -\frac{\gamma}{2}(1+\lambda) \;, \quad \gamma_2 \;=\; -\frac{\gamma}{2}(1-\lambda) \;, \quad \gamma_3 \;=\; \gamma \;, \tag{3}$$

where $\gamma > 0$ measures the *intensity* and $\lambda \in [0,1)$ the *asymmetry* of the strain.

At this point, it is convenient to rewrite the Navier-Stokes equation in non-dimensional form. This will simplify the forthcoming expressions, at the expense of eliminating the physical parameters $\nu, \bar{\rho}, \gamma$. We thus replace the variables x, t and the functions $\mathbf{u}, p$ with the dimensionless quantities

$$\tilde{x} \;=\; \Bigl(\frac{\gamma}{\nu}\Bigr)^{1/2} x \;, \quad \tilde{t} \;=\; \gamma t \;, \quad \tilde{\mathbf{u}} \;=\; \frac{\mathbf{u}}{(\gamma\nu)^{1/2}} \;, \quad \tilde{p} \;=\; \frac{p}{\bar{\rho}\gamma\nu} \;.$$

Dropping the tildes for simplicity, we see that the new functions $\mathbf{u}, p$ satisfy the Navier-Stokes equation (1) with $\nu = \bar{\rho} = 1$. Similarly the new straining flow $\mathbf{u}^s$ is given by (2), (3) with $\gamma = 1$.

Setting $\mathbf{u} = \mathbf{u}^s + \mathbf{U}$ and replacing into (1), we obtain the following evolution equation for the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{U}$:

$$\partial_t \mathbf{\Omega} + (\mathbf{U}\cdot\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\mathbf{U} + (\mathbf{u}^s\cdot\nabla)\mathbf{\Omega} - (\mathbf{\Omega}\cdot\nabla)\mathbf{u}^s \;=\; \Delta\mathbf{\Omega} \;, \quad \nabla\cdot\mathbf{\Omega} \;=\; 0 \;. \tag{4}$$

Under reasonable assumptions which will be satisfied for the solutions we consider, the rotational part $\mathbf{U}$ of the velocity can be recovered from the vorticity $\mathbf{\Omega}$ by means of the Biot-Savart law:

$$\mathbf{U}(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{(\mathbf{x}-\mathbf{y})\times\mathbf{\Omega}(y)}{|x-y|^3}\,\mathrm{d}y \;, \quad x \in \mathbf{R}^3 \;. \tag{5}$$

In the axisymmetric case $\lambda = 0$, it is well-known [1] that (4) has a family of explicit stationary solutions of the form $\mathbf{\Omega} = \rho\hat{\mathbf{\Omega}}^B$, where $\rho \in \mathbf{R}$ is a parameter and

$$\hat{\mathbf{\Omega}}^B(x_\perp) \;=\; \begin{pmatrix} 0 \\ 0 \\ \hat{\Omega}^B(x_\perp) \end{pmatrix} \;, \quad \hat{\Omega}^B(x_\perp) \;=\; \frac{1}{4\pi}\, e^{-|x_\perp|^2/4} \;. \tag{6}$$

Here $x_\perp = (x_1, x_2)$ and $|x_\perp|^2 = x_1^2 + x_2^2$. The velocity field corresponding to $\rho\hat{\mathbf{\Omega}}^B$ is $\rho\hat{\mathbf{U}}^B$, where

$$\hat{\mathbf{U}}^B(x_\perp) \;=\; \frac{1}{2\pi}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \frac{1}{|x_\perp|^2}\Bigl(1 - e^{-|x_\perp|^2/4}\Bigr) \;. \tag{7}$$

These solutions are called the *axisymmetric Burgers vortices*. Observe that $\hat{\Omega}^B$ has been normalized so that its integral over $x_\perp \in \mathbf{R}^2$ is equal to one. It follows that ρ coincides with the integral of $\rho\hat{\Omega}^B$ over $\mathbf{R}^2$, or equivalently with the circulation of the velocity field $\rho\hat{\mathbf{U}}^B$ at infinity (in the horizontal plane $x_3 = 0$). We shall thus refer to the parameter ρ as the *circulation* of the Burgers vortex $\rho\hat{\mathbf{U}}^B$. Following [11], we also define the associated *Reynolds number* as $R = |\rho|$.

Burgers vortices also exist in the asymmetric case $\lambda \in (0,1)$, although no explicit formulas are known [15], [11], [13]. As in the symmetric case, there is in fact a family of vortices (for each value of λ) parametrized by the circulation ρ, but when $\lambda > 0$ these solutions are *not* just multiples of one another. In this paper, we restrict ourselves to small Reynolds numbers, in which case the existence of asymmetric Burgers vortices can be rigorously established by a simple perturbation argument. A complementary result is obtained in [4] where we prove that, if $\lambda > 0$ is sufficiently small, non-axisymmetric vortex solutions exist for *all* values of the Reynolds number.

Before stating our existence result, we introduce the function space in which the asymmetric Burgers vortices will be constructed. Let $b : \mathbf{R}^2 \to \mathbf{R}_+$ be the weight function

$$b(x_\perp) \;=\; (1+x_1^2)^{1/2}(1+x_2^2)^{1/2}\;, \quad x_\perp = (x_1,x_2) \in \mathbf{R}^2\;. \tag{8}$$

Given any $m \ge 0$, we define $L^2(m) \;=\; \{\omega : \mathbf{R}^2 \to \mathbf{R} \,|\, \|\omega\|_{L^2(m)} < \infty\}$, where

$$\|\omega\|_{L^2(m)}^2 \;=\; \int_{\mathbf{R}^2} b(x_\perp)^{2m}|\omega(x_\perp)|^2 \,\mathrm{d}x_\perp\;. \tag{9}$$

In other words, a function ω belongs to $L^2(m)$ if and only if ω, $|x_1|^m\omega$, $|x_2|^m\omega$, and $|x_1x_2|^m\omega$ are square integrable over $\mathbf{R}^2$. For later use, we observe that $L^2(m)$ is continuously embedded into $L^1(\mathbf{R}^2)$ if $m > 1/2$, i.e. there exists $C > 0$ such that $\|\omega\|_{L^1} \le C\|\omega\|_{L^2(m)}$ for all $\omega \in L^2(m)$.

Given $\lambda \in [0,1)$, we define

$$\mathcal{G}_\lambda(x_\perp) \;=\; \frac{\sqrt{1-\lambda^2}}{4\pi}\, e^{-\frac{1}{4}((1+\lambda)x_1^2+(1-\lambda)x_2^2)}\;, \quad x_\perp = (x_1,x_2) \in \mathbf{R}^2\;. \tag{10}$$

If $\lambda = 0$, then $\mathcal{G}_0 = \hat{\Omega}^B$ is just the vorticity field (6) of the symmetric Burgers vortex. As we show below, for any $\lambda \in (0,1)$, $\mathcal{G}_\lambda(x_\perp)$ is still the leading order approximation to the vorticity of a non-axisymmetric Burgers vortex, for small Reynolds number $|\rho|$. Our precise result is:

Theorem 1.1 (Existence of asymmetric Burgers vortices)
Fix $m > 3/2$, $\lambda \in [0,1)$, and assume that $(\gamma_1,\gamma_2,\gamma_3)$ is given by (3) with $\gamma = 1$. There exist $R_1(\lambda) > 0$ and $K_1(\lambda) > 0$ such that, for $|\rho| \le R_1$, the vorticity equation (4) has a stationary solution $\mathbf{\Omega}^B(x_\perp;\rho,\lambda)$ which satisfies

$$\mathbf{\Omega}^B(x_\perp;\rho,\lambda) \;=\; \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp;\rho,\lambda)\end{pmatrix}\;, \quad \int_{\mathbf{R}^2} \Omega^B(x_\perp;\rho,\lambda)\,\mathrm{d}x_\perp \;=\; \rho\;, \tag{11}$$

and

$$\|\Omega^B(\cdot;\rho,\lambda) - \rho\mathcal{G}_\lambda(\cdot)\|_{L^2(m)} \;\le\; K_1\rho^2\;. \tag{12}$$

Furthermore, $\Omega^B(\cdot;\rho,\lambda)$ is a smooth function of ρ and λ, and there is no other stationary solution of (4) of the form (11) satisfying $\|\Omega^B - \rho\mathcal{G}_\lambda\|_{L^2(m)} \le 2R_1$.

Remark 1.2 *The proof shows that $R_1(\lambda) \to 0$ and $K_1(\lambda) \to \infty$ as $\lambda \to 1$. On the other hand, $R_1(0) > 0$ and $K_1(\lambda) = \mathcal{O}(\lambda)$ as $\lambda \to 0$. In particular, setting $\lambda = 0$ in (12), we recover that $\Omega^B(\cdot;\rho,0) = \rho\mathcal{G}_0 = \rho\hat{\Omega}^B$.*

Remark 1.3 *Theorem 2 shows that the asymmetric Burgers vortex $\boldsymbol{\Omega}^B(x_\perp;\rho,\lambda)$ decays rapidly as $|x_\perp| \to \infty$, since the parameter $m > 3/2$ is arbitrary (note, however, that the constants R_1, K_1 depend on m). In fact, proceeding as in [4], it is possible to show that $\boldsymbol{\Omega}^B$ has a Gaussian decay as $|x_\perp| \to \infty$. Moreover, $\boldsymbol{\Omega}^B$ is also a smooth function of $x_\perp$, see Remark 2.3 below.*

The principal result of this paper concerns the evolution of solutions of (4) with initial conditions that are close to a (symmetric or non-symmetric) Burgers vortex. Unlike in much previous work the perturbations we consider do not merely depend on the transverse variables $x_\perp$, but also on x_3. We prove that any solution of (4) starting sufficiently close to the Burgers vortex with circulation ρ converges as $t \to +\infty$ toward a Burgers vortex with circulation ρ' close to ρ, and we give an explicit formula for computing the difference $\rho' - \rho$ in terms of the initial perturbation.

To measure the size of our perturbations, we introduce the three-dimensional analogue of the function space $L^2(m)$ that was used in the construction of asymmetric vortices. More precisely, our main space $X^2(m)$ will be the set of all $\omega : \mathbf{R}^3 \to \mathbf{R}$ such that $x_\perp \mapsto \omega(x_\perp, x_3) \in L^2(m)$ for all $x_3 \in \mathbf{R}$, and such that the map $x_3 \mapsto \omega(\cdot, x_3)$ is bounded and continuous from $\mathbf{R}$ into $L^2(m)$. As is easily verified, $X^2(m) \simeq C_b^0(\mathbf{R}, L^2(m))$ is a Banach space equipped with the norm

$$\|\omega\|_{X^2(m)} \;=\; \sup_{x_3\in\mathbf{R}} \|\omega(\cdot,x_3)\|_{L^2(m)} \;. \tag{13}$$

By definition, a perturbation $\omega \in X^2(m)$ has to decay with some algebraic rate as $|x_\perp| \to \infty$, but need only be bounded in the vertical direction x_3. The choice of such a function space reflects the properties of the Burgers vortex itself.

Remark 1.4 *If $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is a vector field whose components are elements of $X^2(m)$, we shall often write $\boldsymbol{\omega} \in X^2(m)$ instead of $\boldsymbol{\omega} \in X^2(m)^3$, and $\|\boldsymbol{\omega}\|_{X^2(m)}$ instead of $\|(\omega_1^2+\omega_2^2+\omega_3^2)^{1/2}\|_{X^2(m)}$. A similar abuse of notation will occur for other function spaces too.*

Consider initial conditions for the vorticity equation which are a perturbation of the Burgers vortex:

$$\boldsymbol{\Omega}^0(x) \;=\; \boldsymbol{\Omega}^B(x_\perp;\rho,\lambda) + \boldsymbol{\omega}^0(x) \;,$$

with $\rho \in \mathbf{R}$ and $\boldsymbol{\omega}^0 \in X^2(m)^3$. If $|\rho|$ and $\|\boldsymbol{\omega}^0\|_{X^2(m)}$ are sufficiently small, it will be shown that (4) has a unique global solution $\boldsymbol{\Omega}(x,t)$ with initial data $\boldsymbol{\Omega}^0(x)$ such that, for any $t \ge 0$,

$$\boldsymbol{\Omega}(x,t) \;=\; \boldsymbol{\Omega}^B(x_\perp;\rho,\lambda) + \boldsymbol{\omega}(x,t) \;, \tag{14}$$

with $\boldsymbol{\omega}(\cdot,t) \in X^2(m)^3$. If we define

$$\varphi(x_3,t) \;=\; \int_{\mathbf{R}^2} \omega_3(x_\perp,x_3,t)\,\mathrm{d}x_\perp \;, \quad x_3 \in \mathbf{R} \;, \quad t \ge 0 \;, \tag{15}$$

then a direct calculation shows that $\varphi(x_3,t)$ satisfies the remarkably simple equation

$$\partial_t\varphi + x_3\partial_3\varphi \;=\; \partial_3^2\varphi \;,$$

which can be solved explicitly, see (48) below. From the solution formula we see that $\varphi(x_3,t)$ converges uniformly on compact sets to the constant $\delta\rho$ as $t\to+\infty$, where

$$\delta\rho \;=\; \frac{1}{(2\pi)^{1/2}}\int_{\mathbf{R}} e^{-x_3^2/2}\varphi^0(x_3)\,\mathrm{d}x_3 \;=\; \frac{1}{(2\pi)^{1/2}}\int_{\mathbf{R}^3} e^{-x_3^2/2}\omega_3^0(x_\perp,x_3)\,\mathrm{d}x_\perp\,\mathrm{d}x_3 \;. \qquad (16)$$

This argument suggests that the solution $\mathbf{\Omega}(x,t)$ of (4) will not converge to the original vortex $\mathbf{\Omega}^B(\cdot;\rho,\lambda)$ as $t\to+\infty$, but to the modified vortex $\mathbf{\Omega}^B(\cdot;\rho+\delta\rho,\lambda)$, where $\delta\rho$ is given by (16). Moreover, we only expect to have uniform convergence on compact sets in the vertical variable x_3. This is precisely the content of our main result:

Theorem 1.5 (Stability of the family of Burgers vortices)
Fix $m>3/2$, $\lambda\in[0,1)$, and assume that $(\gamma_1,\gamma_2,\gamma_3)$ is given by (3) with $\gamma=1$. For any $\mu\in(0,\frac{1}{2}(1{-}\lambda))$, there exist $R_2(\lambda)>0$ and $\varepsilon_2(\lambda)>0$ such that, if $|\rho|\le R_2$ and if $\mathbf{\Omega}^0(x)=\mathbf{\Omega}^B(x_\perp;\rho,\lambda)+\boldsymbol{\omega}^0(x)$ with $\boldsymbol{\omega}^0\in X^2(m)^3$ satisfying

$$\|\boldsymbol{\omega}^0\|_{X^2(m)} + \lambda\|\partial_3\varphi^0\|_{L^\infty}^2 \;\le\; \varepsilon_2 \;,\quad \textit{where}\quad \varphi^0(x_3) \;=\; \int_{\mathbf{R}^2}\omega_3^0(x_\perp,x_3)\,\mathrm{d}x_\perp \;, \qquad (17)$$

then the solution $\mathbf{\Omega}(x,t)$ of (4) with initial data $\mathbf{\Omega}^0$ converges as $t\to+\infty$ to the vortex solution $\mathbf{\Omega}^B(x_\perp;\rho+\delta\rho,\lambda)$, where $\delta\rho$ is given by (16). More precisely, for any compact interval $I\subset\mathbf{R}$, we have

$$\sup_{x_3\in I}\|\mathbf{\Omega}(\cdot,x_3,t)-\mathbf{\Omega}^B(\cdot;\rho+\delta\rho,\lambda)\|_{L^2(m)} \;=\; \mathcal{O}(e^{-\mu t}) \;,\quad t\to+\infty \;. \qquad (18)$$

Theorem 1.5 is already new (and not really easier to prove) in the symmetric case $\lambda=0$. Note however that the assumptions on the initial data are less restrictive if $\lambda=0$, because no condition on $\partial_3\varphi^0$ is needed in (17). This is due to the fact that symmetric Burgers vortices with different circulations are just multiples of one another.

The proof of Theorem 1.5 uses ideas from our analysis of the stability of the two-dimensional Oseen vortices in [5]. The main observation is that, if we linearize equation (4) at the Burgers vortex $\mathbf{\Omega}^B(\cdot;\rho,\lambda)$ for small ρ, we obtain a small perturbation of a non-constant coefficient differential operator for which we can explicitly compute an integral representation of the associated semigroup. This semigroup decays exponentially when acting on functions $\boldsymbol{\omega}\in X^2(m)^3$ *provided* $\omega_3\in X_0^2(m)$, where

$$X_0^2(m) \;=\; \Bigl\{\omega\in X^2(m)\;\Big|\;\int_{\mathbf{R}^2}\omega(x_\perp,x_3)\,\mathrm{d}x_\perp=0\quad\text{for all } x_3\in\mathbf{R}\Bigr\}\;. \qquad (19)$$

Thus an important step in the proof consists in decomposing the solution (14) of (4) as

$$\mathbf{\Omega}(x,t) \;=\; \mathbf{\Omega}^B(x_\perp;\rho+\varphi(x_3,t),\lambda)+\tilde{\boldsymbol{\omega}}(x,t)\;,$$

where $\varphi(x_3,t)$ is as in (15). By construction, the new perturbation satisfies $\tilde{\omega}_3(\cdot,t)\in X_0^2(m)$ for all $t\ge0$, hence $\tilde{\boldsymbol{\omega}}(\cdot,t)$ will decay exponentially to zero by the remark above. Since in addition $\varphi(x_3,t)$ converges uniformly on compact sets toward the constant $\delta\rho$ as $t\to+\infty$, we obtain (18).

Remark 1.6 *It might seem more natural to use a more symmetric weight like* $\tilde{b}(x_\perp) = (1 + x_1^2 + x_2^2)^{1/2}$ *in defining our function spaces. This would also work and we used such weight functions in [3], [5]. However, as we remark in Section 4 below, an advantage of the choice (8) is that the space* $L^2(m)$ *is just a tensor product of two simpler spaces (consisting of functions of only one variable). Moreover, when we linearize the vorticity equation about the Burgers vortex, the linearized operator can also be written as a tensor product of one-dimensional operators. This observation allows to compute explicitly the spectrum and simplifies the analysis slightly.*

The Burgers vortices are not the only type of vortex solutions that exist in the three-dimensional Navier-Stokes equations. In fact, Lundgren [10] discovered a general transformation relating any two-dimensional Navier-Stokes flow to a particular three-dimensional flow, and he used this relation to construct "swirling vortices" which can be used to model the Kolmogorov energy spectrum for turbulent flows. These ideas were further generalized by Gibbon, Fokas and Doering [6] to construct more complicated stretched vortex solutions in which all three components of the vorticity field are non-zero (as opposed to the Burgers vortex which has only one component of the vorticity non-zero.) The question of whether or not these other types of vortex solutions are stable seems to be completely open and we think it would be interesting to investigate whether or not their stability can be determined using the methods developed here.

The rest of the text is organized as follows. In Section 2, we prove the existence of non-axisymmetric Burgers vortices for small Reynolds numbers. The core of the paper is Section 3, where we show that these families of vortices are asymptotically stable with shift. Section 4 is an appendix where we collect various estimates on the semigroup associated to the linearized vorticity equation, together with a few remarks concerning the Biot-Savart law.

2 Existence of non-axisymmetric Burgers vortices

The properties of non-axisymmetric Burgers vortices seem first to have been studied by Robinson and Saffman [15] who used perturbative methods to investigate their existence for small values of the Reynolds number. There were many further investigations in the intervening years - we mention particularly the perturbative study of the large Reynolds number limit of these vortices by Moffatt, Kida and Ohkitani [11], and the numerical work of Prochazka and Pullin [13]. However, as far as we know there has been no rigorous proof of the existence of these types of solutions and so in this section we present a simple argument which proves the existence of non-symmetric vortices in the case of small Reynolds number.

Fix $\lambda \in [0, 1)$ and assume that $\gamma_1, \gamma_2, \gamma_3$ are given by (3) with $\gamma = 1$. Motivated by the perturbative calculations of [15] we look for stationary solutions of (4) of the form

$$\mathbf{\Omega}^B(x_\perp) \,=\, \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp) \end{pmatrix} \,, \qquad \int_{\mathbf{R}^2} \Omega^B(x_\perp)\,\mathrm{d}x_\perp \,=\, \rho \,,$$

for some $\rho \in \mathbf{R}$ (recall that $|\rho|$ is the Reynolds number). Since $\mathbf{\Omega}^B$ depends only on the horizontal variable $x_\perp = (x_1, x_2)$ and has only the third component nonzero, the associated velocity field $\mathbf{U}^B$ depends only on $x_\perp$ and has only the first two components nonzero. Thus $\mathbf{U}^B$ is naturally identified with a two-dimensional velocity field $\bar{\mathbf{U}}^B$ which can be computed using the two-dimensional version of the Biot-Savart law:

$$\bar{\mathbf{U}}^B(x_\perp) \;=\; \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{|x_\perp - y_\perp|^2}\begin{pmatrix} y_2 - x_2 \\ x_1 - y_1\end{pmatrix}\Omega^B(y_\perp)\,\mathrm{d}y_\perp\;. \tag{20}$$

Inserting these expressions into (4), we see that Ω^B satisfies the scalar equation

$$\bar{\mathbf{U}}^B\cdot\nabla_\perp\Omega^B \;=\; (\mathcal{L}_\perp + \lambda\mathcal{M})\Omega^B\;, \tag{21}$$

where $\mathcal{L}_\perp$ and $\mathcal{M}$ are the differential operators

$$\mathcal{L}_\perp \;=\; \Delta_\perp + \frac{1}{2}(x_\perp\cdot\nabla_\perp) + 1\;,\quad \mathcal{M} \;=\; \frac{1}{2}(x_1\partial_1 - x_2\partial_2)\;. \tag{22}$$

Here we have used the natural notations $\nabla_\perp = (\partial_1,\partial_2)$ and $\Delta_\perp = \partial_1^2 + \partial_2^2$.

We shall solve (21) in the weighted space $L^2(m)$ defined by (9). Our approach rests on the fact that the spectrum of the linear operator $\mathcal{L}_\perp + \lambda\mathcal{M}$ in $L^2(m)$ can be explicitly computed, see Section 4.2. If $m > 1/2$, this operator turns out to be invertible on the invariant subspace $L^2_0(m)$ defined by

$$L^2_0(m) \;=\; \Big\{\omega\in L^2(m)\,\Big|\, \int_{\mathbf{R}^2}\omega(x_\perp)\,\mathrm{d}x_\perp = 0\Big\}\;. \tag{23}$$

This allows to rewrite (21) as a fixed point problem which is easily solved by a contraction argument.

As a preliminary step, let $\bar{\mathbf{V}}_\lambda(x_\perp)$ be the two-dimensional velocity field obtained from $\mathcal{G}_\lambda(x_\perp)$ by the Biot-Savart law (20). Using (10) and (22) one can easily verify that

$$(\mathcal{L}_\perp + \lambda\mathcal{M})\mathcal{G}_\lambda = 0\;,\quad\text{and}\quad \int_{\mathbf{R}^2}\mathcal{G}_\lambda(x_\perp)\,\mathrm{d}x_\perp = 1\;.$$

If we are given $\Omega^B\in L^2(m)$ with $m > 1/2$ and if $\rho = \int_{\mathbf{R}^2}\Omega^B\,\mathrm{d}x_\perp$, we can decompose

$$\Omega^B \;=\; \rho\mathcal{G}_\lambda + \omega\;,\quad \bar{\mathbf{U}}^B \;=\; \rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}}\;, \tag{24}$$

where $\omega\in L^2_0(m)$ and $\bar{\mathbf{u}}$ is the velocity obtained from ω by the Biot-Savart law (20). With these notations, finding a solution to (21) is equivalent to solving

$$(\mathcal{L}_\perp + \lambda\mathcal{M})\omega \;=\; (\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}})\cdot\nabla_\perp(\rho\mathcal{G}_\lambda + \omega)\;,\quad \omega\in L^2_0(m)\;. \tag{25}$$

Note that $(\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}})\cdot\nabla_\perp(\rho\mathcal{G}_\lambda + \omega) = \nabla_\perp\cdot((\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}})(\rho\mathcal{G}_\lambda + \omega))$ since $\bar{\mathbf{V}}_\lambda$ and $\bar{\mathbf{u}}$ are divergence-free. Thus the right-hand side of the (25) has zero mean as expected.

The next proposition ensures that the operator $\mathcal{L}_\perp + \lambda\mathcal{M}$ is invertible on $L^2_0(m)$ and that $(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1}\nabla_\perp$ defines a bounded operator from $L^p(m)$ into $L^2_0(m)$ if $p\in(1,2]$. Here $L^p(m)$ is the weighted L^p space defined in analogy with (9) by

$$L^p(m) \;=\; \{f\in L^p(\mathbf{R}^2)\,|\,b^m f\in L^p(\mathbf{R}^2)\}\;,\quad \|f\|_{L^p(m)} = \|b^m f\|_{L^p}\;.$$

Proposition 2.1 *Fix $m > 3/2$ and $\lambda \in [0,1)$. There exists $C(m,\lambda) > 0$ such that, for all $f \in L^2_0(m)$,*

$$\|(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1} f\|_{L^2(m)} \le C\|f\|_{L^2(m)} \ . \tag{26}$$

Moreover, if $p \in (1,2]$, there exists $C(m,\lambda,p) > 0$ such that, for all $g \in L^p(m)$,

$$\|(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1}\partial_i g\|_{L^2(m)} \le C\|g\|_{L^p(m)} \ , \quad i = 1,2 \ . \tag{27}$$

Proof: Let $\mathcal{T}_\lambda(t)$ denote the strongly continuous semigroup generated by $\mathcal{L}_\perp + \lambda\mathcal{M}$. In Section 4.2 we prove that $\mathcal{T}_\lambda(t)$ is exponentially contracting on $L^2_0(m)$. More precisely there exists $C > 0$ such that, if $f \in L^2_0(m)$,

$$\|\mathcal{T}_\lambda(t) f\|_{L^2(m)} \ \le\ C\, e^{-\frac{1}{2}(1-\lambda)t}\|f\|_{L^2(m)} \ , \quad t \ge 0 \ , \tag{28}$$

see (70). Thus the Laplace formula

$$(\mathcal{L}_\perp + \lambda\mathcal{M})^{-1} f \ =\ -\int_0^\infty \mathcal{T}_\lambda(t) f \,\mathrm{d}t \ , \quad f \in L^2_0(m) \ , \tag{29}$$

shows that $\mathcal{L}_\perp + \lambda\mathcal{M}$ is invertible on $L^2_0(m)$. Combining (28), (29), we easily obtain (26). The semigroup $\mathcal{T}_\lambda(t)$ is not analytic but it does possess some smoothing propoerties. In Section 4.2 we also prove that, if $f = \partial_i g$ for some $i \in \{1,2\}$ and some $g \in L^p(m)$, then

$$\|\mathcal{T}_\lambda(t)\partial_i g\|_{L^2(m)} \ \le\ Ca(t)^{-\frac{1}{p}}\, e^{-\frac{1}{2}(1-\lambda)t}\|g\|_{L^p(m)} \ , \quad t > 0 \ ,$$

where $a(t) = 1 - e^{-t}$, see (71) and (72). Since $p > 1$, the singularity at $t = 0$ in the right-hand side of this estimate is integrable, and we can again apply the Laplace formula to obtain (27). □

We now rewrite (25) as $\omega = F_{\lambda,\rho}(\omega)$, where $F_{\lambda,\rho} : L^2_0(m) \to L^2_0(m)$ is defined by

$$F_{\lambda,\rho}(\omega) \ =\ (\mathcal{L}_\perp + \lambda\mathcal{M})^{-1}\nabla_\perp \cdot ((\rho\bar{\mathbf{V}}_\lambda + \bar{\mathbf{u}}) \cdot (\rho\mathcal{G}_\lambda + \omega)) \ . \tag{30}$$

For any $r > 0$, let $B_m(0,r)$ denote the closed ball of radius r centered at the origin in $L^2_0(m)$. The main result of this section is:

Proposition 2.2 *Fix $m > 3/2$ and $\lambda \in [0,1)$. There exist $R_1(\lambda) > 0$ and $K_1(\lambda) > 0$ such that, if $|\rho| \le R_1$, then $F_{\lambda,\rho}$ has a unique fixed point $\omega_{\lambda,\rho}$ in $B_m(0,2R_1)$. Moreover $\omega_{\lambda,\rho}$ is contained in $B_m(0,K_1\rho^2)$ and $\omega_{\lambda,\rho}$ is a smooth function of both λ and ρ.*

Proof: Let $\mathcal{B}_\lambda : L^2(m) \times L^2(m) \to L^2_0(m)$ be the bilinear map defined by

$$\mathcal{B}_\lambda(\Omega_1,\Omega_2) \ =\ (\mathcal{L}_\perp + \lambda\mathcal{M})^{-1}\nabla_\perp \cdot (\bar{\mathbf{U}}_1\Omega_2) \ ,$$

where $\bar{\mathbf{U}}_1$ is the velocity field obtained from Ω_1 by the Biot-Savart law (20). If $p \in (1,2)$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, we know from Proposition 4.4 that $\|\mathbf{U}_1\|_{L^q} \le C\|\Omega_1\|_{L^p}$. This, combined with Hölder's inequality, readily implies that $\|\bar{\mathbf{U}}_1\Omega_2\|_{L^p(m)} \le C\|\Omega_1\|_{L^2(m)}\|\Omega_2\|_{L^2(m)}$, see Corollary 4.5 for more details. Using this estimate together with (27), we conclude that there exists $C_1(\lambda) > 0$ such that

$$\|\mathcal{B}_\lambda(\Omega_1,\Omega_2)\|_{L^2(m)} \ \le\ C_1(\lambda)\|\Omega_1\|_{L^2(m)}\|\Omega_2\|_{L^2(m)} \ , \quad \Omega_1,\Omega_2 \in L^2(m) \ .$$

Since $F_{\lambda,\rho}(\omega) = \mathcal{B}_\lambda(\rho\mathcal{G}_\lambda + \omega, \rho\mathcal{G}_\lambda + \omega)$, we obtain, for all $\omega \in L^2_0(m)$,

$$\|F_{\lambda,\rho}(\omega)\|_{L^2(m)} \,\le\, C_2(\lambda)\rho^2 + C_3(\lambda)(2|\rho|\|\omega\|_{L^2(m)} + \|\omega\|^2_{L^2(m)}) \;, \tag{31}$$

where $C_2(\lambda) = \|\mathcal{B}_\lambda(\mathcal{G}_\lambda, \mathcal{G}_\lambda)\|_{L^2(m)}$ and $C_3(\lambda) = C_1(\lambda)\max(1, \|\mathcal{G}_\lambda\|_{L^2(m)})$. Similarly, for all $\omega_1, \omega_2 \in L^2_0(m)$,

$$\|F_{\lambda,\rho}(\omega_1) - F_{\lambda,\rho}(\omega_2)\|_{L^2(m)} \,\le\, C_3(\lambda)\|\omega_1 - \omega_2\|_{L^2(m)}(2|\rho| + \|\omega_1\|_{L^2(m)} + \|\omega_2\|_{L^2(m)}) \;. \tag{32}$$

When $\lambda = 0$, $\mathcal{G}_0$ is radially symmetric and $\bar{\mathbf{V}}_0$ is azimuthal, hence $\bar{\mathbf{V}}_0 \cdot \nabla_\perp \mathcal{G}_0 = 0$. Thus $C_2(0) = 0$, so that $C_2(\lambda) = \mathcal{O}(\lambda)$ as $\lambda \to 0$.

Now, choose $R_1 > 0$ sufficiently small so that

$$C_2 R_1 \le 1 \;, \quad \text{and} \quad 8C_3R_1 \,\le\, 1 \;.$$

If $|\rho| \le R_1$ and $2C_2\rho^2 \le r \le 2R_1$, estimates (31) and (32) imply that $F_{\lambda,\rho}$ maps the ball $B_m(0,r)$ into itself and is a strict contraction there. More precisely, if $\omega_1, \omega_2 \in B_m(0,r)$, then

$$\|F_{\lambda,\rho}(\omega_1)\|_{L^2(m)} \,\le\, r \;, \quad \text{and} \quad \|F_{\lambda,\rho}(\omega_1) - F_{\lambda,\rho}(\omega_2)\|_{L^2(m)} \,\le\, \frac{3}{4}\|\omega_1 - \omega_2\|_{L^2(m)} \;.$$

By the contraction mapping theorem, $F_{\lambda,\rho}$ has a unique fixed point $\omega_{\lambda,\rho}$ in $B_m(0,r)$. Choosing $r = 2R_1$, we obtain the existence and uniqueness claim in Proposition 2.2. Then setting $r = K_1\rho^2$ with $K_1 = 2C_2$, we see that $\omega_{\lambda,\rho} \in B_m(0, K_1\rho^2)$. Finally, the smoothness property is a immediate consequence of the implicit function theorem. Indeed, the map $(\omega, \lambda, \rho) \mapsto F_{\lambda,\rho}(\omega)$ is obviously C^∞ from $L^2_0(m) \times [0,1) \times \mathbf{R}$ into $L^2_0(m)$, and the partial differential

$$D_\omega F_{\lambda,\rho} \,=\, \tilde{\omega} \mapsto \mathcal{B}_\lambda(\tilde{\omega}, \rho\mathcal{G}_\lambda + \omega) + \mathcal{B}_\lambda(\rho\mathcal{G}_\lambda + \omega, \tilde{\omega})$$

satisfies $\|D_\omega F_{\lambda,\rho}\| \le 3/4$ whenever $|\rho| \le R_1$ and $\omega \in B_m(0, 2R_1)$. Thus $\mathbf{1} - D_\omega F_{\lambda,\rho}$ is invertible at $\omega = \omega_{\lambda,\rho}$, and the implicit function theorem implies that $\omega_{\lambda,\rho}$ is a smooth function of both λ and ρ. □

Theorem 1.1 is a direct consequence of Proposition 2.2. Indeed, if $|\rho| \le R_1(\lambda)$, we set $\Omega^B(x_\perp; \rho, \lambda) = \rho\mathcal{G}_\lambda(x_\perp) + \omega_{\lambda,\rho}(x_\perp)$, where $\omega_{\lambda,\rho}$ is as in Proposition 2.2, and we denote by $\bar{\mathbf{U}}^B(x_\perp; \rho, \lambda)$ the two-dimensional velocity field obtained from Ω^B by the Biot-Savart law (20). Then

$$\boldsymbol{\Omega}^B(x_\perp; \rho, \lambda) \,=\, \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp; \rho, \lambda) \end{pmatrix} \;, \quad \mathbf{U}^B(x_\perp; \rho, \lambda) \,=\, \begin{pmatrix} \bar{U}^B_1(x_\perp; \rho, \lambda) \\ \bar{U}^B_2(x_\perp; \rho, \lambda) \\ 0 \end{pmatrix} \tag{33}$$

is a stationary solution of (4) which has all the desired properties. In particular, since $\omega_{\lambda,\rho} \in L^2_0(m)$, we have

$$\int_{\mathbf{R}^2} \Omega^B(x_\perp; \rho, \lambda)\,\mathrm{d}x_\perp \,=\, \rho \;, \tag{34}$$

while the fact that $\omega_{\lambda,\rho} \in B_m(0, K_1\rho^2)$ implies that (12) holds. For later use, we observe that there exists $C(\lambda, m) > 0$ such that, for $|\rho| \le R_1$,

$$\|\Omega^B(\cdot; \rho, \lambda)\|_{L^2(m)} \le C|\rho| \;, \quad \text{and} \quad \|\partial_\rho\Omega^B(\cdot; \rho, \lambda)\|_{L^2(m)} \le C \;. \tag{35}$$

Moreover $\|\partial^2_\rho\Omega^B(\cdot; \rho, \lambda)\|_{L^2(m)} \le C\lambda$, because in the symmetric case $\Omega^B(\cdot; \rho, 0) = \rho\mathcal{G}_0$ so that $\partial^2_\rho\Omega^B(\cdot; \rho, 0) = 0$.

Remark 2.3 *We chose to solve (21) in $L^2(m)$ because this is basically the space we shall use in Section 3 to study the stability of the vortices. But it is clear from the proof of Proposition 2.2 that nothing important changes if we replace $L^2(m)$ with the corresponding Sobolev space*

$$H^k(m) \;=\; \left\{ f \in L^2(m) \,\middle|\, \partial_1^i \partial_2^j f \in L^2(m) \text{ for all } i,j \in \mathbf{N} \text{ with } i+j \le k \right\} ,$$

for any $k \in \mathbf{N}$. This shows that the asymmetric Burgers vortex $\Omega^B(x_\perp;\rho,\lambda)$ is a smooth function of $x_\perp$ too. In particular, by Sobolev embedding, $b^m\Omega^B(\cdot;\rho,\lambda) \in C_b^0(\mathbf{R}^2)$ (the space of all continuous and bounded functions on $\mathbf{R}^2$) and we have the analogue of (35):

$$\sup_{x_\perp\in\mathbf{R}^2} b(x_\perp)^m|\Omega^B(x_\perp;\rho,\lambda)| \le C|\rho| \;, \qquad \sup_{x_\perp\in\mathbf{R}^2} b(x_\perp)^m|\partial_\rho\Omega^B(x_\perp;\rho,\lambda)| \le C \;. \tag{36}$$

Moreover, since $\Omega^B(\cdot;\rho,\lambda) \in L^p(\mathbf{R}^2)$ for all $p \in [1,\infty]$, Proposition 4.4 implies that $\mathbf{U}(\cdot;\rho,\lambda) \in L^q(\mathbf{R}^2) \cap C_b^0(\mathbf{R}^2)$ for all $q \in (2,\infty]$, and there exists $C(q,m,\lambda) > 0$ such that

$$\|\mathbf{U}^B(\cdot;\rho,\lambda)\|_{L^q(\mathbf{R}^2)} \le C|\rho| \;, \quad \text{and} \quad \|\partial_\rho\mathbf{U}^B(\cdot;\rho,\lambda)\|_{L^q(\mathbf{R}^2)} \le C \;. \tag{37}$$

3 Stability with respect to three-dimensional perturbations

We now prove that the family of vortices constructed in the previous section is asymptotically stable with shift, provided the Reynolds number is sufficiently small, depending on the asymmetry parameter $\lambda \in [0,1)$. In particular, our result applies to the classical family of symmetric Burgers vortices ($\lambda = 0$).

Throughout this section we fix some $\lambda \in [0,1)$. For $|\rho|$ sufficiently small we denote by $\mathbf{\Omega}^B(x_\perp;\rho)$, $\mathbf{U}^B(x_\perp;\rho)$ the asymmetric vortex (33) with circulation ρ (to simplify the notation, we omit the dependence on λ). As we explained in the introduction, if we slightly perturb the vortex $\mathbf{\Omega}^B(\cdot;\rho)$ the solution of the vorticity equation will converge toward another vortex with a possibly different circulation. This means that we must allow the parameter ρ to depend on time. Also, since the perturbations we consider may depend on the axial variable x_3, it turns out to be convenient to approximate the solutions by vortices with different circulation numbers in different x_3 cross-sections. In other words, we will consider solutions of (4) of the form

$$\mathbf{\Omega}(x,t) \;=\; \begin{pmatrix} 0 \\ 0 \\ \Omega^B(x_\perp;\rho+\varphi(x_3,t)) \end{pmatrix} + \begin{pmatrix} \omega_1(x,t) \\ \omega_2(x,t) \\ \omega_3(x,t) \end{pmatrix} \;, \tag{38}$$

where $\varphi(x_3,t)$ is determined so that $\int_{\mathbf{R}^2}\omega_3(x_\perp,x_3,t)\,\mathrm{d}x_\perp = 0$ for all x_3 and t. In view of (34), it is obvious that any pertubation of $\mathbf{\Omega}^B(\cdot;\rho)$ that is integrable with respect to the transverse variables $x_\perp$ can be decomposed in a unique way as in (38). Similarly, we write the rotational part of the velocity field as

$$\mathbf{U}(x,t) = \begin{pmatrix} \tilde U_1^B(x,t;\rho,\varphi) \\ \tilde U_2^B(x,t;\rho,\varphi) \\ 0 \end{pmatrix} + \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix} \;, \tag{39}$$

where $\tilde{\mathbf{U}}^B(x,t;\rho,\varphi)$ is the velocity field obtained from the vorticity $\mathbf{\Omega}^B(x_\perp;\rho+\varphi(x_3,t))$ by the Biot-Savart law (5). It will be shown in Proposition 4.11 that $\tilde{\mathbf{U}}^B(x,t;\rho,\varphi)$ is a small perturbation of $\mathbf{U}^B(x_\perp;\rho+\varphi(x_3,t))$ if φ varies slowly in the x_3 direction, namely there exists $C(\lambda)>0$ such that

$$\sup_{x\in\mathbf{R}^3} |\tilde{\mathbf{U}}^B(x,t;\rho,\varphi) - \mathbf{U}^B(x_\perp;\rho+\varphi(x_3,t))| \;\le\; C\|\partial_3\varphi(\cdot,t)\|_{L^\infty}\ . \tag{40}$$

Remark 3.1 *Due to the similarity in the notation it is perhaps worth emphasizing the difference between $\mathbf{U}^B(x_\perp;\rho+\varphi(x_3,t))$ and $\tilde{\mathbf{U}}^B(x,t;\rho,\varphi)$. First, $\mathbf{U}^B(x_\perp;\rho+\varphi(x_3,t))$ is the vector field which, for each fixed value of x_3, agrees with the velocity field of the (asymmetric) Burgers vortex with circulation $\rho+\varphi(x_3,t)$. On the other hand, $\tilde{\mathbf{U}}^B(x,t;\rho,\varphi)$ is the velocity field constructed from the vorticity $\mathbf{\Omega}^B(\cdot;\rho+\varphi(x_3,t))$ via the* three-dimensional *Biot-Savart law (5). If $\varphi(x_3,t)$ is independent of x_3, then $\mathbf{U}^B$ and $\tilde{\mathbf{U}}^B$ coincide (as implied by (40)): in this case, the integral over the vertical variable y_3 in the Biot-Savart law can be performed explicitly, and (5) reduces to (20). However, when φ varies with x_3, as it does in general if we enforce the condition that $\int \omega_3(x_\perp,x_3,t)dx_\perp = 0$, then $\mathbf{U}^B$ and $\tilde{\mathbf{U}}^B$ can no longer be expected to coincide.*

Let $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)^T$ and $\mathbf{u} = (u_1,u_2,u_3)^T$ denote the remainder terms in (38) and (39) respectively. By construction, $\mathbf{u}$ is the velocity field obtained from $\boldsymbol{\omega}$ by the Biot-Savart law (5). Remark that $\nabla\cdot\mathbf{u}=0$, but $\nabla\cdot\boldsymbol{\omega} = -(\partial_\rho\Omega^B)\partial_3\varphi \neq 0$, hence $\boldsymbol{\omega}\neq\nabla\times\mathbf{u}$. In broadest terms, our strategy is to show that $\boldsymbol{\omega}(x,t)$ and $\partial_3\varphi(x,t)$ converge to zero as time tends to infinity, so that the vorticity $\mathbf{\Omega}(x,t)$ approaches one of the vortices $\mathbf{\Omega}^B(\cdot;\rho')$ constructed in Section 2. With that in mind, we now write out the evolution equations for $\boldsymbol{\omega}$ and φ.

Inserting (38), (39) into (4) and using the identity $(\mathbf{U}\cdot\nabla)\mathbf{\Omega}-(\mathbf{\Omega}\cdot\nabla)\mathbf{U} = \nabla\times(\mathbf{\Omega}\times\mathbf{U})$, we obtain after straightforward calculations:

$$\partial_t\boldsymbol{\omega} \;=\; \mathbb{L}\boldsymbol{\omega} + \mathbb{P}_\varphi\boldsymbol{\omega} + \mathbb{N}(\boldsymbol{\omega}) + \mathbb{H}(\varphi)\ , \tag{41}$$

where the various terms in the right-hand side are defined as follows.

• The linear operator $\mathbb{L}$ is the leading order part of the equation, which takes into account the diffusion and the effects of the background strain:

$$\mathbb{L}\boldsymbol{\omega} \;=\; \Delta\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}^s - (\mathbf{u}^s\cdot\nabla)\boldsymbol{\omega} \;=\; \begin{pmatrix} (\mathcal{L}+\gamma_1)\omega_1 \\ (\mathcal{L}+\gamma_2)\omega_2 \\ (\mathcal{L}+\gamma_3)\omega_3 \end{pmatrix}\ .$$

Here $\gamma_1,\gamma_2,\gamma_3$ are given by (3) with $\gamma=1$, and

$$\mathcal{L} \;=\; \Delta - (\mathbf{u}^s\cdot\nabla) \;=\; \Delta + \frac{1}{2}(x_\perp\cdot\nabla_\perp) + \frac{\lambda}{2}(x_1\partial_1 - x_2\partial_2) - x_3\partial_3\ . \tag{42}$$

• The term $\mathbb{P}_\varphi\boldsymbol{\omega} = \nabla\times(\tilde{\mathbf{U}}^B\times\boldsymbol{\omega} + \mathbf{u}\times\mathbf{\Omega}^B)$ describes the linear interaction between the perturbation and the modulated vortex, namely:

$$\mathbb{P}_\varphi\boldsymbol{\omega} \;=\; \begin{pmatrix} \partial_2(\tilde{U}_1^B\omega_2 - \tilde{U}_2^B\omega_1) + \partial_3(\tilde{U}_1^B\omega_3 + u_1\Omega^B) \\ \partial_1(\tilde{U}_2^B\omega_1 - \tilde{U}_1^B\omega_2) + \partial_3(\tilde{U}_2^B\omega_3 + u_2\Omega^B) \\ -\partial_1(\tilde{U}_1^B\omega_3 + u_1\Omega^B) - \partial_2(\tilde{U}_2^B\omega_3 + u_2\Omega^B) \end{pmatrix}\ . \tag{43}$$

Here and in the sequel, to simplify the notation, we write $\mathbf{\Omega}^B$ instead of $\mathbf{\Omega}^B(\cdot;\rho+\varphi)$ and $\tilde{\mathbf{U}}^B$ instead of $\tilde{\mathbf{U}}^B(\cdot;\rho,\varphi)$.

• The term $\mathbb{N}(\boldsymbol{\omega}) = \nabla\times(\mathbf{u}\times\boldsymbol{\omega})$ collects all the nonlinear contributions in $\boldsymbol{\omega}$, specifically:

$$\mathbb{N}(\boldsymbol{\omega}) \;=\; \begin{pmatrix} \partial_2(u_1\omega_2 - u_2\omega_1) + \partial_3(u_1\omega_3 - u_3\omega_1) \\ \partial_1(u_2\omega_1 - u_1\omega_2) + \partial_3(u_2\omega_3 - u_3\omega_2) \\ \partial_1(u_3\omega_1 - u_1\omega_3) + \partial_2(u_3\omega_2 - u_2\omega_3) \end{pmatrix}~. \tag{44}$$

• Finally, $\mathbb{H}(\varphi) = \mathbb{L}\mathbf{\Omega}^B + \nabla\times(\tilde{\mathbf{U}}^B\times\mathbf{\Omega}^B) - \partial_t\mathbf{\Omega}^B$ is an inhomogeneous term which is due to the fact that $\mathbf{\Omega}^B(\cdot;\rho+\varphi)$ fails to be a solution of (4) if φ is not identically constant. A simple calculation shows that $\mathbb{H}_i(\varphi) = \partial_3(\tilde{U}_i^B\Omega^B)$ for $i=1,2$. The third component of $\mathbb{H}(\varphi)$ has a more complicated expression:

$$\begin{aligned}\mathbb{H}_3(\varphi) \;&=\; (\mathcal{L}_\perp + \lambda\mathcal{M})\Omega^B - \nabla_\perp\cdot(\tilde{\mathbf{U}}^B\Omega^B) + (\partial_\rho^2\Omega^B)(\partial_3\varphi)^2 \\ &\quad -(\partial_\rho\Omega^B)(\partial_t\varphi + x_3\partial_3\varphi - \partial_3^2\varphi)~,\end{aligned} \tag{45}$$

where $\mathcal{L}_\perp$ and $\mathcal{M}$ are defined in (22).

Equation (41) governs the evolution of both φ and $\boldsymbol{\omega}$. To separate out the evolution equation for φ, we recall that $\boldsymbol{\omega}$ satisfies the constraint $\int_{\mathbf{R}^2}\omega_3(x_\perp,x_3,t)\,\mathrm{d}x_\perp = 0$. If we integrate the third component of the vectorial equation (41) with respect to the transverse variables $x_\perp$, the first three terms in the right-hand side give no contribution, as can be seen from the formulas (42), (43), (44). So we must impose

$$\int_{\mathbf{R}^2}\mathbb{H}_3(\varphi)\,\mathrm{d}x_\perp \;=\; 0~, \quad \text{for all } x_3 \text{ and } t~. \tag{46}$$

As is clear from (21), the first term in the right-hand side of (45) has zero mean with respect to $x_\perp$, and so does the second term because it is explicitly in divergence form. On the other hand, differentiating (34) with respect to ρ, we obtain the identities $\int_{\mathbf{R}^2}\partial_\rho\Omega^B\,\mathrm{d}x_\perp = 1$ and $\int_{\mathbf{R}^2}\partial_\rho^2\Omega^B\,\mathrm{d}x_\perp = 0$. Thus (46) gives the evolution equation for φ:

$$\partial_t\varphi + x_3\partial_3\varphi \;=\; \partial_3^2\varphi~. \tag{47}$$

Remarkably, this equation is *linear* and completely *decoupled* from the rest of the system. As is easily verified, the solution of (47) with initial data $\varphi(x_3,0) = \varphi^0(x_3)$ is given by the explicit formula

$$\varphi(x_3,t) \;=\; (G_t * \varphi^0)(x_3e^{-t})~, \quad x_3\in\mathbf{R}~, \quad t>0~, \tag{48}$$

where

$$G_t(x_3) \;=\; \sqrt{\frac{1}{2\pi(1-e^{-2t})}}\,\exp\Bigl(-\frac{x_3^2}{2(1-e^{-2t})}\Bigr)~, \quad x_3\in\mathbf{R}~, \quad t>0~. \tag{49}$$

The following simple estimates will be useful:

Proposition 3.2 *If $\varphi^0 \in C_b^0(\mathbf{R})$, the solution of (47) with initial data φ^0 satisfies*

$$\|\varphi(\cdot,t)\|_{L^\infty} \;\le\; \|\varphi^0\|_{L^\infty}~, \quad \|\partial_3\varphi(\cdot,t)\|_{L^\infty} \;\le\; \frac{e^{-t}}{\sqrt{1-e^{-2t}}}\|\varphi^0\|_{L^\infty}~, \quad t>0~. \tag{50}$$

If moreover $\partial_3\varphi^0 \in L^\infty(\mathbf{R})$, we also have

$$\|\partial_3\varphi(\cdot,t)\|_{L^\infty} \;\le\; e^{-t}\|\partial_3\varphi^0\|_{L^\infty}~, \quad t\ge 0~. \tag{51}$$

Proof: Since $\|G_t\|_{L^1} = 1$, it follows immediately from (48) that $\|\varphi(\cdot,t)\|_{L^\infty} \le \|\varphi^0\|_{L^\infty}$. If $\partial_3\varphi^0 \in L^\infty(\mathbf{R})$, the same argument gives (51), because

$$\partial_3\varphi(x_3,t) \,=\, e^{-t}(G_t * \partial_3\varphi^0)(x_3 e^{-t}) \,=\, e^{-t}(\partial_3 G_t * \varphi^0)(x_3 e^{-t}) \;, \quad t > 0 \;. \tag{52}$$

To prove the second estimate in (50), we use the last expression in (52) and observe that $\|\partial_3 G_t\|_{L^1} = C/\sqrt{1-e^{-2t}}$, where $C = \sqrt{2/\pi} < 1$. □

Remark 3.3 *Proposition 3.2 shows in particular that (48) defines a semigroup of bounded linear operators on $C_b^0(\mathbf{R})$, the space of all bounded and continuous functions on $\mathbf{R}$ equipped with the L^∞ norm. It is easy to verify that this semigroup is* not strongly continuous *in time, due to the dilation factor e^{-t} in (48) which in turn originates in the unbounded advection term $x_3\partial_3$ in (47). However, if we equip $C_b^0(\mathbf{R})$ with the (weaker) topology of uniform convergence on compact sets, then (48) defines a continuous function of time.*

We now return to the evolution equation for $\boldsymbol{\omega}$. Using (45), equation (47) for φ, and equation (21) satisfied by the asymmetric vortex Ω^B, we obtain for the inhomogeneous term $\mathbb{H}(\varphi)$ the simpler expression

$$\mathbb{H}(\varphi) \,=\, \begin{pmatrix} \partial_3(\tilde U_1^B \Omega^B) \\ \partial_3(\tilde U_2^B \Omega^B) \\ \nabla_\perp \cdot ((\mathbf{U}^B - \tilde{\mathbf{U}}^B)\Omega^B) + (\partial_\rho^2 \Omega^B)(\partial_3\varphi)^2 \end{pmatrix} \;, \tag{53}$$

where as usual $\mathbf{U}^B = \mathbf{U}^B(\cdot;\rho+\varphi)$. Before starting the rigorous analysis, let us briefly comment here on why we expect solutions of (41) to go to zero as t goes to infinity. Given $m > 3/2$, we assume that $\omega_i \in X^2(m)$ for $i = 1,2,3$, where $X^2(m)$ is the space defined in (13). By construction, ω_3 then belongs to the subspace $X_0^2(m)$ given by (19). As we show in Section 4.3, the linear operator $\mathbb{L}$ has spectrum that lies in the half-plane $\{z \in \mathbf{C} \mid \mathrm{Re}\, z \le -\frac{1}{2}(1-\lambda)\}$ when acting on $X^2(m)\times X^2(m)\times X_0^2(m)$. Thus, the semigroup generated by this operator can be expected to decay like $\exp(-\frac{1}{2}(1-\lambda)t)$. The remaining linear terms in the equation, namely $\mathbb{P}_\varphi \omega$, contain a factor of either the velocity or the vorticity field of the vortex, and on the basis of (36) and (37) we expect this factor to be proportional to the circulation $\rho+\varphi$ of the vortex. Hence, for small Reynolds number, the operator $\mathbb{P}_\varphi$ will be a small perturbation of $\mathbb{L}$ and will not destroy the exponential decay. The same is true for the nonlinear terms $\mathbb{N}(\omega)$ provided we restrict ourselves to sufficiently small perturbations. Finally, the inhomogeneous term $\mathbb{H}(\varphi)$ decays at least like e^{-t} by (40) and Proposition 3.2, so we expect the solution $\boldsymbol{\omega}(x,t)$ of (41) to converge exponentially to zero if the initial data are sufficiently small.

We now put these heuristic arguments into a rigorous form. Let $\mathbb{X}(m)$ be the Banach space $X^2(m)\times X^2(m)\times X_0^2(m)$ equipped with the norm $\|\boldsymbol{\omega}\|_{\mathbb{X}(m)} = \|\omega_1\|_{X^2(m)} + \|\omega_2\|_{X^2(m)} + \|\omega_3\|_{X^2(m)}$. As is shown in Proposition 4.6, the linear operator $\boldsymbol{\mathcal{L}}$ is the generator of a semigroup $e^{t\boldsymbol{\mathcal{L}}}$ of bounded operators on $X^2(m)$, hence the same is true for the operator $\mathbb{L}$ acting on $\mathbb{X}(m)$. A natural idea is then to use Duhamel's formula to rewrite (41) as an integral equation:

$$\boldsymbol{\omega}(t) \,=\, e^{t\mathbb{L}}\boldsymbol{\omega}^0 + \int_0^t e^{(t-s)\mathbb{L}}\Big(\mathbb{P}_{\varphi(s)}\boldsymbol{\omega}(s) + \mathbb{N}(\boldsymbol{\omega}(s)) + \mathbb{H}(\varphi(s))\Big)\,\mathrm{d}s \;, \quad t \ge 0 \;, \tag{54}$$

which can then be solved by a fixed point argument. A problem with this approach is that the semigroup $e^{t\mathcal{L}}$ fails to be strongly continuous on $X^2(m)$, essentially for the reason mentioned in Remark 3.3. To restore continuity, it is thus necessary to equip $X^2(m)$ with a weaker topology. For any $n \in \mathbf{N}^*$ we define the seminorm

$$|\omega|_{X^2_n(m)} \;=\; \sup_{|x_3|\le n} \|\omega(\cdot,x_3)\|_{L^2(m)}~, \tag{55}$$

and we denote by $X^2_{\rm loc}(m)$ the space $X^2(m)$ equipped with the topology defined by the family of seminorms (55) for $n \in \mathbf{N}^*$, i.e. the topology of the Fréchet space $C^0(\mathbf{R}, L^2(m))$. In other words, a sequence ω_k converges to zero in $X^2_{\rm loc}(m)$ if and only if $|\omega_k|_{X^2_n(m)} \to 0$ as $k\to\infty$ for all $n \in \mathbf{N}^*$, namely if $\omega_k(x_3)$ converges to zero in $L^2(m)$ *uniformly on compact sets* in x_3. We define the product space $\mathbb{X}_{\rm loc}(m)$ in a similar way. Then Proposition 4.6 shows that the semigroup $e^{t\mathbb{L}}$ is strongly continuous on $\mathbb{X}_{\rm loc}(m)$, and the integrals in (54) can be defined as $\mathbb{X}_{\rm loc}(m)$-valued Riemann integrals, see Corollary 4.7 and Remark 4.8.

Since we expect $\boldsymbol{\omega}(t)$ to converge exponentially to zero as $t \to +\infty$, we shall solve (54) in the Banach space

$$\mathbb{Y}_\mu(m) \;=\; \{\boldsymbol{\omega} \in C^0([0,+\infty), \mathbb{X}_{\rm loc}(m)) \,|\, \|\boldsymbol{\omega}\|_{\mathbb{Y}_\mu(m)} < \infty\}~,$$

for some $\mu > 0$, where

$$\|\boldsymbol{\omega}\|_{\mathbb{Y}_\mu(m)} \;=\; \sup_{t\ge 0} e^{\mu t}\|\boldsymbol{\omega}(t)\|_{\mathbb{X}(m)}~.$$

Given initial data $\varphi^0 \in C^0_b(\mathbf{R})$ and $\boldsymbol{\omega}^0 \in \mathbb{X}(m)$, we first define $\varphi(x_3,t)$ by (48), and then use the integral equation (54) to determine $\boldsymbol{\omega}(t)$ for all $t \ge 0$. Our main result is:

Proposition 3.4 *Fix $\lambda \in [0,1)$, $m > 3/2$, and $0 < \mu < \frac{1}{2}(1{-}\lambda)$. There exist positive constants $\rho_2 > 0$, $\varepsilon_2 > 0$, and $K_2 > 0$ such that, if $|\rho| \le \rho_2$, $\varepsilon \le \varepsilon_2$, and if $\varphi^0 \in C^0_b(\mathbf{R})$ satisfies $\|\varphi^0\|_{L^\infty} + \lambda\|\partial_3\varphi^0\|^2_{L^\infty} \le \varepsilon$, then for all $\boldsymbol{\omega}^0 \in \mathbb{X}(m)$ with $\|\boldsymbol{\omega}^0\|_{\mathbb{X}(m)} \le \varepsilon$ equation (54) has a unique solution $\boldsymbol{\omega} \in \mathbb{Y}_\mu(m)$ with $\|\boldsymbol{\omega}\|_{Y_\mu(m)} \le K_2\varepsilon$.*

Proof: Fix $\lambda \in [0,1)$, $m > 3/2$, and $0 < \mu < \frac{1}{2}(1{-}\lambda)$. To simplify the notations, we shall write $\mathbb{X}$ instead of $\mathbb{X}(m)$ and $\mathbb{Y}$ instead of $\mathbb{Y}_\mu(m)$. For any $r > 0$, we denote by $B_X(0,r)$ (respectively, $B_Y(0,r)$) the closed ball of radius $r > 0$ centered at the origin in $\mathbb{X}$ (respectively, $\mathbb{Y}$). Let $\varphi^0 \in C^0_b(\mathbf{R})$ and denote by $\varphi(x_3,t)$ the solution of (47) with initial data φ^0. Given $\rho \in \mathbf{R}$, $\boldsymbol{\omega}^0 \in \mathbb{X}$, and $\boldsymbol{\omega} \in \mathbb{Y}$, we estimate the various terms in the right-hand side of (54).

We begin with the linear term $e^{t\mathbb{L}}\boldsymbol{\omega}^0$. From Proposition 4.6 we know that the linear operator $\mathcal{L}+1 \equiv \hat{\mathcal{L}}_{1+\lambda,1-\lambda}$ generates a semigroup $S_t = e^{t(\mathcal{L}+1)}$ which is strongly continuous on $X^2_{\rm loc}(m)$, uniformly bounded on $X^2(m)$, and which decays like $e^{-\frac{1}{2}(1-\lambda)t}$ on $X^2_0(m)$. Since

$$e^{t\mathbb{L}}\boldsymbol{\omega}^0 \;=\; \Bigl(e^{t(\mathcal{L}+\gamma_1)}\omega_1^0\,,\, e^{t(\mathcal{L}+\gamma_2)}\omega_2^0\,,\, e^{t(\mathcal{L}+\gamma_3)}\omega_3^0\Bigr)^T~,$$

where $\gamma_1,\gamma_2,\gamma_3$ are given by (3) with $\gamma = 1$, we deduce that $t \mapsto e^{t\mathbb{L}}\boldsymbol{\omega}^0$ is continuous in $\mathbb{X}_{\rm loc}$ and satisfies

$$\begin{aligned}\|e^{t\mathbb{L}}\boldsymbol{\omega}^0\|_{\mathbb{X}} &\le C\Bigl(e^{-\frac{3+\lambda}{2}t}\|\omega_1^0\|_{X^2(m)} + e^{-\frac{3-\lambda}{2}t}\|\omega_2^0\|_{X^2(m)} + e^{-\frac{1-\lambda}{2}t}\|\omega_3^0\|_{X^2(m)}\Bigr)\\ &\le C_1 e^{-\frac{1-\lambda}{2}t}\|\boldsymbol{\omega}^0\|_{\mathbb{X}}~.\end{aligned} \tag{56}$$

Note that it is crucial here that $\omega_3^0 \in X_0^2(m)$, otherwise we do not get any decay at all.

We next consider the linear term $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{P}_{\varphi(s)}\boldsymbol{\omega}(s)\,\mathrm{d}s$. We know from Proposition 4.6 that the linear operator $e^{t(\mathcal{L}+1)}\partial_k$ is bounded in $X^2(m)$ for all $t>0$ and all $k\in\{1,2,3\}$. Thus, in view of (43), it is sufficient to bound the products $\tilde U_i^B\omega_j$ and $u_i\Omega^B$ in $X^2(m)$. First, for $s\ge 0$ and $i\in\{1,2\}$, we know from Proposition 4.11 that $\tilde U_i^B(s)\equiv \tilde U_i^B(\cdot,s;\rho,\varphi(s))\in C_b^0(\mathbf{R}^3)$ and $\|\tilde U_i^B(s)\|_{L^\infty}\le C(|\rho|+\|\varphi(s)\|_{L^\infty})\le C(|\rho|+\|\varphi^0\|_{L^\infty})$, see Proposition 3.2. Since $\omega_j(s)\in X^2(m)$ for $j\in\{1,2,3\}$, it follows that $\tilde U_i^B(s)\omega_j(s)\in X^2(m)$ and

$$\|\tilde U_i^B(s)\omega_j(s)\|_{X^2(m)} \;\le\; \|\tilde U_i^B(s)\|_{L^\infty(\mathbf{R}^3)}\|\omega_j(s)\|_{X^2(m)} \;\le\; C\rho'\|\omega_j(s)\|_{X^2(m)}\;,$$

where $\rho'=|\rho|+\|\varphi^0\|_{L^\infty}$. Moreover, it is not difficult to verify that $s\mapsto \tilde U_i^B(s)\omega_j(s)$ is continuous in $X^2_{\mathrm{loc}}(m)$. Similarly, if $X^p(m)$ denotes the function space introduced in (75), we have $u_i(s)\in X^q(0)$ for $q\in(2,\infty)$ by Proposition 4.9 and $\Omega^B(s)\equiv\Omega^B(\cdot;\rho+\varphi(s))\in X^p(m)$ for $p\in[1,\infty]$ by Remark 2.3, hence $u_i(s)\Omega^B(s)\in X^2(m)$ by Hölder's inequality and

$$\|u_i(s)\Omega^B(s)\|_{X^2(m)} \;\le\; \|u_i(s)\|_{X^4(0)}\|\Omega^B(s)\|_{X^4(m)} \;\le\; C\rho'\|\omega_i(s)\|_{X^2(m)}\;.$$

Again $s\mapsto u_i(s)\Omega^B(s)$ is continuous in $X^2_{\mathrm{loc}}(m)$. Using these estimates, we are now able to define $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{P}_{\varphi(s)}\boldsymbol{\omega}(s)\,\mathrm{d}s$ as an $\mathbb{X}_{\mathrm{loc}}$-valued Riemann integral. Since $X^2_{\mathrm{loc}}(m)$ is not a Banach space, the construction of the integral is somewhat non-standard, but in Corollary 4.7 and Remark 4.8 we show that the three components of this vector are well defined and continuous in $X^2_{\mathrm{loc}}(m)$ for $t\ge 0$. Using the bounds on $e^{t(\mathcal{L}+1)}\partial_k$ given in Proposition 4.6, we can estimate the first component as follows:

$$\begin{aligned}
&\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_1)}\Big(\partial_2(\tilde U_1^B\omega_2-\tilde U_2^B\omega_1)+\partial_3(\tilde U_1^B\omega_3+u_1\Omega^B)\Big)(s)\,\mathrm{d}s\Big\|_{X^2(m)}\\
&\quad\le C\int_0^t \frac{e^{-2(t-s)}}{a(t-s)^{1/2}}\Big(\|\tilde U_1^B(s)\omega_2(s)\|_{X^2(m)}+\|\tilde U_2^B(s)\omega_1(s)\|_{X^2(m)}\Big)\,\mathrm{d}s\\
&\quad+C\int_0^t \frac{e^{-\frac{3+\lambda}{2}(t-s)}}{a(t-s)^{1/2}}\Big(\|\tilde U_1^B(s)\omega_3(s)\|_{X^2(m)}+\|u_1(s)\Omega^B(s)\|_{X^2(m)}\Big)\,\mathrm{d}s\;,
\end{aligned}$$

where $a(t)=1-e^{-t}$. (This estimate could be sharpened somewhat by using the functions $a_1(t)$, $a_2(t)$, and $1-e^{-2t}$ which appear in Proposition 4.6, but they would lead to no qualitative improvement in the final result and so we use this somewhat simpler form.) The other two components can be estimated in exactly the same way except for a slower exponential decay of the linear semigroup, see (56). Summarizing, we obtain:

$$\Big\|\int_0^t e^{(t-s)\mathbb{L}}\mathbb{P}_{\varphi(s)}\boldsymbol{\omega}(s)\,\mathrm{d}s\Big\|_{\mathbb{X}} \;\le\; C\rho'\int_0^t \frac{e^{-\frac{1-\lambda}{2}(t-s)}}{a(t-s)^{1/2}}\,\|\boldsymbol{\omega}(s)\|_{\mathbb{X}}\,\mathrm{d}s \;\le\; C_2\rho' e^{-\mu t}\|\boldsymbol{\omega}\|_{\mathbb{Y}}\;. \qquad (57)$$

We now consider the nonlinear term $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{N}(\boldsymbol{\omega}(s))\,\mathrm{d}s$. Let $1<p<2$. For $s\ge 0$ and $i,j\in\{1,2,3\}$, we know from Corollary 4.10 that $u_i(s)\omega_j(s)\in X^p(m)$ with

$$\|u_i(s)\omega_j(s)\|_{X^p(m)} \;\le\; C\|\omega_i(s)\|_{X^2(m)}\|\omega_j(s)\|_{X^2(m)}\;.$$

Moreover $s\mapsto u_i(s)\omega_j(s)$ is continuous in $X^p_{\mathrm{loc}}(m)$. Using again the definition and properties of integrals of $X^p_{loc}(m)$-valued functions (see Remark 4.8), we deduce that

$\int_0^t e^{(t-s)\mathbb{L}}\mathbb{N}(\boldsymbol{\omega}(s))\,\mathrm{d}s$ is well defined and continuous in $\mathbb{X}_{\mathrm{loc}}$ for $t \ge 0$. Proceeding as above we can estimate the first component as follows:

$$\begin{aligned}
&\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_1)}\Big(\partial_2(u_1\omega_2 - u_2\omega_1) + \partial_3(u_1\omega_3 - u_3\omega_1)\Big)(s)\,\mathrm{d}s\Big\|_{X^2(m)} \\
&\quad \le C\int_0^t \frac{e^{-2(t-s)}}{a(t-s)^{1/p}}\Big(\|u_1(s)\omega_2(s)\|_{X^p(m)} + \|u_2(s)\omega_1(s)\|_{X^p(m)}\Big)\,\mathrm{d}s \\
&\quad + C\int_0^t \frac{e^{-\frac{3+\lambda}{2}(t-s)}}{a(t-s)^{1/p}}\Big(\|u_1(s)\omega_3(s)\|_{X^p(m)} + \|u_3(s)\omega_1(s)\|_{X^p(m)}\Big)\,\mathrm{d}s\ .
\end{aligned}$$

Repeating the same arguments for the other two components, we thus find

$$\Big\|\int_0^t e^{(t-s)\mathbb{L}}\mathbb{N}(\boldsymbol{\omega}(s))\,\mathrm{d}s\Big\|_{\mathbb{X}} \le C\int_0^t \frac{e^{-\frac{1-\lambda}{2}(t-s)}}{a(t-s)^{1/p}}\,\|\boldsymbol{\omega}(s)\|_{\mathbb{X}}^2\,\mathrm{d}s \;\le\; C_3 e^{-\mu t}\|\boldsymbol{\omega}\|_{\mathbb{Y}}^2\ . \tag{58}$$

Finally, we turn our attention to the inhomogeneous term $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{H}(\varphi(s))\,\mathrm{d}s$. To bound the first two components, we observe that $\partial_3(\tilde{U}_i^B(s)\Omega^B(s)) \in X^2(m)$ with

$$\|\partial_3(\tilde{U}_i^B(s)\Omega^B(s))\|_{X^2(m)} \;\le\; C\rho'\|\partial_3\varphi(s)\|_{L^\infty}\ , \quad i = 1,2. \tag{59}$$

Indeed, $\partial_3(\tilde{U}_i^B\Omega^B) = \tilde{U}_i^B\partial_3\Omega^B + \Omega^B\partial_3\tilde{U}_i^B$. Since $\Omega^B = \Omega^B(x_\perp;\rho+\varphi(x_3,t))$, one has $\partial_3\Omega^B = (\partial_\rho\Omega^B)(\partial_3\varphi)$. The factor of $\partial_\rho\Omega^B$ is bounded by (35), while $\tilde{U}_i^B$ is bounded in (97) and we find

$$\|\tilde{U}_i^B\partial_3\Omega^B\|_{X^2(m)} \;\le\; \|\tilde{U}_i^B\|_{L^\infty}\|\partial_3\Omega^B\|_{X^2(m)} \;\le\; C\rho'\|\partial_3\varphi\|_{L^\infty}\ .$$

Moreover, as $\partial_3\tilde{\mathbf{U}}^B$ is the velocity field obtained from $\partial_3\boldsymbol{\Omega}^B = (\partial_\rho\boldsymbol{\Omega}^B)(\partial_3\varphi)$ by the Biot-Savart law, the proof of Proposition 4.11 shows that $\|\partial_3\tilde{U}_i^B\|_{L^\infty} \le C\|\partial_3\varphi\|_{L^\infty}$, hence

$$\|\Omega^B\partial_3\tilde{U}_i^B\|_{X^2(m)} \;\le\; \|\Omega^B\|_{X^2(m)}\|\partial_3\tilde{U}_i^B\|_{L^\infty} \;\le\; C\rho'\|\partial_3\varphi\|_{L^\infty}\ ,$$

which proves (59). As usual, one checks that $s \mapsto \partial_3(\tilde{U}_i^B(s)\Omega^B(s))$ is continuous in $X^2_{\mathrm{loc}}(m)$ for $s > 0$. Using (50), (59), the first component of the inhomogeneous term can be estimated as follows:

$$\begin{aligned}
\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_1)}\partial_3(\tilde{U}_1^B\Omega^B)(s)\,\mathrm{d}s\Big\|_{X^2(m)} &\le C\rho'\int_0^t e^{-\frac{3+\lambda}{2}(t-s)}\frac{e^{-s}}{a(s)^{1/2}}\|\varphi^0\|_{L^\infty}\,\mathrm{d}s \\
&\le Ce^{-\mu t}\rho'\|\varphi^0\|_{L^\infty}\ ,
\end{aligned}$$

and the second one is bounded in exactly the same way. To bound the third component, we first remark that $(U_i^B(s) - \tilde{U}_i^B(s))\Omega^B(s)$ belongs to $X^2(m)$ for $i = 1,2$ and depends continuously on $s > 0$ in $X^2_{\mathrm{loc}}(m)$. By (35), (40),

$$\|(U_i^B - \tilde{U}_i^B)\Omega^B\|_{X^2(m)} \;\le\; \|U_i^B - \tilde{U}_i^B\|_{L^\infty}\|\Omega^B\|_{X^2(m)} \;\le\; C\rho'\|\partial_3\varphi\|_{L^\infty}\ ,$$

hence using (50) we find

$$\begin{aligned}
&\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_3)}\nabla_\perp\cdot(\mathbf{U}^B(s) - \tilde{\mathbf{U}}^B(s))\Omega^B(s)\,\mathrm{d}s\Big\|_{X^2(m)} \\
&\qquad \le C\rho'\int_0^t \frac{e^{-\frac{1-\lambda}{2}(t-s)}}{a(t-s)^{1/2}}\frac{e^{-s}}{a(s)^{1/2}}\|\varphi^0\|_{L^\infty}\,\mathrm{d}s \;\le\; Ce^{-\mu t}\rho'\|\varphi^0\|_{L^\infty}\ .
\end{aligned}$$

On the other hand, $\partial_\rho^2\Omega^B(s)(\partial_3\varphi(s))^2$ lies in $X^2(m)$ and depends continuously on $s>0$ in $X^2_{\rm loc}(m)$. As was mentioned before Remark 2.3, $\|\partial_\rho^2\Omega^B(s)\|_{X^2(m)} \le C\lambda$. If $\lambda>0$, we assume that $\partial_3\varphi^0 \in L^\infty(\mathbf{R})$ and using (51) we obtain

$$\begin{aligned}\Big\|\int_0^t e^{(t-s)(\mathcal{L}+\gamma_3)}(\partial_\rho^2\Omega^B(s))(\partial_3\varphi(s))^2\,\mathrm{d}s\Big\|_{X^2(m)} &\le C\lambda\int_0^t e^{-\frac{1-\lambda}{2}(t-s)}e^{-2s}\|\partial_3\varphi^0\|^2_{L^\infty}\,\mathrm{d}s\\ &\le C\lambda e^{-\mu t}\|\partial_3\varphi^0\|^2_{L^\infty}\ .\end{aligned}$$

Thus we have shown that $\int_0^t e^{(t-s)\mathbb{L}}\mathbb{H}(\varphi(s))\,\mathrm{d}s$ is well defined and continuous in $\mathbb{X}_{\rm loc}$ for $t\ge 0$. Moreover,

$$\Big\|\int_0^t e^{(t-s)\mathbb{L}}\mathbb{H}(\varphi(s))\,\mathrm{d}s\Big\|_{\mathbb{X}} \le C_4 e^{-\mu t}\rho'\|\varphi^0\|_{L^\infty} + C_5\lambda e^{-\mu t}\|\partial_3\varphi^0\|^2_{L^\infty}\ . \tag{60}$$

Given $\boldsymbol{\omega}\in\mathbb{Y}$, we denote by $(\mathbb{F}\boldsymbol{\omega})(t)$ the right-hand side of (54). Estimates (56), (57), (58), (60) show that $t\mapsto(\mathbb{F}\boldsymbol{\omega})(t)\in\mathbb{Y}$. Thus $\mathbb{F}$ maps $\mathbb{Y}$ into itself and

$$\|\mathbb{F}\boldsymbol{\omega}\|_{\mathbb{Y}} \le C_1\|\boldsymbol{\omega}^0\|_{\mathbb{X}} + C_2\rho'\|\boldsymbol{\omega}\|_{\mathbb{Y}} + C_3\|\boldsymbol{\omega}\|^2_{\mathbb{Y}} + C_4\rho'\|\varphi^0\|_{L^\infty} + C_5\lambda\|\partial_3\varphi^0\|^2_{L^\infty}\ , \tag{61}$$

where $\rho' = |\rho| + \|\varphi^0\|_{L^\infty}$. Moreover, if $\boldsymbol{\omega}_1,\boldsymbol{\omega}_2\in\mathbb{Y}$, the same estimates show that

$$\|\mathbb{F}\boldsymbol{\omega}_1 - \mathbb{F}\boldsymbol{\omega}_2\|_{\mathbb{Y}} \le \|\boldsymbol{\omega}_1-\boldsymbol{\omega}_2\|_{\mathbb{Y}}\Big(C_2\rho' + C_3(\|\boldsymbol{\omega}_1\|_{\mathbb{Y}} + \|\boldsymbol{\omega}_2\|_{\mathbb{Y}})\Big)\ , \tag{62}$$

because the linear term $e^{t\mathbb{L}}\boldsymbol{\omega}^0$ and the inhomogeneous term depending on $\mathbb{H}(\varphi)$ drop out when we consider the difference $\mathbb{F}\boldsymbol{\omega}_1-\mathbb{F}\boldsymbol{\omega}_2$. Now, choose $\rho_2>0$ and $\varepsilon_2>0$ small enough so that

$$\rho_2+\varepsilon_2 \le \min\Big(1, R_1, \frac{1}{2C_2}\Big)\ ,\quad\text{and}\quad \varepsilon_2 \le \frac{1}{32C_3(C_1+C_4+C_5)}\ ,$$

where R_1 is as in Proposition 2.2. Assume that $|\rho|\le\rho_2$, $\varepsilon\le\varepsilon_2$, $\|\varphi^0\|_{L^\infty}+\lambda\|\partial_3\varphi^0\|^2_{L^\infty}\le\varepsilon$, and $\|\boldsymbol{\omega}^0\|_{\mathbb{X}}\le\varepsilon$. If $4(C_1+C_4+C_5)\varepsilon\le r\le 1/(8C_3)$, then (61) shows that $\mathbb{F}$ maps the ball $B_Y(0,r)$ into itself. Indeed, under the assumptions above we have $C_2\rho'\le 1/2$ and $C_3 r\le 1/4$, hence if $\boldsymbol{\omega}\in B_Y(0,r)$ then (61) implies

$$\|\mathbb{F}\boldsymbol{\omega}\|_{\mathbb{Y}} \le C_1\varepsilon + \frac{r}{2} + \frac{r}{4} + C_4\varepsilon + C_5\varepsilon = (C_1+C_4+C_5)\varepsilon + \frac{3r}{4} \le r\ .$$

Similarly, $\|\mathbb{F}\boldsymbol{\omega}_1-\mathbb{F}\boldsymbol{\omega}_2\|_{\mathbb{Y}}\le\frac{3}{4}\|\boldsymbol{\omega}_1-\boldsymbol{\omega}_2\|_{\mathbb{Y}}$ if $\boldsymbol{\omega}_1,\boldsymbol{\omega}_2\in B_Y(0,r)$. By the contraction mapping theorem, $\mathbb{F}$ has thus a unique fixed point $\boldsymbol{\omega}$ in $B_Y(0,r)$. Choosing $r=K_2\varepsilon$ with $K_2 = 4(C_1+C_4+C_5)$, we see that $\boldsymbol{\omega}$ is the unique solution of (54) such that $\|\boldsymbol{\omega}\|_{\mathbb{Y}}\le K_2\varepsilon$. □

Theorem 1.5 is a direct consequence of Proposition 3.4. Indeed, suppose that the initial condition for the vorticity is $\boldsymbol{\Omega}^0(x) = \boldsymbol{\Omega}^B(x_\perp;\rho)+\boldsymbol{\omega}^0(x)$, where $\boldsymbol{\omega}^0\in X^2(m)^3$ satisfies (17). Then we can decompose $\boldsymbol{\Omega}^0(x) = \boldsymbol{\Omega}^B(x_\perp;\rho+\varphi^0(x_3))+\tilde{\boldsymbol{\omega}}^0(x)$, where φ^0 is as in (17) and $\tilde{\boldsymbol{\omega}}^0$ belongs to $\mathbb{X}(m)$, namely $\tilde{\omega}_3^0\in X_0^2(m)$. Moreover, there exists $C(m,\lambda)>0$ such that

$$\|\tilde{\boldsymbol{\omega}}^0\|_{\mathbb{X}(m)} + \|\varphi^0\|_{L^\infty} + \lambda\|\partial_3\varphi^0\|^2_{L^\infty} \le C\varepsilon_2\ ,$$

and so the smallness conditions on the perturbation in Theorem 1.5 imply those in Proposition 3.4. We deduce that the solution of (4) with initial data $\boldsymbol{\Omega}^0$ satisfies $\boldsymbol{\Omega}(x,t) =$

$\boldsymbol{\Omega}^B(x_\perp;\rho+\varphi(x_3,t))+\tilde{\boldsymbol{\omega}}(x,t)$ for some $\tilde{\boldsymbol{\omega}} \in \mathbb{Y}_\mu(m)$, hence $\boldsymbol{\Omega}(x,t)$ converges exponentially in $X^2(m)^3$ toward the modulated vortex $\boldsymbol{\Omega}^B(x_\perp;\rho+\varphi(x_3,t))$. On the other hand, from (48), (49), we see that, for any $x_3 \in \mathbf{R}$, $\varphi(x_3,t)$ converges toward the limiting value

$$\lim_{t\to+\infty} \varphi(x_3,t) \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-\frac{1}{2}y^2}\varphi^0(y)\,\mathrm{d}y \;\equiv\; \delta\rho\ ,$$

and that $\sup_{x_3\in I}|\varphi(x_3,t)-\delta\rho| = \mathcal{O}(e^{-t})$ for any compact interval $I \subset \mathbf{R}$. Thus the difference $\|\boldsymbol{\Omega}^B(\cdot;\rho+\varphi(x_3,t))-\boldsymbol{\Omega}^B(\cdot;\rho+\delta\rho)\|_{L^2(m)}$ will converge exponentially to zero as $t\to+\infty$, uniformly for x_3 in any compact interval. Combining these estimates, we obtain (18).

4 Appendix

In this appendix we collect a number of technical estimates used in the main body of the paper. They relate mostly to the behavior of the semigroup generated by the linearization of the vorticity equation around the Burgers vortex. We also prove some estimates relating the vorticity field to the corresponding velocity field defined by the Biot-Savart law.

4.1 The one-dimensional Fokker-Planck operator

Fix $\alpha > 0$, and consider the one-dimensional linear equation

$$\partial_t\omega \;=\; \mathcal{L}_\alpha\omega \;\equiv\; \partial_x^2\omega + \frac{\alpha}{2}x\partial_x\omega + \frac{\alpha}{2}\omega\ , \tag{63}$$

where $x\in\mathbf{R}$ and $t\ge 0$. If $\omega(x,t)=\tilde\omega(\sqrt{\alpha}x,\alpha t)$, then $\partial_t\tilde\omega=\mathcal{L}_1\tilde\omega$ hence we could assume without loss of generality that $\alpha=1$. However for our purposes it is more convenient to keep $\alpha>0$ arbitrary.

The linear operator $\mathcal{L}_\alpha$ is formally conjugated to the Hamiltonian of the harmonic operator in quantum mechanics:

$$e^{\frac{\alpha x^2}{8}}\,\mathcal{L}_\alpha\,e^{-\frac{\alpha x^2}{8}} \;=\; L_\alpha \;\equiv\; \partial_x^2 - \frac{\alpha^2x^2}{16} + \frac{\alpha}{4}\ .$$

As is well-known, the spectrum of L_α in $L^2(\mathbf{R})$ is a sequence of simple eigenvalues:

$$\sigma(L_\alpha) \;=\; \Big\{-\frac{n\alpha}{2}\,\Big|\, n=0,1,2,\dots\Big\}\ ,$$

and the associated eigenfunctions are the Hermite functions $h_n(x)=e^{\alpha x^2/8}\partial_x^n e^{-\alpha x^2/4}$. This observation, however, is not sufficient to determine the whole spectrum of $\mathcal{L}_\alpha$ because we want to consider this operator acting on a space of functions with algebraic (rather than Gaussian) decay at infinity.

For any $m\ge 0$ and $p\ge 1$ we define the space $\mathrm{L}^p(m)=\{f\in L^p(\mathbf{R})\,|\,w^mf\in L^p(\mathbf{R})\}$, where $w(x)=(1+x^2)^{1/2}$. This Banach space is equipped with the natural norm

$$\|f\|_{\mathrm{L}^p(m)} \;=\; \|w^mf\|_{L^p} \;=\; \Big(\int_{\mathbf{R}}|w(x)^mf(x)|^p\,\mathrm{d}x\Big)^{1/p}\ .$$

The parameter m determines the decay rate at infinity of functions in $\mathrm{L}^p(m)$. For instance, it is easy to verify that $\mathrm{L}^2(m) \hookrightarrow L^1(\mathbf{R})$ if (and only if) $m > 1/2$, because in that case $w^{-m} \in L^2(\mathbf{R})$ so that any $f \in \mathrm{L}^2(m)$ satisfies

$$\int_{\mathbf{R}} |f(x)| \,\mathrm{d}x \;=\; \int_{\mathbf{R}} w(x)^m |f(x)| w(x)^{-m} \,\mathrm{d}x \;\le\; \|w^m f\|_{L^2} \|w^{-m}\|_{L^2} \;=\; C\|f\|_{\mathrm{L}^2(m)} \;, \tag{64}$$

by Hölder's inequality. If $m > 1/2$, we thus define

$$\mathrm{L}^2_0(m) \;=\; \Big\{ f \in \mathrm{L}^2(m) \,\Big|\, \int_{\mathbf{R}} f(x) \,\mathrm{d}x = 0 \Big\} \;.$$

This closed subspace of $\mathrm{L}^2(m)$ is clearly invariant under the evolution defined by (63).

In ([3], Appendix A) it is shown that the spectrum of $\mathcal{L}_\alpha$ in $\mathrm{L}^2(m)$ is

$$\sigma_m(\mathcal{L}_\alpha) \;=\; \Big\{ -\frac{n\alpha}{2} \,\Big|\, n = 0, 1, 2, \dots \Big\} \cup \Big\{ z \in \mathbf{C} \,\Big|\, \mathrm{Re}\,(z) \le \alpha\Big(\frac{1}{4} - \frac{m}{2}\Big) \Big\} \;. \tag{65}$$

Thus, in addition to the discrete spectrum of the harmonic oscillator, the operator $\mathcal{L}_\alpha$ also has essential spectrum due to the slow spatial decay of functions in $\mathrm{L}^2(m)$. Note however that this essential spectrum can be pushed far away from the imaginary axis by taking $m \ge 0$ sufficiently large. Therefore, if m is large, the relevant part of the spectrum of $\mathcal{L}_\alpha$ is still given by the first few eigenvalues of L_α. In particular zero is an isolated eigenvalue of $\mathcal{L}_\alpha$ if $m > 1/2$, and the rest of the spectrum is strictly contained in the left-half plane. If we restrict ourselves to the invariant subspace $\mathrm{L}^2_0(m)$, the spectrum of $\mathcal{L}_\alpha$ is unchanged except for the zero eigenvalue (which is absent).

Equation (63) can be explicitly solved as $\omega(t) = e^{t\mathcal{L}_\alpha}\omega(0)$, where

$$\Big(e^{t\mathcal{L}_\alpha} f\Big)(x) \;=\; \frac{e^{\alpha t/2}}{(4\pi a(t))^{1/2}} \int_{\mathbf{R}} e^{-\frac{(x-y)^2}{4a(t)}} f(y e^{\alpha t/2}) \,\mathrm{d}y \;, \quad x \in \mathbf{R} \;, \quad t > 0 \;,$$

and $a(t) = (1 - e^{-\alpha t})/\alpha$. Using this expression, it is straightforward to verify that $e^{t\mathcal{L}_\alpha}$ defines a strongly continuous semigroup in $\mathrm{L}^2(m)$ for any $m \ge 0$. Moreover, $e^{t\mathcal{L}_\alpha}$ maps $\mathrm{L}^2_0(m)$ into $\mathrm{L}^2_0(m)$ if $m > 1/2$, and the following estimates hold (see [3], Appendix A):

Proposition 4.1 *If $m > 1/2$, the semigroup $e^{t\mathcal{L}_\alpha}$ is uniformly bounded in $\mathrm{L}^2(m)$ for all $t \ge 0$. Moreover, if $m > 3/2$, there exists $C(m, \alpha) > 0$ such that, for all $f \in \mathrm{L}^2_0(m)$,*

$$\|e^{t\mathcal{L}_\alpha} f\|_{\mathrm{L}^2(m)} \;\le\; C\, e^{-\alpha t/2} \|f\|_{\mathrm{L}^2(m)} \;, \quad t \ge 0 \;. \tag{66}$$

Finally, if $1 \le p \le 2$ and $m > 3/2$, then $e^{t\mathcal{L}_\alpha}\partial_x$ defines a bounded operator from $\mathrm{L}^p(m)$ into $\mathrm{L}^2_0(m)$ and there exists $C(m, \alpha, p) > 0$ such that, for all $f \in \mathrm{L}^p(m)$,

$$\|e^{t\mathcal{L}_\alpha}\partial_x f\|_{\mathrm{L}^2(m)} \;\le\; C \frac{e^{-\alpha t/2}}{a(t)^{\frac{1}{2p}+\frac{1}{4}}} \|f\|_{\mathrm{L}^p(m)} \;, \quad t > 0 \;, \tag{67}$$

where $a(t) = (1 - e^{-\alpha t})/\alpha$.

4.2 Two-dimensional estimates

We next consider the two-dimensional equation

$$\partial_t \omega \;=\; \mathcal{L}_{\alpha_1,\alpha_2}\omega \;\equiv\; \Delta\omega + \frac{\alpha_1}{2}x_1\partial_1\omega + \frac{\alpha_2}{2}x_2\partial_2\omega + \frac{\alpha_1+\alpha_2}{2}\omega\;, \tag{68}$$

where $x \in \mathbf{R}^2$, $t \ge 0$, and $\alpha_1 \ge \alpha_2 > 0$. In the particular case where $\alpha_1 = 1+\lambda$ and $\alpha_2 = 1-\lambda$ for some $\lambda \in [0,1)$, we see that $\mathcal{L}_{\alpha_1,\alpha_2} = \mathcal{L}_\perp + \lambda\mathcal{M}$, where $\mathcal{L}_\perp, \mathcal{M}$ are defined in (22). Note that the parameters α_1, α_2 cannot be eliminated by a rescaling, unless $\alpha_1 = \alpha_2$.

We study the operator $\mathcal{L}_{\alpha_1,\alpha_2}$ in the weighted space $L^p(m)$, which we recall is defined by:

$$L^p(m) \;=\; \{f \in L^p(\mathbf{R}^2)\,|\, b^m f \in L^p(\mathbf{R}^2)\}\;, \qquad \|f\|_{L^p(m)} = \|b^m f\|_{L^p}\;, \tag{69}$$

where $p \ge 1$, $m \ge 0$, and $b(x_1,x_2) = w(x_1)w(x_2) = (1+x_1^2)^{1/2}(1+x_2^2)^{1/2}$. It is clear that $L^2(m) = \mathrm{L}^2(m) \otimes \mathrm{L}^2(m)$, where $\mathrm{L}^2(m)$ is the one-dimensional space defined in the previous paragraph and $\otimes$ denotes the tensor product of Hilbert spaces, see [14]. Comparing the definitions (63), (68), we see that our operator can be decomposed as $\mathcal{L}_{\alpha_1,\alpha_2} = \mathcal{L}_{\alpha_1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_{\alpha_2}$, where $\mathbf{1}$ denotes the identity operator. It follows that the spectrum of $\mathcal{L}_{\alpha_1,\alpha_2}$ in $L^2(m)$ is just the sum

$$\sigma_m(\mathcal{L}_{\alpha_1,\alpha_2}) \;=\; \sigma_m(\mathcal{L}_{\alpha_1}) + \sigma_m(\mathcal{L}_{\alpha_2})\;,$$

where $\sigma_m(\mathcal{L}_{\alpha_i})$ is given by (65) for $i = 1,2$. In particular, zero is an isolated eigenvalue of $\mathcal{L}_{\alpha_1,\alpha_2}$ if $m > 1/2$, and if $m > 3/2$ there exists $\mu > \alpha_2/2$ such that

$$\sigma_m(\mathcal{L}_{\alpha_1,\alpha_2}) \;\subset\; \Big\{0\,,\,-\frac{\alpha_2}{2}\Big\} \,\cup\, \Big\{z \in \mathbf{C}\,\Big|\,\mathrm{Re}\,(z) \le -\mu\Big\}\;.$$

(Recall that we assumed $\alpha_1 \ge \alpha_2$.) Moreover, if $m > 1/2$, the subspace $L^2_0(m)$ defined by (23) is invariant under the action of $\mathcal{L}_{\alpha_1,\alpha_2}$, and the restriction of $\mathcal{L}_{\alpha_1,\alpha_2}$ to $L^2_0(m)$ has spectrum $\sigma_m(\mathcal{L}_{\alpha_1,\alpha_2}) \setminus \{0\}$. Thus $\mathcal{L}_{\alpha_1,\alpha_2}$ is invertible in $L^2_0(m)$ if $m > 1/2$, with bounded inverse.

The semigroup generated by $\mathcal{L}_{\alpha_1,\alpha_2}$ satisfies $e^{t\mathcal{L}_{\alpha_1,\alpha_2}} = e^{t\mathcal{L}_{\alpha_1}} \otimes e^{t\mathcal{L}_{\alpha_2}}$. Thus, using Proposition 4.1, we immediately obtain the following estimates:

Proposition 4.2 *If $m > 1/2$, the semigroup $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is uniformly bounded in $L^2(m)$ for all $t \ge 0$. Moreover, if $m > 3/2$, there exists $C(m,\alpha_1,\alpha_2) > 0$ such that, for all $f \in L^2_0(m)$,*

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}} f\|_{L^2(m)} \;\le\; C\,e^{-\alpha_2 t/2}\|f\|_{L^2(m)}\;, \quad t \ge 0\;. \tag{70}$$

Finally, if $1 \le p \le 2$ and $m > 3/2$, then $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k$ defines a bounded operator from $L^p(m)$ into $L^2_0(m)$ for $k = 1,2$, and there exists $C(m,\alpha_1,\alpha_2,p) > 0$ such that, for all $f \in L^p(m)$,

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_1 f\|_{L^2(m)} \;\le\; C\frac{e^{-\alpha_1 t/2}}{a_1(t)^{\frac{1}{2p}+\frac{1}{4}}a_2(t)^{\frac{1}{2p}-\frac{1}{4}}}\|f\|_{L^p(m)}\;, \quad t > 0\;, \tag{71}$$

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_2 f\|_{L^2(m)} \;\le\; C\frac{e^{-\alpha_2 t/2}}{a_1(t)^{\frac{1}{2p}-\frac{1}{4}}a_2(t)^{\frac{1}{2p}+\frac{1}{4}}}\|f\|_{L^p(m)}\;, \quad t > 0\;, \tag{72}$$

where

$$a_i(t) \;=\; \frac{1-e^{-\alpha_i t}}{\alpha_i} \;=\; \int_0^t e^{-\alpha_i s}\,\mathrm{d}s\;, \quad i = 1,2\;.$$

Remark 4.3 *For $p \in [1,2]$ and $m > 1/2$, we also have the following bound:*

$$\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}} f\|_{L^2(m)} \;\le\; \frac{C}{a_1(t)^{\frac{1}{2p}-\frac{1}{4}} a_2(t)^{\frac{1}{2p}-\frac{1}{4}}} \|f\|_{L^p(m)} \;, \quad t > 0 \;.$$

We conclude this paragraph with a short discussion of the two-dimensional Biot-Savart law:

$$\mathbf{u}(x) \;=\; \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{1}{|x-y|^2} \begin{pmatrix} y_2 - x_2 \\ x_1 - y_1 \end{pmatrix} \omega(y)\,\mathrm{d}y \;, \quad x \in \mathbf{R}^2 \;. \tag{73}$$

Proposition 4.4 *Let $\mathbf{u}$ be the velocity field defined from ω via the Biot-Savart law (73).*
i) If $\omega \in L^p(\mathbf{R}^2)$ for some $p \in (1,2)$, then $\mathbf{u} \in L^q(\mathbf{R}^2)$ where $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, and there exists $C(p) > 0$ such that $\|\mathbf{u}\|_{L^q} \le C\|\omega\|_{L^p}$.
ii) If $\omega \in L^p(\mathbf{R}^2) \cap L^q(\mathbf{R}^2)$ for some $p \in [1,2)$ and some $q \in (2,\infty]$ then $\mathbf{u} \in C_b^0(\mathbf{R}^2)$ and there exists $C(p,q) > 0$ such that

$$\|\mathbf{u}\|_{L^\infty} \;\le\; C\|\omega\|_{L^p}^{\alpha} \|\omega\|_{L^q}^{1-\alpha} \;, \quad \textit{where} \quad \frac{1}{2} \;=\; \frac{\alpha}{p} + \frac{1-\alpha}{q} \;.$$

Proof: Assertion i) is a direct consequence of the Hardy-Littlewood-Sobolev inequality [9]. For a proof of ii), see for instance ([3], Lemma 2.1). □

We deduce from Proposition 4.4 the following useful bound on the product $\mathbf{u}\omega$:

Corollary 4.5 *Assume that $\omega_1, \omega_2 \in L^2(m)$ for some $m > 1/2$, and let $\mathbf{u}_1$ be the velocity field obtained from ω_1 via the Biot-Savart law (73). Then $\mathbf{u}_1\omega_2 \in L^p(m)$ for all $p \in (1,2)$, and there exists $C(m,p) > 0$ such that*

$$\|\mathbf{u}_1\omega_2\|_{L^p(m)} \le C\|\omega_1\|_{L^2(m)} \|\omega_2\|_{L^2(m)} \;.$$

Proof: Assume that $1 < p < 2$. By Hölder's inequality

$$\|\mathbf{u}_1\omega_2\|_{L^p(m)} \;=\; \|b^m \mathbf{u}_1\omega_2\|_{L^p} \;\le\; \|\mathbf{u}_1\|_{L^q} \|b^m\omega_2\|_{L^2} \;, \quad \text{where} \quad \frac{1}{q} \;=\; \frac{1}{p} - \frac{1}{2} \;.$$

Now $\|\mathbf{u}_1\|_{L^q} \le C\|\omega_1\|_{L^p}$ by Proposition 4.4, and $\|\omega_1\|_{L^p} \le C\|\omega_1\|_{L^2(m)}$ because $L^2(m) \hookrightarrow L^p(\mathbf{R}^2)$ for $p \in [1,2]$ if $m > 1/2$. This gives the desired result. □

4.3 The three-dimensional semigroup

This section is devoted to the three-dimensional equation

$$\partial_t \omega \;=\; \hat{\mathcal{L}}_{\alpha_1,\alpha_2}\omega \;\equiv\; \Delta\omega + \frac{\alpha_1}{2} x_1\partial_1\omega + \frac{\alpha_2}{2} x_2\partial_2\omega - x_3\partial_3\omega + \frac{\alpha_1+\alpha_2}{2}\omega \;, \tag{74}$$

where $x \in \mathbf{R}^3$, $t \ge 0$, and $\alpha_1 \ge \alpha_2 > 0$. In the particular case where $\alpha_1 = 1+\lambda$ and $\alpha_2 = 1-\lambda$, we have $\hat{\mathcal{L}}_{\alpha_1,\alpha_2} = \mathcal{L} + 1$ where $\mathcal{L}$ is defined in (42).

It is important to realize that the evolution defined by (74) is essentially contracting in the transverse variables $x_\perp = (x_1, x_2)$ and expanding in the axial variable x_3. This is due to the signs of the advection terms, which in turn originate in our choice of the straining flow (2). For this reason we can assume that the solutions of (74) decay to zero as $|x_\perp| \to \infty$, but we cannot impose any decay in the x_3 variable (otherwise the solutions will not stay uniformly bounded for all times in the corresponding norm). This motivates the following choice of our function space. For $p \ge 1$ and $m \ge 0$, we introduce the Banach space

$$X^p(m) \equiv C_b^0(\mathbf{R}, L^p(m)) = \{\omega : \mathbf{R} \to L^p(m) \,|\, \omega \text{ is bounded and continuous}\} \tag{75}$$

equipped with the norm

$$\|\omega\|_{X^p(m)} = \sup_{x_3 \in \mathbf{R}} \|\omega(x_3)\|_{L^p(m)} .$$

For any $n \in \mathbf{N}^*$ we also define the seminorm

$$|\omega|_{X^p_n(m)} = \sup_{|x_3| \le n} \|\omega(x_3)\|_{L^p(m)} , \tag{76}$$

and we denote by $X^p_{\mathrm{loc}}(m)$ the space $X^p(m)$ equipped with the topology defined by the family of seminorms (76) for $n \in \mathbf{N}^*$. For later use, we observe that the ball $\{\omega \in X^p(m) \,|\, \|\omega\|_{X^p(m)} \le R\}$ is closed in $X^p_{\mathrm{loc}}(m)$ for any $R > 0$.

At least formally, the space $X^2(m)$ can be thought of as the tensor product $C_b^0(\mathbf{R}) \otimes L^2(m)$, i.e. the space generated by linear combinations of elements of the form $\omega(x_\perp, x_3) = f(x_3)g(x_\perp)$, with $f \in C_b^0(\mathbf{R})$ and $g \in L^2(m)$. In this picture, the linear operator defined by (74) can be decomposed as $\hat{\mathcal{L}}_{\alpha_1,\alpha_2} = \hat{\mathcal{L}}_3 \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_{\alpha_1,\alpha_2}$, where $\mathcal{L}_{\alpha_1,\alpha_2}$ is defined in (68) and $\hat{\mathcal{L}}_3$ is the one-dimensional operator $\hat{\mathcal{L}}_3 f = \partial_3^2 f - x_3 \partial_3 f$. It is readily verified that $\hat{\mathcal{L}}_3$ generates a semigroup in $C_b^0(\mathbf{R})$ given by the explicit formula

$$(e^{t\hat{\mathcal{L}}_3} f)(x_3) = (G_t * f)(x_3 e^{-t}) , \quad x_3 \in \mathbf{R} , \quad t > 0 , \tag{77}$$

where G_t is defined in (49), and we know from Section 4.2 that $\mathcal{L}_{\alpha_1,\alpha_2}$ generates a strongly continuous semigroup in $L^2(m)$. Thus we expect that $\hat{\mathcal{L}}_{\alpha_1,\alpha_2}$ will generate a semigroup $\{S_t\}_{t\ge 0}$ in $X^2(m)$ given by $S_t = e^{t\hat{\mathcal{L}}_3} \otimes e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$, or explicitly

$$(S_t\omega)(x_3) = \int_{\mathbf{R}} G_t(x_3 e^{-t} - y_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}} \omega(y_3)\Big) \,\mathrm{d}y_3 , \quad x_3 \in \mathbf{R} , \quad t > 0 . \tag{78}$$

We shall prove that these heuristic considerations are indeed correct in the sense that (78) defines a semigroup of bounded operators in $X^2(m)$ with the property that $\omega(t) = S_t\omega$ is the solution of (74) with initial data $\omega \in X^2(m)$. However, the map $t \mapsto S_t\omega$ is not continuous in the topology of $X^2(m)$, but only in the (weaker) topology of $X^2_{\mathrm{loc}}(m)$. This is due to the fact that equation (74) has "infinite speed of propagation" in the sense that the advection term in the vertical variable is unbounded, see Remark 3.3.

Proposition 4.6 *For any $m \ge 0$, the family $\{S_t\}_{t\ge 0}$ defined by (78) and $S_0 = \mathbf{1}$ is a semigroup of bounded linear operators on $X^2(m)$. If $\omega_0 \in X^2(m)$ and $\omega(t) = S_t\omega_0$, then $\omega : [0, +\infty) \to X^2_{\mathrm{loc}}(m)$ is continuous, and $\omega(t)$ solves (74) for $t > 0$. For any $R > 0$, if $B_R = \{f \in X^2(m) \,|\, \|f\|_{X^2(m)} \le R\}$ is equipped with the topology of $X^2_{\mathrm{loc}}(m)$, then*

$S_t : B_R \to X^2_{\rm loc}(m)$ is continuous, uniformly in time on compact intervals. Moreover:
i) If $m > 1/2$ then S_t is uniformly bounded on $X^2(m)$ for all $t \ge 0$.
ii) If $m > 3/2$ there exists $C(m,\alpha_1,\alpha_2) > 0$ such that, for all ω in the subspace $X^2_0(m)$ defined in (19),

$$\|S_t\omega\|_{X^2(m)} \le C\, e^{-\frac{\alpha_2}{2}t}\|\omega\|_{X^2(m)}\ , \quad t \ge 0\ . \tag{79}$$

iii) If $p \in [1,2]$ and $m > 3/2$, $S_t\partial_k$ defines a bounded operator from $X^p(m)$ into $X^2(m)$ for $t > 0$ and $k = 1,2,3$, and there exists $C(m,\alpha_1,\alpha_2,p) > 0$ such that

$$\|S_t\partial_1\omega\|_{X^2(m)} \le C\,\frac{e^{-\alpha_1 t/2}}{a_1(t)^{\frac{1}{2p}+\frac14}a_2(t)^{\frac{1}{2p}-\frac14}}\,\|\omega\|_{X^p(m)}\ , \tag{80}$$

$$\|S_t\partial_2\omega\|_{X^2(m)} \le C\,\frac{e^{-\alpha_2 t/2}}{a_1(t)^{\frac{1}{2p}-\frac14}a_2(t)^{\frac{1}{2p}+\frac14}}\,\|\omega\|_{X^p(m)}\ , \tag{81}$$

$$\|S_t\partial_3\omega\|_{X^2(m)} \le \frac{C}{\sqrt{1-e^{-2t}}}\,\frac{1}{a_1(t)^{\frac{1}{2p}-\frac14}a_2(t)^{\frac{1}{2p}-\frac14}}\,\|\omega\|_{X^p(m)}\ , \tag{82}$$

where $a_1(t), a_2(t)$ are as in Proposition 4.2.

Proof: We first rewrite (78) in a slightly more convenient form. By (49) we have $G_t(y) = c(t)^{-1/2}G(c(t)^{-1/2}y)$, where $c(t) = 1 - e^{-2t}$ and $G(z) = (2\pi)^{-1/2}e^{-z^2/2}$. Thus setting $y_3 = x_3e^{-t} + c(t)^{1/2}z_3$ in (78), we obtain the equivalent formula

$$(S_t\omega)(x_3) = \int_{\mathbf R} G(z_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3e^{-t} + c(t)^{1/2}z_3)\Big)\,{\rm d}z_3\ , \quad x_3 \in \mathbf{R}\ , \quad t \ge 0\ . \tag{83}$$

Fix $m \ge 0$. If $\omega \in X^2(m)$, then for any $t \ge 0$ the map $x_3 \mapsto e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3)$ also belongs to $X^2(m)$, because $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is a bounded operator on $L^2(m)$ by Proposition 4.2. Thus it follows immediately from (83) that $S_t\omega \in X^2(m)$ and

$$\|S_t\omega\|_{X^2(m)} \le \sup_{x_3\in\mathbf R}\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3)\|_{L^2(m)} \le N_m(t)\|\omega\|_{X^2(m)}\ , \tag{84}$$

where $N_m(t) = \|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\|_{L^2(m)\to L^2(m)}$. The semigroup formula $S_{t_1+t_2} = S_{t_1}S_{t_2}$ is easily verified using (78), Fubini's theorem, and the identity

$$\int_{\mathbf R} G_{t_1}(xe^{-t_1} - y)G_{t_2}(ye^{-t_2} - z)\,{\rm d}y = G_{t_1+t_2}(xe^{-(t_1+t_2)} - z)\ .$$

Thus $\{S_t\}_{t\ge0}$ is a semigroup of bounded operators in $X^2(m)$.

On the other hand, by (83), we have for all $k \in \mathbf{N}$:

$$\|S_t\omega(x_3)\|_{L^2(m)} \le \int_{-k}^{k} G(z_3)N_m(t)\|\omega(x_3e^{-t} + c(t)^{1/2}z_3)\|_{L^2(m)}\,{\rm d}z_3 + N_m(t)\|\omega\|_{X^2(m)}\varepsilon_k\ ,$$

where $\varepsilon_k = \int_{|z|\ge k} G(z)\,{\rm d}z \to 0$ as $k \to \infty$. Since $|x_3e^{-t} + c(t)^{1/2}z_3| \le n + k$ whenever $|x_3| \le n$ and $|z_3| \le k$, we deduce that for all $n \in \mathbf{N}^*$:

$$|S_t\omega|_{X^2_n(m)} \le N_m(t)\Big(|\omega|_{X^2_{n+k}(m)} + \varepsilon_k\|\omega\|_{X^2(m)}\Big)\ . \tag{85}$$

This bound implies that $S_t : B_R \to X^2_{\rm loc}(m)$ is continuous, uniformly in time on compact intervals.

Furthermore, if $\omega \in X^2(m)$ and $t > 0$, we have

$$\begin{aligned}(S_t\omega - \omega)(x_3) &= \int_{\mathbf{R}} G(z_3) e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\Big(\omega(x_3 e^{-t} + c(t)^{1/2} z_3) - \omega(x_3)\Big)\,\mathrm{d}z_3 \\ &+ e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3) - \omega(x_3)\ ,\end{aligned}$$

hence proceeding as above we find for all $n, k \in \mathbf{N}^*$:

$$\begin{aligned}|S_t\omega - \omega|_{X^2_n(m)} &\le \int_{-k}^{k} G(z_3) N_m(t) \sup_{|x_3|\le n} \|\omega(x_3 e^{-t} + c(t)^{1/2} z_3) - \omega(x_3)\|_{L^2(m)}\,\mathrm{d}z_3 \\ &+ 2N_m(t)\varepsilon_k \|\omega\|_{X^2(m)} + \sup_{|x_3|\le n} \|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3) - \omega(x_3)\|_{L^2(m)}\ .\end{aligned}$$

The last term goes to zero as $t \to 0+$ because $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is a strongly continuous semigroup on $L^2(m)$ and $\omega : [-n, n] \to L^2(m)$ is continuous (hence has compact range). Similarly, for each $k \in \mathbf{N}$, the integral goes to zero as $t \to 0+$ by Lebesgue's dominated convergence theorem, because $\omega : [-n-k, n+k] \to L^2(m)$ is uniformly continuous. Since $\varepsilon_k \to 0$ as $k \to \infty$, we conclude that $S_t\omega \to \omega$ in $X^2_{\rm loc}(m)$ as $t \to 0+$. Then, using the semigroup property, we deduce that the map $t \mapsto S_t\omega$ is continuous to the right at any $t \ge 0$. Finally, if $t > \varepsilon > 0$, we have $S_t\omega - S_{t-\varepsilon}\omega = S_{t-\varepsilon}(S_\varepsilon\omega - \omega)$, hence by (85)

$$|S_t\omega - S_{t-\varepsilon}\omega|_{X^2_n(m)} \le N_m(t-\varepsilon)\Big(|S_\varepsilon\omega - \omega|_{X^2_{n+k}(m)} + \varepsilon_k(1 + N_m(\varepsilon))\|\omega\|_{X^2(m)}\Big)\ .$$

Since $|S_\varepsilon\omega - \omega|_{X^2_{n+k}(m)} \to 0$ as $\varepsilon \to 0+$ for all $k, n \in \mathbf{N}^*$, this shows that $t \mapsto S_t\omega$ is also continuous to the left at any $t > 0$.

Next, using (83), (77), and the explicit formula for $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$, it is rather straightforward to verify that, for any $\omega \in X^2(m)$, the map $(x_\perp, x_3, t) \mapsto \omega(x_\perp, x_3, t) = ((S_t\omega)(x_3))(x_\perp)$ is smooth and satisfies (74) for $t > 0$. Thus $S_t\omega$ is indeed the solution of (74) with initial data ω.

It remains to establish the decay properties of S_t:

i) If $m > 1/2$, we know from Proposition 4.2 that $N_m(t) \le C$ for all $t \ge 0$, hence $\{S_t\}_{t\ge 0}$ is uniformly bounded on $X^2(m)$ by (84).

ii) If $m > 3/2$ and $\omega \in X^2_0(m)$, then $\omega(x_3) \in L^2_0(m)$ for all $x_3 \in \mathbf{R}$ and (79) follows immediately from (84) and (70).

iii) If $k = 1, 2$, we define $S_t\partial_k$ by

$$(S_t\partial_k\omega)(x_3) = \int_{\mathbf{R}} G(z_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k\omega(x_3 e^{-t} + c(t)^{1/2} z_3)\Big)\,\mathrm{d}z_3\ . \tag{86}$$

If $m > 3/2$ and $p \in [1, 2]$, we know from Proposition 4.2 that $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k$ is a bounded operator from $L^p(m)$ into $L^2_0(m)$ satisfying (71) or (72). Thus the formula (86) defines a bounded operator from $X^p(m)$ into $X^2_0(m)$ and

$$\|S_t\partial_k\omega\|_{X^2(m)} \le \sup_{x_3\in\mathbf{R}} \|\mathrm{e}^{t\mathcal{L}_{\alpha_1,\alpha_2}}\partial_k\omega(x_3)\|_{L^2(m)}\ .$$

Thus (80), (81) follow immediately from (71), (72). Finally we define $S_t\partial_3$ by

$$(S_t\partial_3\omega)(x_3) \;=\; -\frac{1}{c(t)^{1/2}}\int_{\mathbf{R}} \partial_3 G(z_3)\Big(e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3 e^{-t} + c(t)^{1/2}z_3)\Big)\,\mathrm{d}z_3\;. \tag{87}$$

We know that $e^{t\mathcal{L}_{\alpha_1,\alpha_2}}$ is a bounded operator from $L^p(m)$ into $L^2(m)$, see Remark 4.3. Since $\partial_3 G \in L^1(\mathbf{R}^2)$, we thus find

$$\begin{aligned}\|S_t\partial_3\omega\|_{X^2(m)} \;&\le\; \frac{C}{c(t)^{1/2}}\sup_{x_3\in\mathbf{R}}\|e^{t\mathcal{L}_{\alpha_1,\alpha_2}}\omega(x_3)\|_{L^2(m)}\\ &\le\; \frac{C}{c(t)^{1/2}}\frac{1}{a_1(t)^{\frac{1}{2p}-\frac{1}{4}}a_2(t)^{\frac{1}{2p}-\frac{1}{4}}}\|\omega\|_{X^p(m)}\;,\end{aligned}$$

which is (82). This concludes the proof. □

Corollary 4.7 *Let $T > 0$ and let $f : [0,T] \to X^2(m)$ be a bounded function satisfying $f \in C^0([0,T], X^2_{\mathrm{loc}}(m))$. Then the map $F : [0,T] \to X^2(m)$ defined by*

$$F(t) \;=\; \int_0^t S_{t-s}f(s)\,\mathrm{d}s\;,\quad 0 \le t \le T\;,$$

satisfies $F \in C^0([0,T], X^2_{\mathrm{loc}}(m))$, and $\|F(t)\|_{X^2(m)} \le \int_0^t N_m(t-s)\|f(s)\|_{X^2(m)}\,\mathrm{d}s$, where N_m is as in (84).

Proof: For any $t \in [0,T]$, we define $\psi_t : [0,t] \to X^2(m)$ by $\psi_t(s) = S_{t-s}f(s)$. Since $f \in C^0([0,T], X^2_{\mathrm{loc}}(m))$ and since the semigroup S_t is continuous on $X^2_{\mathrm{loc}}(m)$ as described in Proposition 4.6, it is easy to verify that the map $\psi_t : [0,t] \to X^2_{\mathrm{loc}}(m)$ is also continuous. As $X^2_{\mathrm{loc}}(m)$ is a subspace of the Fréchet space $C^0(\mathbf{R}, L^2(m))$, the integral $\int_0^t \psi_t(s)\,\mathrm{d}s$ can be defined as in ([17], Theorem 3.17). However, in the present case, we can also use the following "pedestrian" construction (which agrees with the general one). For any $n \in \mathbf{N}^*$, we define

$$F_n(t) \;=\; \int_0^t \chi_n\psi_t(s)\,\mathrm{d}s \;=\; \int_0^t \chi_n S_{t-s}f(s)\,\mathrm{d}s\;,$$

where χ_n denotes the map $x_3 \mapsto \mathbf{1}_{[-n,n]}(x_3)$. Clearly $\chi_n\psi_t(s)$ is a continuous function of $s \in [0,t]$ with values in the Banach space $C^0([-n,n], L^2(m))$, hence $F_n(t) \in C^0([-n,n], L^2(m))$ can be defined for any $t \in [0,T]$ as a Banach-valued Riemann integral. Using again the continuity of the semigroup S_t one finds that $F_n : [0,T] \to C^0([-n,n], L^2(m))$ is continuous and satisfies

$$|F_n(t)|_{X^2_n(m)} \;\le\; \int_0^t N_m(t-s)\|f(s)\|_{X^2(m)}\,\mathrm{d}s \;\le\; C(T)\int_0^t \|f(s)\|_{X^2(m)}\,\mathrm{d}s\;.$$

(Note that $t \mapsto N_m(t)$ and $t \mapsto \|f(t)\|_{X^2(m)}$ are lower semicontinuous, hence measurable.) Now, for each $t \in [0,T]$, it is clear that $(F_m(t))(x_3) = (F_n(t))(x_3)$ if $|x_3| \le n \le m$, hence there is a unique $F(t) \in C^0(\mathbf{R}, L^2(m))$ such that $(F(t))(x_3) = (F_n(t))(x_3)$ whenever $|x_3| \le n$. By construction, $\|F(t)\|_{X^2(m)} \le \int_0^t N_m(t-s)\|f(s)\|_{X^2(m)}\,\mathrm{d}s$ for all $t \in [0,T]$, and $F \in C^0([0,T], X^2_{\mathrm{loc}}(m))$. □

Remark 4.8 *Similarly, if $p \in (1,2]$ and $k \in \{1,2,3\}$, Proposition 4.6 implies that, if $f : [0,T] \to X^p(m)$ is bounded in $X^p(m)$ and continuous in $X^p_{\mathrm{loc}}(m)$, the map $F : [0,T] \to X^2(m)$ defined by*

$$F(t) \;=\; \int_0^t S_{t-s}\partial_k f(s)\,\mathrm{d}s\;, \quad 0 \le t \le T\;,$$

is bounded in $X^2(m)$ and continuous in $X^2_{\mathrm{loc}}(m)$. In that case, for each $n \in \mathbf{N}^*$ and each $t \in (0,T]$, $F_n(t) \equiv \chi_n F(t)$ is defined by a "generalized" Riemann integral, because the integrand has a singularity at $s = t$.

4.4 The three-dimensional Biot-Savart law

In this final section, we discuss the three-dimensional Biot-Savart law, namely

$$\mathbf{u}(x) \;=\; -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{(\mathbf{x}-\mathbf{y})\times\boldsymbol{\omega}(y)}{|x-y|^3}\,\mathrm{d}y\;, \quad x \in \mathbf{R}^3\;. \tag{88}$$

We first prove the analogue of Proposition 4.4 in the spaces $X^p(m)$ defined by (75).

Proposition 4.9 *Fix $m > 1/2$. If $\boldsymbol{\omega} \in X^2(m)$, the velocity field given by (88) satisfies $\mathbf{u} \in X^q(0)$ for all $q \in (2,\infty)$, and there exists $C(m,q) > 0$ such that $\|\mathbf{u}\|_{X^q(0)} \le C\|\boldsymbol{\omega}\|_{X^2(m)}$.*

Proof: Assume that $\omega \in X^2(m)$ for some $m > 1/2$. For all $x = (x_\perp, x_3) \in \mathbf{R}^3$, we have by Fubini's theorem

$$|\mathbf{u}(x_\perp,x_3)| \;\le\; C\int_{\mathbf{R}^3} \frac{|\boldsymbol{\omega}(y_\perp,y_3)|}{|x_\perp-y_\perp|^2+(x_3-y_3)^2}\,\mathrm{d}y_\perp\,\mathrm{d}y_3 \;=\; C\int_{\mathbf{R}} F(x_\perp;x_3,y_3)\,\mathrm{d}y_3\;,$$

where

$$F(x_\perp;x_3,y_3) \;=\; \int_{\mathbf{R}^2} \frac{|\boldsymbol{\omega}(y_\perp,y_3)|}{|x_\perp-y_\perp|^2+(x_3-y_3)^2}\,\mathrm{d}y_\perp\;.$$

By Minkowski's inequality, it follows that

$$\|\mathbf{u}(\cdot,x_3)\|_{L^q(\mathbf{R}^2)} \;\le\; C\int_{\mathbf{R}} \|F(\cdot;x_3,y_3)\|_{L^q(\mathbf{R}^2)}\,\mathrm{d}y_3\;. \tag{89}$$

If $2 < q < \infty$, we shall show that there exists $H_q \in L^1(\mathbf{R})$ and $C > 0$ such that

$$\|F(\cdot;x_3,y_3)\|_{L^q(\mathbf{R}^2)} \;\le\; C\|\boldsymbol{\omega}(\cdot,y_3)\|_{L^2(m)}H_q(x_3-y_3)\;, \quad x_3,y_3 \in \mathbf{R}\;. \tag{90}$$

Together with (89), this gives $\|\mathbf{u}(\cdot,x_3)\|_{L^q(\mathbf{R}^2)} \le C\|\boldsymbol{\omega}\|_{X^2(m)}$ for all $x_3 \in \mathbf{R}$, which is the desired bound. Since the Biot-Savart law is invariant under spatial translations, the same arguments show that, for all $x_3 \in \mathbf{R}$,

$$\|\mathbf{u}(\cdot,x_3+\varepsilon)-\mathbf{u}(\cdot,x_3)\|_{L^q(\mathbf{R}^2)} \;\le\; C\int_{\mathbf{R}} \|\boldsymbol{\omega}(\cdot,y_3+\varepsilon)-\boldsymbol{\omega}(\cdot,y_3)\|_{L^2(m)}H_q(x_3-y_3)\,\mathrm{d}y_3\;.$$

As $\varepsilon \to 0$ the right-hand side converges to zero by Lebesgue's dominated convergence theorem, thus $\mathbf{u} \in C^0_b(\mathbf{R}, L^q(\mathbf{R}^2)) \equiv X^q(0)$.

To prove (90), for any $a \in \mathbf{R}$ we define $f_a(y_\perp) = (|y_\perp|^2 + a^2)^{-1}$. If $a \neq 0$, then $f_a \in L^r(\mathbf{R}^2)$ for all $r > 1$, and there exists $C_r > 0$ such that

$$\|f_a\|_{L^r(\mathbf{R}^2)} \le \frac{C_r}{|a|^{2-\frac{2}{r}}} .$$

Since $F(\cdot; x_3, y_3) = |\boldsymbol{\omega}(\cdot, y_3)| * f_{x_3-y_3}$, Young's inequality implies

$$\|F(\cdot; x_3, y_3)\|_{L^q(\mathbf{R}^2)} \le C\|\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(\mathbf{R}^2)}\|f_{x_3-y_3}\|_{L^p(\mathbf{R}^2)} \le \frac{C\|\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(\mathbf{R}^2)}}{|x_3 - y_3|^{2-\frac{2}{p}}} , \tag{91}$$

where $1 + \frac{1}{q} = \frac{1}{2} + \frac{1}{p}$, hence $2 - \frac{2}{p} = 1 - \frac{2}{q} < 1$. On the other hand, by Hölder's inequality,

$$\begin{aligned} F(x_\perp; x_3, y_3) &= \int_{\mathbf{R}^2} b^m(y_\perp)|\boldsymbol{\omega}(y_\perp, y_3)| \frac{1}{b^m(y_\perp)(|x_\perp - y_\perp|^2 + (x_3 - y_3)^2)} \, \mathrm{d}y_\perp \\ &\le \|b^m\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(\mathbf{R}^2)} \Big(\frac{1}{b^{2m}} * f^2_{x_3-y_3}\Big)^{1/2}(x_\perp) . \end{aligned}$$

Using Young's inequality again, we obtain

$$\begin{aligned} \|F(\cdot; x_3, y_3)\|_{L^q(\mathbf{R}^2)} &\le C\|\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(m)}\|b^{-m}\|_{L^2(\mathbf{R}^2)}\|f_{x_3-y_3}\|_{L^q(\mathbf{R}^2)} \\ &\le \frac{C\|\boldsymbol{\omega}(\cdot, y_3)\|_{L^2(m)}}{|x_3 - y_3|^{2-\frac{2}{q}}} , \end{aligned} \tag{92}$$

where $2 - \frac{2}{q} > 1$. Combining (91), (92), we obtain (90). □

An easy consequence is the analogue of Corollary 4.5:

Corollary 4.10 *Assume that $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in X^2(m)$ for some $m > 1/2$, and let $\mathbf{u}_1$ be the velocity field obtained from $\boldsymbol{\omega}_1$ via the Biot-Savart law (88). Then $\mathbf{u}_1\boldsymbol{\omega}_2 \in X^p(m)$ for all $p \in (1, 2)$, and there exists $C(m, p) > 0$ such that*

$$\|\mathbf{u}_1\boldsymbol{\omega}_2\|_{X^p(m)} \le C\|\boldsymbol{\omega}_1\|_{X^2(m)}\|\boldsymbol{\omega}_2\|_{X^2(m)} . \tag{93}$$

Proof: Assume that $1 < p < 2$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. By Hölder's inequality, we have for all $x_3 \in \mathbf{R}$

$$\|b(\cdot)^m\mathbf{u}_1(\cdot, x_3)\boldsymbol{\omega}_2(\cdot, x_3)\|_{L^p(\mathbf{R}^2)} \le \|\mathbf{u}_1(\cdot, x_3)\|_{L^q(\mathbf{R}^2)}\|b(\cdot)^m\boldsymbol{\omega}_2(\cdot, x_3)\|_{L^2(\mathbf{R}^2)} .$$

Taking the supremum over x_3 and using Proposition 4.9, we obtain (93). Moreover, since $\mathbf{u}_1 \in X^q(0)$ and $\boldsymbol{\omega}_2 \in X^2(m)$, it is clear that $x_3 \mapsto \mathbf{u}_1(\cdot, x_3)\boldsymbol{\omega}_2(\cdot, x_3)$ is continuous from $\mathbf{R}$ into $L^p(m)$. □

In the rest of this section, we fix some $\lambda \in [0, 1)$. Given $\rho \in \mathbf{R}$ and $\varphi \in C^1_b(\mathbf{R})$, our goal is to compare the velocity field $\mathbf{U}^B(x_\perp; \rho + \varphi(x_3))$ defined by (33) with the velocity field $\tilde{\mathbf{U}}^B(x; \rho, \varphi)$ obtained from $\boldsymbol{\Omega}^B(x_\perp; \rho + \varphi(x_3))$ via the Biot-Savart law. (As in Section 3 we omit the dependence on λ for simplicity.) Since $\boldsymbol{\Omega}^B$ has only the third component nonzero, (88) implies that $\tilde{\mathbf{U}}^B$ has only the first two components nonzero:

$$\begin{pmatrix} \tilde{U}^B_1(x; \rho, \varphi) \\ \tilde{U}^B_2(x; \rho, \varphi) \end{pmatrix} = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{1}{|x - y|^3} \begin{pmatrix} y_2 - x_2 \\ x_1 - y_1 \end{pmatrix} \Omega^B(y_\perp; \rho + \varphi(y_3)) \, \mathrm{d}y_\perp \, \mathrm{d}y_3 . \tag{94}$$

On the other hand, for any $x_3 \in \mathbf{R}$, $\mathbf{U}^B(x_\perp;\rho+\varphi(x_3))$ is obtained from $\mathbf{\Omega}^B(x_\perp;\rho+\varphi(x_3))$ via the two-dimensional Biot-Savart law (73), which can be written in the form

$$\begin{pmatrix} U_1^B(x_\perp;\rho+\varphi(x_3)) \\ U_2^B(x_\perp;\rho+\varphi(x_3)) \end{pmatrix} = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{1}{|x-y|^3}\begin{pmatrix} y_2-x_2 \\ x_1-y_1 \end{pmatrix} \Omega^B(y_\perp;\rho+\varphi(x_3))\,\mathrm{d}y_\perp\,\mathrm{d}y_3\ , \quad (95)$$

because

$$\int_{\mathbf{R}} \frac{1}{|x-y|^3}\,\mathrm{d}y_3 \equiv \int_{\mathbf{R}} \frac{1}{(|x_\perp-y_\perp|^2+(x_3-y_3)^2)^{3/2}}\,\mathrm{d}y_3 = \frac{2}{|x_\perp-y_\perp|^2}\ . \quad (96)$$

Using these representation formulas, it is easy to show that the velocity fields $\tilde{\mathbf{U}}^B$, $\mathbf{U}^B$ are close if the function φ varies sufficiently slowly.

Proposition 4.11 *Fix $\lambda \in [0,1)$, and assume that $\rho \in \mathbf{R}$ and $\varphi \in C_b^1(\mathbf{R})$ satisfy $|\rho| + \|\varphi\|_{L^\infty} \le R_1(\lambda)$, where R_1 is defined in Proposition 2.2. Then $\tilde{\mathbf{U}}^B(\cdot;\rho,\varphi) \in C_b^0(\mathbf{R}^3)$, and there exists $C(\lambda) > 0$ such that*

$$\sup_{x\in\mathbf{R}^3} |\tilde{\mathbf{U}}^B(x;\rho,\varphi)| \le C(|\rho| + \|\varphi\|_{L^\infty})\ , \quad (97)$$

$$\sup_{x\in\mathbf{R}^3} |\tilde{\mathbf{U}}^B(x;\rho,\varphi) - \mathbf{U}^B(x_\perp;\rho+\varphi(x_3))| \le C\|\varphi'\|_{L^\infty}\ . \quad (98)$$

Proof: Since $\Omega^B(x_\perp;\rho)$ is a continuous function of $x_\perp \in \mathbf{R}^2$ which decays rapidly as $|x_\perp| \to \infty$, uniformly in $\rho \in [-R_1, R_1]$, it is not difficult to verify that the velocity field $\tilde{\mathbf{U}}^B(x;\rho,\varphi)$ defined by (94) depends continuously on $x \in \mathbf{R}^3$. Next using (96) and (36) with $m = 1$, we find

$$|\tilde{\mathbf{U}}^B(x;\rho,\varphi)| \le \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{|x_\perp-y_\perp|}\frac{C(|\rho|+\|\varphi\|_{L^\infty})}{b(y_\perp)}\,\mathrm{d}y_\perp\ .$$

Since $b^{-1} \in L^p(\mathbf{R}^2)$ for all $p \in (1,\infty]$, the above integral is uniformly bounded for all $x_\perp \in \mathbf{R}^2$ (see Proposition 4.4), and we obtain (97).

Finally, taking the difference of (94) and (95), we see that $|\tilde{\mathbf{U}}^B - \mathbf{U}^B| \le C(D_1 + D_2)$, where

$$D_i(x) = \int_{\mathbf{R}^3} \frac{|x_i-y_i|}{|x-y|^3}\Big|\Omega^B(y_\perp;\rho+\varphi(y_3)) - \Omega^B(y_\perp;\rho+\varphi(x_3))\Big|\,\mathrm{d}y_\perp\,\mathrm{d}y_3\ , \quad i = 1,2\ .$$

But

$$\begin{aligned}\Big|\Omega^B(y_\perp;\rho+\varphi(y_3)) - \Omega^B(y_\perp;\rho+\varphi(x_3))\Big| &\le \Big|\int_{x_3}^{y_3} \partial_\rho\Omega^B(y_\perp;\rho+\varphi(z))\varphi'(z)\,\mathrm{d}z\Big| \\ &\le |x_3-y_3|\|\varphi'\|_{L^\infty}\sup_{|\rho|\le R_1}|\partial_\rho\Omega^B(y_\perp;\rho)|\ .\end{aligned}$$

Since

$$\int_{\mathbf{R}} \frac{|x_3-y_3|}{|x-y|^3}\,\mathrm{d}y_3 = \frac{2}{|x_\perp-y_\perp|}\ , \quad \text{and} \quad \frac{|x_i-y_i|}{|x_\perp-y_\perp|} \le 1\ ,$$

we thus find

$$\|D_i\|_{L^\infty} \le 2\|\varphi'\|_{L^\infty}\int_{\mathbf{R}^2}\sup_{|\rho|\le R_1}|\partial_\rho\Omega^B(y_\perp;\rho)|\,\mathrm{d}y_\perp\ , \quad i = 1,2\ . \quad (99)$$

Using now (36) with $m = 2$, we see that the integrand is bounded by $C/b(y_\perp)^2$, hence the integral in (99) is finite. This gives (98). □

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