Uniqueness of axisymmetric viscous flows originating from circular vortex filaments

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Abstract

The incompressible Navier-Stokes equations in $\mathbb{R}^3$ are shown to admit a unique axisymmetric solution without swirl if the initial vorticity is a circular vortex filament with arbitrarily large circulation Reynolds number. The emphasis is on uniqueness, as existence has already been established in [10]. The main difficulty which has to be overcome is that the nonlinear regime for such flows is outside of applicability of standard perturbation theory, even for short times. The solutions we consider are archetypal examples of viscous vortex rings, and can be thought of as axisymmetric analogues of the self-similar Lamb-Oseen vortices in two-dimensional flows. Our method provides the leading term in a fixed-viscosity short-time asymptotic expansion of the solution, and may in principle be extended so as to give a rigorous justification, in the axisymmetric situation, of higher-order formal asymptotic expansions that can be found in the literature [7].

1 Introduction

In three-dimensional ideal fluids, a *vortex ring* is an axisymmetric flow with the property that the vorticity is entirely concentrated in a solid torus, which moves with constant speed along the symmetry axis. The vortex lines form large circles that fill the torus, whereas fluid particles spin around the vortex core within perpendicular cross sections. If $\bar{r}, r$ denote the major and minor radii of the torus, respectively, and if Γ is the flux of the vorticity vector through any cross section, the "local induction approximation" gives the following expression for the translation speed along the axis

$$V \;=\; \frac{\Gamma}{4\pi\bar{r}}\Bigl(\log\frac{1}{\epsilon} + \mathcal{O}(1)\Bigr)\,, \tag{1.1}$$

which is valid in the asymptotic regime where the aspect ratio $\epsilon = r/\bar{r}$ is small. For the three-dimensional Euler equations, existence of large families of uniformly translating vortex ring solutions has been obtained using fixed point methods [12] or variational techniques [2, 13, 14], and formula (1.1) has been rigorously justified when $\epsilon \ll 1$ [12, 14]. In addition, for general initial data that are close enough to a vortex ring with small aspect ratio, it is known that the solution evolves in such a way that the vorticity remains sharply concentrated, for a relatively long time, near a vortex ring whose speed is given by (1.1), see [4].

The situation is quite different for viscous fluids, in which uniformly translating vortex rings cannot exist because all localized structures are eventually spread out by diffusion. In that case, however, it is quite natural to consider the initial value problem with a *vortex filament* as initial data, namely a vortex ring with infinitesimal cross section and yet nonzero circulation Γ, so that the initial vorticity is a measure supported by a circle of radius $\bar{r}$. It is then expected that the solution at time $t > 0$ will be close to a vortex ring with Gaussian vorticity profile and minor radius $r = \sqrt{\nu t}$, where ν is the kinematic viscosity. Moreover, this vortex will move along its symmetry axis at a speed given by (1.1), as long as the time-dependent aspect ratio $\epsilon = \sqrt{\nu t}/\bar{r}$ is sufficiently small.

Justifying these heuristic considerations requires some work. For singular initial data such as vortex filaments, the best available results on the Cauchy problem for the three-dimensional Navier-Stokes equations provide existence of a (unique and global) solution only if the circulation parameter Γ is small enough compared to viscosity, see [21, 25]. For larger values of Γ/ν, existence of a (global) axisymmetric solution without swirl has been recently obtained by H. Feng and the second author [10], using approximation techniques that do not give any information about uniqueness, even within the axisymmetric class. In this paper, our main purpose is to fill this gap and to prove that, if one starts from a circular vortex filament with arbitrary strength Γ, the Navier-Stokes equations have a *unique* axisymmetric solution without swirl, which is global and smooth for positive times. This axisymmetric solution is the archetype of a viscous vortex ring, just as the two-dimensional Lamb-Oseen solution is the archetype of a viscous columnar vortex [19]. Our approach is constructive and allows us to determine the leading term in the short-time asymptotic expansion of the vortex ring for a fixed viscosity. In principle, performing the calculations to higher orders in the spirit of Callegari and Ting's paper [7], one should be able to obtain more precise approximations of the solution that remain valid as long as the aspect ratio $\epsilon = \sqrt{\nu t}/\bar{r}$ is small enough. In particular, computing the next order after the leading term, we should recover the asymptotic formula (1.1) for the translation speed if $|\Gamma|/\nu \gg 1$. We leave this extension for future work.

To state our results in a more precise way, we start from the Navier-Stokes equations

$$\partial_t u + (u\cdot\nabla)u \,=\, \nu\Delta u - \frac{1}{\rho}\nabla p\,, \qquad \operatorname{div} u \,=\, 0\,, \tag{1.2}$$

in the whole space $\mathbb{R}^3$, where $u = u(x,t) \in \mathbb{R}^3$ denotes the velocity field and $p = p(x,t) \in \mathbb{R}$ is the internal pressure. Both the kinematic viscosity $\nu > 0$ and the fluid density $\rho > 0$ are assumed to be constant. We restrict ourselves to *axisymmetric solutions without swirl* for which the velocity field u and the vorticity $\omega = \operatorname{curl} u$ have the particular form:

$$u(x,t) \,=\, u_r(r,z,t)e_r + u_z(r,z,t)e_z\,, \qquad \omega(r,z,t) \,=\, \omega_\theta(r,z,t)e_\theta\,. \tag{1.3}$$

Here (r,θ,z) are the usual cylindrical coordinates in $\mathbb{R}^3$, such that $x = (r\cos\theta, r\sin\theta, z)$ for any $x \in \mathbb{R}^3$, and e_r, e_θ, e_z denote the unit vectors in the radial, toroidal, and vertical directions, respectively. The axisymmetric vorticity $\omega_\theta = \partial_z u_r - \partial_r u_z$ satisfies the evolution equation

$$\partial_t\omega_\theta + u\cdot\nabla\omega_\theta - \frac{u_r}{r}\omega_\theta \,=\, \nu\Big(\Delta\omega_\theta - \frac{\omega_\theta}{r^2}\Big)\,, \tag{1.4}$$

where $u\cdot\nabla = u_r\partial_r + u_z\partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$ is the axisymmetric Laplace operator in cylindrical coordinates. The velocity u can be expressed in terms of the axisymmetric vorticity ω_θ by solving the linear elliptic system

$$\partial_r u_r + \frac{1}{r}u_r + \partial_z u_z \,=\, 0\,, \qquad \partial_z u_r - \partial_r u_z \,=\, \omega_\theta\,, \tag{1.5}$$

in the half-plane $\Omega = \{(r,z) \in \mathbb{R}^2 \,|\, r > 0\,,\; z \in \mathbb{R}\}$. Boundary conditions for the quantities u_r, u_z, and ω_θ are prescribed by requiring that the vectorial functions u, ω in (1.3) be smooth across the symmetry axis $r = 0$. One finds that the radial velocity u_r and the axisymmetric vorticity ω_θ should satisfy the homogeneous Dirichlet condition on $\partial\Omega$, whereas the vertical velocity u_z satisfies the homogeneous Neumann condition.

Since the pioneering work of Ukhovskii and Yudovitch [32], and of Ladyzhenskaya [26], it is well known that the axisymmetric Navier-Stokes equations without swirl are globally well-posed for velocities in (appropriate subspaces of) the energy class, see also [1, 27] for further results in this direction. In the recent work [17], the Cauchy problem for the vorticity equation (1.4) is studied using scale invariant function spaces which emphasize the analogy with the two-dimensional vorticity equation. Following [17], we equip the half-plane Ω with the two-dimensional measure $\mathrm{d}r\,\mathrm{d}z$, as opposed to the three-dimensional measure $r\,\mathrm{d}r\,\mathrm{d}z$ which appears more naturally in cylindrical coordinates. In particular, for any $p \in [1,\infty)$, we denote by $L^p(\Omega)$ the space of measurable functions $\omega_\theta : \Omega \to \mathbb{R}$ such that

$$\|\omega_\theta\|_{L^p(\Omega)} \,:=\, \left(\int_\Omega |\omega_\theta(r,z)|^p \,\mathrm{d}r\,\mathrm{d}z\right)^{1/p} \,<\, \infty\,.$$

As usual, the limiting space $L^\infty(\Omega)$ is equipped with the essential supremum norm. We also denote by $\mathcal{M}(\Omega)$ the set of all real-valued finite regular measures on Ω, equipped with the total variation norm

$$\|\mu\|_{\mathrm{tv}} \,=\, \sup\left\{\int_\Omega \phi \,\mathrm{d}\mu \,\Big|\, \phi \in C_0(\Omega)\,,\; \|\phi\|_{L^\infty(\Omega)} \le 1\right\}\,,$$

where $C_0(\Omega)$ denotes the set of all real-valued continuous functions on Ω that vanish at infinity and on the boundary $\partial\Omega$. Clearly $L^1(\Omega)$ is a closed subspace of $\mathcal{M}(\Omega)$, and $\|\mu\|_{\mathrm{tv}} = \|\omega_\theta\|_{L^1(\Omega)}$ if $\mu = \omega_\theta\,\mathrm{d}r\,\mathrm{d}z$ for some $\omega_\theta \in L^1(\Omega)$.

As is proved in [17, Theorem 1.3], the Cauchy problem for the axisymmetric vorticity equation (1.4) is globally well-posed if the initial vorticity $\mu \in \mathcal{M}(\Omega)$ is a finite measure whose atomic part μ_{pp} satisfies $\|\mu_{pp}\|_{\mathrm{tv}} \le C_0\nu$, where $C_0 > 0$ is a universal constant. We are especially interested here in the particular situation where $\mu = \Gamma\,\delta_{(\bar r,\bar z)}$, which corresponds to a circular vortex filament of strength $\Gamma \in \mathbb{R}$ and radius $\bar r > 0$, centered at the origin in the affine plane $x_3 = \bar z \in \mathbb{R}$. In that case, we have $\|\mu_{pp}\|_{\mathrm{tv}} = \|\mu\|_{\mathrm{tv}} = |\Gamma|$, so that the results of [17] assert the existence of a unique global solution if $|\Gamma| \le C_0\nu$. On the other hand, for arbitrary values of the circulation parameter Γ, existence of a global solution to (1.4) was recently obtained by H. Feng and the second author [10], using approximation techniques which however do not give any information about uniqueness.

With this perspective in mind, our main result can now be stated as follows:

Theorem 1.1. *Fix $\Gamma \in \mathbb{R}$, $\bar r > 0$, $\bar z \in \mathbb{R}$, and $\nu > 0$. Then the axisymmetric vorticity equation* (1.4) *has a unique global mild solution $\omega_\theta \in C^0((0,\infty), L^1(\Omega) \cap L^\infty(\Omega))$ such that*

$$\sup_{t>0} \|\omega_\theta(t)\|_{L^1(\Omega)} \,<\, \infty\,, \qquad \text{and} \quad \omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \Gamma\,\delta_{(\bar r,\bar z)} \quad \text{as } t \to 0\,. \tag{1.6}$$

In addition, there exists a constant $C_1 > 0$, depending only on the ratio $|\Gamma|/\nu$, such that the following estimate holds:

$$\int_\Omega \left|\omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\, e^{-\frac{(r-\bar r)^2+(z-\bar z)^2}{4\nu t}}\right| \mathrm{d}r\,\mathrm{d}z \,\le\, C_1\,|\Gamma|\,\frac{\sqrt{\nu t}}{\bar r}\,\log\frac{\bar r}{\sqrt{\nu t}}\,, \tag{1.7}$$

as long as $\sqrt{\nu t} \le \bar r/2$.

We recall that a mild solution to (1.4) is a solution of the associated integral equation, see Definition 2.1 below. To clarify the scope of our result, a few comments are in order.

1. Theorem 1.1 can be seen as the axisymmetric counterpart of Proposition 1.3 in [19], which characterizes the Lamb-Oseen vortices among all viscous planar flows. However, unlike in the two-dimensional case, the vortex rings defined by (1.7) do not play any special role in the long-time dynamics of the vorticity equation (1.4).

2. As was already mentioned, existence of a global mild solution to (1.4) satisfying (1.6) was established in [10]. Uniqueness is thus the main new assertion in Theorem 1.1, together with the small time asymptotic expansion (1.7). It should be mentioned, however, that the techniques developed in Section 4, when properly adapted, can provide existence of a solution to (1.4) satisfying (1.6), using a standard fixed point argument which also gives uniqueness in a restricted class.

3. Assumptions (1.6) are the weakest ones under which the conclusions of Theorem 1.1 are expected to hold. Indeed, we recall that the L^1 norm of any solution to (1.4) is a nonincreasing function of time, see [17, Lemma 5.1], and it follows from (1.7) that $\|\omega_\theta(t)\|_{L^1(\Omega)} \to |\Gamma|$ as $t \to 0$, hence the first condition in (1.6) is clearly necessary. The second hypothesis states that $\omega_\theta(t)$ converges to $\Gamma\,\delta_{(\bar r,\bar z)}$ as $t \to 0$ in the weak-star topology of $\mathcal{M}(\Omega)$, which is usually referred to as the "weak convergence of measures". But since $\omega_\theta(t)$ is uniformly bounded in $L^1(\Omega)$, it is equivalent to suppose that convergence holds in the sense of distributions on Ω, and this is arguably the weakest way to specify the initial data.

4. The short time estimate (1.7) is sharp in the sense that the right-hand side cannot be replaced by $C_1|\Gamma|\epsilon$, where $\epsilon = \sqrt{\nu t}/\bar r$ is the aspect ratio at time t. This is because, in (1.7), we compare the solution $\omega_\theta(t)$ to a viscous vortex ring located at a fixed point $(\bar r, \bar z)$ in cylindrical coordinates, whereas we know that any vortex ring should move in the vertical direction at a speed given approximately by (1.1). In fact, it is possible to show that, if we replace in (1.7) the fixed vertical coordinate $\bar z$ by

$$\bar z(t) \;=\; \bar z + \frac{\Gamma t}{4\pi \bar r}\,\log\frac{\bar r}{\sqrt{\nu t}}\,,$$

then estimate (1.7) holds without the logarithmic term in the right-hand side. More generally, the Gaussian vorticity profile in (1.7) is only the first term in an asymptotic expansion of the solution $\omega_\theta(t)$ which, in principle, can be computed to arbitrary order in ϵ, see also the next comment below.

5. In estimate (1.7) convergence is expressed in the L^1 norm for simplicity, but in the proof we use a weighted L^2 norm in self-similar variables, which is considerably stronger and also implies approximation results for the velocity field associated with $\omega_\theta(t)$. On the other hand, we emphasize that (1.7) is a short time result at fixed viscosity, which cannot be used to describe the solution at fixed time $t > 0$ in the vanishing viscosity limit $\nu \to 0$, because the constant C_1 in the right-hand side strongly depends on the ratio $|\Gamma|/\nu$. Controlling the weakly viscous vortex ring over a fixed time interval is a different problem, which requires in particular constructing a much more precise approximation of the solution $\omega_\theta(t)$. We hope to address this interesting question in a future work.

6. It is worth emphasizing that the uniqueness statement is proved only withing the class of axisymmetric solutions. A natural question is whether uniqueness remains true among all (reasonable) solutions of (1.2) that approach the initial vortex filament in a suitable sense as $t \to 0$. For instance, one may assume that the velocity field $u(x,t)$ is smooth in $\mathbb{R}^3 \times (0,\infty)$, and satisfies the scale-invariant estimates listed in (2.14) below. As for the associated vorticity

$\omega(x,t)$, one may suppose (motivated by [21], see also Remark 5.2 below) that a natural quantity such as

$$\sup_{x\in\mathbb{R}^3}\sup_{R>0}\frac{1}{R}\int_{B_{x,R}}|\omega(y,t)|\,\mathrm{d}y \tag{1.8}$$

is uniformly bounded for $t>0$, and that $\omega(t)$ approaches the vortex filament in the sense of distributions as $t\to 0$. But even under these strong assumptions, it seems to be a difficult open problem to decide whether $u(x,t)$ has to be axisymmetric. It is conceivable that the symmetry of the initial data can be broken and, in addition to the axisymmetric solution, there is another solution which is not axisymmetric. In fact, the same question already arises for rectilinear vortices: the uniqueness problem when the initial vorticity is a (vertical) straight vortex filament, considered within the class of x_3-independent velocity fields of the form $(u_1,u_2,0)$, is the same as the 2d uniqueness and has been solved in [15, 16], but uniqueness among reasonable classes of 3d vector fields remains open.

The difficulties arise because the initial data do not belong to functions spaces where perturbation theory gives existence and uniqueness of local-in-time solutions for large data. Typical examples of function spaces (for the velocity field) where large data can be handled, locally in time, are the Lebesgue space $L^3(\mathbb{R}^3)$ or the Besov space $\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ for $p\in(3,\infty)$ and $q<\infty$. For the solutions we consider here the initial velocity field u_0 in $\mathbb{R}^3$ corresponding to the vortex ring given by $\mu=\Gamma\delta_{(\bar r,\bar z)}$ does not belong to spaces where local-in-time well-posedness can be established by existing perturbation results, unless Γ is small. It is easy to see that u_0 belongs to the Besov space $B^{-1}_{\infty,\infty}$, which is invariant under the Navier-Stokes scaling $u_0(x)\to\lambda u_0(\lambda x)$. However, this is not a good space for perturbation theory, even for small data, as shown in [6, 20]. With slightly more work one can see that $u_0\in\mathrm{BMO}^{-1}$. We show in Section 5.2 that, in fact, $u_0\in(L^\infty)^{-1}$. Therefore for small Γ one can obtain global existence and uniqueness (in suitable classes of functions) by applying the well-known results of [25]. One can also use [21], where the authors work with a Morrey-type space $M^{3/2}(\mathbb{R}^3)$ for the vorticity $\operatorname{curl} u_0$ (which gives BMO^{-1} for u_0, see Remark 5.2).

The case of large Γ is not covered by such considerations, as the perturbation theory in BMO^{-1} and similar spaces requires smallness of the initial data u_0, even for the local-in-time results. It is conjectured that this is not just due to some technical issues of the method, and that the Navier-Stokes equations are in fact not well-posed locally in time for general $u_0\in\mathrm{BMO}^{-1}$. In [23, 22] some evidence is given that local-in-time well-posedness, and, indeed, uniqueness, may fail already for initial data u_0 that are compactly supported, smooth away from the origin, and (-1)-homogeneous near the origin. Hence the uniqueness question (with respect to the 3d perturbations) can be raised a fortiori for vortex filaments, where the singularity of the initial data is not located at a single point but is spread over a whole curve.

Although the proof of Theorem 1.1 is the main purpose of this paper, we establish on the way several auxiliary results that have their own interest. As the existence part of the theorem is already settled in [10], we concentrate on uniqueness and short time asymptotics. Our presentation is structured as follows. In Section 2, we assume that we are given a mild solution $\omega_\theta\in C^0((0,T),L^1(\Omega)\cap L^\infty(\Omega))$ of (1.4) which is uniformly bounded in the space $L^1(\Omega)$. We recall some a priori estimates that were obtained in [17], and we show that $\omega_\theta(t)$ converges weakly, as $t\to 0$, to some (uniquely defined) Radon measure $\mu\in\mathcal{M}(\Omega)$. Next, using recent results on linear parabolic equations with singular divergence-free drifts [30], we prove that the solutions of the adjoint equation to (1.4) are continuous all the way to the initial time $t=0$, even at the symmetry axis $r=0$. This nontrivial result allows us to deduce that the family of measures $\omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z$ remains tight as $t\to 0$, so that no mass can escape to infinity nor concentrate on the symmetry axis.

In Section 3 we focus on the particular case where $\mu = \Gamma\,\delta_{(\bar r,\bar z)}$ for some $(\bar r,\bar z) \in \Omega$, assuming without loss of generality that $\Gamma > 0$. We prove that the solution $\omega_\theta(t)$ is strictly positive and satisfies, for any $\eta \in (0,1)$, the Gaussian bound

$$\omega_\theta(r,z,t) \,\le\, C\,\frac{\Gamma}{\nu t}\,\exp\Bigl(-\frac{1-\eta}{4\nu t}\bigl((r-\bar r)^2 + (z-\bar z)^2\bigr)\Bigr)\,, \tag{1.9}$$

where $C > 0$ depends only on η and on the ratio Γ/ν. In the two-dimensional case, estimates of the form (1.9) were obtained by Osada [28], see also [8]. Reproducing them in the axisymmetric case is not straightforward, because the left-hand side of (1.4) contains the zero order term $u_r\omega_\theta/r$ which is harmless only if one can prove that $\|u_r(t)/r\|_{L^\infty(\Omega)}$ is integrable in time. That property does not follow from the scale invariant a priori estimates on the solution, but we can show that it holds as soon as the support of the initial measure μ is bounded away from the symmetry axis, which is of course the case in our problem. Thus a minor modification of the method presented in [9] allows us to establish the Aronson type estimate (1.9).

Section 4 is devoted to the actual proof of Theorem 1.1. To study the behavior of the solution near the location $(\bar r,\bar z)$ of the initial vortex filament, we introduce self-similar variables via the transformation

$$\omega_\theta(r,z,t) \,=\, \frac{\Gamma}{\nu t}\,f\Bigl(\frac{r-\bar r}{\sqrt{\nu t}}\,,\,\frac{z-\bar z}{\sqrt{\nu t}}\,,\,t\Bigr)\,. \tag{1.10}$$

The rescaled vorticity $f(R,Z,t)$ defined by (1.10) is positive and, in view of (1.9), bounded from above by a Gaussian function. Using a compactness argument and a Liouville theorem established in [19], we show that $f(t)$ converges as $t \to 0$ to the Gaussian G defined by

$$G(R,Z) \,=\, \frac{1}{4\pi}e^{-(R^2+Z^2)/4}\,, \qquad (R,Z) \in \mathbb{R}^2\,. \tag{1.11}$$

Convergence holds in the weighted space $X = L^2(\mathbb{R}^2, G^{-1}\,\mathrm{d}R\,\mathrm{d}Z)$ which is continuously embedded in $L^1(\mathbb{R}^2)$, hence returning to the original variables we deduce that the left-hand side of (1.7) vanishes as $t \to 0$. We next use energy estimates to show that the difference $\|f(t) - G\|_X$ is $\mathcal{O}(\epsilon|\log\epsilon|)$, where $\epsilon = \sqrt{\nu t}/\bar r$, and this concludes the proof of (1.7). Finally, repeating the energy estimates for the difference of two solutions satisfying the assumptions of Theorem 1.1, we prove that $\|f_1(t) - f_2(t)\|_X = 0$ for sufficiently small times, and invoking a well-posedness result from [17] we conclude that the solutions coincide for all times, which gives uniqueness.

The final Section 5 is an appendix were the proofs of a few auxiliary results are collected for easy reference in the text.

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2 General properties of L^1-bounded solutions

In this section, we establish some preliminary results concerning mild solutions of (1.4) that are uniformly bounded in $L^1(\Omega)$. The class of solutions we consider is thus larger than what is necessary to prove Theorem 1.1, but the results presented here have their own interest, and are most naturally stated in this general framework. We first recall a few notations and estimates from the earlier works [10, 17].

2.1 The linear semigroup and the axisymmetric Biot-Savart law

As in [17], we denote by $(S(t))_{t\geq 0}$ the evolution semigroup defined by the linearized equation (1.4) with unit viscosity:

$$\partial_t \omega_\theta \;=\; \Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\omega_\theta\,, \tag{2.1}$$

which is considered in the half-plane $\Omega = \{(r,z) \in \mathbb{R}^2 \,|\, r > 0\,,\; z \in \mathbb{R}\}$ with homogeneous Dirichlet boundary condition on $\partial\Omega$. Using the explicit representation formula given in [17, Section 3], one can show that the semigroup $(S(t))_{t\geq 0}$ is strongly continuous in $L^p(\Omega)$ for all $p \in [1,\infty)$, and satisfies the same $L^p - L^q$ estimates as the heat semigroup in $\mathbb{R}^2$. In particular, if $\omega_0 \in L^p(\Omega)$ for some $p \in [1,\infty]$, then $S(t)\omega_0 \in L^q(\Omega)$ for all $t > 0$ and all $q \in [p,\infty]$, and there exists a constant $C_2 > 0$ such that

$$\|S(t)\omega_0\|_{L^q(\Omega)} \;\leq\; \frac{C_2}{t^{\frac{1}{p}-\frac{1}{q}}}\,\|\omega_0\|_{L^p(\Omega)}\,, \qquad t > 0\,, \tag{2.2}$$

see [17, Proposition 3.4]. Similarly, if $w = (w_r, w_z) \in L^p(\Omega)^2$, we have

$$\|S(t)\,\mathrm{div}_*\, w\|_{L^q(\Omega)} \;\leq\; \frac{C_2}{t^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}\,\|w_0\|_{L^p(\Omega)}\,, \qquad t > 0\,, \tag{2.3}$$

where $\mathrm{div}_*\, w = \partial_r w_r + \partial_z w_z$ denotes the two-dimensional divergence of the vector field w. Note that, when $1 \leq p < 2$, estimate (2.3) gives a better decay rate for large times than what is known for the heat semigroup in the same domain Ω with Dirichlet boundary condition. This illustrates the fact that the symmetry axis $r = 0$ is not a material boundary in our problem, but an artificial boundary resulting from a particular choice of coordinates. Also, to see the optimal decay rates, it is often useful to revert to the 3d picture and use that in our situation $\int_{\mathbb{R}^3} \omega(x,0)\,\mathrm{d}x = 0$, which gives some additional cancellation which may not be immediatelly transparent in the 2d picture.

On the other hand, if $\omega_\theta \in L^1(\Omega) \cap L^\infty(\Omega)$, it is shown in [10, 17] that the linear elliptic system (1.5), with homogeneous Dirichlet boundary condition for u_r and homogeneous Neumann condition for u_z, has a unique solution $u = (u_r, u_z) \in C^0(\Omega)^2$ vanishing at infinity. Moreover $u \in L^q(\Omega)$ for all $q > 2$, and there exists a constant $C_3 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \;\leq\; C_3 \|\omega_\theta\|_{L^1(\Omega)}^{1/2}\,\|\omega_\theta\|_{L^\infty(\Omega)}^{1/2}\,, \tag{2.4}$$

see [17, Proposition 2.3]. We call the map $\omega_\theta \mapsto u$ the axisymmetric Biot-Savart law, and we occasionally denote $u = \mathrm{BS}[\omega_\theta]$. Explicit formulas for u in terms of ω_θ can be found in Section 2 of both references [10, 17]. We also recall the following useful estimate: if $\omega_\theta \in L^1(\Omega)$ and $\omega_\theta/r \in L^\infty(\Omega)$, then $u_r/r \in L^\infty(\Omega)$ and

$$\Big\|\frac{u_r}{r}\Big\|_{L^\infty(\Omega)} \;\leq\; C_3 \|\omega_\theta\|_{L^1(\Omega)}^{1/3}\,\|\omega_\theta/r\|_{L^\infty(\Omega)}^{2/3}\,, \tag{2.5}$$

see [17, Proposition 2.6]. Needless to say, both inequalities (2.4), (2.5) are scale invariant.

Finally, it is important to note that, due to the divergence-free condition in (1.5), the evolution equation (1.4) can be written in the equivalent "conservation form"

$$\partial_t \omega_\theta + \mathrm{div}_*(u\,\omega_\theta) \;=\; \nu\Big(\partial_r^2 \omega_\theta + \partial_z^2 \omega_\theta + \partial_r \frac{\omega_\theta}{r}\Big)\,, \tag{2.6}$$

where again $\mathrm{div}_*(u\,\omega_\theta) = \partial_r(u_r\omega_\theta) + \partial_z(u_z\omega_\theta)$. We can thus define mild solutions of (1.4) in the following way:

Definition 2.1. *Given $T > 0$ and $\nu > 0$, we say that $\omega_\theta \in C^0((0,T), L^1(\Omega) \cap L^\infty(\Omega))$ is a* mild solution of (1.4) on $(0,T)$ *if the integral equation*

$$\omega_\theta(t) \;=\; S(\nu(t-t_0))\omega_\theta(t_0) - \int_{t_0}^t S(\nu(t-s))\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s \tag{2.7}$$

is satisfied whenever $0 < t_0 < t < T$. Here $u(s) = \mathrm{BS}[\omega_\theta(s)]$ for all $s \in (0,T)$.

In view of estimates (2.2), (2.3) and (2.4), it is clear that the integrand in (2.7) is an integrable function of $s \in [t_0, t]$ with values in $L^1(\Omega)\cap L^\infty(\Omega)$, and it follows that both sides of (2.7) belong to $C^0\big((t_0,T), L^1(\Omega)\cap L^\infty(\Omega)\big)$ for any $t_0 \in (0,T)$.

2.2 A priori estimates

From now on, we always assume that $\omega_\theta \in C^0((0,T), L^1(\Omega)\cap L^\infty(\Omega))$ is a mild solution of (1.4) on $(0,T)$ in the sense of Definition 2.1. We know from [17, Lemma 5.1] that the norm $\|\omega_\theta(t)\|_{L^1(\Omega)}$ is a nonincreasing function of time, and is even strictly decreasing unless ω_θ vanishes identically. We make the crucial assumption that ω_θ is *uniformly bounded in* $L^1(\Omega)$, so that we can define

$$M \;=\; \frac{1}{\nu}\lim_{t\to 0}\|\omega_\theta(t)\|_{L^1(\Omega)} \;<\; \infty\,. \tag{2.8}$$

We thus have $\|\omega_\theta(t)\|_{L^1(\Omega)} \le M\nu$ for all $t \in (0,T)$. Under this hypothesis, we can state the following a priori estimates, which exploit the particular structure of Eq. (1.4) and are only valid for axisymmetric flows without swirl.

Lemma 2.2. *For any mild solution of* (1.4) *on* $(0,T)$ *satisfying* (2.8)*, we have for all* $t \in (0,T)$*:*

$$\Big\|\frac{\omega_\theta(t)}{r}\Big\|_{L^\infty(\Omega)} \;\le\; \frac{C_4 M}{t\sqrt{\nu t}}\,, \qquad \text{and} \quad \|\omega_\theta(t)\|_{L^\infty(\Omega)} \;\le\; \frac{C_5(M)M}{t}\,, \tag{2.9}$$

where $C_4 > 0$ is a universal constant and $C_5 > 0$ depends on M.

Remark 2.3. *Here and in what follows we denote by $C_k(M)$ various quantities that are increasing functions of M satisfying $C_k(0) > 0$ and $C_k(M) \le C(1+M)^\sigma$ for some universal constants $C > 0$ and $\sigma > 0$. The precise value of the exponent σ does not play any role in our arguments.*

Proof. It is sufficient to prove (2.9) when $\nu = 1$, because the general case then follows by a simple rescaling argument. Due to parabolic smoothing, if $\omega_\theta \in C^0((0,T), L^1(\Omega)\cap L^\infty(\Omega))$ is a mild solution of (1.4), then ω_θ is smooth on $\Omega \times (0,T)$ and satisfies (1.4) in the classical sense. Applying Nash's method to the evolution equation satisfied by the quantity ω_θ/r, one obtains the following estimate:

$$\Big\|\frac{\omega_\theta(t)}{r}\Big\|_{L^\infty(\Omega)} \;\le\; \frac{C_4}{(t-t_0)^{3/2}}\,\|\omega_\theta(t_0)\|_{L^1(\Omega)} \;\le\; \frac{C_4 M}{(t-t_0)^{3/2}}\,, \tag{2.10}$$

for all $t \in (0,T)$ and all $t_0 \in (0,t)$, see [10, Lemma 3.8]. Thus taking the limit $t_0 \to 0$ we arrive at the first inequality in (2.9). Similarly, it follows from [17, Proposition 5.3] that

$$\|\omega_\theta(t)\|_{L^\infty(\Omega)} \;\le\; \frac{C_5(\|\omega_\theta(t_0)\|_{L^1(\Omega)})}{t-t_0}\,\|\omega_\theta(t_0)\|_{L^1(\Omega)} \;\le\; \frac{C_5(M)M}{t-t_0}\,, \tag{2.11}$$

for some function $C_5 > 0$ as specified in Remark 2.3. Taking again the limit $t_0 \to 0$ yields the second inequality in (2.9). $\square$

Combining Lemma 2.2 with estimates (2.4), (2.5), we easily obtain:

Corollary 2.4. *Under the assumptions of Lemma 2.2, we have for all $t \in (0,T)$:*

$$\Big\|\frac{u_r(t)}{r}\Big\|_{L^\infty(\Omega)} \le \frac{C_6 M}{t}\,, \qquad \textit{and} \quad \|u(t)\|_{L^\infty(\Omega)} \le C_7(M)M\sqrt{\frac{\nu}{t}}\,, \tag{2.12}$$

where $C_6 = C_3 C_4^{2/3}$ and $C_7(M) = C_3 C_5(M)^{1/2}$.

We also have scale-invariant estimates on the derivatives of the vorticity or the velocity. For instance, Proposition 5.5 in [17] asserts that

$$\|\nabla\omega_\theta(t)\|_{L^\infty(\Omega)} \le \frac{C_8(M)M}{t\sqrt{\nu t}}\,, \qquad 0 < t < T\,. \tag{2.13}$$

More generally, the velocity $u = u_r e_r + u_z e_z$ (considered as a function of $x \in \mathbb{R}^3$) satisfies, for all $k, \ell \in \mathbb{N}$,

$$\|\partial_t^k \nabla_x^\ell u(t)\|_{L^\infty(\mathbb{R}^3)} \le \frac{C_{k\ell}(M)M}{t^k(\nu t)^{\ell/2}}\sqrt{\frac{\nu}{t}}\,, \qquad 0 < t < T\,. \tag{2.14}$$

This bound can be deduced from the second estimate in (2.12) using general regularity results for the three-dimensional Navier-Stokes equations, as in the proof of [17, Proposition 5.5].

2.3 The trace of the solution at initial time

Using the a priori estimates established in the previous section, we now prove that any mild solution satisfying (2.8) converges as $t \to 0$ to some finite measure $\mu \in \mathcal{M}(\Omega)$.

Proposition 2.5. *If $\omega_\theta \in C^0((0,T), L^1(\Omega) \cap L^\infty(\Omega))$ is a mild solution of (1.4) on $(0,T)$ satisfying (2.8), there exists a unique measure $\mu \in \mathcal{M}(\Omega)$ such that $\omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \mu$ as $t \to 0$.*

Proof. We assume without loss of generality that $\nu = 1$. We first show that $\omega_\theta(t)$ has a limit as $t \to 0$ in $D'(\Omega)$, the space of all distributions on Ω. Let $\phi \in C_c^2(\Omega)$ be a C^2 function with compact support in Ω. Using (2.6) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \phi\,\omega_\theta \,\mathrm{d}r\,\mathrm{d}z = \int_\Omega \Big(u\cdot\nabla\phi + \partial_r^2\phi + \partial_z^2\phi - \frac{1}{r}\partial_r\phi\Big)\omega_\theta\,\mathrm{d}r\,\mathrm{d}z\,,$$

for all $t \in (0,T)$. As $\|\omega_\theta(t)\|_{L^1(\Omega)} \le M$ we have

$$\Big|\int_\Omega \Big(\partial_r^2\phi + \partial_z^2\phi - \frac{1}{r}\partial_r\phi\Big)\omega_\theta\,\mathrm{d}r\,\mathrm{d}z\Big| \le CM\|\nabla^2\phi\|_{L^\infty(\Omega)}\,,$$

for some universal constant $C > 0$, and using estimate (2.12) we also obtain

$$\Big|\int_\Omega u\cdot\nabla\phi\,\mathrm{d}r\,\mathrm{d}z\Big| \le \frac{C_7(M)M}{t^{1/2}}\,\|\nabla\phi\|_{L^\infty(\Omega)}\,.$$

This shows that the quantity $\int_\Omega \phi(r,z)\,\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z$ has a limit as $t \to 0$ for any $\phi \in C_c^2(\Omega)$, hence $\omega_\theta(t)$ converges in $D'(\Omega)$ to some limit which we denote by μ.

On the other hand, since $\omega_\theta(t)$ is uniformly bounded in $L^1(\Omega)$ by assumption, the Banach-Alaoglu theorem asserts that, for any sequence $t_m \to 0$, there exists a subsequence $t'_m \to 0$ and a measure $\bar\mu \in \mathcal{M}(\Omega)$ such that $\omega_\theta(t'_m)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \bar\mu$ as $m \to \infty$. But weak-star convergence in $\mathcal{M}(\Omega)$ implies convergence in $D'(\Omega)$, so we necessarily have $\bar\mu = \mu$, hence $\mu \in \mathcal{M}(\Omega)$. Moreover, this shows that the weak-star limit is independent of the choice of the subsequence $t'_m \to 0$, so that in fact $\omega_\theta(t_m)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \mu$ as $m \to \infty$. Since the sequence $t_m \to 0$ was arbitrary, this is the desired result. $\square$

Corollary 2.6. *Under the assumptions of Proposition 2.5, one has*

$$\omega_\theta(t) \;=\; S(\nu t)\mu - \int_0^t S(\nu(t-s))\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s\,, \qquad 0 < t < T\,. \tag{2.15}$$

Proof. We again assume that $\nu = 1$. For any fixed $t \in (0,T)$, our goal is to take the limit $t_0 \to 0$ in the integral representation (2.7), where both sides are considered as elements of $L^1(\Omega)$. The integral term is easily controlled using estimates (2.3), (2.4), and (2.12). We find

$$\begin{aligned}\int_{t_0}^t \|S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\|_{L^1(\Omega)}\,\mathrm{d}s \;&\le\; \int_{t_0}^t \frac{C_2}{(t-s)^{1/2}}\,\|u(s)\|_{L^\infty(\Omega)}\|\omega_\theta(s)\|_{L^1(\Omega)}\,\mathrm{d}s \\ &\le\; \int_0^t \frac{C_2 M}{(t-s)^{1/2}}\,\frac{C_7(M)M}{s^{1/2}}\,\mathrm{d}s \;=\; \pi C_2 C_7(M) M^2 \;<\; \infty\,,\end{aligned}$$

hence the integral term in (2.7) has a limit in $L^1(\Omega)$ as $t_0 \to 0$. To treat the other term, we decompose

$$S(t-t_0)\omega_\theta(t_0) \;=\; \Big(S(t-t_0) - S(t)\Big)\omega_\theta(t_0) + S(t)\omega_\theta(t_0)\,.$$

Using the explicit representation formula for the semigroup $S(t)$ given in [17, Section 3], it is quite straightforward to verify that

$$\|(S(t-t_0) - S(t))\omega_\theta(t_0)\|_{L^1(\Omega)} \;\le\; C\,\frac{t_0}{t}\,\|\omega_\theta(t_0)\|_{L^1(\Omega)} \;\xrightarrow[t_0\to 0]{}\; 0\,.$$

Moreover, it follows from Proposition 2.5 that

$$\Big(S(t)\omega_\theta(t_0)\Big)(r,z) \;\xrightarrow[t_0\to 0]{}\; (S(t)\mu)(r,z)\,, \qquad \text{for all } (r,z) \in \Omega\,,$$

and since the left-hand side of (2.7) does not depend on t_0 we deduce that convergence holds in $L^1(\Omega)$ too. So taking the limit $t_0 \to 0$ in (2.7) we obtain (2.15). □

Remark 2.7. *In view of Proposition 2.5, a natural question is whether a mild solution of* (1.4) *on* $(0,T)$ *satisfying* (2.8) *is uniquely determined by its "trace at initial time", namely by the measure* μ. *Using the results established in [17], one can show that the answer is positive if the atomic part of* μ *is small enough compared to viscosity. In the present paper, we focus on the particular case where* μ *is a single Dirac mass. The general case is still open.*

2.4 The adjoint equation

The aim of the section is to establish some important relations between a mild solution of (1.4) satisfying (2.8) and its initial trace given by Proposition 2.5. To do that, the idea is to consider Eq. (1.4) or (2.6) as a *linear* evolution equation for the axisymmetric vorticity ω_θ with a given advection field u, and to take the adjoint equation with respect to the scalar product in $L^2(\Omega,\,\mathrm{d}r\,\mathrm{d}z)$, namely

$$\partial_t\phi + u\cdot\nabla\phi + \nu\Big(\Delta\phi - \frac{2}{r}\,\partial_r\phi\Big) \;=\; 0\,. \tag{2.16}$$

We recall that $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$ is the axisymmetric Laplace operator. Eq. (2.16) is again considered as a linear equation, where the velocity field u is given and satisfies the bounds (2.12).

It is important to realize that the adjoint equation (2.16) can be solved backwards in time, imposing simultaneously Dirichlet and Neumann boundary conditions on $\partial\Omega$, and one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega_\theta(r,z,t)\phi(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \;=\; 0\,, \tag{2.17}$$

so that the solutions of (2.16) can be used as convenient test functions. If one thinks of the equation for ω_θ as a Kolmogorov-type equation for some stochastic process, then the adjoint equation (2.16) is the corresponding backward Kolmogorov equation.

A natural way to introduce the adjoint equation is to start from the three-dimensional vorticity equation

$$\partial_t \omega + [u, \omega] - \nu \Delta \omega \;=\; 0\,, \qquad x \in \mathbb{R}^3\,, \tag{2.18}$$

where we use the Lie bracket notation $[u, \omega] = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u$, and consider it as a linear equation for (general) vector fields ω, with u given. If we take the adjoint equation to (2.18) for (general) vector fields Φ, given by the requirement that $\int_{\mathbb{R}^3} \omega(x,t) \cdot \Phi(x,t)\,\mathrm{d}x$ be constant in time, we obtain

$$\partial_t \Phi + L_u \Phi + \nu \Delta \Phi \;=\; 0\,, \qquad x \in \mathbb{R}^3\,, \tag{2.19}$$

where $L_u \Phi$ is the Lie derivative of Φ along the vector field u when Φ is considered as a 1-form. In coordinates we have $(L_u \Phi)_i = u_j \partial_j \Phi_i + \Phi_j \partial_i u_j$, where we sum over repeated indices.

Due to estimates (2.14) we see from the standard linear parabolic theory that Eq. (2.19) can be solved backwards in time, for any bounded divergence-free "terminal data" Φ_1 at time $t_1 \in (0, T)$, and the solution Φ will be smooth in the open set $\mathbb{R}^3 \times (0, t_1)$. When u is axisymmetric with no swirl, then the fields ω of the form $\omega = \omega_\theta(r, z, t) e_\theta$ are preserved by (2.18) (considered as a linear equation for ω), and the same is true for (2.19) if $\Phi = \Phi_\theta(r, z, t)\, e_\theta$. For ω and Φ of this form we have

$$\int_{\mathbb{R}^3} \omega(x,t) \cdot \Phi(x,t)\;\mathrm{d}x \;=\; \int_\Omega \omega_\theta(r,z,t) \Phi_\theta(r,z,t) 2\pi r \,\mathrm{d}r\,\mathrm{d}z\,,$$

and therefore in (2.16) we should take

$$\phi \;=\; 2\pi r \Phi_\theta\,. \tag{2.20}$$

For the solutions we consider here, equation (2.16) is the same as (2.19) after the change of variables (2.20). Now, as Φ is smooth in $\mathbb{R}^3 \times (0, t_1)$, we must have $\Phi_\theta = rg$ for some smooth function $g = g(r, z, t)$ that is bounded on $\bar{\Omega}$ for any $t \in (0, t_1)$, and we conclude that the natural boundary condition for ϕ at $r = 0$ is that both ϕ and $\partial_r \phi$ vanish.

Alternatively, it is easy to verify that Eq. (2.16) is well-posed (backwards in time) under the Neumann boundary condition $\partial_r \phi(0, z, t) = 0$, and that the boundary data $a(z, t) = \phi(0, z, t)$ satisfy the equation

$$\partial_t a(z,t) + u_z(0,z,t) \partial_z a(z,t) + \nu \partial_z^2 a(z,t) \;=\; 0\,, \qquad z \in \mathbb{R}\,, \qquad t \in (0,T)\,.$$

In particular, if ϕ vanishes on the boundary $\partial\Omega$ at terminal time t_1, the same property holds for all $t \in (0, t_1)$, and as demonstrated above this is the natural condition under which (2.16) can be considered as the adjoint equation to (1.4) in Ω.

From now on, given $0 < t_1 < T$ and $\phi_1 \in C_0(\Omega)$, we denote by $\phi(r, z, t)$, for $(r, z) \in \Omega$ and $t \in (0, t_1)$, the unique solution of (2.16) with "terminal condition" $\phi(\cdot, \cdot, t_1) = \phi_1$. The main result of this subsection is:

Proposition 2.8. *Assume u is the velocity field associated with a mild solution ω_θ of* (1.4) *satisfying* (2.8). *Given $t_1 \in (0, T)$ and $\phi_1 \in C_0(\Omega)$, the unique solution ϕ of the (linear) adjoint equation* (2.16) *with terminal condition $\phi(\cdot, \cdot, t_1) = \phi_1$ can be extended to a continuous function on $\bar{\Omega} \times [0, t_1]$ satisfying $\phi(0, z, 0) = 0$ for all $z \in \mathbb{R}$. Moreover one has $\phi(\cdot, \cdot, t) \in C_0(\Omega)$ for all $t \in [0, t_1]$, and*

$$\sup_{(r,z) \in \Omega} |\phi(r,z,t) - \phi(r,z,0)| \;\xrightarrow[t \to 0]{}\; 0\,. \tag{2.21}$$

Proof. We can assume that $\nu = 1$ without loss of generality. As we have seen above, the standard parabolic theory applied to the form (2.19) of (2.16), together with estimates (2.14) for u, give the smoothness of ϕ for $t \in (0, t_1)$. The only issue is the possible deterioration of the estimates as $t \to 0$. We will use optimal regularity theory for linear parabolic equations with rough coefficients to overcome the difficulty.

To explain our strategy, consider the linear equation

$$\partial_t h + b(x,t) \cdot \nabla h + \Delta h \,=\, 0\,, \tag{2.22}$$

in $Q = B \times (0,1)$, where $B \subset \mathbb{R}^n$ is a unit ball and b is a drift term. Assume that

$$|\partial_t^k \nabla_x^\ell b| \,\le\, C_{k,\ell}\, t^{-k-\frac{\ell}{2}-\frac{1}{2}} \quad \text{in } Q\,, \quad \text{for } k,\ell = 0,1,2,\dots \tag{2.23}$$

This is a critical case for the regularity theory: if we could increase the exponent on the right-hand side by any positive number, no matter how small, the classical linear theory would imply that any bounded solution h is uniformly Hölder continuous in $Q_r = B_r \times (0, r^2)$ for any $r < 1$ (with estimates depending on r). On the other hand, without additional assumptions the condition (2.23) by itself may not be enough to arrive at that conclusion.

Luckily, the velocity field u in (2.16) has additional properties. First, it is divergence-free. Second, it is bounded in the space $L^\infty_t \mathrm{BMO}^{-1}_x$. It turns out that these two properties are sufficient to ensure the Hölder-continuity estimates we need. This is one of the main results in [30], see also [11]. Strictly speaking, the claim in [30, Theorem 1.1] is the parabolic Harnack inequality, but it is well-known that Hölder continuity is one of the easy consequences of the Harnack inequality. In fact, in the present situation, one can even prove that u is bounded in the space $L^\infty_t (L^\infty_x)^{-1}$, namely that $u = \operatorname{div} \Psi$ for some matrix-valued function Ψ that is bounded in space and time. More precisely, we have the following result, whose proof is postponed to Section 5.2.

Lemma 2.9. *Let $\omega = \omega_\theta e_\theta$ with $\omega_\theta \in L^1(\Omega)$, and let u be the velocity field obtained from ω via the three-dimensional Biot-Savart law. Then there exists $c > 0$ such that*

$$\|u\|_{(L^\infty)^{-1}(\mathbb{R}^3)} \,\le\, c \|\omega_\theta\|_{L^1(\Omega)}\,. \tag{2.24}$$

Estimate (2.24) is more than we need if we use the sharp results of [30], but it has its own interest and it shows that the more classical results of Osada [28] also apply to our situation.

In what follows we denote

$$\mathcal{O} \,=\, \bigl\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,\big|\, x_1^2 + x_2^2 > 0\bigr\}\,, \tag{2.25}$$

namely $\mathcal{O}$ is obtained from $\mathbb{R}^3$ be removing the symmetry axis $x_1 = x_2 = 0$. We observe that the vector field $\frac{1}{r} e_r$ is divergence-free and smooth in $\mathcal{O}$. Together with (2.24) this implies that in any parabolic ball $B_{x,\rho} \times (0,\rho^2) \subset \mathcal{O} \times (0,t_1)$ with $\rho < (x_1^2 + x_2^2)^{1/2}$ the adjoint equation (2.16) is of the form (2.22) with $b \in L^\infty_t \mathrm{BMO}^{-1}(B_{x,\rho})$ and $\operatorname{div} b = 0$, so that Theorem 1.1 in [30] can be applied.[1] This remark will be used freely in the proof below. Here and what follows, we consider ϕ and ϕ_1 as functions on $\mathbb{R}^3 \times (0,t_1)$ and $\mathbb{R}^3$, respectively.

From the above considerations we see that our solution ϕ satisfies the maximum principle:

$$|\phi(x,t)| \,\le\, \max_{y \in \mathbb{R}^3} |\phi_1(y)|\,, \qquad x \in \mathbb{R}^3\,, \quad t \in (0,t_1]\,, \tag{2.26}$$

[1] Theorem 1.1 in [30] is purely local, even though in the introduction of [30] a global condition $b \in L^\infty_t \mathrm{BMO}^{-1}_x$ is mentioned. In the proof one only needs the local condition. Also, when we are interested in the solution only in $B_{x,\rho} \times (0,\rho^2)$, we can change the field $\frac{1}{r} e_r$ outside $B_{x,\rho}$ to a smooth div-free vector field in $\mathbb{R}^3$, so that the global condition will in fact be satisfied (even though it is not needed).

and can be extended to a continuous function on $(\mathcal{O} \times [0,t_1]) \cup (\mathbb{R}^3 \times (0,t_1])$. The main point now is to prove its continuity at any point $(x,0)$ with $x \in \mathbb{R}^3 \setminus \mathcal{O}$. For any sufficiently small $\rho > 0$, we define

$$A(\rho) \,=\, \sup\bigl\{\phi(x,t) \,\big|\, x \in \mathcal{C}_\rho\,,~ 0 < t < \rho^2\bigr\}\,,$$

where $\mathcal{C}_\rho = \bigl\{x \in \mathbb{R}^3 \,\big|\, x_1^2 + x_2^2 \le \rho^2\bigr\}$ is the cylinder of radius ρ centered on the x_3–axis. Clearly A is an increasing function, so that we can consider the limit

$$a \,=\, \lim_{\rho\to 0} A(\rho) \,\ge\, 0\,.$$

We need to show that $a = 0$. (Once we have this, we can repeat the same argument for $-\phi$, and we conclude that ϕ can be extended to a continuous function on $\mathbb{R}^3 \times [0,t_1]$ satisfying $\phi(x,0) = 0$ for all $x \in \mathbb{R}^3 \setminus \mathcal{O}$.) As in many other critical problems, it is natural to argue by contradiction using the scaling invariance, see for example [24] for a situation where related issues arise in the context of the Navier-Stokes equations.

Assume thus that $a > 0$ and choose a sequence of points $(x^{(m)}, t^{(m)})$ such that $x^{(m)}$ approaches the x_3–axis, $t^{(m)} \searrow 0$, and

$$\lim_{m\to\infty} \phi(x^{(m)}, t^{(m)}) \,=\, a\,.$$

For $m \in \mathbb{N}$ we denote

$$\lambda_m \,=\, \sqrt{\bigl(x_1^{(m)}\bigr)^2 + \bigl(x_2^{(m)}\bigr)^2 + t^{(m)}}\,,$$

and we define, for $y \in \mathbb{R}^3$ and $0 < s < \lambda_m^{-2} t_1$,

$$u^{(m)}(y,s) \,=\, \lambda_m u(\lambda_m y_1, \lambda_m y_2, \lambda_m y_3 + x_3^{(m)}, \lambda_m^2 s)\,, \tag{2.27}$$

$$\phi^{(m)}(y,s) \,=\, \phi(\lambda_m y_1, \lambda_m y_2, \lambda_m y_3 + x_3^{(m)}, \lambda_m^2 s)\,. \tag{2.28}$$

Note that $x_3^{(m)}$ may not converge as $m \to \infty$, but this is unimportant for what follows. Setting

$$y^{(m)} \,=\, \bigl(\lambda_m^{-1} x_1^{(m)}, \lambda_m^{-1} x_2^{(m)}, 0\bigr)\,, \qquad s^{(m)} \,=\, \lambda_m^{-2} t^{(m)}\,,$$

we have $|y^{(m)}|^2 + s^{(m)} = 1$ for all m, and we can therefore assume (after extracting a subsequence, if necessary) that

$$(y^{(m)}, s^{(m)}) \xrightarrow[m\to\infty]{} (\bar{y}, \bar{s})\,, \qquad \text{where} \quad |\bar{y}|^2 + \bar{s} \,=\, 1\,.$$

Note that the operator $D := \frac{2}{r}\partial_r$ in (2.16) has the same scaling as the Laplacian, and is also invariant under translations along the x_3–axis. Using this observation, it is straightforward to verify that the functions $u^{(m)}, \phi^{(m)}$ defined in (2.27), (2.28) satisfy the equation

$$\partial_s \phi^{(m)} + u^{(m)} \cdot \nabla_y \phi^{(m)} + (\Delta_y - D_y)\phi^{(m)} \,=\, 0\,, \tag{2.29}$$

in the region $\mathbb{R}^3 \times (0, \lambda_m^{-2} t_1)$. Moreover, in view of (2.14) and (2.27), we have the a priori estimates

$$\|\partial_s^k \nabla_y^\ell u^{(m)}(s)\|_{L^\infty(\mathbb{R}^3)} \,\le\, C_{k\ell}(M) M\, s^{-k-\ell/2-1/2}\,, \qquad 0 < s < \lambda_m^{-2} t_1\,, \tag{2.30}$$

which are similar to (2.23) and hold uniformly in m. Finally, uniform bounds on $\phi^{(m)}$ and its derivatives are easily obtained by applying standard linear parabolic theory to Eq. (2.29), taking into account (2.26) and (2.30). Thus we can assume (again after choosing subsequences, if necessary) that

$$u^{(m)} \xrightarrow[m\to\infty]{} \bar u\,, \qquad \phi^{(m)} \xrightarrow[m\to\infty]{} \bar\phi\,, \tag{2.31}$$

for suitable functions $\bar u, \bar\phi$, where the convergence is uniform, with all derivatives, on compact subsets of $\mathbb{R}^3\times(0,\infty)$. (Note that $u^{(m)}, \phi^{(m)}$ are well defined on any such set once m is sufficiently large.) By construction, the functions $\bar u, \bar\phi$ satisfy $\partial_s\bar\phi + \bar u\cdot\nabla\bar\phi + (\Delta - D)\bar\phi = 0$ in $\mathbb{R}^3\times(0,\infty)$. Due to (2.24) and scale-invariance of the relevant norms we also have the uniform bound

$$\|u^{(m)}\|_{L^\infty_t \mathrm{BMO}^{-1}_x} \,\le\, c\|\omega_\theta\|_{L^\infty(0,T;L^1(\Omega))} \,\le\, cM\,,$$

which means, as we have seen above, that the functions $\phi^{(m)}$ are in fact uniformly continuous up to $t=0$ on any compact set $B\times[0,t_2]$ as long as $\bar B\subset\mathcal{O}$. Hence the function $\bar\phi$ is continuous on $(\mathcal{O}\times[0,t_1])\cup(\mathbb{R}^3\times(0,t_1])$, and it is clear from the definitions that $\bar\phi\le a$ in that domain. At the same time, we know that

$$\bar\phi(\bar y,\bar s) \,=\, \lim_{m\to\infty}\phi^{(m)}(y^{(m)},s^{(m)}) \,=\, \lim_{m\to\infty}\phi(x^{(m)},t^{(m)}) \,=\, a\,.$$

Finally, we have $\bar\phi(y,s) = \lim_{m\to\infty}\phi^{(m)}(y,s) = 0$ when $y\in\mathbb{R}^3\setminus\mathcal{O}$ and $s>0$.

Since we assumed that $a>0$, these observations immediately lead to a contradiction with the strong maximum principle when $\bar s>0$. It thus remains to deal with the case where $\bar s = 0$ and $|\bar y| = 1$. In that situation, the Harnack inequality from [30, Theorem 1.1] applied to the parabolic ball $Q = B_{\bar y,1/2}\times[0,1/4)$ shows that $\bar\phi = a$ in a neighborhood of $(\bar y,0)$ in Q, and we again obtain a contradiction with the strong maximum principle, as in the case $\bar s>0$. This concludes the proof of the assertion that $\phi(x,t)$ extends to a continuous function on $\mathbb{R}^3\times[0,t_1]$ satisfying $\phi(0,0)=0$.

To conclude the proof of Proposition 2.8, it remains to verify that $\phi(x,t)$ vanishes as $|x|\to\infty$ uniformly for all $t\in(0,t_1]$, which implies in particular (2.21) in view of the previous results. Since $\phi_1\in C_0(\Omega)$, this property is intuitively obvious because the drift term in Eq. (2.16) satisfies $\int_0^{t_1}\|u(\cdot,t)\|_{L^\infty}\,\mathrm{d}t<\infty$, and therefore can move "diffusion particles" over finite distances only, during the time interval $(0,t_1)$. This heuristic argument can easily be made rigorous if one proceeds as in [17, Proposition 6.1], see also Proposition 3.3 below. Alternatively, it is possible to reach the same conclusion using the parabolic Harnack inequality and the conservation of the mass $\int_{\mathbb{R}^3}\phi(x,t)\,\mathrm{d}x$, which can be checked by a direct calculation. We leave the details to the reader. □

In the rest of this section, we derive a few important consequences of Proposition 2.8. In view of (2.17), if ϕ is as in the statement, we have

$$\int_\Omega \omega_\theta(r,z,t)\phi(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, \int_\Omega \omega_\theta(r,z,t_0)\phi(r,z,t_0)\,\mathrm{d}r\,\mathrm{d}z\,, \qquad 0<t_0\le t\le t_1\,.$$

To take the limit $t_0\to 0$, we decompose the right-hand side as

$$\int_\Omega \omega_\theta(r,z,t_0)\big(\phi(r,z,t_0)-\phi(r,z,0)\big)\,\mathrm{d}r\,\mathrm{d}z + \int_\Omega \omega_\theta(r,z,t_0)\phi(r,z,0)\,\mathrm{d}r\,\mathrm{d}z\,,$$

and we observe that the first term tends to zero in view of (2.8), (2.21) while the second one converges to $\int_\Omega\phi(\cdot,\cdot,0)\,\mathrm{d}\mu$ by Proposition 2.5. We thus have

$$\int_\Omega \omega_\theta(r,z,t)\phi(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, \int_\Omega \phi(\cdot,\cdot,0)\,\mathrm{d}\mu\,, \qquad 0<t\le t_1\,. \tag{2.32}$$

Corollary 2.10. *If $\omega_\theta \in C^0((0,T), L^1(\Omega) \cap L^\infty(\Omega))$ is a mild solution of* (1.4) *on* $(0,T)$ *satisfying* (2.8)*, then $\|\omega_\theta(t)\|_{L^1(\Omega)} \le \|\mu\|_{\mathrm{tv}}$ for all $t \in (0,T)$, where $\mu \in \mathcal{M}(\Omega)$ is the measure introduced in Proposition 2.5. In particular, one has $M = \|\mu\|_{\mathrm{tv}}/\nu$ if M is defined by* (2.8)*.*

Proof. Fix $t_1 \in (0,T)$, and take $\phi_1 \in C_0(\Omega)$ such that $\|\phi_1\|_{L^\infty(\Omega)} \le 1$. Let $\phi : \Omega \times [0,t_1] \to \mathbb{R}$ be the solution of the adjoint equation (2.16) with terminal condition $\phi(\cdot,\cdot,t_1) = \phi_1$ given by Proposition 2.8. By the parabolic maximum principle, we know that $|\phi(r,z,t)| \le 1$ for all $(r,z) \in \Omega$ and all $t \in [0,t_1]$. It thus follows from (2.32) with $t = t_1$ that

$$\Big|\int_\Omega \omega_\theta(r,z,t_1)\phi_1(r,z)\,\mathrm{d}r\,\mathrm{d}z\Big| ~=~ \Big|\int_\Omega \phi(\cdot,\cdot,0)\,\mathrm{d}\mu\Big| ~\le~ \|\mu\|_{\mathrm{tv}}\,,$$

and taking the supremum over all $\phi_1 \in C_0(\Omega)$ satisfying the bound $\|\phi_1\|_{L^\infty(\Omega)} \le 1$ we conclude that $\|\omega_\theta(t_1)\|_{L^1(\Omega)} \le \|\mu\|_{\mathrm{tv}}$. Thus $M\nu = \lim_{t\to 0}\|\omega_\theta(t)\|_{L^1(\Omega)} \le \|\mu\|_{\mathrm{tv}}$, and the converse inequality directly follows from Proposition 2.5. ☐

Corollary 2.11. *If the measure μ given by Proposition 2.5 is positive, then the solution ω_θ of* (1.4) *satisfies $\omega_\theta(r,z,t) \ge 0$ for all $(r,z) \in \Omega$ and all $t \in (0,T)$.*

Proof. Assume on the contrary that $\omega_\theta(r_1,z_1,t_1) < 0$ for some $(r_1,z_1) \in \Omega$ and some $t_1 \in (0,T)$. Take a nonnegative function $\phi_1 \in C_0(\Omega)$ such that $\phi_1(r_1,z_1,t_1) = 1$ and ϕ_1 is supported in a small neighborhood of (r_1,z_1) where $\omega(\cdot,\cdot,t_1)$ takes negative values only. If ϕ denotes the solution of the adjoint equation (2.16) with terminal condition $\phi(\cdot,\cdot,t_1) = \phi_1$, we obtain a contradiction from Eq. (2.32) with $t = t_1$ because the left-hand side is strictly negative by construction, whereas the right-hand side is nonnegative since $\phi \ge 0$ and μ is a positive measure. ☐

Corollary 2.12. *The family of signed measures $(\omega_\theta(\cdot,\cdot,t)\,\mathrm{d}r\,\mathrm{d}z)_{t\in(0,T)}$ is tight under the assumptions of Proposition 2.5. In particular, the convergence*

$$\int_\Omega \phi(r,z)\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \xrightarrow[t\to 0]{} \int_\Omega \phi\,\mathrm{d}\mu \tag{2.33}$$

holds for any bounded and continuous function ϕ on Ω, and not just for any $\phi \in C_0(\Omega)$.

Proof. We use here some notions from measure theory that are recalled in Section 5.1, for the reader's convenience. The family of measures $(\omega_\theta(\cdot,\cdot,t)\,\mathrm{d}r\,\mathrm{d}z)_{t\in(0,T)}$ converges weakly to some measure $\mu \in \mathcal{M}(\Omega)$ by Proposition 2.5, and Corollary 2.11 implies that $\|\omega_\theta(t)\|_{L^1(\Omega)} \to \|\mu\|_{\mathrm{tv}}$ as $t \to 0$. Applying Proposition 5.1, we thus obtain the desired result. ☐

3 Gaussian estimates

As in the previous section, we assume that $\omega_\theta \in C^0((0,T), L^1(\Omega) \cap L^\infty(\Omega))$ is a mild solution of the axiymmetric vorticity equation (1.4) on the time interval $(0,T)$ satisfying (2.8), and we denote by $\mu \in \mathcal{M}(\Omega)$ the initial measure defined by Proposition 2.5. Our goal here is to give accurate estimates on the axisymmetric vorticity ω_θ and the associated velocity field $u = (u_r, u_z)$ under the additional hypotheses that μ is a *positive* measure whose support is *bounded away from the symmetry axis* $r = 0$ and *localized in the radial direction.* Of course, the application we have in mind is the case where μ is a Dirac mass located at some point $(\bar r, \bar z) \in \Omega$, which is the situation considered in Theorem 1.1.

3.1 L^1 estimates near the symmetry axis

The goal of this section is to show that the L^1norm of the axisymmetric vorticity ω_θ is extremely small near the symmetry axis for short times, if the initial measure μ is positive and supported away from the axis. The precise statement is:

Proposition 3.1. *Assume that $\mu \in \mathcal{M}(\Omega)$ is a positive measure whose support is contained in the set $[2\rho,\infty)\times\mathbb{R}\subset\Omega$ for some $\rho>0$. Then the solution ω_θ of* (1.4) *satisfies*

$$0 \le \int_0^\rho \left\{\int_{\mathbb{R}} \omega_\theta(r,z,t)\,\mathrm{d}z\right\}\mathrm{d}r \le C_9(M)\|\mu\|_{\mathrm{tv}}\, e^{-\frac{\rho^2}{16\nu t}}\,, \qquad t\in(0,T)\,, \tag{3.1}$$

for some positive constant C_9 depending only on $M=\|\mu\|_{\mathrm{tv}}/\nu$.

Proof. Without loss of generality we suppose that $\nu = 1$. Since μ is a positive measure, Corollary 2.11 asserts that the solution of (1.4) satisfies $\omega_\theta(r,z,t)\ge 0$ for all $(r,z)\in\Omega$ and all $t\in(0,T)$. As in [17, Section 6.1], we define

$$f(R,t) = \int_R^\infty \left\{\int_{\mathbb{R}} \omega_\theta(r,z,t)\,\mathrm{d}z\right\}\mathrm{d}r\,, \qquad R>0\,,\quad t\in(0,T)\,. \tag{3.2}$$

Then $f(R,t)$ is a nonincreasing function of R which converges to $\|\omega_\theta(t)\|_{L^1(\Omega)}$ as $R\to 0$ and to zero as $R\to\infty$. Moreover f satisfies the evolution equation

$$\partial_t f(R,t) = \partial_R^2 f(R,t) + \frac{1}{R}\,\partial_R f(R,t) + \int_{\mathbb{R}} u_r(R,z,t)\omega_\theta(R,z,t)\,\mathrm{d}z\,, \tag{3.3}$$

which follows easily from (2.6). Our goal is to obtain a lower bound on $f(\rho,t)$ under the assumption that the initial measure μ is supported in the set $[2\rho,\infty)\times\mathbb{R}$. This hypothesis already implies that $f(R,t)\to M=\|\mu\|_{\mathrm{tv}}$ as $t\to 0$ for any $R<2\rho$, because if $\phi:\Omega\to[0,1]$ is a continuous function equal to zero for $r\le R$ and to 1 for $r\ge 2\rho$, we have

$$M \ge f(R,t) \ge \int_\Omega \phi(r,z)\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \xrightarrow[t\to 0]{} \int_\Omega \phi\,\mathrm{d}\mu = M\,,$$

where the convergence follows from Corollary 2.12.

Using the bound $\|u_r(t)\|_{L^\infty(\Omega)} \le C_7(M)Mt^{-1/2}$, which comes from Corollary 2.4, and observing that $\partial_R f(R,t) = -\int_{\mathbb{R}}\omega_\theta(R,z,t)\,\mathrm{d}z \le 0$, we deduce from (3.3) that

$$\partial_t f(R,t) \ge \partial_R^2 f(R,t) + \frac{1}{R}\,\partial_R f(R,t) + C_7(M)\frac{M}{\sqrt{t}}\,\partial_R f(R,t)\,. \tag{3.4}$$

To eliminate the drift terms in (3.4), we fix $t_1\in(0,T)$ and we define $g(y,t)=f(y+a(t),t)$ for $y\ge 0$ and $t\in(0,t_1]$, where

$$a(t) = \rho + \frac{t_1 - t}{\rho} + 2C_7(M)M\bigl(\sqrt{t_1}-\sqrt{t}\bigr)\,, \qquad t\in[0,t_1]\,. \tag{3.5}$$

Note that $a(t)\ge\rho$ for $t\in[0,t_1]$ and $a(t_1)=\rho$. Using (3.4) and (3.5), it is easy to verify that

$$\partial_t g(y,t) \ge \partial_y^2 g(y,t)\,, \qquad y\ge 0\,,\quad t\in(0,t_1]\,,$$

and we obviously have $\partial_y g(0,t) = \partial_R f(a(t),t) \le 0$ for $t\in(0,t_1]$. In physical terms, the function $g(y,t)$ is a solution of the heat equation on the positive half-line with a nonnegative source term in the bulk and a nonnegative influx through the boundary. By the parabolic maximum principle, given any $t_0 \in (0, t_1)$, we thus have $g(y, t) \geq h(y, t)$ for all $y \geq 0$ and all $t \in [t_0, t_1]$, where h is defined by

$$\begin{cases} \partial_t h(y,t) \;=\; \partial_y^2 h(y,t)\,, & y \geq 0\,,\ t \geq t_0\,, \\ \partial_y h(0,t) \;=\; 0\,, & t \geq t_0\,, \\ h(y,t_0) \;=\; g(y,t_0) \;\equiv\; f(y + a(t_0), t_0)\,, & y \geq 0\,. \end{cases} \tag{3.6}$$

Solutions of (3.6) are easily computed by symmetrizing the initial data and solving the heat equation on the whole real line. In particular, this gives the desired lower bound on the quantity $f(\rho, t_1) = g(0, t_1)$.

To be more explicit, we first assume that the observation time t_1 is small enough so that

$$4t_1 \;\leq\; \rho^2\,, \qquad \text{and} \qquad 8C_7(M)M\sqrt{t_1} \;\leq\; \rho\,. \tag{3.7}$$

In view of (3.5) we then have $a(t_0) \leq a(0) \leq 3\rho/2$ for any $t_0 \in (0, t_1)$, and this in turn implies that $h(y, t_0) = f(y + a(t_0), t_0) \geq f(y + 3\rho/2, t_0)$ for all $y \geq 0$. Using the representation formula

$$h(0,t_1) \;=\; \frac{1}{\sqrt{\pi(t_1 - t_0)}} \int_0^\infty e^{-\frac{y^2}{4(t_1-t_0)}}\, h(y,t_0)\,\mathrm{d}y\,,$$

and recalling that $f(y + 3\rho/2, t_0) \to M$ as $t_0 \to 0$ for all $y < \rho/2$, we deduce that

$$f(\rho,t_1) \;\geq\; h(0,t_1) \;\geq\; \frac{M}{\sqrt{\pi t_1}} \int_0^{\rho/2} e^{-\frac{y^2}{4t_1}}\,\mathrm{d}y \;\geq\; M\Big(1 - e^{-\frac{\rho^2}{16t_1}}\Big)\,. \tag{3.8}$$

In the last inequality we used the elementary bound

$$\mathrm{erfc}(x) \;=\; \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2}\,\mathrm{d}y \;\leq\; e^{-x^2}\,, \qquad x \geq 0\,. \tag{3.9}$$

Since $\|\omega_\theta(t_1)\|_{L^1(\Omega)} \leq M = \|\mu\|_{\mathrm{tv}}$, we conclude that

$$\int_0^\rho \Big\{\int_{\mathbb{R}} \omega_\theta(r,z,t_1)\,\mathrm{d}z\Big\}\,\mathrm{d}r \;\leq\; M - f(\rho,t_1) \;\leq\; M\,e^{-\frac{\rho^2}{16t_1}}\,,$$

which gives the desired bound (3.1) with $t = t_1$ and $\nu = 1$, provided (3.7) holds. If condition (3.7) is not satisfied, one can take $C_9 = C_9(M) \geq e^{\rho^2/(16t_1)}$, in which case estimate (3.1) is obvious. ☐

Corollary 3.2. *Under the assumptions of Proposition 3.1 we have*

$$\Big\|\frac{u_r(t)}{r}\Big\|_{L^\infty(\Omega)} \;\leq\; \frac{C_{10}(M)M}{t}\Big(\frac{\nu t}{\rho^2}\Big)^{1/3}\,, \qquad t \in (0,T)\,, \tag{3.10}$$

where C_{10} is a positive constant depending only on $M = \|\mu\|_{\mathrm{tv}}/\nu$.

Proof. Fix $t \in (0, T)$. We decompose $\omega_\theta(r, z, t) = \omega_\theta^-(r, z, t) + \omega_\theta^+(r, z, t)$, where

$$\omega_\theta^-(r,z,t) \;=\; \omega_\theta(r,z,t)\,\mathbf{1}_{\{r\leq\rho\}}\,, \qquad \omega_\theta^+(r,z,t) \;=\; \omega_\theta(r,z,t)\,\mathbf{1}_{\{r>\rho\}}\,.$$

By linearity of the axisymmetric Biot-Savart law, there is a corresponding decomposition for the velocity field $u(r, z, t) = u^-(r, z, t) + u^+(r, z, t)$, where $u^\pm$ is the velocity associated with $\omega_\theta^\pm$, respectively. Using estimate (2.5), Proposition 3.1, and the first inequality in (2.9), we find

$$\begin{aligned} \Big\|\frac{u_r^-(t)}{r}\Big\|_{L^\infty(\Omega)} &\leq\; C_3\|\omega_\theta^-(t)\|_{L^1(\Omega)}^{1/3}\,\|\omega_\theta^-(t)/r\|_{L^\infty(\Omega)}^{2/3} \\ &\leq\; C_3\,C_4^{2/3}\,C_9(M)^{1/3}\,\frac{M}{t}\,e^{-\frac{\rho^2}{48\nu t}} \;\leq\; \frac{C(M)M}{t}\Big(\frac{\nu t}{\rho^2}\Big)^{1/3}\,. \end{aligned}$$

Similarly, using the second inequality in (2.9), we obtain

$$\Big\|\frac{u_r^+(t)}{r}\Big\|_{L^\infty(\Omega)} \le C_3\|\omega_\theta^+(t)\|_{L^1(\Omega)}^{1/3}\,\rho^{-2/3}\,\|\omega_\theta^+(t)\|_{L^\infty(\Omega)}^{2/3} \le C_3\,C_5(M)^{2/3}\,\frac{M}{t}\Big(\frac{\nu t}{\rho^2}\Big)^{1/3}.$$

Combining both estimates we arrive at (3.10). □

3.2 L^1 estimates away from the axis

We next consider the opposite case where the support of the initial measure μ is bounded in the radial direction. The analogue of Proposition 3.1 is:

Proposition 3.3. *Assume that $\mu \in \mathcal{M}(\Omega)$ is a positive measure whose support is contained in the set $(0, 2\rho] \times \mathbb{R} \subset \Omega$ for some $\rho > 0$. Then the solution ω_θ of (1.4) satisfies*

$$0 \le \int_{3\rho}^{\infty}\Big\{\int_{\mathbb{R}}\omega_\theta(r,z,t)\,\mathrm{d}z\Big\}\,\mathrm{d}r \le C_{11}(M)\|\mu\|_{\mathrm{tv}}\,e^{-\frac{\rho^2}{16\nu t}}, \qquad t \in (0,T), \tag{3.11}$$

for some positive constant C_{11} depending only on $M = \|\mu\|_{\mathrm{tv}}/\nu$.

Proof. We proceed as in the proof of Proposition 3.1, assuming again that $\nu = 1$. We observe that the function $f(R,t)$ defined in (3.2) satisfies the differential inequality

$$\partial_t f(R,t) \le \partial_R^2 f(R,t) - C_7(M)\frac{M}{\sqrt{t}}\,\partial_R f(R,t), \qquad R > 0, \tag{3.12}$$

which is obtained in the same way as the lower bound (3.4). Arguing as in [17, Section 6.1], we deduce from (3.12) that, for any $t_0 \in (0,T)$,

$$f(R,t) \le g(R - 2C_7(M)M\sqrt{t}, t), \qquad R > 0, \quad t_0 \le t < T,$$

where $g(y,t)$ is the solution of the heat equation $\partial_t g = \partial_y^2 g$ on the real line $\mathbb{R}$ with initial data satisfying $g(y,t_0) = f(y,t_0)$ if $y \ge 0$ and $g(y,t_0) = f(0,t_0)$ if $y < 0$. Taking the limit $t_0 \to 0$ in the representation formula

$$g(y,t) = \frac{1}{\sqrt{4\pi(t-t_0)}}\Big(\int_{-\infty}^{0} e^{-\frac{(y-r)^2}{4(t-t_0)}} f(0,t_0)\,\mathrm{d}r + \int_0^\infty e^{-\frac{(y-r)^2}{4(t-t_0)}} f(r,t_0)\,\mathrm{d}r\Big),$$

and using the bound $f(R,t_0) \le M$ together with the fact that $f(R,t_0) \to 0$ as $t_0 \to 0$ if $R > 2\rho$, which can be established by applying (2.33) to a continuous function $\phi : \Omega \to [0,1]$ equal to 0 for $r \le 2\rho$ and to 1 for $r \ge R$, we deduce that

$$g(y,t) \le \frac{M}{\sqrt{4\pi t}}\int_{-\infty}^{2\rho} e^{-(y-r)^2/(4t)}\,\mathrm{d}y, \qquad y \in \mathbb{R}, \quad t \in (0,T),$$

hence

$$f(R,t) \le \frac{M}{\sqrt{4\pi t}}\int_{-\infty}^{2\rho} e^{-(R-2C_7(M)M\sqrt{t}-r)^2/(4t)}\,\mathrm{d}r, \qquad R > 0, \quad t \in (0,T). \tag{3.13}$$

If $t > 0$ is small enough so that $2C_7(M)M\sqrt{t} \le \rho/2$, it follows from (3.13), (3.9) that

$$f(3\rho,t) \le \frac{M}{\sqrt{4\pi t}}\int_{-\infty}^{2\rho} e^{-(5\rho/2-r)^2/(4t)}\,\mathrm{d}r \le M\,e^{-\rho^2/(16t)},$$

which is (3.11). If $2C_7(M)M\sqrt{t} > \rho/2$, then (3.11) follows from the trivial bound $f(3\rho,t) \le M$, provided the constant C_{11} is chosen appropriately. □

Corollary 3.4. *Under the assumptions of Proposition 3.3, the axisymmetric vorticity ω_θ has a finite impulse*

$$\mathcal{I} \,=\, \int_\Omega r^2\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, \int_\Omega r^2\,\mathrm{d}\mu(r,z)\,, \qquad t\in(0,T)\,. \tag{3.14}$$

In particular, the impulse $\mathcal{I}$ is a conserved quantity.

Proof. We assume that $\nu = 1$. Let $\chi : [0,\infty) \to \mathbb{R}$ be a smooth, nonincreasing function such that $\chi(r) = 1$ for $r \in [0,1]$ and $\chi(r) = 0$ for $r \ge 2$. Using definition (3.2) and integrating by parts we obtain the identity

$$\int_\Omega r^2\chi(r/R)\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, \int_0^\infty r\Big(2\chi(r/R) + (r/R)\chi'(r/R)\Big)f(r,t)\,\mathrm{d}r\,, \tag{3.15}$$

which holds for all $R > 0$ and all $t \in (0,T)$. For any fixed $t \in (0,T)$, we know from (3.13) that $f(R,t)$ decays rapidly to zero at infinity, thus taking the limit $R \to \infty$ in (3.15) we obtain

$$\int_\Omega r^2\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, 2\int_0^\infty rf(r,t)\,\mathrm{d}r \,<\, \infty\,, \qquad t\in(0,T)\,. \tag{3.16}$$

The left-hand side of (3.16) is the total impulse $\mathcal{I}$ of the axisymmetric vorticity ω_θ, which is known to be conserved under the evolution defined by (1.4), see e.g. [17, Lemma 6.4].

On the other hand, for any fixed $R > 2\rho$, the left-hand side of (3.15) converges as $t \to 0$ to the quantity

$$\mathcal{I}_0 \,=\, \int_\Omega r^2\chi(r/R)\,\mathrm{d}\mu(r,z) \,\equiv\, \int_\Omega r^2\,\mathrm{d}\mu(r,z)\,.$$

Convergence holds by Corollary 2.12, and the limit does not depend on $R > 2\rho$ since the measure μ is supported in $(0,2\rho] \times \mathbb{R}$. In fact $\mathcal{I}_0 = \mathcal{I}$, because the convergence of (3.15) to (3.16) as $R \to \infty$ holds *uniformly in time* if $t > 0$ is sufficiently small. Indeed, if $2C_7(M)M\sqrt{t} \le \rho$, it follows from (3.13), (3.9) that $f(R,t) \le M\,e^{-(R-3\rho)^2/(4t)}$ for all $R \ge 3\rho$, which in turns implies that the quantity $\int_R^\infty rf(r,t)\,\mathrm{d}r$ converges to zero uniformly in time as $R \to \infty$. This proves the uniform convergence of the right-hand side of (3.15) to that of (3.16) as $R \to \infty$. □

The next step is a general estimate for nonnegative solutions of (2.6) with finite impulse.

Proposition 3.5. *Assume that $\omega_\theta \in C^0((0,T), L^1(\Omega)\cap L^\infty(\Omega))$ is a nonnegative solution of* (2.6) *which is uniformly bounded in $L^1(\Omega)$ and has finite impulse $\mathcal{I}$. Then*

$$\|\omega_\theta(t)\|_{L^1(\Omega)} \,\le\, \frac{C_{12}(M)\mathcal{I}}{\nu t}\,, \qquad \textit{for all } t\in(0,T)\,, \tag{3.17}$$

where C_{12} is a positive constant depending only on the quantity M defined in (2.8).

Proof. The proof is essentially contained in [17, Section 6.2], although estimate (3.17) is not explicitly stated there. For completeness we provide here the missing details, assuming as usual that $\nu = 1$. We first observe that it is sufficient to establish (3.17) for $t \ge T_* = \mathcal{I}/M$, because for smaller times we obviously have $\|\omega_\theta(t)\|_{L^1(\Omega)} \le M \le \mathcal{I}/t$. We start from the integral equation (2.7) with $t_0 = t/2$, namely

$$\omega_\theta(t) \,=\, S(t/2)\omega_\theta(t/2) - \int_{t/2}^t S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s\,, \qquad t\ge T_*\,. \tag{3.18}$$

To bound the first term in the right-hand side, we use the linear estimate

$$\|S(t)\omega_0\|_{L^1(\Omega)} \le \frac{C}{t}\int_\Omega r^2\omega_0(r,z)\,\mathrm{d}r\,\mathrm{d}z\,, \qquad t>0\,,$$

which holds for all nonnegative $\omega_0 \in L^1(\Omega)$ with finite impulse, and can be established using the explicit formula for the linear semigroup $S(t)$ given in [17, Section 3], see [17, Lemma 6.5] for a similar calculation. We thus have $\|S(t/2)\omega_\theta(t/2)\|_{L^1(\Omega)} \le C\mathcal{I}/t$ for some $C>0$. On the other hand, applying the weighted inequality given in [17, Proposition 3.5], which has no analogue in the two-dimensional case, we find

$$\|S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\|_{L^1(\Omega)} \le \frac{C}{(t-s)^{3/4}}\,\|u(s)\|_{L^\infty(\Omega)}\|r^{1/2}\omega_\theta(s)\|_{L^1(\Omega)}\,,$$

for $s\in(0,t)$. If we now interpolate $\|r^{1/2}\omega_\theta\|_{L^1} \le \|r^2\omega_\theta\|_{L^1(\Omega)}^{1/4}\|\omega_\theta\|_{L^1(\Omega)}^{3/4}$ and use the estimate

$$\|u\|_{L^\infty(\Omega)} \le C\|r^2\omega_\theta\|_{L^1(\Omega)}^{1/4}\|\omega_\theta\|_{L^1(\Omega)}^{1/4}\|\omega_\theta/r\|_{L^\infty(\Omega)}^{1/2}\,,$$

which is established in [10, Section 2], we obtain

$$\|S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\|_{L^1(\Omega)} \le \frac{C}{(t-s)^{3/4}}\,\|r^2\omega_\theta(s)\|_{L^1(\Omega)}^{1/2}\|\omega_\theta(s)\|_{L^1(\Omega)}\|\omega_\theta(s)/r\|_{L^\infty(\Omega)}^{1/2}\,.$$

As $\|r^2\omega_\theta(s)\|_{L^1(\Omega)} = \mathcal{I}$ and $\|\omega_\theta(s)/r\|_{L^\infty(\Omega)} \le C_4 M s^{-3/2}$ by Lemma 2.2, we deduce from (3.18) that

$$\|\omega_\theta(t)\|_{L^1(\Omega)} \le \frac{C\mathcal{I}}{t} + CM^{1/2}\mathcal{I}^{1/2}\int_{t/2}^t \frac{\|\omega_\theta(s)\|_{L^1(\Omega)}}{(t-s)^{3/4}s^{3/4}}\,\mathrm{d}s\,, \qquad t\ge T_*\,. \tag{3.19}$$

The end of the proof is a straightforward bootstrap argument. First, since $\|\omega_\theta(t)\|_{L^1(\Omega)} \le M$, estimate (3.19) shows that $\|\omega_\theta(t)\|_{L^1(\Omega)} \le C(M)M^{1/2}\mathcal{I}^{1/2}t^{-1/2}$ for $t\ge T_*$, hence also for all $t>0$. Inserting this bound into the right-hand side of (3.19), we conclude that $\|\omega_\theta(t)\|_{L^1(\Omega)} \le C(M)\mathcal{I}/t$, which is the desired result. □

Corollary 3.6. *Under the assumptions of Proposition 3.3 we have*

$$\Big\|\frac{u_r(t)}{r}\Big\|_{L^\infty(\Omega)} \le \frac{C_{13}(M)M}{t}\Big(\frac{\rho^2}{\nu t}\Big)^{1/3}\,, \qquad t\in(0,T)\,, \tag{3.20}$$

where C_{13} is a positive constant depending only on $M = \|\mu\|_{\mathrm{tv}}/\nu$.

Proof. Since $\mathrm{supp}(\mu)\subset(0,2\rho]\times\mathbb{R}$, Corollary 3.4 shows that $\mathcal{I}\le 4\rho^2\|\mu\|_{\mathrm{tv}} = 4\rho^2 M\nu$. Thus estimate (3.20) immediately follows from (2.5), (3.17), and the first inequality in (2.9). □

3.3 Gaussian estimates for the viscous vortex ring

Finally, we consider the particular case where the initial measure μ is a vortex filament located at some point $(\bar r,\bar z)\in\Omega$, namely $\mu = \Gamma\delta_{(\bar r,\bar z)}$ for some $\Gamma>0$. We of course have $\|\mu\|_{\mathrm{tv}} = \Gamma$, hence $M=\Gamma/\nu$, and $\mathcal{I} = \Gamma\bar r^2$. The goal of this section is to prove the following Gaussian estimate on the axisymmetric vorticity:

Proposition 3.7. *Assume that $\omega_\theta\in C^0((0,T),L^1(\Omega)\cap L^\infty(\Omega))$ is a mild solution of (1.4) which is uniformly bounded in $L^1(\Omega)$, and such that $\omega_\theta(\cdot,t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \Gamma\delta_{(\bar r,\bar z)}$ as $t\to 0$ for some $\Gamma>0$ and some $(\bar r,\bar z)\in\Omega$. For any $\eta\in(0,1)$ we have the pointwise estimate*

$$0 < \omega_\theta(r,z,t) \le K_\eta(M)\,\frac{\Gamma}{\nu t}\,\exp\Big(-\frac{1-\eta}{4\nu t}\big((r-\bar r)^2+(z-\bar z)^2\big)\Big)\,, \tag{3.21}$$

for all $t\in(0,T)$ and all $(r,z)\in\Omega$, where the constant $K_\eta(M)$ depends only on η and $M=\Gamma/\nu$.

As a first step in the proof of Proposition 3.7, we apply the results of Sections 3.1 and 3.2 with $\rho = \bar{r}/2$ and obtain the following integral estimate:

Lemma 3.8. *Under the assumptions of Proposition 3.7 we have*

$$\int_0^T \|u_r(t)/r\|_{L^\infty(\Omega)} \, \mathrm{d}t \;\le\; C_{14}(M)M\,, \tag{3.22}$$

where C_{14} is a positive constant depending only on $M = \Gamma/\nu$.

Proof. Let $T_* = \bar{r}^2/\nu = 4\rho^2/\nu$. Using estimate (3.10) for $t \in (0, T_*)$ and, if necessary, estimate (3.20) for $t \in (T_*, T)$, we immediately obtain (3.22). $\square$

To derive estimate (3.21) it is convenient to abandon the cylindrical coordinates and to return for a moment to the vector valued vorticity $\omega(x,t) = \omega_\theta(r,z,t)e_\theta$, which is considered as a function of $x = (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3$ and $t \in (0,T)$. The evolution equation (1.4) is equivalent to

$$\partial_t\omega + (U\cdot\nabla)\omega - V\omega \;=\; \nu\Delta\omega\,, \qquad x \in \mathbb{R}^3\,, \qquad t \in (0,T)\,, \tag{3.23}$$

where $U = u_r e_r + u_z e_z$ is the velocity field associated with ω via the three-dimensional Biot-Savart law, and $V = u_r/r$. Since the pioneering work of Aronson [3], which relied itself on previous results by Nash, De Giorgi, and Moser, it is well known that solutions of advection-diffusion equations such as (3.23) can be represented in terms of a (uniquely defined) fundamental solution Φ, which is Hölder continuous in space and time and satisfies Gaussian upper and lower bounds. In our problem we only have limited information on the advection field U and the potential V, and we need an upper bound on the fundamental solution with explicit dependence on the data U, V, and ν. For that reason, we state here a particular case of Aronson's estimates which is tailored to our purposes.

Proposition 3.9. *Assume that $U : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n$ and $V : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n$ are continuous functions such that $\mathrm{div}\, U(\cdot,t) = 0$ for all $t \in (0,T)$ and*

$$\sup_{0<t<T}\Bigl(\frac{t}{\nu}\Bigr)^{1/2}\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \;=\; K_1 \;<\; \infty\,, \qquad \int_0^T \|V(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}\,\mathrm{d}t \;=\; K_2 \;<\; \infty\,. \tag{3.24}$$

Then the (regular) solutions of the advection-diffusion equation

$$\partial_t f + (U\cdot\nabla)f - Vf \;=\; \nu\Delta f\,, \qquad x \in \mathbb{R}^n\,, \qquad t \in (0,T)\,, \tag{3.25}$$

can be represented in the following way:

$$f(x,t) \;=\; \int_{\mathbb{R}^n} \Phi_{U,V,\nu}(x,t;y,s) f(y,s)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^n\,, \qquad 0 < s < t < T\,,$$

where the fundamental solution $\Phi_{U,V,\nu}(x,t;y,s)$ satisfies, for $x, y \in \mathbb{R}^n$ and $0 < s < t < T$,

$$0 \;<\; \Phi_{U,V,\nu}(x,t;y,s) \;\le\; \frac{C_n}{(\nu(t-s))^{n/2}}\exp\Bigl(-\frac{|x-y|^2}{4\nu(t-s)} + K_1\frac{|x-y|}{\sqrt{\nu(t-s)}} + K_2\Bigr)\,. \tag{3.26}$$

Here the constant C_n depends only on the space dimension n.

For completeness, we give a short proof of Proposition 3.9 in Section 5.3 below, but our purpose here is to apply it to the vorticity equation (3.23), for which $n = 3$. In view of Corollary 2.4 and Lemma 3.8, both assumptions in (3.24) are satisfied, and the constants K_1, K_2 depend only on $M = \Gamma/\nu$. Solutions of (3.23) can thus be represented in the following way:

$$\omega(x,t) \,=\, \int_{\mathbb{R}^3} \Phi(x,t;y,s)\omega(y,s)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^3\,, \qquad 0 < s < t < T\,,$$

and the fundamental solution Φ satisfies (3.26) with $n = 3$. As $\omega(x,t) = \omega_\theta(r,z,t)e_\theta$, we deduce that the axisymmetric vorticity ω_θ satisfies

$$\omega_\theta(r,z,t) \,=\, \int_\Omega \tilde{\Phi}(r,z,t;r',z',s)\omega_\theta(r',z',s)\,\mathrm{d}r'\,\mathrm{d}z'\,, \tag{3.27}$$

for $(r,z) \in \Omega$ and $0 < s < t < T$, where

$$\tilde{\Phi}(r,z,t;r',z',s) \,=\, \int_{-\pi}^{\pi} \Phi([r,0,z],t;[r'\cos\theta, r'\sin\theta, z'],s)r'\cos\theta\,\mathrm{d}\theta\,. \tag{3.28}$$

Here $[r,0,z]$ denotes the point $x \in \mathbb{R}^3$ with coordinates $x_1 = r$, $x_2 = 0$, $x_3 = z$, and similarly $[r'\cos\theta, r'\sin\theta, z']$ denotes the point $y \in \mathbb{R}^3$ such that $y_1 = r'\cos\theta$, $y_2 = r'\sin\theta$, $y_3 = z'$.

Lemma 3.10. *For any $\eta \in (0,1)$ there exists a positive constant $K_\eta(M)$, depending only on η and M, such that the fundamental solution $\tilde{\Phi}$ defined in* (3.28) *satisfies*

$$0 \,<\, \tilde{\Phi}(r,z,t;r',z',s) \,\le\, \frac{K_\eta(M)}{\nu(t-s)}\,\frac{r'^{1/2}}{r^{1/2}}\,\tilde{H}\Big(\frac{\nu(t-s)}{(1-\eta)rr'}\Big)\,e^{-\frac{1-\eta}{4\nu(t-s)}\left((r-r')^2+(z-z')^2\right)}\,, \tag{3.29}$$

for $(r,z), (r',z') \in \Omega$ and $0 < s < t < T$, where

$$\tilde{H}(\tau) \,=\, \frac{1}{\sqrt{\pi\tau}}\int_{-\pi/4}^{\pi/4} e^{-\frac{\sin^2\phi}{\tau}}\cos(2\phi)\,\mathrm{d}\phi\,, \qquad \tau > 0\,. \tag{3.30}$$

Proof. The positivity of the fundamental solution $\tilde{\Phi}$ of equation (1.4) is a consequence of the strong maximum principle. To obtain the upper bound (3.29), we start from (3.28) and first observe that

$$\tilde{\Phi}(r,z,t;r',z',s) \,\le\, \int_{-\pi/2}^{\pi/2} \Phi([r,0,z],t;[r'\cos\theta, r'\sin\theta, z'],s)r'\cos\theta\,\mathrm{d}\theta\,, \tag{3.31}$$

because $\cos\theta \le 0$ when $\pi/2 \le |\theta| \le \pi$. Next, we estimate the integrand using (3.26) with $n = 3$. Applying Young's inequality we obtain, for any $\eta \in (0,1)$,

$$\Phi(x,t;y,s) \,\le\, \frac{C}{(\nu(t-s))^{3/2}}\,e^{-(1-\eta)\frac{|x-y|^2}{4\nu(t-s)}+\frac{K_1^2}{\eta}+K_2} \,=\, \frac{K_\eta(M)}{(\nu(t-s))^{3/2}}\,e^{-(1-\eta)\frac{|x-y|^2}{4\nu(t-s)}}\,.$$

Here we take $x = [r,0,z]$ and $y = [r'\cos\theta, r'\sin\theta, z']$, so that

$$|x-y|^2 \,=\, |r-r'|^2 + |z-z'|^2 + 4rr'\sin^2(\theta/2)\,.$$

Thus we deduce from (3.31) that

$$\begin{aligned}\tilde{\Phi}(r,z,t;r',z',s) \,&\le\, \int_{-\pi/2}^{\pi/2} \Phi([r,0,z],t;[r'\cos\theta, r'\sin\theta, z'],s)r'\cos\theta\,\mathrm{d}\theta \\ &\le\, \frac{K_\eta(M)}{(\nu(t-s))^{3/2}}\,e^{-\frac{1-\eta}{4\nu(t-s)}\left((r-r')^2+(z-z')^2\right)}\int_{-\pi/2}^{\pi/2} e^{-\frac{(1-\eta)rr'}{\nu(t-s)}\sin^2(\theta/2)}r'\cos\theta\,\mathrm{d}\theta\,.\end{aligned}$$

Setting $\theta = 2\phi$ and using definition (3.30), we arrive at (3.29) with a modified constant $K_\eta(M)$. ∎

Remark 3.11. *The function $\tilde H$ in Lemma 3.10 is not the same as the function H defined in [17, Section 3]. One can show that $\tilde H : (0,\infty) \to \mathbb{R}$ is decreasing with $\tilde H(\tau) \to 1$ as $\tau \to 0$ and $\tilde H(\tau) \sim 1/\sqrt{\pi\tau}$ as $\tau \to \infty$. Moreover $\tilde H(\tau) \le 1/\sqrt{\pi\tau}$ for all $\tau > 0$.*

Proof of Proposition 3.7. Fix $(r,z) \in \Omega$ and $t \in (0,T)$. Using the representation (3.27) and the bound (3.29), we obtain for all $s \in (0,t)$:

$$\omega_\theta(r,z,t) \,\le\, \frac{K_\eta(M)}{\nu(t-s)} \int_\Omega \frac{r'^{1/2}}{r^{1/2}}\, \tilde H\Bigl(\frac{\nu(t-s)}{(1-\eta)rr'}\Bigr)\, e^{-\frac{1-\eta}{4\nu(t-s)}\bigl((r-r')^2+(z-z')^2\bigr)}\omega_\theta(r',z',s)\,\mathrm{d}r'\,\mathrm{d}z'\,.$$

If $r' \le 2r$ in the right-hand side, we bound the function $\tilde H$ by 1. If $r' \ge 2r$ we use the fact that $\tilde H(\tau) \le 1/\sqrt{\pi\tau}$, so that

$$\frac{r'^{1/2}}{r^{1/2}}\, \tilde H\Bigl(\frac{\nu(t-s)}{(1-\eta)rr'}\Bigr) \,\le\, \frac{r'}{\sqrt{\pi}}\Bigl(\frac{1-\eta}{\nu(t-s)}\Bigr)^{1/2} \,\le\, C_\eta\, e^{\frac{\eta(1-\eta)}{4\nu(t-s)}(r-r')^2}\,,$$

because $r'^2 \le 4(r-r')^2$ and $x \le C_\eta\, e^{\eta x^2/4}$ for any $x \ge 0$. We thus obtain the simpler estimate

$$\begin{aligned}\omega_\theta(r,z,t) \,&\le\, \frac{K_\eta(M)}{\nu(t-s)}\int_\Omega e^{-\frac{(1-\eta)^2}{4\nu(t-s)}\bigl((r-r')^2+(z-z')^2\bigr)}\omega_\theta(r',z',s)\,\mathrm{d}r'\,\mathrm{d}z'\\ &\le\, \frac{K_\eta(M)}{\nu(t-s)}\int_\Omega e^{-\frac{(1-\eta)^2}{4\nu t}\bigl((r-r')^2+(z-z')^2\bigr)}\omega_\theta(r',z',s)\,\mathrm{d}r'\,\mathrm{d}z'\,,\end{aligned}$$

with possibly a different constant $K_\eta(M)$. We now take the limit $s \to 0$ and use the assumption that $\omega_\theta(\cdot,\cdot,s)\,\mathrm{d}r'\,\mathrm{d}z' \rightharpoonup \Gamma\delta_{(\bar r,\bar z)}$, together with Corollary 2.12. We thus obtain an upper bound of the form (3.21), where η is replaced by $\tilde\eta = 2\eta - \eta^2$. Finally, as was already observed, the positivity of ω_θ is a consequence of the strong maximum principle. □

4 Self-similar variables and energy estimates

This section is devoted to the actual proof of Theorem 1.1. Using the existence result established in [10], we can assume that $\omega_\theta \in C^0((0,T), L^1(\Omega)\cap L^\infty(\Omega))$ is a mild solution of (1.4) which is uniformly bounded in $L^1(\Omega)$ and converges weakly to $\Gamma\delta_{(\bar r,\bar z)}$ as $t \to 0$, for some $\Gamma > 0$ and some $(\bar r,\bar z) \in \Omega$. If M is defined by (2.8), we recall that $M = \Gamma/\nu$ by Corollary 2.10. The Gaussian estimate in Proposition 3.7 indicates that, for short times, the axisymmetric vorticity $\omega_\theta(r,z,t)$ concentrates in a self-similar way around the initial position $(\bar r,\bar z)$ of the vortex filament. A natural idea is thus to introduce self-similar variables, in order to analyze more accurately the short-time behavior of the solution.

4.1 Definitions and a priori estimates

Motivated by (3.21), we set

$$\omega_\theta(r,z,t) \,=\, \frac{\Gamma}{\nu t}\, f\Bigl(\frac{r-\bar r}{\sqrt{\nu t}}\,,\,\frac{z-\bar z}{\sqrt{\nu t}}\,,\,t\Bigr), \qquad (r,z)\in\Omega\,, \quad t \in (0,T)\,. \tag{4.1}$$

We also introduce the important notation

$$\epsilon \,=\, \frac{\sqrt{\nu t}}{\bar r}\,, \qquad \gamma \,=\, \frac{\Gamma}{\nu}\,, \qquad R \,=\, \frac{r-\bar r}{\sqrt{\nu t}}\,, \qquad Z \,=\, \frac{z-\bar z}{\sqrt{\nu t}}\,. \tag{4.2}$$

The dimensionless quantity ϵ is the ratio of the typical core thickness of the vortex ring at time t to the radius of the initial vortex filament. We are interested in the regime where ϵ is small, and most of our analysis actually deals with the limit as $\epsilon \to 0$. The ratio γ of the vortex strength Γ to the viscosity ν is sometimes called the "circulation Reynolds number" in the physical literature. It is also dimensionless, and coincides in the present case with the quantity M defined in (2.8), but we find it natural to keep both symbols γ, M in what follows because, conceptually, they denote rather different quantities. Finally, the dimensionless variables (R,Z) are new coordinates centered at the position of the vortex filament, where distances are measured in units of the core thickness $\sqrt{\nu t}$. Note that the domain constraint $r > 0$ translates into $1 + \epsilon R > 0$, which means that the rescaled vorticity $f(R,Z,t)$ given by (4.1) is actually defined in the time-dependent domain $\Omega_\epsilon = \{(R,Z) \in \mathbb{R}^2 \,|\, 1 + \epsilon R > 0\}$, which converges to $\mathbb{R}^2$ as $\epsilon \to 0$. However, since the function $f(R,Z,t)$ satisfies the homogeneous Dirichlet condition at the boundary $R = -1/\epsilon$, we can extend it by zero outside that domain and thereby identify it with a function $\bar{f}(R,Z,t)$ which is now defined on the whole plane $\mathbb{R}^2$, for any $t \in (0,T)$.

In view of (3.21), given any $\eta \in (0,1)$, the rescaled vorticity $f(R,Z,t)$ satisfies

$$0 \,<\, f(R,Z,t) \,\le\, K_\eta(M)\, e^{-\frac{1-\eta}{4}(R^2+Z^2)}\,, \tag{4.3}$$

for all $(R,Z) \in \Omega_\epsilon$ and all $t \in (0,T)$. Moreover, it follows from (2.13) that the spatial derivatives of f are uniformly bounded:

$$|\nabla f(R,Z,t)| \,\le\, C_8(M)\,. \tag{4.4}$$

Finally, using (2.8) and (4.1), we obtain

$$\int_{\Omega_\epsilon} f(R,Z,t)\,\mathrm{d}R\,\mathrm{d}Z \,=\, \frac{1}{\Gamma}\int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,\xrightarrow[t\to 0]{}\, \frac{M\nu}{\Gamma} \,=\, 1\,. \tag{4.5}$$

It is also useful to express the velocity field u associated with ω_θ in self-similar variables. The correct ansatz is:

$$u(r,z,t) \,=\, \frac{\Gamma}{\sqrt{\nu t}}\, U^\epsilon\Big(\frac{r-\bar{r}}{\sqrt{\nu t}}\,,\, \frac{z-\bar{z}}{\sqrt{\nu t}}\,,\, t\Big)\,, \qquad (r,z) \in \Omega\,, \quad t \in (0,T)\,, \tag{4.6}$$

where $U^\epsilon = U_r^\epsilon e_r + U_z^\epsilon e_z$ denotes the rescaled velocity field. We use here the superscript ϵ to keep in mind that, in the new variables, the Biot-Savart law depends explicitly on time through the parameter $\epsilon = \sqrt{\nu t}/\bar{r}$. Indeed, for any $t \in (0,T)$, the velocity U^ϵ satisfies the elliptic system

$$\partial_Z U_r^\epsilon - \partial_R U_z^\epsilon \,=\, f\,, \qquad \partial_R U_r^\epsilon + \frac{\epsilon U_r^\epsilon}{1+\epsilon R} + \partial_Z U_z^\epsilon \,=\, 0\,, \tag{4.7}$$

in the domain Ω_ϵ, together with the boundary conditions $U_r^\epsilon = \partial_R U_z^\epsilon = 0$ on $\partial\Omega_\epsilon$. In view of (2.12), we have the following uniform a priori estimate

$$|U^\epsilon(R,Z,t)| \,\le\, C_7(M)\,, \qquad (R,Z) \in \Omega_\epsilon\,, \quad t \in (0,T)\,. \tag{4.8}$$

In fact, estimate (4.8) can be improved as follows.

Lemma 4.1. *The rescaled velocity field defined in* (4.6) *satisfies*

$$(1 + |R| + |Z|)\,|U^\epsilon(R,Z,t)| \,\le\, C_{15}(M)\,, \qquad (R,Z) \in \Omega_\epsilon\,, \quad t \in (0,T)\,, \tag{4.9}$$

where C_{15} depends only on M.

Proof. If u is the velocity field associated with the vorticity ω_θ via the axisymmetric Biot-Savart law, it is shown in [17, Proposition 2.3] that

$$|u(r,z)| \,\le\, \int_\Omega \frac{C}{\sqrt{(r-r')^2+(z-z')^2}}\,|\omega_\theta(r',z')|\,\mathrm{d}r'\,\mathrm{d}z'\,, \qquad (r,z)\in\Omega\,,$$

where $C>0$ is a universal constant. Using the change of variables (4.1) and (4.6), we deduce that, for any $\epsilon>0$,

$$|U^\epsilon(R,Z)| \,\le\, \int_{\Omega_\epsilon} \frac{C}{\sqrt{(R-R')^2+(Z-Z')^2}}\,|f(R',Z')|\,\mathrm{d}R'\,\mathrm{d}Z'\,, \qquad (R,Z)\in\Omega_\epsilon\,. \tag{4.10}$$

In view of (4.10), estimate (4.9) follows easily from the Gaussian bound (4.3). $\square$

In Section 4.4 below we need accurate estimates on the difference $U^\epsilon - U^0$, where U^0 denotes the velocity field obtained from f via the Biot-Savart law on $\mathbb{R}^2$. To prove such bounds, we use a rather explicit representation for the solution of (4.7), which we now derive.

4.2 The parametrized Biot-Savart law

We look for a solution of (4.7) in the form

$$U_r^\epsilon \,=\, -\frac{\partial_Z\phi^\epsilon}{1+\epsilon R}\,, \qquad U_z^\epsilon \,=\, \frac{\partial_R\phi^\epsilon}{1+\epsilon R}\,, \tag{4.11}$$

where $\phi^\epsilon : \Omega_\epsilon \to \mathbb{R}$ is the axisymmetric stream function, which satisfies the second-order elliptic equation

$$-\frac{\partial_R^2\phi^\epsilon}{1+\epsilon R} + \frac{\epsilon\partial_R\phi^\epsilon}{(1+\epsilon R)^2} - \frac{\partial_Z^2\phi^\epsilon}{1+\epsilon R} \,=\, f\,, \tag{4.12}$$

in the domain Ω_ϵ, with both Dirichlet and Neumann conditions on the boundary $\partial\Omega_\epsilon$. The solution of (4.12) can be computed as in [10, Section 2] and is found to be

$$\phi^\epsilon(R,Z) \,=\, \frac{1}{2\pi}\int_{\Omega_\epsilon}\sqrt{(1{+}\epsilon R)(1{+}\epsilon R')}\,F\biggl(\epsilon^2\frac{(R{-}R')^2+(Z{-}Z')^2}{(1{+}\epsilon R)(1{+}\epsilon R')}\biggr)f(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,, \tag{4.13}$$

where $F:(0,\infty)\to(0,\infty)$ is defined by

$$F(s) \,=\, \int_0^{\pi/2}\frac{1-2\sin^2\psi}{\sqrt{\sin^2\psi+s/4}}\,\mathrm{d}\psi \,=\, \begin{cases}\log\frac{8}{\sqrt{s}} - 2 + \mathcal{O}(s\log s) & \text{as } s\to 0\,,\\ \frac{\pi}{2s^{3/2}} + \mathcal{O}(s^{-5/2}) & \text{as } s\to\infty\,.\end{cases} \tag{4.14}$$

Differentiating (4.13) with respect to R and Z, and using (4.11), we obtain

$$\begin{aligned} U_r^\epsilon(R,Z) \,&=\, \frac{1}{2\pi}\int_{\Omega_\epsilon}\sqrt{\frac{1{+}\epsilon R'}{1{+}\epsilon R}}\,\tilde F(\xi^2)\,\frac{Z-Z'}{(R{-}R')^2+(Z{-}Z')^2}\,f(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,,\\ U_z^\epsilon(R,Z) \,&=\, -\frac{1}{2\pi}\int_{\Omega_\epsilon}\sqrt{\frac{1{+}\epsilon R'}{1{+}\epsilon R}}\,\tilde F(\xi^2)\,\frac{R-R'}{(R{-}R')^2+(Z{-}Z')^2}\,f(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\\ &\quad + \frac{\epsilon}{4\pi}\int_{\Omega_\epsilon}\frac{\sqrt{1{+}\epsilon R'}}{(1+\epsilon R)^{3/2}}\bigl(F(\xi^2)+\tilde F(\xi^2)\bigr)f(R',Z')\,\mathrm{d}R'\,\mathrm{d}Z'\,, \end{aligned} \tag{4.15}$$

where ξ^2 is a shorthand notation for the quantity

$$\xi^2 \,=\, \epsilon^2\frac{(R{-}R')^2+(Z{-}Z')^2}{(1{+}\epsilon R)(1{+}\epsilon R')}\,, \tag{4.16}$$

and $\tilde{F} : (0,\infty) \to (0,\infty)$ is defined by

$$\tilde{F}(s) \;=\; -2sF'(s) \;=\; \begin{cases} 1 + \mathcal{O}(s\log s) & \text{as } s \to 0\,, \\ \frac{3\pi}{2s^{3/2}} + \mathcal{O}(s^{-5/2}) & \text{as } s \to \infty\,. \end{cases} \tag{4.17}$$

For simplicity we write $U^\epsilon = \mathrm{BS}^\epsilon[f]$ when (4.15) holds.

When $\epsilon \to \infty$, the domain Ω_ϵ shrinks to the half-plane Ω, and (4.15) coincides with the axisymmetric Biot-Savart law, which is studied e.g. in [10, Section 2]. In contrast, as $\epsilon \to 0$, the domain Ω_ϵ expands to the full plane $\mathbb{R}^2$, and in this limit (4.15) reduces to the usual two-dimensional Biot-Savart law:

$$U^0(R,Z) \;=\; \begin{pmatrix} U_r^0(R,Z) \\ U_z^0(R,Z)\end{pmatrix} \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \begin{pmatrix} Z-Z' \\ R'-R\end{pmatrix} \frac{f(R',Z')}{(R{-}R')^2 + (Z{-}Z')^2}\,\mathrm{d}R'\,\mathrm{d}Z'\,, \tag{4.18}$$

which we denote $U^0 = \mathrm{BS}^0[f]$. Thus the ϵ-dependent Biot-Savart law defined by (4.7) or (4.15) nicely interpolates between the axisymmetric case and the two-dimensional case.

We now compare the velocity fields U^ϵ and U^0 obtained from the same vorticity distribution.

Lemma 4.2. *Assume that f vanishes outside Ω_ϵ. If $U^\epsilon = \mathrm{BS}^\epsilon[f]$ and $U^0 = \mathrm{BS}^0[f]$, we have, for all $(R,Z) \in \Omega_\epsilon$,*

$$|U^\epsilon(R,Z) - U^0(R,Z)| \;\le\; \int_{\Omega_\epsilon} \frac{C\epsilon}{1+\epsilon R}\Bigl(1 + \log_+\frac{1+\epsilon R}{\epsilon\rho}\Bigr)|f(R',Z')|\,\mathrm{d}R'\,\mathrm{d}Z'\,, \tag{4.19}$$

where $\rho = \sqrt{(R{-}R')^2 + (Z{-}Z')^2}$ and $\log_+(x) = \max(\log(x),0)$.

Proof. Since f is supported in Ω_ϵ by assumption, the integrals in (4.15), (4.18) are taken over the same domain. Thus, all we need is to subtract (4.18) from (4.15) and to estimate the various terms in the difference, using the following elementary bounds

$$\Bigl|\sqrt{\frac{1{+}\epsilon R'}{1{+}\epsilon R}}\,\tilde{F}(\xi^2) - 1\Bigr| \;\le\; \sqrt{\frac{1{+}\epsilon R'}{1{+}\epsilon R}}\Bigl|\tilde{F}(\xi^2) - 1\Bigr| + \Bigl|\sqrt{\frac{1{+}\epsilon R'}{1{+}\epsilon R}} - 1\Bigr| \;\le\; C\frac{\epsilon\rho}{1{+}\epsilon R}\,, \tag{4.20}$$

$$\frac{\epsilon\sqrt{1{+}\epsilon R'}}{(1{+}\epsilon R)^{3/2}}\Bigl|F(\xi^2) + \tilde{F}(\xi^2)\Bigr| \;\le\; C\frac{\epsilon}{1{+}\epsilon R}\Bigl(1 + \log_+\frac{1{+}\epsilon R}{\epsilon\rho}\Bigr)\,. \tag{4.21}$$

Estimate (4.20) easily follows from the bound $|\tilde{F}(\xi^2){-}1| \le C|\xi|$, which is a direct consequence of (4.17). The proof of (4.21) requires a little more work. In the region where $1+\epsilon R' \le 2(1+\epsilon R)$, we obtain (4.21) using the facts that $\tilde{F}(\xi^2)$ is bounded and $F(\xi^2) \le C(1 + \log_+\xi^{-1})$, see (4.14). When $1 + \epsilon R' \ge 2(1+\epsilon R)$, we observe that $2\epsilon\rho \ge 2\epsilon(R'-R) \ge 1+\epsilon R'$, and using the bounds $F(\xi^2) + \tilde{F}(\xi^2) \le C\xi^{-1}$ we obtain (4.21) (without the logarithmic term in that case). $\square$

4.3 Characterization of the α-limit set

The evolution equation satisfied by the rescaled vorticity f defined in (4.1) reads

$$t\partial_t f + \gamma\partial_R(U_r^\epsilon f) + \gamma\partial_Z(U_z^\epsilon f) \;=\; \mathcal{L}f + \partial_R\Bigl(\frac{\epsilon f}{1+\epsilon R}\Bigr)\,, \tag{4.22}$$

for $(R,Z) \in \Omega_\epsilon$ and $t \in (0,T)$, where $\gamma = \Gamma/\nu$ and $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L}f \;=\; (\partial_R^2 + \partial_Z^2)f + \frac{1}{2}(R\partial_R f + Z\partial_Z f) + f\,. \tag{4.23}$$

The homogeneous Dirichlet boundary condition for f reads $f(-1/\epsilon, Z, t) = 0$ for all $Z \in \mathbb{R}$ and all $t \in (0,T)$. If we formally take the limit $\epsilon \to 0$ in (4.22), (4.7) and introduce the logarithmic time $\tau = \log(t/T)$, so that $\partial_\tau = t\partial_t$, we arrive at the evolution equation

$$\partial_\tau f + \gamma U \cdot \nabla f \,=\, \mathcal{L} f\,, \qquad (R,Z) \in \mathbb{R}^2\,, \tag{4.24}$$

where $\partial_R U_r + \partial_Z U_z = 0$ and $\partial_Z U_r - \partial_R U_z = f$. In other words, we obtain in that limit the two-dimensional vorticity equation in self-similar variables, which was thoroughly studied, for instance, in [18, 19].

We now introduce the weighted L^2 space $X = \{f \in L^2(\mathbb{R}^2) \,|\, \|f\|_X < \infty\}$ where

$$\|f\|_X^2 \,=\, \int_{\mathbb{R}^2} |f(R,Z)|^2 \, e^{(R^2+Z^2)/4} \,\mathrm{d}R\,\mathrm{d}Z\,. \tag{4.25}$$

For later use we also denote

$$w(R,Z) \,=\, e^{(R^2+Z^2)/4}\,, \qquad G(R,Z) \,=\, \frac{1}{4\pi} e^{-(R^2+Z^2)/4}\,, \qquad (R,Z) \in \mathbb{R}^2\,. \tag{4.26}$$

The aim of this section is to prove the following result:

Proposition 4.3. *The solution of* (4.22) *defined by* (4.1) *satisfies* $\|\bar f(t) - G\|_X \to 0$ *as* $t \to 0$, *where* $\bar f$ *denotes the extension of* f *by zero outside* Ω_ϵ.

Proposition 4.3 means that the axisymmetric vorticity $\omega_\theta(r,z,t)$ is not only bounded from above by a self-similar function with Gaussian profile, as asserted in (4.3), but actually approaches a uniquely determined self-similar solution of the 2d vorticity equation as $t \to 0$. Before giving a detailed proof, we make some preliminary remarks. Let $X_0 \subset X$ be the Banach space defined by the norm

$$\|f\|_{X_0} \,=\, \sup_{(R,Z)\in\mathbb{R}^2} |f(R,Z)|\, e^{\frac{1-\eta}{4}(R^2+Z^2)} \,+\, \sup_{(R,Z)\in\mathbb{R}^2} |\nabla f(R,Z)|\,, \tag{4.27}$$

where $\eta \in (0,1/2)$ is any fixed real number. We have the following elementary result:

Lemma 4.4. *The space* X_0 *is compactly embedded in* X, *and the unit ball in* X_0 *is closed for the topology induced by* X.

According to (4.3) and (4.4) the trajectory $(\bar f(t))_{t\in(0,T)}$ is bounded in X_0, hence relatively compact in X. We can thus consider the α-limit set

$$\mathcal{A} \,=\, \bigl\{h \in X \,\big|\, \text{there exists a sequence } t_m \to 0 \text{ such that } \|\bar f(t_m) - h\|_X \to 0 \text{ as } m \to \infty\bigr\}\,,$$

which is of course nonempty. We know from Lemma 4.4 that $\mathcal{A}$ is bounded in X_0, and in view of (4.5) any $h \in \mathcal{A}$ satisfies $\int h \,\mathrm{d}R\,\mathrm{d}Z = 1$. Proposition 4.3 asserts that $\mathcal{A}$ is a *singleton*, namely $\mathcal{A} = \{G\}$. The intuition behind this result is that the α-limit set $\mathcal{A}$ is (positively and negatively) invariant under the evolution defined on the whole plane $\mathbb{R}^2$ by the limiting equation (4.24), which is obtained by formally taking the limit $\epsilon \to 0$ in (4.22). But it is proved in [19] that the only solutions of (4.24) that are uniformly bounded in X for all negative times τ are equilibria of the form $f = \alpha G$, for some $\alpha \in \mathbb{R}$. Since we have the normalization condition $\int h \,\mathrm{d}R\,\mathrm{d}Z = 1$ for any $h \in \mathcal{A}$, we conclude that $\mathcal{A} = \{G\}$.

Making this argument rigorous requires a detailed comparison of the evolutions defined by equations (4.22) and (4.24), which is rather delicate. We thus prefer using a different argument to establish Proposition 4.3.

Proof of Proposition 4.3. Let $h_* \in \mathcal{A}$, and let (t_m) be a sequence in $(0,T)$ such that $t_m \to 0$ and $\|\bar f(t_m) - h_*\|_X \to 0$ as $m \to \infty$. Our goal is to show that $h_* = G$. To prove that, it is convenient to return to the three-dimensional formulation of the vorticity equation. As in (1.3), we denote by $u(x,t)$ and $\omega(x,t)$ the three-dimensional velocity and vorticity fields, respectively. For any $m \in \mathbb{N}$, any $y \in \mathbb{R}^3$, and any $s \in (0, T\epsilon_m^{-2})$, we define

$$\left\{\begin{aligned} u^{(m)}(y,s) &= \epsilon_m\, u(\bar x + \epsilon_m y\,,\, \epsilon_m^2 s)\,, \\ \omega^{(m)}(y,s) &= \epsilon_m^2\, \omega(\bar x + \epsilon_m y\,,\, \epsilon_m^2 s)\,, \end{aligned}\right. \tag{4.28}$$

where $\bar x = (\bar x_1, \bar x_2, \bar x_3) = (\bar r, 0, \bar z) \in \mathbb{R}^3$ and, in agreement with (4.2),

$$\epsilon_m = \frac{\sqrt{\nu t_m}}{\bar r}\,, \qquad m \in \mathbb{N}\,.$$

In other words, the vector fields $u^{(m)}, \omega^{(m)}$ are defined by a self-similar blow-up of the original quantities u, ω near the point $\bar x \in \mathbb{R}^3$ and near the initial time $t = 0$.

It is clear that $u^{(m)}, \omega^{(m)}$ satisfy the three-dimensional vorticity equation

$$\partial_s \omega^{(m)} + [u^{(m)}, \omega^{(m)}] - \nu\Delta\omega^{(m)} = 0\,, \tag{4.29}$$

for $y \in \mathbb{R}^3$ and $0 < s < T\epsilon_m^{-2}$, together with the constraints $\operatorname{div} u^{(m)} = 0$ and $\operatorname{curl} u^{(m)} = \omega^{(m)}$. This is due to the scaling and translational symmetries of the equations. Note that, in (4.29) and in the rest of the proof, all spatial derivatives act on the variable $y \in \mathbb{R}^3$. In view of (2.14) and (4.28), we have the a priori estimates

$$\|\partial_s^k \nabla_y^\ell u^{(m)}(s)\|_{L^\infty(\mathbb{R}^3)} \le \frac{C_{k\ell}(M) M}{s^k (\nu s)^{\ell/2}} \sqrt{\frac{\nu}{s}}\,, \qquad 0 < s < T\epsilon_m^{-2}\,,$$

which hold for all indices $k, \ell \in \mathbb{N}$, uniformly in $m \in \mathbb{N}$. Up to extracting a subsequence, we can therefore assume that

$$u^{(m)} \to \bar u\,, \qquad \omega^{(m)} \to \bar\omega\,, \qquad \text{as} \quad m \to \infty\,,$$

with uniform convergence of both vector fields and all their derivatives on any compact subset of $\mathbb{R}^3 \times (0,\infty)$. The limiting fields $\bar u, \bar\omega$ are smooth on $\mathbb{R}^3 \times (0,\infty)$ and satisfy

$$\partial_s \bar\omega + [\bar u, \bar\omega] - \nu\Delta\bar\omega = 0\,, \tag{4.30}$$

together with $\operatorname{div} \bar u = 0$ and $\operatorname{curl} \bar u = \bar\omega$.

We now relate the limiting vorticity $\bar\omega$ to the α-limit points of the rescaled vorticity f. In view of (4.1) and (4.28), we have, for all $m \in \mathbb{N}$, all $y \in \mathbb{R}^3$, and all $s \in (0, T\epsilon_m^{-2})$,

$$\omega^{(m)}(y,s) = \frac{\Gamma}{\nu s} f\Biggl(\frac{\sqrt{(\bar r + \epsilon_m y_1)^2 + \epsilon_m^2 y_2^2} - \bar r}{\epsilon_m \sqrt{\nu s}}\,, \frac{y_3}{\sqrt{\nu s}}\,, \epsilon_m^2 s\Biggr)\, e_\theta(\bar x + \epsilon_m y)\,. \tag{4.31}$$

For any fixed $s > 0$, we can assume (up to extracting another subsequence) that $f(\cdot,\cdot,\epsilon_m^2 s)$ converges in the topology of X to some $h_s \in \mathcal{A}$ as $m \to \infty$. Since $f(\cdot,\cdot,t)$ is bounded in X_0, the convergence also holds uniformly on any compact set of $\mathbb{R}^3$. Thus taking the limit $m \to \infty$ in (4.31) and observing that $e_\theta(\bar x) = e_2 = (0,1,0)$, we obtain

$$\bar\omega(y,s) = \frac{\Gamma}{\nu s} h_s\Bigl(\frac{y_1}{\sqrt{\nu s}}\,, \frac{y_3}{\sqrt{\nu s}}\Bigr)\, e_2 =: \bigl(0, \bar\omega_2(y_1, y_3, s), 0\bigr)\,. \tag{4.32}$$

We deduce in particular that

$$|\bar\omega_2(y_1,y_3,s)| \,\le\, K_\eta(M)\frac{\Gamma}{\nu s}\,e^{-\frac{1-\eta}{4\nu s}(y_1^2+y_3^2)}\,, \quad \text{and} \quad \int_{\mathbb{R}^2}\bar\omega_2(y_1,y_3,s)\,\mathrm{d}y_1\,\mathrm{d}y_3 \,=\, \Gamma\,. \tag{4.33}$$

Similarly, in view of (4.6) and (4.28), we have the relation

$$u^{(m)}(y,s) \,=\, \frac{\Gamma}{\sqrt{\nu s}}\,U^{\tilde\epsilon_m}\biggl(\frac{\sqrt{(\bar r+\epsilon_m y_1)^2+\epsilon_m^2 y_2^2}-\bar r}{\epsilon_m\sqrt{\nu s}}\,,\frac{y_3}{\sqrt{\nu s}}\,,\epsilon_m^2 s\biggr)\,,$$

where $\tilde\epsilon_m = \epsilon_m\sqrt{\nu s/\bar r}$. Taking the limit $m\to\infty$, we infer as above that the limiting velocity $\bar u$ has the particular form

$$\bar u(y,s) \,=\, \bar u_1(y_1,y_3,s)e_1 + \bar u_3(y_1,y_3,s)e_3 \,=\, \bigl(\bar u_1(y_1,y_3,s),0,\bar u_3(y_1,y_3,s)\bigr)\,, \tag{4.34}$$

and using Lemma 4.1 we also obtain the pointwise estimate

$$|\bar u(y,s)| \,\le\, \frac{C_{15}(M)\Gamma}{\sqrt{\nu s}+|y_1|+|y_3|}\,, \qquad y\in\mathbb{R}^3\,, \quad s>0\,. \tag{4.35}$$

As $\operatorname{div}\bar u = 0$ and $\operatorname{curl}\bar u = \bar\omega$, we deduce from (4.32), (4.34) that $\partial_1\bar u_1+\partial_3\bar u_3 = 0$ and $\partial_3\bar u_1-\partial_1\bar u_3 = \bar\omega_2$. Since $\bar u$ vanishes at infinity by (4.35), we conclude that $(\bar u_1,\bar u_3)$ is the two-dimensional velocity field obtained from the scalar vorticity $\bar\omega_2$ via the Biot-Savart law in $\mathbb{R}^2$.

Summarizing, we have shown that the limiting vorticity $\bar\omega_2$, together with the associated velocity $(\bar u_1,\bar u_3)$, solves the Navier-Stokes equations in $\mathbb{R}^2\times(0,\infty)$, and it follows from (4.33) that $\bar\omega_2(\cdot,s)$ is uniformly bounded in $L^1(\mathbb{R}^2)$ and converges weakly to the Dirac measure $\Gamma\delta_0$ as $s\to 0$. Invoking [19, Proposition 1.3], we deduce that, for any $s>0$,

$$\bar\omega_2(y_1,y_3,s) \,=\, \frac{\Gamma}{\nu s}\,G\Bigl(\frac{y_1}{\sqrt{\nu s}}\,,\frac{y_3}{\sqrt{\nu s}}\Bigr) \,=\, \frac{\Gamma}{4\pi\nu s}\,e^{-(y_1^2+y_3^2)/(4\nu s)}\,, \qquad (y_1,y_3)\in\mathbb{R}^2\,. \tag{4.36}$$

In particular, setting $s = s_* = \bar r^2/\nu$, so that $\epsilon_m^2 s = t_m$, and comparing (4.32) with (4.36), we conclude that $h_{s_*}\equiv h_* = G$, which is the desired result. $\square$

4.4 Short time asymptotics

The goal of this section is to establish the short time estimate (1.7). Let $\chi:[0,\infty)\to[0,1]$ be a smooth nonincreasing function such that $\chi(x)=1$ for $x\in[0,1/4]$ and $\chi(x)=0$ for $x\ge 1/2$. We define

$$f_0(R,Z,t) \,=\, G(R,Z)\chi(\epsilon^2(R^2{+}Z^2))\,, \qquad (R,Z)\in\mathbb{R}^2\,, \quad t\in(0,T)\,, \tag{4.37}$$

where $\epsilon = \sqrt{\nu t}/\bar r$ and $G(R,Z) = (4\pi)^{-1}e^{-(R^2+Z^2)/4}$, see (4.2), (4.26). Due to the localization function χ, it is clear that $f_0(R,Z,t)$ vanishes when $\epsilon R < -1/\sqrt{2}$. In particular, f_0 satisfies the Dirichlet boundary condition in the time-dependent domain $\Omega_\epsilon = \{(R,Z)\in\mathbb{R}^2\,|\,1+\epsilon R>0\}$.

Lemma 4.5. *There exists $C_{16}>0$ such that, for any $t\in(0,T)$, the velocity field $U_0^\epsilon = \mathrm{BS}^\epsilon[f_0]$ associated with f_0 satisfies*

$$\|U_0^\epsilon\|_{L^\infty(\Omega_\epsilon)} \,\le\, C_{16}\,, \qquad \|\operatorname{div}_* U_0^\epsilon\|_{L^\infty(\Omega_\epsilon)} \,\le\, C_{16}\,(\epsilon+\epsilon^2)\,. \tag{4.38}$$

Proof. Since $|f_0| \le G$, the first bound in (4.38) is a direct consequence of estimate (4.10) in Lemma 4.1. In view of the identity

$$\operatorname{div}_* U_0^\epsilon \;=\; \partial_R U_{0,r}^\epsilon + \partial_Z U_{0,z}^\epsilon \;=\; -\frac{\epsilon U_{0,r}^\epsilon}{(1+\epsilon R)}\,, \tag{4.39}$$

it follows that $|\operatorname{div}_* U_0^\epsilon| \le C\epsilon$ whenever the denominator $1+\epsilon R$ is bounded away from zero. The proof of (4.38) is completed using the improved estimate

$$|U_{0,r}^\epsilon(R,Z,t)| \;\le\; C(1+\epsilon R)\epsilon\,, \qquad (R,Z) \in \tilde\Omega_\epsilon\,, \quad t \in (0,T)\,, \tag{4.40}$$

which holds in the subdomain $\tilde\Omega_\epsilon = \{(R,Z) \in \mathbb{R}^2 \,|\, 0 < 1+\epsilon R < 1/4\}$. To establish (4.40), we start from the representation (4.15) for $U_{0,r}^\epsilon$, where f is replaced by f_0. Using the bound $\tilde F(\xi^2) \le C|\xi|^{-3}$, which follows from (4.17), we easily obtain

$$\frac{|U_{0,r}^\epsilon(R,Z,t)|}{1+\epsilon R} \;\le\; C\epsilon \int_{\Omega_\epsilon} \frac{(1+\epsilon R')^2}{\epsilon^4\rho^4}\, f_0(R',Z',t)\,\mathrm{d}R'\,\mathrm{d}Z'\,, \tag{4.41}$$

for all $(R,Z) \in \Omega_\epsilon$ and all $t \in (0,T)$, where $\rho = \sqrt{(R-R')^2 + (Z-Z')^2}$. The integrand in (4.41) is nonzero only in the region where $\epsilon^2(R'^2 + Z'^2) \le 1/2$. Thus $1 - 1/\sqrt{2} \le 1 + \epsilon R' \le 1 + 1/\sqrt{2}$, and if $1+\epsilon R < 1/4$ it follows that $\epsilon\rho \ge \epsilon(R'-R) \ge 3/4 - 1/\sqrt{2} > 0$. With these observations in mind, estimate (4.40) is a direct consequence of (4.41). □

If f is the solution of (4.22) given by (4.1), and if U^ϵ is the associated velocity field, we decompose

$$\begin{cases} f(R,Z,t) \;=\; f_0(R,Z,t) + \tilde f(R,Z,t)\,, \\ U^\epsilon(R,Z,t) \;=\; U_0^\epsilon(R,Z,t) + \tilde U^\epsilon(R,Z,t)\,, \end{cases} \qquad (R,Z) \in \Omega_\epsilon\,, \quad t \in (0,T)\,, \tag{4.42}$$

where $U_0^\epsilon = \mathrm{BS}^\epsilon[f_0]$ and $\tilde U^\epsilon = \mathrm{BS}^\epsilon[\tilde f]$. The equation satisfied by the perturbation $\tilde f$ is

$$t\partial_t \tilde f + \gamma \operatorname{div}_*\bigl(U_0^\epsilon \tilde f + \tilde U^\epsilon f_0\bigr) + \gamma \operatorname{div}_*\bigl(\tilde U^\epsilon \tilde f\bigr) \;=\; \mathcal{L}\tilde f + \partial_R\Bigl(\frac{\epsilon \tilde f}{1+\epsilon R}\Bigr) + \mathcal{H}\,, \tag{4.43}$$

where $\mathcal{H}$ is a source term which quantifies by how much f_0 fails to be an exact solution of (4.22). Explicitly,

$$\mathcal{H} \;=\; \mathcal{L} f_0 + \partial_R\Bigl(\frac{\epsilon f_0}{1+\epsilon R}\Bigr) - t\partial_t f_0 - \gamma \operatorname{div}_*\bigl(U_0^\epsilon f_0\bigr)\,. \tag{4.44}$$

Here and in what follows, if $V = (V_r, V_z)$ is a vector field on Ω_ϵ or on the whole plane $\mathbb{R}^2$, we denote $\operatorname{div}_* V = \partial_R V_r + \partial_Z V_z$. Note that the perturbation $\tilde f$ still satisfies the Dirichlet boundary condition on $\partial\Omega_\epsilon$.

It is clear from definition (4.37) that f_0 belongs for all times to the space X introduced in (4.25), and that $\|f_0(t) - G\|_X \to 0$ as $t \to 0$. Thus the perturbation $\tilde f$ (implicitly extended by zero outside Ω_ϵ) belongs to X for all $t \in (0,T)$, and Proposition 4.3 implies that $\|\tilde f(t)\|_X \to 0$ as $t \to 0$. In the rest of this section, using appropriate energy estimates, we prove that $\|\tilde f(t)\|_X = \mathcal{O}(\epsilon|\log\epsilon|)$ as $t \to 0$, and this implies (1.7) in view of the continuous injection $X \hookrightarrow L^1(\mathbb{R}^2)$.

For any $t \in (0,T)$, we define

$$E(t) \;=\; \frac{1}{2}\int_{\Omega_\epsilon} \tilde f(R,Z,t)^2 w(R,Z)\,\mathrm{d}R\,\mathrm{d}Z\,, \tag{4.45}$$

where $w(R,Z) = e^{(R^2+Z^2)/4}$, see (4.26). Although the integral in (4.45) is taken over the time-dependent domain Ω_ϵ, there is no contribution from the boundary when we differentiate with respect to time, because $\tilde f$ satisfies the homogeneous Dirichlet condition on $\partial\Omega_\epsilon$. Using (4.43), we thus obtain

$$\begin{aligned} tE'(t) &= \int_{\Omega_\epsilon} \tilde f(R,Z,t)\big(t\partial_t \tilde f(R,Z,t)\big)w(R,Z)\,\mathrm{d}R\,\mathrm{d}Z \\ &= D_1(t) + D_2(t) + H(t) - \gamma\Big(A_1(t)+A_2(t)+N(t)\Big)\,, \end{aligned} \tag{4.46}$$

where

$$\begin{aligned} D_1(t) &= \int_{\Omega_\epsilon} \tilde f\,(\mathcal{L}\tilde f)w\,\mathrm{d}R\,\mathrm{d}Z\,, & D_2(t) &= \int_{\Omega_\epsilon} \tilde f\,\partial_R\Big(\frac{\epsilon\tilde f}{1+\epsilon R}\Big)w\,\mathrm{d}R\,\mathrm{d}Z\,, \\ A_1(t) &= \int_{\Omega_\epsilon} \tilde f\ \mathrm{div}_*\big(U_0^\epsilon \tilde f\big)w\,\mathrm{d}R\,\mathrm{d}Z\,, & A_2(t) &= \int_{\Omega_\epsilon} \tilde f\ \mathrm{div}_*\big(\tilde U^\epsilon f_0\big)w\,\mathrm{d}R\,\mathrm{d}Z\,, \\ H(t) &= \int_{\Omega_\epsilon} \tilde f\,\mathcal{H}w\,\mathrm{d}R\,\mathrm{d}Z\,, & N(t) &= \int_{\Omega_\epsilon} \tilde f\ \mathrm{div}_*\big(\tilde U^\epsilon \tilde f\big)w\,\mathrm{d}R\,\mathrm{d}Z\,. \end{aligned}$$

The main result of this section is

Proposition 4.6. *There exists $\delta > 0$ and, for any $\gamma > 0$, there exist $\epsilon_0 \in (0,1/2)$ and $\kappa > 0$ such that, if $t > 0$ is small enough so that $\epsilon \le \epsilon_0$, then*

$$tE'(t) \le -2\delta\mathcal{E}(t) + \kappa\epsilon|\log\epsilon|\,E(t)^{1/2} + \kappa E(t)^{1/2}\mathcal{E}(t) + \mathcal{R}(t)\,, \tag{4.47}$$

where $\mathcal{R}(t) \le e^{-1/(36\epsilon^2)}$ and

$$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega_\epsilon}\Big(|\nabla\tilde f|^2 + (1+R^2+Z^2)\tilde f^2\Big)w\,\mathrm{d}R\,\mathrm{d}Z \ge E(t)\,. \tag{4.48}$$

Proof. We proceed in six steps.

Step 1: *Control of the mass.* For any $t \in (0,T)$ we denote

$$m(t) = \int_{\Omega_\epsilon} \tilde f(R,Z,t)\,\mathrm{d}R\,\mathrm{d}Z = \int_{\Omega_\epsilon}\big(f(R,Z,t) - f_0(R,Z,t)\big)\,\mathrm{d}R\,\mathrm{d}Z\,. \tag{4.49}$$

We shall show that $m(t)$ is extremely small for short times. Indeed, since $\int_{\mathbb{R}^2} G\,\mathrm{d}R\,\mathrm{d}Z = 1$, it follows from definition (4.37) that

$$0 \le 1 - \int_{\Omega_\epsilon} f_0(R,Z,t)\,\mathrm{d}R\,\mathrm{d}Z = \int_{\mathbb{R}^2}\Big(1-\chi(\epsilon^2(R^2+Z^2))\Big)G\,\mathrm{d}R\,\mathrm{d}Z \le e^{-1/(16\epsilon^2)}\,, \tag{4.50}$$

because, in the last integral, the integrand vanishes when $R^2+Z^2 \le 1/(4\epsilon^2)$. On the other hand, estimate (3.8) (where one can take $\rho = \bar r/2$) shows that

$$\Gamma \ge \int_{\bar r/2}^\infty \Big\{\int_{\mathbb{R}} \omega_\theta(r,z,t)\,\mathrm{d}z\Big\}\,\mathrm{d}r \ge \Gamma\Big(1 - e^{-\bar r^2/(64\nu t)}\Big)\,,$$

and in view of (4.1) this implies that

$$0 \le 1 - \int_{\Omega_\epsilon} f(R,Z,t)\,\mathrm{d}R\,\mathrm{d}Z \le e^{-1/(64\epsilon^2)}\,. \tag{4.51}$$

Combining (4.50) and (4.51), we deduce that

$$|m(t)| \le e^{-1/(64\epsilon^2)}, \qquad \text{where} \quad \epsilon = \frac{\sqrt{\nu t}}{\bar{r}}. \tag{4.52}$$

Step 2: *The diffusive terms.* After this preliminary step, we estimate separately the various terms in the right-hand side of (4.46), starting with $D_1(t)$ and $D_2(t)$ which originate from the diffusion operator in (4.43). Using the identity $(\mathcal{L}\tilde{f})w = \mathrm{div}_*(w\nabla\tilde{f}) + w\tilde{f}$ and integrating by parts, we first obtain

$$D_1 = \int \tilde{f}(\mathcal{L}\tilde{f})w \,\mathrm{d}R\,\mathrm{d}Z = \int \Big(-|\nabla\tilde{f}|^2 + \tilde{f}^2\Big) w \,\mathrm{d}R\,\mathrm{d}Z. \tag{4.53}$$

Here and in what follows, all integrals are taken over the domain Ω_ϵ, or over the whole plane $\mathbb{R}^2$ if one extends the integrands by zero outside Ω_ϵ (as we implicitly do when necessary). For simplicity we also write $\tilde{f}$ instead of $\tilde{f}(R,Z,t)$, and similarly for other quantities.

Estimate (4.53) is not sufficient for our purposes, because it is not clear if the right-hand side is negative. To improve it, we observe that $\tilde{f}(\mathcal{L}\tilde{f})w = \tilde{g}(L\tilde{g})$ where $\tilde{g} = w^{1/2}\tilde{f}$ and L is the linear operator defined by

$$L\tilde{g} = \Delta\tilde{g} - \frac{R^2+Z^2}{16}\tilde{g} + \frac{1}{2}\tilde{g}.$$

We thus have the alternative formula

$$D_1 = \int \tilde{g}(L\tilde{g}) \,\mathrm{d}R\,\mathrm{d}Z = \int \Big(-|\nabla\tilde{g}|^2 - \frac{R^2+Z^2}{16}\tilde{g}^2 + \frac{1}{2}\tilde{g}^2\Big) \,\mathrm{d}R\,\mathrm{d}Z. \tag{4.54}$$

The operator L is related to the quantum harmonic oscillator in $\mathbb{R}^2$. With the normalization above, it is self-adjoint in $L^2(\mathbb{R}^2)$ with spectrum $\sigma(L) = \{-n/2 \,|\, n = 0,1,2\dots\}$, see e.g. [18, Appendix A], and this observation already implies that $D_1 \le 0$. Moreover, the kernel of L is one-dimensional and spanned by the function $w^{1/2}G$. As a consequence, if $\tilde{f}$ has zero mean over $\mathbb{R}^2$, then $\tilde{g} = w^{1/2}\tilde{f}$ is orthogonal to $w^{1/2}G$ in $L^2(\mathbb{R}^2)$, hence belongs to the invariant subspace where $L \le -1/2$. Thus

$$D_1 = \int \tilde{f}(\mathcal{L}\tilde{f})w \,\mathrm{d}R\,\mathrm{d}Z \le -\frac{1}{2}\int \tilde{f}^2 w \,\mathrm{d}R\,\mathrm{d}Z, \qquad \text{if} \quad \int \tilde{f} \,\mathrm{d}R\,\mathrm{d}Z = 0. \tag{4.55}$$

In the general case, we can decompose $\tilde{f} = m(t)G + \hat{f}$, so that $\hat{f}$ has zero mean by construction. As $\mathcal{L}G = 0$, we have $\int \tilde{f}(\mathcal{L}\tilde{f})w \,\mathrm{d}R\,\mathrm{d}Z = \int \hat{f}(\mathcal{L}\hat{f})w \,\mathrm{d}R\,\mathrm{d}Z$ and applying (4.55) to $\hat{f}$ we obtain

$$D_1 = \int \hat{f}(\mathcal{L}\hat{f})w \,\mathrm{d}R\,\mathrm{d}Z \le -\frac{1}{2}\int \hat{f}^2 w \,\mathrm{d}R\,\mathrm{d}Z = -\frac{1}{2}\int \tilde{f}^2 w \,\mathrm{d}R\,\mathrm{d}Z + \frac{m(t)^2}{8\pi}. \tag{4.56}$$

We now take a convex combination of estimates (4.53), (4.54), and (4.56), for instance with coefficients 1/6, 1/6, and 2/3. This gives our improved bound

$$D_1 \le -\int \Big(\frac{1}{6}|\nabla\tilde{f}|^2 + \frac{R^2+Z^2}{96}\tilde{f}^2 + \frac{1}{12}\tilde{f}^2\Big) w \,\mathrm{d}R\,\mathrm{d}Z + \frac{m(t)^2}{12\pi} \le -\frac{\mathcal{E}}{48} + \frac{m(t)^2}{12\pi}. \tag{4.57}$$

Next, we consider the second diffusive term D_2. Integrating by parts and using the fact that $\partial_R w = Rw/2$, we find

$$\begin{aligned} D_2 &= -\frac{\epsilon}{2}\int \Big(\frac{\tilde{f}}{1+\epsilon R}\Big)^2 \partial_R\big((1+\epsilon R)w\big) \,\mathrm{d}R\,\mathrm{d}Z \\ &= -\frac{\epsilon^2}{2}\int \Big(\frac{\tilde{f}}{1+\epsilon R}\Big)^2 w \,\mathrm{d}R\,\mathrm{d}Z - \frac{\epsilon}{4}\int \Big(\frac{\tilde{f}^2}{1+\epsilon R}\Big) Rw \,\mathrm{d}R\,\mathrm{d}Z. \end{aligned} \tag{4.58}$$

The last term in (4.58) has no sign, but is obviously harmless when $1 + \epsilon R \geq 1/4$. In the subdomain $\tilde{\Omega}_\epsilon = \{(R, Z) \,|\, 0 < 1 + \epsilon R < 1/4\}$, we can apply Young's inequality to obtain

$$\frac{\epsilon}{4} \int_{\tilde{\Omega}_\epsilon} \Bigl(\frac{\tilde{f}^2}{1+\epsilon R}\Bigr)|R| w \,\mathrm{d}R \,\mathrm{d}Z \,\leq\, \frac{\epsilon^2}{2} \int_{\tilde{\Omega}_\epsilon} \Bigl(\frac{\tilde{f}}{1+\epsilon R}\Bigr)^2 w \,\mathrm{d}R \,\mathrm{d}Z + \frac{1}{32} \int_{\tilde{\Omega}_\epsilon} f^2 R^2 w \,\mathrm{d}R \,\mathrm{d}Z \,, \qquad (4.59)$$

where we replaced $\tilde{f}$ by f in the last integrand because f_0 vanishes identically in $\tilde{\Omega}_\epsilon$. Using the upper bound (4.3) with (for instance) $\eta = 1/4$, we see that the last integral in (4.59) is transcendentally small. Summarizing, we have shown that

$$D_2 \,\leq\, \epsilon \int \tilde{f}^2 |R| w \,\mathrm{d}R \,\mathrm{d}Z + C\, e^{-1/(16\epsilon^2)} \,\leq\, 2\epsilon E^{1/2} \mathcal{E}^{1/2} + C\, e^{-1/(16\epsilon^2)} \,, \qquad (4.60)$$

where the constant $C > 0$ depends only on $M = \gamma$. Note that the first term in the right-hand side of (4.60) is bounded by $2\epsilon\mathcal{E}$ and can therefore be controlled by the negative terms in (4.57), if ϵ is small enough.

Step 3: *The source term.* We turn our attention to the source term $\mathcal{H}$ defined in (4.44). We claim that

$$\|\mathcal{H}(t)\|_X \,\leq\, C\epsilon + C\gamma\epsilon |\log \epsilon| \,, \qquad (4.61)$$

whenever $\epsilon \leq 1/2$, where $C > 0$ is a universal constant. To prove (4.61) we consider separately the various terms in (4.44). First, as $\partial_t G = \mathcal{L} G = 0$, it is straightforward to verify that both quantities $t\partial_t f_0$ and $\mathcal{L} f_0$ are transcendentally small in X as $\epsilon \to 0$. Next, since $1+\epsilon R$ is bounded away from zero on the support of f_0, it is clear that the second-term in the right-hand side of (4.44) is $\mathcal{O}(\epsilon)$ in X. So the main contribution comes from the last term $\gamma\, \mathrm{div}_*\bigl(U_0^\epsilon f_0\bigr)$, which requires a more careful analysis. We observe that

$$\mathrm{div}_*\bigl(U_0^\epsilon f_0\bigr) \,=\, U_0^\epsilon \cdot \nabla f_0 + \mathrm{div}_*(U_0^\epsilon) f_0 \,=\, (U_0^\epsilon - U_0^0) \cdot \nabla f_0 + \mathrm{div}_*(U_0^\epsilon) f_0 \,, \qquad (4.62)$$

where $U_0^0 = \mathrm{BS}^0[f_0]$ denotes the velocity field obtained from f_0 via the two-dimensional Biot-Savart law (4.18). Note that $U_0^0 \cdot \nabla f_0 = 0$, because f_0 is radially symmetric in $\mathbb{R}^2$, but we included that term in (4.62) so that the right-hand side contains the difference $U_0^\epsilon - U_0^0$, which can be estimated using inequality (4.19) (with f replaced by f_0). We thus find

$$|U_0^\epsilon(R, Z, t) - U_0^0(R, Z, t)| \,\leq\, \frac{C\epsilon}{1+\epsilon R}\Bigl(1 + \log_+ \frac{1+\epsilon R}{\epsilon}\Bigr) \,, \qquad (4.63)$$

and it follows easily that $\|(U_0^\epsilon - U_0^0) \cdot \nabla f_0\|_X \,\leq\, C\epsilon |\log \epsilon|$ if ϵ is small enough. Moreover, by Lemma 4.5, the last term in (4.62) is $\mathcal{O}(\epsilon)$ in X. This concludes the proof of (4.61), and we deduce that

$$H \,=\, \int \tilde{f}\, \mathcal{H} w \,\mathrm{d}R \,\mathrm{d}Z \,\leq\, C E^{1/2} \Bigl(\epsilon + \gamma\epsilon |\log \epsilon|\Bigr) \,, \qquad (4.64)$$

whenever $\epsilon \leq 1/2$.

Step 4: *The advection terms.* We now consider the terms produced by the advection operator $\tilde{f} \mapsto \mathrm{div}_*(U_0^\epsilon \tilde{f})$ and the nonlocal operator $\tilde{f} \mapsto \mathrm{div}_*(\tilde{U}^\epsilon f_0)$ in (4.43). Integrating by parts, we find

$$A_1 \,=\, \int \tilde{f}\, \mathrm{div}_*\bigl(U_0^\epsilon \tilde{f}\bigr) w \,\mathrm{d}R \,\mathrm{d}Z \,=\, \frac{1}{2} \int \tilde{f}^2 (\mathrm{div}_*\, U_0^\epsilon) w \,\mathrm{d}R \,\mathrm{d}Z - \frac{1}{2} \int \tilde{f}^2 (U_0^\epsilon \cdot \nabla w) \,\mathrm{d}R \,\mathrm{d}Z \,.$$

By Lemma 4.5, the first integral in the right-hand side is bounded by $C\epsilon \|\tilde{f}\|_X^2$ if ϵ is small. The second integral is decomposed as $A_{11} + A_{12}$, where

$$A_{11} \,=\, \int_{\Omega_\epsilon \setminus \tilde{\Omega}_\epsilon} \tilde{f}^2 \bigl((U_0^\epsilon - U_0^0) \cdot \nabla w\bigr) \,\mathrm{d}R \,\mathrm{d}Z \,, \qquad A_{12} \,=\, \int_{\tilde{\Omega}_\epsilon} \tilde{f}^2 (U_0^\epsilon \cdot \nabla w) \,\mathrm{d}R \,\mathrm{d}Z \,. \qquad (4.65)$$

We recall that $\tilde\Omega_\epsilon = \{(R,Z)\in\mathbb{R}^2\,|\,0<1+\epsilon R<1/4\}$ and $U_0^0 = \mathrm{BS}^0[f_0]$. Note that $U_0^0\cdot\nabla w = 0$, because w is radially symmetric in $\mathbb{R}^2$, but it is useful to make the difference $U_0^\epsilon - U_0^0$ appear in the term A_{11}. Using (4.63) and the obvious bound $|\nabla w| \le C(|R|+|Z|)w$, we obtain

$$|A_{11}| \,\le\, \int_{\Omega_\epsilon\setminus\tilde\Omega_\epsilon} \frac{C\epsilon}{1+\epsilon R}\Bigl(1+\log_+\frac{1+\epsilon R}{\epsilon}\Bigr)\tilde f^2(|R|+|Z|)w\,\mathrm{d}R\,\mathrm{d}Z \,\le\, C\epsilon|\log\epsilon|\,E^{1/2}\mathcal{E}^{1/2}\,.$$

In the subdomain $\tilde\Omega_\epsilon$, we use the estimate $|U_0^\epsilon(R,Z,t)|\le C\epsilon$, which is similar to (4.40) and can be established by exactly the same argument. This gives

$$|A_{12}| \,\le\, C\epsilon\int_{\tilde\Omega_\epsilon}\tilde f^2(|R|+|Z|)w\,\mathrm{d}R\,\mathrm{d}Z \,\le\, C\epsilon\,E^{1/2}\mathcal{E}^{1/2}\,.$$

Altogether we have shown that

$$|A_1| \,\le\, C\epsilon|\log\epsilon|\,E^{1/2}\mathcal{E}^{1/2}\,. \tag{4.66}$$

As for the nonlocal term A_2, we observe that

$$A_2 \,=\, \int\tilde f\,\mathrm{div}_*\bigl([\tilde U^\epsilon-\tilde U^0]f_0\bigr)w\,\mathrm{d}R\,\mathrm{d}Z + \int\tilde f\,(\tilde U^0\cdot\nabla[f_0-G])w\,\mathrm{d}R\,\mathrm{d}Z\,, \tag{4.67}$$

where $\tilde U^0 = \mathrm{BS}^0[\tilde f]$ is the velocity field obtained from the vorticity $\tilde f$ via the two-dimensional Biot-Savart law (4.18). In deriving (4.67) we used the nontrivial observation

$$\int\tilde f\,\bigl(\tilde U^0\cdot\nabla G\bigr)w\,\mathrm{d}R\,\mathrm{d}Z \,\equiv\, \int_{\mathbb{R}^2}\tilde f\,\bigl(\mathrm{BS}^0[\tilde f]\cdot\nabla G\bigr)w\,\mathrm{d}R\,\mathrm{d}Z \,=\, 0\,,$$

which was first made in [19, Lemma 4.8]. Let A_{21} denote the first term in the right-hand side of (4.67). Integrating by parts, we find

$$A_{21} \,=\, -\int f_0\,\bigl(\tilde U^\epsilon-\tilde U^0\bigr)\cdot\nabla(\tilde f w)\,\mathrm{d}R\,\mathrm{d}Z\,. \tag{4.68}$$

Note once again that f_0 is supported in the region where $1+\epsilon R\ge 1/4$, and in that domain we infer from (4.19) that $|\tilde U^\epsilon-\tilde U^0|\le C\epsilon|\log\epsilon|\|\tilde f\|_{L^1\cap L^2}$. Using Hölder's inequality and the continuous injection $X\hookrightarrow L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, we deduce that

$$\begin{aligned}
|A_{21}| \,&\le\, C\epsilon|\log\epsilon|\|\tilde f\|_X\int\Bigl(|\nabla\tilde f| + (|R|^2+|Z^2|)^{1/2}|\tilde f|\Bigr)\,\mathrm{d}R\,\mathrm{d}Z\\
&\le\, C\epsilon|\log\epsilon|E^{1/2}\Bigl\{\int\Bigl(|\nabla\tilde f|^2+(|R|^2+|Z^2|)|\tilde f|^2\Bigr)w\,\mathrm{d}R\,\mathrm{d}Z\Bigr\}^{1/2}\\
&\le\, C\epsilon|\log\epsilon|E^{1/2}\mathcal{E}^{1/2}\,.
\end{aligned} \tag{4.69}$$

Again the right-hand side can be controlled by the negative terms in (4.57) if ϵ is sufficiently small.

Finally, let A_{22} denote the last integral term in (4.67). Here the integral is taken over the domain $\hat\Omega_\epsilon = \{(R,Z)\,|\,\epsilon^2(R^2+Z^2)\ge 1/4\}$, because $f_0 = G$ on $\Omega_\epsilon\setminus\hat\Omega_\epsilon$. Using Hölder's inequality, we obtain

$$[A_{22}| \,\le\, C\int_{\hat\Omega_\epsilon}|\tilde U^0||\tilde f|(R^2{+}Z^2)^{1/2}\,\mathrm{d}R\,\mathrm{d}Z \,\le\, C\|\tilde U^0\|_{L^4}\Bigl\{\int_{\hat\Omega_\epsilon}|\tilde f|^{4/3}(R^2{+}Z^2)^{2/3}\,\mathrm{d}R\,\mathrm{d}Z\Bigr\}^{3/4}.$$

As $\tilde U^0$ is the velocity field obtained from $\tilde f$ via the two-dimensional Biot-Savart law, the Hardy-Littlewood-Sobolev inequality implies that $\|\tilde U^0\|_{L^4} \le C\|\tilde f\|_{L^{4/3}} \le C\|\tilde f\|_X$. On the other hand, using Hölder's inequality again, we find

$$\int_{\hat\Omega_\epsilon} |\tilde f|^{4/3}(R^2{+}Z^2)^{2/3}\,\mathrm{d}R\,\mathrm{d}Z \,\le\, \Big\{\int |\tilde f|^2 w\,\mathrm{d}R\,\mathrm{d}Z\Big\}^{2/3}\Big\{\int_{\hat\Omega_\epsilon}(R^2{+}Z^2)^2\frac{1}{w^2}\,\mathrm{d}R\,\mathrm{d}Z\Big\}^{1/3},$$

where the last integral can be explicitly computed and is found to be transcendentally small as $\epsilon \to 0$. Summarizing, we have shown that $|A_{22}| \le C\,e^{-1/(32\epsilon^2)}\,\|\tilde f\|_X^2$, hence

$$|A_2| \,\le\, |A_{21}| + |A_{22}| \,\le\, C\epsilon|\log\epsilon|\,E^{1/2}\mathcal{E}^{1/2}\,. \tag{4.70}$$

Step 5: *The nonlinear term.* Finally we consider the nonlinear term N in (4.46). Integrating by part, we find

$$N(t) \,=\, -\int \tilde f\,\tilde U^\epsilon\cdot\big(w\nabla\tilde f + \tilde f\nabla w\big)\,\mathrm{d}R\,\mathrm{d}Z\,, \tag{4.71}$$

so that

$$|N(t)| \,\le\, C\int |\tilde U^\epsilon|\,\big(|\tilde f|w^{1/2}\big)\,\Big(\big(|\nabla\tilde f| + (R^2{+}Z^2)^{1/2}|\tilde f|\big)w^{1/2}\Big)\,\mathrm{d}R\,\mathrm{d}Z\,.$$

We apply the trilinear Hölder inequality to the right-hand side, with exponents 4, 4, and 2. Since $\tilde U^\epsilon = \mathrm{BS}^\epsilon[\tilde f]$, it follows from (4.10) (using again the Hardy-Littlewood-Sobolev inequality) that $\|\tilde U^\epsilon\|_{L^4} \le C\|\tilde f\|_{L^{4/3}} \le C\|\tilde f\|_X$. On the other hand, using Sobolev's interpolation inequality, we see that

$$\|\tilde f w^{1/2}\|_{L^4} \,\le\, C\|\tilde f w^{1/2}\|_{L^2}^{1/2}\|\nabla(\tilde f w^{1/2})\|_{L^2}^{1/2} \,\le\, CE^{1/4}\mathcal{E}^{1/4}\,.$$

Finally,

$$\|\big(|\nabla\tilde f| + (R^2{+}Z^2)^{1/2}|\tilde f|\big)w^{1/2}\|_{L^2} \,\le\, C\mathcal{E}^{1/2}\,.$$

Altogether, we have shown that

$$|N| \,\le\, CE^{3/4}\mathcal{E}^{3/4} \,\le\, CE^{1/2}\mathcal{E}\,. \tag{4.72}$$

Alternatively, one can apply the trilinear Hölder inequality with exponents ∞, 2, 2, and deduce from (4.10) that $\|\tilde U^\epsilon\|_{L^\infty} \le C\|\tilde f\|_{L^{4/3}}^{1/2}\|\tilde f\|_{L^4}^{1/2} \le CE^{1/4}\mathcal{E}^{1/4}$. This also leads to (4.72).

Step 6: *Conclusion.* Combining estimates (4.52), (4.57), (4.60), (4.64), (4.66), (4.70), and (4.72), we obtain (4.47). Note that the negative term $-2\delta\mathcal{E}$ and the remainder $\mathcal{R}$ in (4.47) are entirely produced by the diffusion terms D_1 and D_2, whereas the quantities $\kappa\epsilon|\log\epsilon|\,E^{1/2}$ and $\kappa E^{1/2}\mathcal{E}$ originate from the source term H and the cubic term N, respectively. As for the advections terms A_1 and A_2, their contributions can be controlled by the negative term if ϵ is small enough. □

Proof of estimate (1.7) **in Theorem 1.1.** We know from Proposition 4.3 that $f(t)$ converges to G in X as $t \to 0$, and so does $f_0(t)$ in view of definition (4.37). Thus $E(t) \to 0$ as $t \to 0$. As long as t is small enough so that $\epsilon \le \epsilon_0$ and $\kappa E(t)^{1/2} \le \delta/2$, where $\epsilon_0, \kappa, \delta$ are as in Proposition 4.6, it follows from (4.47) and Young's inequality that

$$tE'(t) \,\le\, -\delta\mathcal{E}(t) + \mathcal{R}_1(t) \,\le\, -\delta E(t) + \mathcal{R}_1(t)\,, \tag{4.73}$$

where $\mathcal{R}_1 \le C\epsilon^2|\log\epsilon|^2$. Integrating that differential inequality, we obtain

$$E(t) \,\le\, t^{-\delta}\int_0^t s^{\delta-1}\mathcal{R}_1(s)\,\mathrm{d}s \,=:\, \mathcal{R}_2(t)\,, \tag{4.74}$$

where again $\mathcal{R}_2 \le C\epsilon^2|\log\epsilon|^2$. From that bound, we see that there exists $\epsilon_1 \in (0,\epsilon_0)$ such that our assumption $\kappa E^{1/2} \le \delta/2$ is satisfied whenever $\epsilon \le \epsilon_1$. So, for $\epsilon \le \epsilon_1$, we have

$$\|f(t) - f_0(t)\|_{L^1(\Omega_\epsilon)} \,=\, \|\tilde f(t)\|_{L^1(\Omega_\epsilon)} \,\le\, C\|\tilde f(t)\|_X \,\le\, CE(t)^{1/2} \,\le\, C\epsilon|\log\epsilon|\,,$$

and since f_0 is extremely close to G this proves exactly (1.7), after returning to the original variables. When $\epsilon_1 \le \epsilon \le 1/2$, estimate (1.7) obviously holds (for some appropriate constant C_1), because the left-hand side is trivially smaller than $2|\Gamma|$. $\square$

4.5 Uniqueness

This final section is devoted to the uniqueness claim in Theorem 1.1. Assume for this purpose that $\omega_\theta^{(1)}, \omega_\theta^{(2)} \in C^0((0,T), L^1(\Omega)\cap L^\infty(\Omega))$ are two mild solutions of equation (1.4) which are uniformly bounded in $L^1(\Omega)$ and converge weakly to $\Gamma\,\delta_{(\bar r,\bar z)}$ as $t \to 0$. Introducing self-similar variables as in (4.1), we obtained two rescaled vorticities $f_1(R,Z,t), f_2(R,Z,t)$ which can both be decomposed as in (4.42):

$$f_1(R,Z,t) \,=\, f_0(R,Z,t) + \tilde f_1(R,Z,t)\,, \qquad f_2(R,Z,t) \,=\, f_0(R,Z,t) + \tilde f_2(R,Z,t)\,.$$

The associated velocity fields are decomposed in a similar way:

$$U_1^\epsilon(R,Z,t) \,=\, U_0^\epsilon(R,Z,t) + \tilde U_1^\epsilon(R,Z,t)\,, \qquad U_2^\epsilon(R,Z,t) \,=\, U_0^\epsilon(R,Z,t) + \tilde U_2^\epsilon(R,Z,t)\,.$$

We take the difference of both solutions and denote

$$\begin{aligned} \tilde f(R,Z,t) \,&=\, f_1(R,Z,t) - f_2(R,Z,t) \,=\, \tilde f_1(R,Z,t) - \tilde f_2(R,Z,t)\,,\\ \tilde U^\epsilon(R,Z,t) \,&=\, U_1^\epsilon(R,Z,t) - U_2^\epsilon(R,Z,t) \,=\, \tilde U_1^\epsilon(R,Z,t) - \tilde U_2^\epsilon(R,Z,t)\,. \end{aligned}$$

The evolution equation for $\tilde f$ reads

$$t\partial_t \tilde f + \gamma\,\mathrm{div}_*\big(U_0^\epsilon \tilde f + \tilde U^\epsilon f_0\big) + \gamma\,\mathrm{div}_*\big(\tilde U^\epsilon \tilde f_1 + \tilde U_2^\epsilon \tilde f\big) \,=\, \mathcal{L}\tilde f + \partial_R\Big(\frac{\epsilon \tilde f}{1+\epsilon R}\Big)\,. \tag{4.75}$$

This is basically the same equation as (4.43), except that the source term $\mathcal{H}$ has disappeared when taking the difference of the equations for $\tilde f_1$ and $\tilde f_2$, and the nonlinear term has been expanded as follows: $\tilde U_1^\epsilon \tilde f_1 - \tilde U_2^\epsilon \tilde f_2 = \big(\tilde U_1^\epsilon - \tilde U_2^\epsilon\big)\tilde f_1 + \tilde U_2^\epsilon\big(\tilde f_1 - \tilde f_2\big)$. In analogy with (4.45) we denote

$$E \,=\, \frac12\int \tilde f^2 w\,\mathrm{d}R\,\mathrm{d}Z\,, \qquad E_1 \,=\, \frac12\int \tilde f_1^2 w\,\mathrm{d}R\,\mathrm{d}Z\,, \qquad E_2 \,=\, \frac12\int \tilde f_2^2 w\,\mathrm{d}R\,\mathrm{d}Z\,,$$

and as in (4.48) we also define

$$\mathcal{E} \,=\, \frac12\Big(|\nabla\tilde f|^2 + (1+R^2+Z^2)\tilde f^2\Big) w\,\mathrm{d}R\,\mathrm{d}Z\,.$$

We claim that for $\epsilon < \epsilon_0 \le 1/2$ we have

$$tE'(t) \,\le\, -2\delta\mathcal{E}(t) + \kappa\big(E_1(t)^{1/2} + E_2(t)^{1/2}\big)\mathcal{E}(t) + \tilde{\mathcal{R}}(t)\,, \tag{4.76}$$

with the remainder $\tilde{\mathcal{R}}(t)$ satisfying $\tilde{\mathcal{R}}(t) \le e^{-1/(36\epsilon^2)}$. This is obtained by repeating the proof of Proposition 4.6, with only minor adjustments. No new estimates are needed, the only change worth mentioning is that the expression (4.71) is replaced by

$$-\int \big(\tilde U^\epsilon \tilde f_1 + \tilde U_2^\epsilon \tilde f\big)\big(w\nabla\tilde f + \tilde f\nabla w\big)\,\mathrm{d}R\,\mathrm{d}Z\,. \tag{4.77}$$

The integral (4.77) is bounded by the cubic term $\kappa(E_1(t)^{1/2} + E_2(t)^{1/2})\mathcal{E}(t)$ in (4.76). To see this, we can control the term produced by $U^\epsilon \tilde{f}_1$ using the alternative approach mentioned at the end of Step 5 above, while the second nonlinear term arising from $\tilde{U}_2^\epsilon \tilde{f}$ can be treated by the original approach in Step 5. Of course inequality (4.76) does not include any term of the form $\kappa\epsilon|\log\epsilon|\, E(t)^{1/2}$, because in (4.47) that term was produced by the source $\mathcal{H}$ which does not appear in (4.75).

As long as t is small enough so that $\kappa(E_1(t)^{1/2} + E_2(t)^{1/2}) \le \delta$, it follows from (4.76) that $tE'(t) \le -\delta E(t) + \tilde{\mathcal{R}}(t)$, hence

$$E(t) \,\le\, t^{-\delta}\int_0^t s^{\delta-1}\tilde{\mathcal{R}}(s)\,\mathrm{d}s \,=\, \mathcal{O}\Big(e^{-1/(36\epsilon^2)}\Big)\,. \tag{4.78}$$

This already shows that $E(t)$ converges extremely rapidly to zero as $t \to 0$, but our actual goal is to prove that $E(t)$ vanishes identically.

To do that, our strategy is to combine (4.78) with another estimate, which is less sophisticated and simply shows that $E(t)$ cannot grow faster than some (large) power of t. As long as $\epsilon \le 1/2$, we claim that

$$tE'(t) \,\le\, -\delta\mathcal{E}(t) + KE(t) + \kappa\big(E_1(t)^{1/2} + E_2(t)^{1/2}\big)\mathcal{E}(t)\,, \tag{4.79}$$

for some positive constants K and κ (depending on γ). Note that there is no remainder term $\tilde{\mathcal{R}}(t)$ in (4.79), but this is obtained at the expense of including the positive term $KE(t)$ with a (possibly large) constant K. To obtain (4.79), the only modifications in the proof of Proposition 4.6 concern the diffusive terms D_1 and D_2. To bound D_1 we forget about (4.56) and only take a convex combination of (4.53), (4.54), with coefficients 1/3 and 2/3. The result is

$$D_1 \,\le\, -\int\Big(\frac{1}{3}|\nabla\tilde{f}|^2 + \frac{R^2{+}Z^2}{24}\tilde{f}^2 + \frac{1}{3}\tilde{f}^2\Big)w\,\mathrm{d}R\,\mathrm{d}Z + \int \tilde{f}^2 w\,\mathrm{d}R\,\mathrm{d}Z\,. \tag{4.80}$$

As for D_2, we use estimate (4.59) on the whole domain Ω_ϵ and add it to (4.58), which gives

$$D_2 \,\le\, \frac{1}{32}\int \tilde{f}^2 R^2 w\,\mathrm{d}R\,\mathrm{d}Z\,. \tag{4.81}$$

When taking the sum $D_1 + D_2$, we observe that the right-hand side of (4.81) is entirely absorbed in the negative terms that appear in (4.80). In particular, no remainder term is produced.

Now, whenever t is small enough so that $\kappa(E_1(t)^{1/2} + E_2(t)^{1/2}) \le \delta$, it follows from (4.79) that $tE'(t) \le KE(t)$, hence

$$E(t) \,\le\, \Big(\frac{t}{t_0}\Big)^K E(t_0)\,, \qquad 0 < t_0 < t\,. \tag{4.82}$$

In view of (4.78), the right-hand side of (4.82) converges to 0 as $t_0 \to 0$. Thus $E(t) = 0$, and we deduce that $f_1(t) = f_2(t)$ for all sufficiently small times. Returning to the original variables, we conclude that

$$\omega_\theta^{(1)}(r,z,t) \,=\, \omega_\theta^{(2)}(r,z,t)\,,$$

for sufficiently small times, hence for all $t \in (0,T)$ in view of the well-posedness result established in [17, Theorem 1.1]. The proof of Theorem 1.1 is now complete. □

5 Appendix

5.1 Convergence of signed measures

For easy reference, we collect here a few remarks on weak convergence of signed measures. The content of this section is probably standard, although most of the classical literature is devoted to the particular case of probability measures. We state the results in a general framework, but in the rest of the paper all measures are defined on the half-plane $\Omega \subset \mathbb{R}^2$. We first recall a few definitions.

1. Given a locally compact metric space X, we denote by $C_0(X)$ the space of all continuous functions $f : X \to \mathbb{R}$ that vanish at infinity in the following sense: for any $\epsilon > 0$, there exists a compact set $K \subset X$ such that $|f(x)| \le \epsilon$ for all $x \in K^c := X \setminus K$. Equipped with the supremum norm, $C_0(X)$ is a real Banach space.

2. Let $\mathcal{M}(X)$ be the set of all finite, signed, regular Borel measures on X. If $\mu \in X$, we denote by $|\mu|$ the total variation of μ [29], which is a nonnegative finite Borel measure on X. The total variation norm of μ is the real number $\|\mu\| = |\mu|(X) \ge 0$. Equipped with the total variation norm, the space $\mathcal{M}(X)$ becomes a real Banach space.

3. By the Riesz-Markov theorem [29], if $\Phi : C_0(X) \to \mathbb{R}$ is any continuous linear functional, there exists a unique measure $\mu \in \mathcal{M}(X)$ such that

$$\Phi(f) = \int_X f \,\mathrm{d}\mu\,, \qquad \text{for all } f \in C_0(X)\,. \tag{5.1}$$

Moreover the total variation norm $\|\mu\|$ is precisely the norm of the linear functional Φ. The space $\mathcal{M}(X)$ can thus be identified via (5.1) to the topological dual $C_0(X)'$.

4. If (μ_n) is a sequence in $\mathcal{M}(X)$, we say that μ_n *converges weakly* to $\mu \in \mathcal{M}(X)$ if

$$\lim_{n\to\infty} \int_X f \,\mathrm{d}\mu_n = \int_X f \,\mathrm{d}\mu\,, \qquad \text{for all } f \in C_0(X)\,. \tag{5.2}$$

We write $\mu_n \rightharpoonup \mu$ as $n \to \infty$. This notion coincides with the weak-$*$ convergence in $\mathcal{M}(X) \simeq C_0(X)'$. We always have

$$\|\mu\| \le \liminf_{n\to\infty} \|\mu_n\|\,.$$

5. A family of measures $\mathcal{F} \subset \mathcal{M}(X)$ is *tight* if, for any $\epsilon > 0$, there exists a compact set $K \subset X$ such that $|\mu|(K^c) \le \epsilon$ for all $\mu \in \mathcal{F}$. Any singleton $\{\mu\}$ is necessarily tight, because the measure $\mu \in \mathcal{M}(X)$ is inner regular. If (μ_n) is a tight sequence in $\mathcal{M}(X)$ that converges weakly to $\mu \in \mathcal{M}(X)$, the convergence in (5.2) holds for all bounded and continuous functions $f : X \to \mathbb{R}$, and not only for all $f \in C_0(X)$. This is the case, for instance, if (μ_n) is a sequence of probability measures that converges to a probability measure μ.

The main purpose of this section is to state the following basic result:

Proposition 5.1. *Let (μ_n) be a sequence in $\mathcal{M}(X)$, and let $\mu \in \mathcal{M}(X)$. We assume that*

$$\mu_n \rightharpoonup \mu \quad \text{and} \quad \|\mu_n\| \to \|\mu\|\,, \qquad \text{as } n \to \infty\,.$$

Then $|\mu_n| \rightharpoonup |\mu|$ as $n \to \infty$, and the sequence $(|\mu_n|)$ is tight.

Proof. The result is obvious if $\mu = 0$, so we assume henceforth that $\mu \neq 0$. Then $\mu_n \neq 0$ for all sufficiently large $n \in \mathbb{N}$, and we can thus define the normalized measures

$$\tilde{\mu}_n = \frac{\mu_n}{\|\mu_n\|}\,, \qquad \text{and} \quad \tilde{\mu} = \frac{\mu}{\|\mu\|}\,.$$

By construction $|\tilde{\mu}_n|$ and $|\tilde{\mu}|$ are now probability measures on X, and $\tilde{\mu}_n \rightharpoonup \tilde{\mu}$ as $n \to \infty$.

Let U be an open subset of X, and take $f \in C_0(U)$ such that $|f(x)| \le 1$ for all $x \in U$. We denote by $\bar{f} : X \to \mathbb{R}$ the extension of f by zero outside U. One verifies that $\bar{f} \in C_0(X)$, so that

$$\Big|\int_U f \,\mathrm{d}\tilde{\mu}\Big| \,=\, \Big|\int_X \bar{f} \,\mathrm{d}\tilde{\mu}\Big| \,=\, \lim_{n\to\infty}\Big|\int_X \bar{f} \,\mathrm{d}\tilde{\mu}_n\Big| \,\le\, \liminf_{n\to\infty} |\tilde{\mu}_n|(U)\ ,$$

because $|\bar{f}| \le 1_U$ (the indicator function of U). It follows that

$$|\tilde{\mu}|(U) \,=\, \sup\Big\{\Big|\int_U f \,\mathrm{d}\tilde{\mu}\Big|\ ;\ f \in C_0(U)\,,\ \|f\|_\infty \le 1\Big\} \,\le\, \liminf_{n\to\infty} |\tilde{\mu}_n|(U)\ . \qquad (3)$$

Since (3) holds for any open set $U \subset X$, the celebrated Portmanteau theorem [5] implies that $|\tilde{\mu}_n| \rightharpoonup |\tilde{\mu}|$ as $n \to \infty$, hence also $|\mu_n| \rightharpoonup |\mu|$ as $n \to \infty$.

As $(|\tilde{\mu}_n|)$ is a sequence of probability measures that converges weakly to the probability measure $|\tilde{\mu}| \in \mathcal{M}(X)$, the sequence $(|\tilde{\mu}_n|)$ is tight (see the discussion above), and so is the sequence $(|\mu_n|)$. □

5.2 Velocity bounds in $L^\infty(\mathbb{R}^3)^{-1}$

This section is devoted to the proof of Lemma 2.9. We first note that it is enough to show that $\|u\|_{(L^\infty)^{-1}} \le c$ when $\omega_\theta = \delta_{(\bar{r},\bar{z})}$ for some $(\bar{r},\bar{z}) \in \Omega$, as the general situation can be thought of as a continuous superposition of these special cases. Moreover, due to the scaling invariance and the translational symmetry along the z–axis, it is enough to consider the particular case where $\bar{r} = 1$, $\bar{z} = 0$.

The proof can be motivated by the following observation, which is as a variant of formula (1.11) in [28]:

$$\partial_i \log|x| \,=\, \mathrm{div}\,(x_i \nabla \log|x|) \ \text{ in } \mathbb{R}^2 \quad (i = 1,2)\,. \qquad (5.3)$$

This shows that, in dimension two, the vector field $\nabla \log|x|$ belongs to $(L^\infty)^{-1}$, and not only to BMO^{-1}. We now consider a three-dimensional analogue of (5.3), which is adapted to our purposes. Let

$$\mathcal{G}(x) \,=\, \frac{1}{4\pi|x|}\,, \qquad x \in \mathbb{R}^3 \setminus \{0\}\,,$$

be the fundamental solution of the Laplacian in $\mathbb{R}^3$, and consider the matrix-valued function

$$P \,=\, \begin{pmatrix} -\partial_3\mathcal{G} & 0 & \partial_1\mathcal{G} \\ 0 & -\partial_3\mathcal{G} & \partial_2\mathcal{G} \\ \partial_1\mathcal{G} & \partial_2\mathcal{G} & \partial_3\mathcal{G} \end{pmatrix}.$$

Note that div $P = 0$ in $\mathbb{R}^3 \setminus \{0\}$, where (as usual) div P is the vector given in coordinates by $(\mathrm{div}\ P)_i = \partial_j P_{ij}$. In the sense of distributions, we have

$$\nabla\mathcal{G} \,=\, \mathrm{div}\,(x_3 P)\,, \quad \text{in } \mathcal{D}'(\mathbb{R}^3)\,. \qquad (5.4)$$

Let us parametrize the vortex filament supported by the circle $\mathcal{C} = \{(x_1,x_2,0)\,|\,x_1^2 + x_2^2 = 1\}$ using $\gamma(s) = (\cos s, \sin s, 0)$ for $s \in (-\pi,\pi]$. The associated velocity field U is

$$U(x) \,=\, \mathrm{curl}\int_{-\pi}^{\pi} \mathcal{G}(x-\gamma(s))\gamma'(s)\,\mathrm{d}s \,=\, \int_{-\pi}^{\pi} \nabla\mathcal{G}(x-\gamma(s)) \wedge \gamma'(s)\,\mathrm{d}s\,. \qquad (5.5)$$

Using (5.4) together with the fact that $|\gamma'(s)| = 1$, we see that to prove our claim, it is enough to establish a uniform bound for the quantity

$$\int_{-\pi}^{\pi} |(x-\gamma(s))_3 P(x-\gamma(s))|\,\mathrm{d}s \,=\, |x_3|\int_{-\pi}^{\pi} |P(x-\gamma(s))|\,\mathrm{d}s\,.$$

As $|P(x)| \le c|x|^{-2}$, we only need to bound the expression

$$I(x) \;=\; \int_{-\pi}^{\pi} \frac{|x_3|}{|x-\gamma(s)|^2}\,\mathrm{d}s \;=\; \frac{2\pi|x_3|}{\sqrt{(1+|x|^2)^2-4(x_1^2+x_2^2)}}\,.$$

But $(1+|x|^2)^2-4(x_1^2+x_2^2) = (1-x_1^2-x_2^2)^2+2(1+x_1^2+x_2^2)x_3^2+x_3^4 \ge 2x_3^2$, hence $I(x) \le \sqrt{2}\pi$. The proof is thus complete. □

Remark 5.2. *If we only wish to prove a* BMO^{-1} *bound for u, which is sufficient to apply the results of [30], we see from* (5.5) *that it is enough to estimate the vector field*

$$A(x) \;=\; \int_{-\pi}^{\pi} \mathcal{G}(x-\gamma(s))\gamma'(s)\,\mathrm{d}s$$

in the space BMO*. This can be done in a number of ways. For example, we note that $\nabla A \in L^p(\mathbb{R}^3)$ for any $p \in (1,2)$, and near the circle $\mathcal{C}$ we have $|\nabla A(x)| \lesssim \mathrm{dist}(x,\mathcal{C})^{-1}$. This easily gives a uniform bound on $R^{-3+p}\int_{B_{x,R}} |\nabla A(y)|^p\,\mathrm{d}y$, which implies that $A \in \mathrm{BMO}$.*

If one is willing to use deeper results in harmonic analysis, one can apply for example Theorem 3 on page 159 of Stein's book [31] and some elementary estimates to see that, for the BMO *bound of $A = \Delta^{-1}\omega$, it is enough to control*

$$\sup_{x\in\mathbb{R}^3}\,\sup_{R>0}\,\frac{1}{R}\int_{B_{x,R}} |\omega(y)|\,\mathrm{d}y\,.$$

That quantity is in turn bounded by $c\|\omega_\theta\|_{L^1(\Omega)}$, as is easily verified.

5.3 Bounds on the fundamental solution

This section is devoted to the proof of Proposition 3.9. Since the existence of a (unique) fundamental solution Φ is known from the work of Aronson, we concentrate on the derivation of the upper bound (3.26), and for that purpose we adapt to our particular situation the efficient approach of Fabes and Stroock [9]. Without loss of generality, we take $\nu = 1$, we assume that the functions U, V are smooth and bounded on $\mathbb{R}^n \times [0,T]$, and we prove estimate (3.26) for $s = 0$.

Let f be a smooth solution to (3.25) on $\mathbb{R}^n \times [0,T]$, with (for instance) compactly supported initial data. Given any fixed vector $\alpha \in \mathbb{R}^n$, we define $g(x,t) = e^{-\alpha\cdot x} f(x,t)$ for $x \in \mathbb{R}^n$ and $t \in [0,T]$. The evolution equation satisfied by g is

$$\partial_t g + U\cdot\nabla g + (U\cdot\alpha + V)g \;=\; \Delta g + 2\alpha\cdot\nabla g + \alpha^2 g\,. \tag{5.6}$$

The proof of the upper bound on the fundamental solution of (3.25) involves four steps:

Step 1: L^1 estimate. Assuming first that g is a nonnegative solution of (5.6), and using the assumption that $\operatorname{div} U = 0$, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int g\,\mathrm{d}x \;&=\; \alpha^2\int g\,\mathrm{d}x - \int (U\cdot\alpha + V)g\,\mathrm{d}x\\ &\le\; \Big(\alpha^2 + |\alpha|\|U(t)\|_{L^\infty} + \|V(t)\|_{L^\infty}\Big)\int g\,\mathrm{d}x\,.\end{aligned}$$

Here and in what follows all integrals are taken over the whole Euclidean space $\mathbb{R}^n$, and for simplicity we write $\|U(t)\|_{L^\infty}$ instead of $\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$. Applying Gronwall's lemma, we obtain the estimate

$$\int |g(x,t)|\,\mathrm{d}x \;\le\; \Big(\int |g(x,0)|\,\mathrm{d}x\Big)\exp\Big(\alpha^2 t + \int_0^t \big(|\alpha|\|U(s)\|_{L^\infty} + \|V(s)\|_{L^\infty}\big)\,\mathrm{d}s\Big)\,, \tag{5.7}$$

for $t \in [0,T]$. Note that (5.7) remains valid in the general case where g changes its sign.

Step 2: L^1–L^2 estimate. By a similar calculation, we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int g^2\,\mathrm{d}x &= \int g\Big(\Delta g + 2\alpha\cdot\nabla g + \alpha^2 g - U\cdot\nabla g - (U\cdot\alpha + V)g\Big)\,\mathrm{d}x \\ &= -\int |\nabla g|^2\,\mathrm{d}x + \Big(\alpha^2 + |\alpha|\|U(t)\|_{L^\infty} + \|V(t)\|_{L^\infty}\Big)\int g^2\,\mathrm{d}x\,.\end{aligned}$$

To estimate the right-hand side we apply Nash's inequality

$$\Big(\int g^2\,\mathrm{d}x\Big)^{1+2/n} \le C_n\Big(\int |g|\,\mathrm{d}x\Big)^{4/n}\int |\nabla g|^2\,\mathrm{d}x\,,$$

which holds for any $g \in L^1(\mathbb{R}^n)\cap H^1(\mathbb{R}^n)$ with a constant $C_n > 0$ depending only on the space dimension n. We thus obtain the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int g^2\,\mathrm{d}x \le -\frac{\Big(\int g^2\,\mathrm{d}x\Big)^{1+2/n}}{C_n\Big(\int |g|\,\mathrm{d}x\Big)^{4/n}} + \Big(\alpha^2 + |\alpha|\|U(t)\|_{L^\infty} + \|V(t)\|_{L^\infty}\Big)\int g^2\,\mathrm{d}x\,, \tag{5.8}$$

which is a differential inequality for the L^2 norm of the solutions of (5.6). To solve (5.8), we temporarily denote

$$\begin{aligned}\Lambda(t) &= \alpha^2 t + \int_0^t \big(|\alpha|\|U(s)\|_{L^\infty} + \|V(s)\|_{L^\infty}\big)\,\mathrm{d}s\,,\\ A(t) &= \exp(-\Lambda(t))\int |g(x,t)|\,\mathrm{d}x \le A(0)\,,\\ B(t) &= \exp(-2\Lambda(t))\int g(x,t)^2\,\mathrm{d}x\,, \qquad t\in[0,T]\,.\end{aligned}$$

Here the bound $A(t) \le A(0)$ is a reformulation of (5.7). Using (5.8), we find

$$\begin{aligned}B'(t) &\le -\frac{2}{C_n}\frac{\Big(\int g^2\,\mathrm{d}x\Big)^{1+2/n}}{\Big(\int |g|\,\mathrm{d}x\Big)^{4/n}}e^{-2\Lambda(t)} \le -\frac{2}{C_n}\frac{\Big(B(t)\,e^{2\Lambda(t)}\Big)^{1+2/n}}{\Big(A(0)\,e^{\Lambda(t)}\Big)^{4/n}}e^{-2\Lambda(t)}\\ &= -\frac{2}{C_n}\frac{B(t)^{1+2/n}}{A(0)^{4/n}}\,, \qquad 0 < t \le T\,.\end{aligned}$$

Integrating this simple differential inequality we obtain $B(t) \le (C_n' A(0))^2 t^{-n/2}$ for $t \in (0,T]$, where $C_n' = (nC_n/4)^{n/4}$. In other words, we have proved the L^1–L^2 estimate

$$\|g(t)\|_{L^2} \le \frac{C_n'}{t^{n/4}}\,\|g(0)\|_{L^1}\,\exp\Big(\alpha^2 t + \int_0^t \big(|\alpha|\|U(s)\|_{L^\infty} + \|V(s)\|_{L^\infty}\big)\,\mathrm{d}s\Big)\,, \tag{5.9}$$

for all $t \in (0,T]$.

Step 3: L^1–L^∞ estimate. We consider the adjoint equation

$$\partial_t\tilde g - U\cdot\nabla\tilde g + (U\cdot\alpha + V)\tilde g = \Delta\tilde g - 2\alpha\cdot\nabla\tilde g + \alpha^2\tilde g\,, \tag{5.10}$$

which has exactly the same structure as (5.6). In particular, the L^1–L^2 bound (5.9) holds for the solutions of (5.10), and using a standard duality argument this implies the following L^2–L^∞ estimate for the solutions of (5.6):

$$\|g(t)\|_{L^\infty} \;\le\; \frac{C_n'}{t^{n/4}}\,\|g(0)\|_{L^2}\,\exp\Bigl(\alpha^2 t + \int_0^t \bigl(|\alpha|\|U(s)\|_{L^\infty} + \|V(s)\|_{L^\infty}\bigr)\,\mathrm{d}s\Bigr)\,, \tag{5.11}$$

for all $t \in (0,T]$. To obtain the L^1–L^∞ bound we estimate $\|g(t/2)\|_{L^2}$ in terms of $\|g(0)\|_{L^1}$ using (5.9), and then $\|g(t)\|_{L^\infty}$ in terms of $\|g(t/2)\|_{L^2}$ using the analogue of (5.11). Denoting $C_n'' = 2^{n/2}C_n'^2$, this gives

$$\begin{aligned}\|g(t)\|_{L^\infty} \;&\le\; \frac{C_n''}{t^{n/2}}\,\|g(0)\|_{L^1}\,\exp\Bigl(\alpha^2 t + \int_0^t \bigl(|\alpha|\|U(s)\|_{L^\infty} + \|V(s)\|_{L^\infty}\bigr)\,\mathrm{d}s\Bigr)\\ &\le\; \frac{C_n''}{t^{n/2}}\,\|g(0)\|_{L^1}\,\exp\Bigl(\alpha^2 t + 2K_1|\alpha|\sqrt{t} + K_2\Bigr)\,,\end{aligned} \tag{5.12}$$

for all $t \in (0,T]$, where in the second inequality we used definitions (3.24).

Step 4: conclusion. By construction the solutions of (5.6) can be represented as

$$g(x,t) \;=\; \int e^{\alpha\cdot(y-x)}\Phi(x,t;y)g(y,0)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^n\,, \qquad 0 < t \le T\,,$$

where $\Phi(x,t;y) = \Phi_{U,V,1}(x,t;y,0)$ is the fundamental solution of equation (3.25) with $\nu = 1$. Estimate (5.12), which holds for all smooth and compactly supported initial data $g(x,0)$, is thus equivalent to the pointwise upper bound

$$\Phi(x,t;y) \;\le\; \frac{C_n''}{t^{n/2}}\,e^{\alpha\cdot(x-y)}\,\exp\Bigl(\alpha^2 t + 2K_1|\alpha|\sqrt{t} + K_2\Bigr)\,, \qquad x,y \in \mathbb{R}^n\,, \quad 0 < t \le T\,. \tag{5.13}$$

The vector $\alpha \in \mathbb{R}^n$ was arbitrary, and the dependence upon α is fully explicit in (5.13). Given $x, y \in \mathbb{R}^n$ and $t > 0$, we can thus choose $\alpha = -(x-y)/(2t)$, in which case (5.13) becomes

$$\Phi(x,t;y) \;\le\; \frac{C_n''}{t^{n/2}}\,\exp\Bigl(-\frac{|x-y|^2}{4t} + K_1\frac{|x-y|}{\sqrt{t}} + K_2\Bigr)\,. \tag{5.14}$$

This proves (3.26) for $\nu = 1$ and $s = 0$, and the general case easily follows. □

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