Stability of Vortices in Ideal Fluids: the Legacy of Kelvin and Rayleigh

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Abstract

The mathematical theory of hydrodynamic stability started in the middle of the 19th century with the study of model examples, such as parallel flows, vortex rings, and surfaces of discontinuity. We focus here on the equally interesting case of columnar vortices, which are axisymmetric stationary flows where the velocity field only depends on the distance to the symmetry axis and has no component in the axial direction. The stability of such flows was first investigated by Kelvin in 1880 for some particular velocity profiles, and the problem benefited from important contributions by Rayleigh in 1880 and 1917. Despite further progress in the 20th century, notably by Howard and Gupta (1962), the only rigorous results so far are necessary conditions for instability under either two-dimensional or axisymmetric perturbations. This note is a non-technical introduction to a recent work in collaboration with D. Smets, where we prove under mild assumptions that columnar vortices are spectrally stable with respect to general three-dimensional perturbations, and that the linearized evolution group has a subexponential growth as $|t| \to \infty$.

1 Introduction to Hydrodynamic Stability Theory

Hydrodynamic stability is the subdomain of fluid dynamics which studies the stability and the onset of instability in fluid flows. These fundamental questions were first addressed in the 19th century, with pioneering contributions by G. Stokes, H. von Helmholtz, W. Thomson (Lord Kelvin), and J. W. Strutt (Lord Rayleigh) on the theoretical side, and by O. Reynolds on the experimental side [9]. In early times the notion of stability still lacked a precise mathematical definition, but its physical meaning was already perfectly understood, as can be seen from the following quote by J. C. Maxwell [19], which dates back to 1873:

> "When the state of things is such that an infinitely small variation of the present state will alter only by an infinitely small quantity the state at some future time, the condition of the system, whether at rest or in motion, is said to be stable; but when an infinitely small variation in the present state may bring about a finite difference in the state of the system in a finite time, the system is said to be unstable."

What is exactly meant by "infinitely small" in this definition is rigorously specified, for instance, in the subsequent memoir by A. M. Lyapunov [26], which was published in 1892. The relevance of stability questions in fluid mechanics cannot be overestimated. As an example, in the idealized situation where the fluid is assumed to be incompressible and inviscid, a plethora of explicit stationary solutions are known which describe shear flows, vortices, or flows past obstacles. However, depending on circumstances, these solutions may or may not be observed in real life, where experimental uncertainties, viscosity effects, and boundary conditions play an important role. To determine the relevance of a given flow, the stability analysis is certainly the first step to perform, but even in an idealized framework this often leads to difficult mathematical problems, a complete solution of which was largely out of reach in the 19th century and is still a serious challenge today.

To make the previous considerations more concrete, we analyze in this introduction three relatively simple cases, of increasing complexity, where stability can be discussed using the techniques introduced by Rayleigh [30]. These examples are classical and thoroughly studied in many textbooks [7, 8, 11, 23, 33], as well as in the excellent review article [10]. The results obtained for these model problems will serve as a guideline for the stability analysis of columnar vortices, which will be presented in Sections 2 and 3.

1.1 The Rayleigh-Taylor Instability

We consider the motion of an incompressible and inviscid fluid in the infinite strip $D = \mathbb{R} \times [0, L]$ with coordinates (x, z), where $x \in \mathbb{R}$ is the horizontal variable and $z \in [0, L]$ the vertical variable. The state of the fluid at time $t \in \mathbb{R}$ is defined by the density distribution $\rho(x, z, t) > 0$, the velocity field $u(x, z, t) \in \mathbb{R}^2$, and the pressure $p(x, z, t) \in \mathbb{R}$. The evolution is determined by the density-dependent incompressible Euler equations

$$\partial_t \rho + u \cdot \nabla \rho = 0\,, \quad \rho\big(\partial_t u + (u \cdot \nabla) u\big) \,=\, -\nabla p - \rho g e_z\,, \quad \operatorname{div} u \,=\, 0\,, \tag{1.1}$$

where g denotes the acceleration due to gravity and e_z is the unit vector in the (upward) vertical direction. Setting $u = (u_x, u_z)$, we impose the impermeability condition $u_z(x, z, t) = 0$ at the bottom and the top of the domain D, namely for $z = 0$ and $z = L$.

The PDE system (1.1) has a family of stationary solutions of the form $\rho = \bar\rho(z)$, $u = 0$, $p = \bar p(z)$, where the density $\bar\rho$ is an arbitrary function of the vertical coordinate z, and the associated pressure is determined (up to an irrelevant additive constant) by the hydrostatic balance $\bar p'(z) = -\bar\rho(z) g$. To study the stability of the equilibrium $(\bar\rho, 0, \bar p)$, we consider perturbed solutions of the form

$$\rho(x, z, t) \,=\, \bar\rho(z) + \tilde\rho(x, z, t)\,, \quad u(x, z, t) \,=\, \tilde u(x, z, t)\,, \quad p(x, z, t) \,=\, \bar p(z) + \tilde p(x, z, t)\,.$$

Inserting this Ansatz into (1.1) and neglecting all quadratic terms in $(\tilde\rho, \tilde u)$, we obtain the *linearized* equations for the perturbations $(\tilde\rho, \tilde u, \tilde p)$:

$$\begin{aligned} \bar\rho(z)\partial_t \tilde u_x \,&=\, -\partial_x \tilde p\,, \qquad & \partial_t \tilde\rho + \bar\rho'(z)\tilde u_z \,&=\, 0\,, \\ \bar\rho(z)\partial_t \tilde u_z \,&=\, -\partial_z \tilde p - \tilde\rho g\,, \qquad & \partial_x \tilde u_x + \partial_z \tilde u_z \,&=\, 0\,. \end{aligned} \tag{1.2}$$

Remark 1.1. It is not obvious at all that considering the linearized perturbation equations (1.2) is sufficient, or even appropriate, to determine the stability of stationary solutions to (1.1). In fact the validity of Lyapunov's linearization method in the context of fluid mechanics is a difficult question [39], which is the object of ongoing research. In particular, for ideal fluids, there is no general result asserting that a linearly stable equilibrium is actually stable in the sense of Lyapunov. However, if the linearized system is exponentially unstable, for instance due to the existence of an eigenvalue with nonzero real part, it is often possible to conclude that the equilibrium under consideration is unstable, see [5, 16, 25, 38] for a few results in this direction. To summarize, the linearization approach may be useful to detect exponential instabilities, but stability results have to be established by a different approach, for instance (in two space dimensions) using variational techniques [2, 3]

The linearized equations (1.2) are invariant under translations in the horizontal direction, so that we can use a Fourier transform to reduce the number of independent variables. A further simplification is made by restricting our attention to *eigenfunctions* of the linearized operator. In other words, we consider solutions of (1.2) of the particular form

$$\tilde{\rho}(x,z,t) \,=\, \rho(z)\, e^{ikx}\, e^{st}\,, \quad \tilde{u}(x,z,t) \,=\, u(z)\, e^{ikx}\, e^{st}\,, \quad \tilde{p}(x,z,t) \,=\, p(z)\, e^{ikx}\, e^{st}\,, \qquad (1.3)$$

where $k \in \mathbb{R}$ is the horizontal wave number and $s \in \mathbb{C}$ is the spectral parameter. The representation (1.3) transforms the linearized equations (1.2) into an ODE system :

$$\begin{aligned} \bar{\rho}(z) s u_x &= -ikp\,, & s\rho + \bar{\rho}'(z) u_z &= 0\,, \\ \bar{\rho}(z) s u_z &= -\partial_z p - \rho g\,, & iku_x + \partial_z u_z &= 0\,, \end{aligned} \qquad (1.4)$$

which (if $s \neq 0$) can in turn be reduced to a single equation for the vertical velocity profile u_z :

$$-\partial_z\big(\bar{\rho}(z)\partial_z u_z\big) + k^2 \bar{\rho}(z) u_z - \frac{k^2 g}{s^2}\, \bar{\rho}'(z) u_z \,=\, 0\,, \qquad z \in [0,L]\,. \qquad (1.5)$$

By construction, the values of the spectral parameter $s \in \mathbb{C} \setminus \{0\}$ for which the ODE (1.5) has a nontrivial solution u_z satisfying the boundary conditions $u_z(0) = u_z(L) = 0$ are *eigenvalues* of the linearized operator (1.2) in the Fourier subspace indexed by the horizontal wavenumber $k \in \mathbb{R}$. Spectral stability is obtained if all eigenvalues are purely imaginary, whereas the existence of an eigenvalue $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$ implies exponential instability of the linearized system in positive or negative times.

Remarks 1.2.
1. The Fourier transform reduces the linearized equations to a one-dimensional PDE system in the bounded domain $[0, L]$, but this does not immediately imply that the spectrum of the full linearized operator is the union of the point spectra obtained for all values of the horizontal wavenumber $k \in \mathbb{R}$. So, even if one can prove that the eigenvalues are purely imaginary for all $k \in \mathbb{R}$, an additional argument is needed to verify that the full linearized operator has indeed no spectrum outside the imaginary axis. This rather technical issue will not be discussed further in this introduction, but we shall come back to it in Section 3.
2. In the literature, the Rayleigh-Taylor equation (1.5) is often derived in the Boussinesq approximation, which consists in neglecting the variations of the density profile $\bar{\rho}(z)$ everywhere except in the buoyancy term. This gives the simplified eigenvalue equation

$$-\partial_z^2 u_z + k^2\Big(1 + \frac{N(z)^2}{s^2}\Big) u_z \,=\, 0\,, \qquad \text{where} \quad N(z)^2 \,=\, -\frac{g\bar{\rho}'(z)}{\rho(z)}\,. \qquad (1.6)$$

When $\bar{\rho}'(z) < 0$, the real number $N(z)$ is called the Brunt-Väisälä frequency. This is the (maximal) oscillation frequency of gravity waves inside a stably stratified fluid.

Assume that, for some $k \in \mathbb{R}$ and some $s \in \mathbb{C} \setminus \{0\}$, the ODE (1.5) has a nontrivial solution u_z satisfying the boundary conditions $u_z(0) = u_z(L) = 0$. Multiplying both sides of (1.5) by the complex conjugate $\bar{u}_z$ and integrating over the vertical domain $[0, L]$, we obtain the integral identity

$$\int_0^L \bar{\rho}(z) |\partial_z u_z|^2 \,\mathrm{d}z + k^2 \int_0^L \bar{\rho}(z) |u_z|^2 \,\mathrm{d}z - \frac{k^2 g}{s^2} \int_0^L \bar{\rho}'(z) |u_z|^2 \,\mathrm{d}z \,=\, 0\,. \qquad (1.7)$$

The first two terms in (1.7) being real and positive, equality can hold only if the third term is real and negative. Thus we must have $k \neq 0$ and $\mathrm{Im}(s^2) = 0$, namely $s \in \mathbb{R}$ or $s \in i\mathbb{R}$. Now, if we assume that the fluid is *stably stratified*, in the sense that $\bar\rho'(z) \le 0$ for all $z \in [0, L]$, the last term in (1.7) is positive only if $s^2 < 0$, which means that $s \in i\mathbb{R}$. Under this assumption, we conclude that all eigenfunctions of the form (1.3) with $k \in \mathbb{R}$ correspond to eigenvalues s on the imaginary axis, so that the equilibrium $(\bar\rho, 0, \bar p)$ of (1.1) is *spectrally stable*, up to the technical issue mentioned in Remark 1.2.1.

On the other hand, if $\bar\rho'(z) > 0$ for some $z \in [0, L]$, a nice argument due to Synge [36] shows that, for any $k \neq 0$, the Rayleigh equation has a nontrivial solution u_z (satisfying the boundary conditions) for a sequence of real eigenvalues $s_n \to 0$. The equilibrium $(\bar\rho, 0, \bar p)$ of (1.1) is thus spectrally unstable. Summarizing, the stability of the rest state $u = 0$ in stratified ideal fluids is reasonably understood, in the sense that Rayleigh's approach provides a *necessary and sufficient* condition for spectral stability in that case.

1.2 Shear Flows in Homogeneous Fluids

For the same equations (1.1) in the domain D, we now consider a different family of equilibria, namely shear flows of the form $\rho = 1$, $u = U(z)e_x$, $p = 0$, where the horizontal velocity profile U is an arbitrary function. For the moment, we assume that the fluid is homogeneous and only allow for perturbations of the velocity field. The perturbed solutions thus take the form

$$\rho(x, z, t) = 1\,, \quad u(x, z, t) = U(z)e_x + \tilde u(x, z, t)\,, \quad p(x, z, t) = \tilde p(x, z, t)\,,$$

and the linearized equations become

$$\begin{aligned} \partial_t \tilde u_x + U(z)\partial_x \tilde u_x + U'(z)\tilde u_z &= -\partial_x \tilde p\,, \\ \partial_t \tilde u_z + U(z)\partial_x \tilde u_z &= -\partial_z \tilde p\,, \end{aligned} \qquad \partial_x \tilde u_x + \partial_z \tilde u_z = 0\,. \tag{1.8}$$

As before, we suppose that $\tilde u(x, z, t) = u(z)\, e^{ikx}\, e^{st}$ and $\tilde p(x, z, t) = p(z)\, e^{ikx}\, e^{st}$ for some $k \in \mathbb{R}$ and some $s \in \mathbb{C}$. The functions u, p are solutions of the ODE system

$$\gamma(z)u_x + U'(z)u_z = -ikp\,, \quad \gamma(z)u_z = -\partial_z p\,, \quad iku_x + \partial_z u_z = 0\,, \tag{1.9}$$

where $\gamma(z) = s + ikU(z)$ is the symbol of the material derivative $\partial_t + U(z)\partial_x$. This function plays an important role in the stability analysis, as it incorporates the spectral parameter s.

Since we are interested in detecting potential instabilities, we assume in what follows that $\mathrm{Re}(s) \neq 0$, which implies in particular that $\gamma(z) \neq 0$ for all $z \in [0, L]$. Under this hypothesis, we can reduce the ODE system (1.9) to the following scalar equation for the vertical velocity:

$$-\partial_z^2 u_z + k^2 u_z + \frac{ikU''(z)}{\gamma(z)}\, u_z = 0\,, \qquad z \in [0, L]\,. \tag{1.10}$$

This equation looks simpler than (1.5), but is in fact substantially harder to analyze. If u_z is a nontrivial solution satisfying the boundary conditions, we have Rayleigh's identity

$$\int_0^L |\partial_z u_z|^2\,\mathrm{d}z + k^2 \int_0^L |u_z|^2\,\mathrm{d}z + ik \int_0^L \frac{U''(z)}{\gamma(z)}\, |u_z|^2\,\mathrm{d}z = 0\,, \tag{1.11}$$

which can be satisfied only if $k \neq 0$ and if $U''(z)$ is not identically zero. Under these assumptions, the imaginary part of (1.11) gives the useful relation

$$\mathrm{Re}(s) \int_0^L \frac{U''(z)}{|\gamma(z)|^2}\, |u_z|^2\,\mathrm{d}z = 0\,. \tag{1.12}$$

If $U''(z)$ does not change sign on $[0, L]$, the integral in (1.12) is nonzero, which contradicts our assumption that $\mathrm{Re}(s) \neq 0$. This gives Rayleigh's *inflection point criterion* [30]: a necessary condition for the shear flow with velocity profile $U(z)$ to be (spectrally) unstable is that the function $z \mapsto U''(z)$ changes sign on the interval $[0, L]$.

Rayleigh's inflection point criterion is not sharp, and can be improved somehow by exploiting both the real and imaginary parts of identity (1.11), see [15]. However, surprisingly enough, it seems difficult to formulate a necessary and sufficient stability condition for shear flows, even in the ideal case considered here. An instructive example is Kolmogorov's flow $U(z) = \sin(z - L/2)$, which is known to be stable if and only if $L \le \pi$ [10, 24], although both Rayleigh's and Fjørtoft's criteria allow for a possible instability for any $L > 0$. In fact, the origin of inertial instabilities in shear flows seems only partially understood from a physical point of view, see [4].

1.3 Shear Flows in Stratified Fluids

Following the same approach as in the previous paragraphs, we now analyze the stability of shear flows in (stably) stratified fluids. We consider the Euler equations (1.1) in the vicinity of a stationary solution of the form $\rho = \bar\rho(z)$, $u = U(z)e_x$, $p = \bar p(z)$, where $\bar p'(z) = -\bar\rho(z)g$ (hydrostatic balance). The perturbed solutions are written in the form

$$\rho(x,z,t) \,=\, \bar\rho(z) + \tilde\rho(x,z,t)\,, \quad u(x,z,t) \,=\, U(z)e_x + \tilde u(x,z,t)\,, \quad p(x,z,t) \,=\, \bar p(z) + \tilde p(x,z,t)\,,$$

so that the linearized equations become

$$\begin{aligned} \bar\rho(z)\big(\partial_t \tilde u_x + U(z)\partial_x \tilde u_x + U'(z)\tilde u_z\big) \,&=\, -\partial_x \tilde p\,, \qquad & \partial_t\tilde\rho + U(z)\partial_x\tilde\rho + \bar\rho'(z)\tilde u_z \,&=\, 0\,, \\ \bar\rho(z)\big(\partial_t \tilde u_z + U(z)\partial_x \tilde u_z\big) \,&=\, -\partial_z\tilde p - \tilde\rho g\,, & \partial_x\tilde u_x + \partial_z \tilde u_z \,&=\, 0\,. \end{aligned} \tag{1.13}$$

For perturbations of the form (1.3), we arrive at the ODE system

$$\begin{aligned} \bar\rho(z)\big(\gamma(z)u_x + U'(z)u_z\big) \,&=\, -ikp\,, \qquad & \gamma(z)\rho + \bar\rho'(z)u_z \,&=\, 0\,, \\ \bar\rho(z)\gamma(z)u_z \,&=\, -\partial_z p - \rho g\,, & iku_x + \partial_z u_z \,&=\, 0\,, \end{aligned} \tag{1.14}$$

where $\gamma(z) = s + ikU(z)$ is the spectral function. If we assume that $\mathrm{Re}(s) \neq 0$, so that $\gamma(z) \neq 0$, we can reduce the system (1.14) to the *Taylor-Goldstein equation*

$$-\partial_z\big(\bar\rho(z)\partial_z u_z\big) + k^2\bar\rho(z)u_z + \frac{ik}{\gamma(z)}\big(\bar\rho U'\big)'(z)u_z - \frac{k^2 g}{\gamma(z)^2}\,\bar\rho'(z)u_z \,=\, 0\,, \qquad z \in [0, L]\,. \tag{1.15}$$

Note that we recover the Rayleigh-Taylor equation (1.5) by setting $U = 0$, hence $\gamma(z) = s$, in (1.14). Similarly, (1.14) reduces to the Rayleigh stability equation (1.10) when $\bar\rho = 1$.

The original approach of Rayleigh does not give much information on the solutions of (1.15). If u_z is a nontrivial solution satisfying the boundary conditions, it is difficult to exploit the integral identity

$$\int_0^L \Big\{\bar\rho(z)|\partial_z u_z|^2 + k^2\bar\rho(z)|u_z|^2 + ik\frac{(\bar\rho U')'(z)}{\gamma(z)}\,|u_z|^2 - \frac{k^2 g}{\gamma(z)^2}\bar\rho'(z)|u_z|^2\Big\}\,\mathrm{d}z \,=\, 0\,, \tag{1.16}$$

because the real or imaginary parts of the last two terms in the integrand have no obvious sign. A solution to this problem was found by Miles [27] and Howard [20] in the early 60's. Following the elegant approach of [20], we perform the change of variables

$$u_z(z) \,=\, \gamma(z)^{1/2}v_z(z)\,, \qquad \text{where} \quad \gamma(z) \,=\, s + ikU(z)\,.$$

The new function v_z satisfies the modified ODE

$$-\partial_z\big(\bar\rho(z)\gamma(z)\partial_z v_z\big) + k^2\bar\rho(z)\gamma(z)v_z + \frac{ik}{2}\,(\bar\rho U')'(z)\,v_z + \Big(\frac{\bar\rho\gamma'^2}{4\gamma} - \frac{k^2 g\bar\rho'}{\gamma}\Big)(z)\,v_z \;=\; 0\,. \qquad (1.17)$$

If v_z is a nontrivial solution satisfying the boundary conditions $v_z(0) = v_z(L) = 0$, we multiply both sides of (1.17) by the complex conjugate $\bar v_z$ and integrate over the domain $[0, L]$. After taking the real part, we obtain the useful identity

$$\mathrm{Re}(s)\int_0^L \Big\{\bar\rho(z)\big(|\partial_z v_z|^2 + k^2|v_z|^2\big) + \frac{k^2\bar\rho(z)U'(z)^2}{|\gamma(z)|^2}\Big(\mathrm{Ri}(z) - \frac14\Big)\,|v_z|^2\Big\}\,\mathrm{d}z \;=\; 0\,, \qquad (1.18)$$

where $\mathrm{Ri}(z)$ is the (local) *Richardson number* defined by

$$\mathrm{Ri}(z) \;=\; \frac{-\bar\rho'(z)\,g}{\bar\rho(z)}\,\frac{1}{U'(z)^2} \;=\; \Big(\frac{N(z)}{U'(z)}\Big)^2\,.$$

We assume here that $\bar\rho'(z) \le 0$ (stable stratification), so that $\mathrm{Ri}(z) \ge 0$, and we denote by $N(z)$ the Brunt-Väisälä frequency (1.6).

The Richardson number compares the stabilizing effect of the stratification, measured by the frequency N of the gravity waves, to the potentially destabilizing effect of the shear flow, which may be proportional to the velocity gradient U' [10]. Clearly, equality (1.18) cannot hold if $\mathrm{Ri}(z) \ge 1/4$ for all $z \in [0, L]$, because the integrand is then positive while we assumed that $\mathrm{Re}(s) \neq 0$. This gives the celebrated *Miles-Howard criterion*: a shear flow in a stratified fluid is spectrally stable if the Richardson number is greater than or equal to 1/4 everywhere in the fluid. The threshold value 1/4 is known to be sharp, in the sense that it cannot be replaced by any smaller real number. However, the Miles-Howard criterion itself is by no means sharp: if $\bar\rho = 1$, any shear flow without inflection point is spectrally stable by Rayleigh's criterion, although $\mathrm{Ri}(z) \equiv 0$ in that case.

Remark 1.3. So far we concentrated on the two-dimensional case, but it is also instructive to investigate the stability of shear flows with respect to three-dimensional perturbations. In that case, we work in the domain $D' = \mathbb{R}^2 \times [0, L]$ with coordinates (x, y, z), and consider perturbations that are plane waves with horizontal wave vector $k = (k_1, k_2) \in \mathbb{R}^2$. For instance, the three-dimensional velocity field takes the form

$$u(x, y, z, t) \;=\; U(z)e_x + u(z)\,e^{i(k_1x+k_2y)}\,e^{\sigma t}\,,$$

where $\sigma \in \mathbb{C}$ is the spectral parameter. Using a similar Ansatz for the density and the pressure, it is easy to derive the 3D perturbation equations which generalize (1.14). Now, in the homogeneous case where $\rho \equiv 1$, a well-know result due to Squire [35] shows that, if the 3D perturbation equations have a nontrivial solution for some $k_1, k_2 \neq 0$ and some $\sigma \in \mathbb{C}$ with $\mathrm{Re}(\sigma) \neq 0$, then the 2D perturbation equations (1.9) also have a nontrivial solution with $k = (k_1^2 + k_2^2)^{1/2}$ and $s = (k/k_1)\sigma$. Note that $|\,\mathrm{Re}(s)| > |\,\mathrm{Re}(\sigma)|$, so that the most unstable modes are always two-dimensional; in other words, it is sufficient to consider the 2D case to detect potential instabilities. A similar result holds in the general situation where the fluid is stratified [10], but in that case Squire's transformation also affects the acceleration due to gravity, replacing g by the larger quantity $(k^2/k_1^2)g$. This means that, to any unstable 3D mode, there corresponds a more unstable 2D mode *in a stronger gravitational field*. Therefore, unless the fluid is stably stratified, this result does not imply that the most unstable modes are necessarily two-dimensional.

2 Classical Stability Results for Vortices in Ideal Fluids

We now discuss our main topic, namely the stability of a family of axisymmetric stationary solutions to the three-dimensional Euler equations which describe steady vortex columns. For symmetry reasons, it is convenient to introduce cylindrical coordinates (r,θ,z), and to decompose the velocity of the fluid as $u = u_r e_r + u_\theta e_\theta + u_z e_z$, where e_r, e_θ, e_z are unit vectors in the radial, azimuthal, and vertical directions, respectively. Assuming that the fluid density is constant and equal to one, the Euler equations become

$$\begin{aligned} \partial_t u_r + (u\cdot\nabla)u_r - \frac{u_\theta^2}{r} &= -\partial_r p\,, \\ \partial_t u_\theta + (u\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{r}\partial_\theta p\,, \\ \partial_t u_z + (u\cdot\nabla)u_z &= -\partial_z p\,, \end{aligned} \tag{2.1}$$

where $u\cdot\nabla = u_r\partial_r + \frac{1}{r}u_\theta\partial_\theta + u_z\partial_z$. In addition, we have the incompressibility condition

$$\operatorname{div} u \equiv \frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z = 0\,. \tag{2.2}$$

Columnar vortices are stationary solutions of (2.1), (2.2) of the form

$$u = V(r)\,e_\theta\,, \qquad p = P(r)\,, \tag{2.3}$$

where $V:\mathbb{R}_+\to\mathbb{R}$ is an arbitrary velocity profile, and the associated pressure $P:\mathbb{R}_+\to\mathbb{R}$ is determined, up to an irrelevant additive constant, by the centrifugal balance $rP'(r) = V(r)^2$. For the moment, we only assume that V is a piecewise differentiable function, and that the vortex (2.3) is localized in the sense that $V(r)\to 0$ as $r\to\infty$, but more restrictive assumptions will be added later. We introduce the angular velocity Ω and the vorticity W, which are defined as follows:

$$\Omega(r) = \frac{V(r)}{r}\,, \qquad W(r) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\bigl(rV(r)\bigr) = r\Omega'(r) + 2\Omega(r)\,. \tag{2.4}$$

Without loss of generality, we normalize the vortex so that $\Omega(0) = 1$, hence $W(0) = 2$. Typical examples we have in mind are

- The *Rankine vortex*: $\Omega(r) = \begin{cases} 1 & \text{if } r\le 1\,, \\ r^{-2} & \text{if } r\ge 1\,, \end{cases} \qquad W(r) = \begin{cases} 2 & \text{if } r<1\,, \\ 0 & \text{if } r>1\,. \end{cases}$
- the *Lamb-Oseen vortex*: $\Omega(r) = \frac{1}{r^2}\Bigl(1-e^{-r^2}\Bigr)\,, \quad W(r) = 2\,e^{-r^2}\,.$
- the *Kaufmann-Scully vortex*: $\Omega(r) = \frac{1}{1+r^2}\,, \quad W(r) = \frac{2}{(1+r^2)^2}\,.$

To study the stability of the vortex (2.3), we consider perturbed solutions of the form

$$u(r,\theta,z,t) = V(r)\,e_\theta + \tilde u(r,\theta,z,t)\,, \qquad p(r,\theta,z,t) = P(r) + \tilde p(r,\theta,z,t)\,.$$

This leads to the linearized evolution equations

$$\begin{aligned} \partial_t\tilde u_r + \Omega(r)\partial_\theta\tilde u_r - 2\Omega(r)\tilde u_\theta &= -\partial_r\tilde p\,, \\ \partial_t\tilde u_\theta + \Omega(r)\partial_\theta\tilde u_\theta + W(r)\tilde u_r &= -\frac{1}{r}\partial_\theta\tilde p\,, \\ \partial_t\tilde u_z + \Omega(r)\partial_\theta\tilde u_z &= -\partial_z\tilde p\,, \end{aligned} \tag{2.5}$$

where the pressure is determined so that the velocity perturbation remains divergence-free. Taking the divergence of both sides in (2.5), we obtain for $\tilde{p}$ the second order elliptic equation

$$-\partial_r^*\partial_r\tilde{p} - \frac{1}{r^2}\partial_\theta^2\tilde{p} - \partial_z^2\tilde{p} \;=\; 2\bigl(\partial_r^*\Omega\bigr)\partial_\theta\tilde{u}_r - 2\partial_r^*\bigl(\Omega\,\tilde{u}_\theta\bigr)\,, \tag{2.6}$$

where we introduced the shorthand notation $\partial_r^* = \partial_r + \frac{1}{r}$.

System (2.5) was first studied by Kelvin [37] for some particular velocity profiles. In [31], Rayleigh drew an interesting analogy between columnar vortices and shear flows in stratified fluids, on the basis of which he obtained a sufficient condition for stability with respect to axisymmetric perturbations. Further progress was made in the 20th century, notably by Howard and Gupta [21], and the state of the art is reviewed in textbooks on vortex dynamics [1, 29] or hydrodynamic stability [7, 11]. In this section we give a brief account of these classical developments, and we postpone the presentation of our own results to Section 3.

2.1 Normal Mode Analysis

As the coefficients in (2.5) only depend on the distance r to the symmetry axis, we can reduce the number of independent variables by using a Fourier series decomposition in the angular variable θ and a Fourier transform in the vertical coordinate z. Moreover, as in Sections 1.1–1.3, we focus our attention to the eigenvalues of the linearized operator. We thus consider velocities and pressures of the following form

$$\tilde{u}(r,\theta,z,t) \;=\; u(r)\,e^{i(m\theta+kz)}\,e^{st}\,, \qquad p(r,\theta,z,t) \;=\; p(r)\,e^{i(m\theta+kz)}\,e^{st}\,, \tag{2.7}$$

where $m \in \mathbb{Z}$ is the angular Fourier mode, $k \in \mathbb{R}$ is the vertical wave number, and $s \in \mathbb{C}$ is the spectral parameter. The velocity $u = (u_r, u_\theta, u_z)$ and the pressure p in (2.7) satisfy the ODE system

$$\gamma(r)u_r - 2\Omega(r)u_\theta \;=\; -\partial_r p\,, \quad \gamma(r)u_\theta + W(r)u_r \;=\; -\frac{im}{r}p\,, \quad \gamma(r)u_z \;=\; -ikp\,, \tag{2.8}$$

where $\gamma(r) = s + im\Omega(r)$ is the spectral function. The incompressibility condition becomes

$$\frac{1}{r}\partial_r(ru_r) + \frac{im}{r}u_\theta + iku_z \;=\; 0\,. \tag{2.9}$$

If $(m,k) \neq (0,0)$ it is possible to reduce the system (2.8), (2.9) to a scalar equation for the radial velocity u_r, by eliminating the pressure p and the velocity components u_θ, u_z, see [11, Section 15] or [17, Section 2]. After straightforward calculations, we obtain the second order ODE

$$-\partial_r\Bigl(\frac{r^2\partial_r^* u_r}{m^2+k^2r^2}\Bigr) + \Bigl\{1 + \frac{imr}{\gamma(r)}\partial_r\Bigl(\frac{W(r)}{m^2+k^2r^2}\Bigr) + \frac{1}{\gamma(r)^2}\frac{k^2r^2\Phi(r)}{m^2+k^2r^2}\Bigr\}u_r \;=\; 0\,, \tag{2.10}$$

where $\partial_r^* = \partial_r + \frac{1}{r}$ and $\Phi(r) = 2\Omega(r)W(r)$ is the Rayleigh function. This equation is well defined if $\gamma(r) \neq 0$ for all $r > 0$, which is the case if $\mathrm{Re}(s) \neq 0$ or, more generally, if $s \neq -imb$ for all b in the range of the angular velocity Ω. Eigenvalues of the linearized operator correspond to those values of the spectral parameter s for which equation (2.10) has a nontrivial solution u_r that is regular at the origin and decays to zero as $r \to \infty$.

It is instructive to notice that the stability equation (2.10) has a very similar structure as the Taylor-Goldstein equation (1.15). Both are second order Schrödinger equations involving a complex-valued potential which is a polynomial of degree two in the inverse spectral function $1/\gamma$.

The coefficient of $1/\gamma(r)^2$ in (2.10) is proportional to the Rayleigh function Φ, and corresponds to the buoyancy term involving $-k^2 g\bar{\rho}'$ in (1.15). Similarly, the coefficient of $1/\gamma(r)$ in (2.10) is proportional to the vorticity W and its derivative, and corresponds to the inertial term involving $ik(\bar{\rho}U')'$ in (1.15). This analogy is grounded in deep physical reasons, which are explained in the pioneering work of Rayleigh [31]. It gives hope that the stability equation (2.10) can be analyzed using the techniques that were developed for shear flows, but we shall see that additional difficulties arise in the case of columnar vortices.

2.2 Kelvin's Vibration Modes

When the spectral parameter s is purely imaginary, the stability equation (2.10) has real-valued coefficients and can be studied using classical methods such as Sturm-Liouville theory. If $m \neq 0$, it is convenient to set $s = -imb$ for some $b \in \mathbb{R}$, so that $\gamma(r) = im(\Omega(r) - b)$. In that case, the equation satisfied by the radial velocity u_r becomes

$$-\partial_r\Bigl(\frac{r^2\partial_r^* u_r}{m^2+k^2r^2}\Bigr) + \Bigl\{1 + \frac{r}{\Omega(r){-}b}\partial_r\Bigl(\frac{W(r)}{m^2+k^2r^2}\Bigr) - \frac{1/m^2}{(\Omega(r){-}b)^2}\frac{k^2r^2\Phi(r)}{m^2+k^2r^2}\Bigr\}u_r \,=\, 0\,. \qquad (2.11)$$

This equation is well-posed if the spectral parameter b does not belong to the range of the angular velocity Ω, so that $\Omega(r) - b \neq 0$ for all $r > 0$.

In the particular case of Rankine's vortex, for which the vorticity distribution W is piecewise constant, Kelvin [37] observed that the stability equation can be explicitly solved in terms of modified Bessel functions in both regions $r < 1$ and $r > 1$. In the generic case where $k \neq 0$, matching conditions at $r = 1$ lead to the "dispersion relation"

$$\frac{I_m'(\beta)}{\beta I_m(\beta)} + \frac{2}{(1-b)\beta^2} \,=\, \frac{K_m'(k)}{kK_m(k)}\,, \qquad \text{where} \quad \beta^2 \,=\, k^2\Bigl(1 - \frac{4}{m^2(1-b)^2}\Bigr)\,. \qquad (2.12)$$

Here I_m, K_m are modified Bessel functions of order m of the first and second kind, respectively. Those values of $b \neq 1$ for which (2.12) holds give purely imaginary eigenvalues of the linearized operator, which correspond to periodic oscillations of the columnar vortex. A careful analysis [37] reveals that the relation (2.12) is satisfied for a decreasing sequence $b_n \to 1$, and also for an increasing sequence $b_n' \to 1$, all solutions being contained in the interval $|b - 1| \leq 2/|m|$. So, for any $m \neq 0$ and $k \neq 0$, Kelvin established the existence of an infinite sequence of purely imaginary eigenvalues for the linearized operator at Rankine's vortex. He was confident that the whole spectrum could be obtained in that way [37]:

> "All possible simple harmonic vibrations are thus found: and summation, after the manner of Fourier, for different values of $[m, k]$, with different amplitudes and different epochs, gives every possible motion, deviating infinitely little from the undisturbed motion in circular orbits."

Unfortunately, the above claim is not substantiated by any argument in Kelvin's paper. Nevertheless, in the case of Rankine's vortex, one can show that the linearized operator has no eigenvalue outside the imaginary axis, so that the whole spectrum can indeed be obtained as demonstrated by Kelvin, see [17, Section 6.2].

The situation is different for a vortex with smooth angular velocity profile, as is the case for the Lamb-Oseen or the Kaufmann-Scully vortex. Assuming that $\Omega(0) = 1$ and $\Omega'(r) < 0$ for $r > 0$, it can be proved that, if $m \neq 0$ and $k \neq 0$, there exists a decreasing sequence $b_n \to 1$ of values of the spectral parameter for which the eigenvalue equation (2.11) has a nontrivial solution satisfying the boundary conditions. Moreover, (2.11) may have a solution for a finite number of negative values of b [17, Section 3.2]. So we still have an infinite number of purely imaginary eigenvalues, but in addition to these Kelvin waves there is also *continuous spectrum* filling the interval where $0 \le b \le 1$. Note that, if $0 < b < 1$, the eigenvalue equation (2.11) has a singularity at $r = \bar{r} := \Omega^{-1}(b)$, which is referred to as a "critical layer" in the physical literature. The interested reader is referred to [6, 14, 22, 32] for a few recent contributions to the study of Kelvin waves.

2.3 Axisymmetric or Two-Dimensional Perturbations

From now on we concentrate on the spectrum of the linearized operator outside the imaginary axis. The stability equation (2.10) is difficult to analyze in general, but important insight can be obtained by considering some particular cases.

To begin with, we restrict our attention to *axisymmetric perturbations* for which $m = 0$. In that case, we have $\gamma(r) = s$ for all $r > 0$, so that (2.10) reduces to the simpler equation

$$-\partial_r \partial_r^* u_r + k^2 \Bigl(1 + \frac{\Phi(r)}{s^2}\Bigr) u_r \,=\, 0\,, \qquad r > 0\,. \tag{2.13}$$

The analogy with the Rayleigh-Taylor equation (1.6) is striking, and we see that the Rayleigh function Φ in (2.13) plays the exact role of the buoyancy term $N^2 = -g\bar{\rho}'/\bar{\rho}$ in (1.6). Following the same approach as in Section 1.1, we conclude that, if Φ is everywhere nonnegative, equation (2.13) has no nontrivial solution satisfying the boundary conditions when $\mathrm{Re}(s) \neq 0$. Moreover, if $\Phi(r) < 0$ for some $r > 0$, Synge's argument [17, 36] shows that equation (2.13) has a nontrivial solution for sequence of real eigenvalues $s_n \to 0$, so that the positivity of the Rayleigh function is a necessary and sufficient condition for stability in the axisymmetric case.

Remark 2.1. The analogy between columnar vortices and shear flows in stratified fluids was already noticed by Rayleigh [31], and can be roughly explained as follows. In a stratified fluid, exchanging the positions of two fluid particles located on the same vertical line results in a gain or a loss of potential energy, depending on whether the fluid density is decreasing or increasing upwards. The first situation is thus stable, and the second unstable. A similar effect occurs in vortices, even if the fluid is homogeneous, because the centrifugal force (which plays the role of gravity) varies as a function of the distance to the vortex center. It turns out that exchanging two fluid particles on the same radial line results in a gain or a loss of energy depending on the sign of the Rayleigh function Φ, and that a stable "stratification" corresponds to $\Phi \ge 0$.

We next consider *two-dimensional perturbations*, which correspond to $k = 0$. In that case, the stability equation (2.10) reduces to

$$-\partial_r(r^2 \partial_r^* u_r) + m^2 u_r + \frac{imrW'(r)}{\gamma(r)}\, u_r \,=\, 0\,, \qquad r > 0\,. \tag{2.14}$$

Here we can compare with the Rayleigh stability equation (1.10), and we see that the vorticity derivative W' in (2.14) plays the role of the second order derivative U'' in (1.10). Thus, proceeding as in Section 1.2, we conclude that, if W' does not change sign, equation (2.14) has no nontrivial solution satisfying the boundary condition if $\mathrm{Re}(s) \neq 0$. The monotonicity of the vorticity profile W is thus a sufficient condition for stability with respect to two-dimensional perturbations, but as in the case of shear flows this condition is not necessary in general (and no sharp stability criterion is known).

Remarks 2.2.

1. For any localized vortex, the monotonicity of the vorticity distribution W *implies* the positivity of the Rayleigh function Φ. Indeed, if W is monotone, then $W(r) \to 0$ as $r \to \infty$ (otherwise the vortex would not be localized), hence W does not change sign, and the reconstruction formula

$$\Omega(r) \,=\, \frac{1}{r^2}\int_0^r W(s) s \,\mathrm{d}s\,, \qquad r > 0\,, \tag{2.15}$$

shows that Ω has the same sign as W. Thus $\Phi = 2\Omega W \geq 0$.

2. In view of the previous remark, if we extrapolate the conclusions obtained in the particular cases considered above, one may be tempted to conjecture that a columnar vortex with monotone vorticity distribution W is (spectrally) stable for all values of the Fourier parameters m, k. That daring claim has not been proved or disproved so far, and it is good to keep in mind that, in the present state of affairs, there is no analog of Squire's theorem for columnar vortices. In other words, there is no argument indicating that the most unstable modes (if any) should always correspond to axisymmetric or two-dimensional perturbations.

2.4 Howard Identities

We assume henceforth that $\Phi(r) > 0$ and $W'(r) < 0$ for all $r > 0$, so that the vortex under consideration is stable with respect to axisymmetric or two-dimensional perturbations. Our goal is now to study the eigenvalue equation (2.10) in the general case where $m \neq 0$ and $k \neq 0$. It is convenient to write the spectral parameter as $s = m(a - ib)$, where $a, b \in \mathbb{R}$, so that

$$\gamma(r) \,=\, s + im\Omega(r) \,=\, im\gamma_\star(r)\,, \qquad \text{where} \quad \gamma_\star(r) \,=\, \Omega(r) - b - ia\,. \tag{2.16}$$

When $a \neq 0$, we have $\gamma_\star(r) \neq 0$ for all $r > 0$, and equation (2.10) can be written in the condensed form

$$-\partial_r\big(\mathcal{A}(r)\partial_r^* u_r\big) + \mathcal{B}(r) u_r \,=\, 0\,, \qquad r > 0\,, \tag{2.17}$$

where $\partial_r^* = \partial_r + \frac{1}{r}$ and

$$\mathcal{A}(r) \,=\, \frac{r^2}{m^2 + k^2 r^2}\,, \qquad \mathcal{B}(r) \,=\, 1 + \frac{r}{\gamma_\star(r)}\partial_r\Big(\frac{W(r)}{m^2 + k^2 r^2}\Big) - \frac{k^2}{m^2}\,\frac{\mathcal{A}(r)\Phi(r)}{\gamma_\star(r)^2}\,. \tag{2.18}$$

If we assume that (2.17) has a nontrivial solution that is regular at the origin and decays to zero at infinity, we can multiply both sides of by $r\bar{u}_r$ and integrate over $\mathbb{R}_+$ to arrive at the identity

$$\int_0^\infty \Big(\mathcal{A}(r)|\partial_r^* u_r|^2 + \mathcal{B}(r)|u_r|^2\Big) r \,\mathrm{d}r \,=\, 0\,. \tag{2.19}$$

As $|\gamma_\star(r)| \geq |a| > 0$ for all $r > 0$, we deduce from (2.18) that

$$|1 - \mathcal{B}(r)| \,\leq\, \frac{C}{m^2}\Big(\frac{1}{|a|} + \frac{1}{|a|^2}\Big)\,, \qquad r > 0\,,$$

for some constant $C > 0$ depending only on the vorticity profile W. In particular, if we suppose that $|a| > M := \max(1, 2C)$, then $\mathrm{Re}\,\mathcal{B}(r) > 0$ for all $r > 0$, and taking the real part of (2.19) we obtain a contradiction. Thus equation (2.17) has no nontrivial solution if $|a| > M$. Similarly, if we take the imaginary part of (2.19) use the definitions (2.16), (2.18), we obtain the relation

$$a\int_0^\infty \Big\{\frac{r}{a^2 + (\Omega - b)^2}\partial_r\Big(\frac{W(r)}{m^2 + k^2 r^2}\Big) + \frac{2(b - \Omega(r))}{(a^2 + (\Omega - b)^2)^2}\,\frac{k^2}{m^2}\,\mathcal{A}(r)\Phi(r)\Big\}|u_r|^2 r \,\mathrm{d}r \,=\, 0\,. \tag{2.20}$$

If $a \neq 0$, the integral in (2.20) must vanish. But the first term in the integrand is negative since $W'(r) < 0$, and the second one is negative too if we suppose that $b \leq 0$, because $\Omega(r) > 0$ for all $r > 0$. Thus we conclude from (2.20) that (2.17) has no nontrivial solution if $a \neq 0$ and $b \leq 0$, see Fig. 1.

To obtain further information on the spectrum outside the imaginary axis, we proceed as in the case of the Taylor-Goldstein equation (1.15), which was analyzed in Section 1.3. Following Howard's approach [20, 21], we first consider the differential equation satisfied by the new function $w_r = u_r/\gamma_\star(r)$. Straightforward calculations that are reproduced in [17, Section 3.4] show that w_r satisfies

$$-\partial_r\Big(\gamma_\star(r)^2\mathcal{A}(r)\partial_r^* w_r\Big) + \mathcal{D}(r)w_r \,=\, 0\,, \qquad r > 0\,, \tag{2.21}$$

where

$$\mathcal{D}(r) \,=\, \gamma_\star(r)^2 + 2r\gamma_\star(r)\partial_r\Big(\frac{\Omega(r)}{m^2+k^2r^2}\Big) - \frac{k^2}{m^2}\,\mathcal{A}(r)\Phi(r)\,.$$

In particular, if we multiply (2.21) by $r\bar{w}_r$, integrate the result over $\mathbb{R}_+$, and take the imaginary part, we obtain the relation

$$2a\int_0^\infty \Big\{(b-\Omega(r))\Big(\mathcal{A}(r)|\partial_r^* w_r|^2 + |w_r|^2\Big) - r\partial_r\Big(\frac{\Omega(r)}{m^2+k^2r^2}\Big)|w_r|^2\Big\} r\,\mathrm{d}r \,=\, 0\,. \tag{2.22}$$

The second term in the integrand is positive, because $\Omega'(r) < 0$, and the first one is positive too if we assume that $b \geq 1$, so that $b - \Omega(r) > 0$ for all $r > 0$. We thus conclude from (2.22) that equation (2.21), hence also equation (2.17), has no nontrivial solution satisfying the boundary conditions if $a \neq 0$ and $b \geq 1$, see Fig. 1.

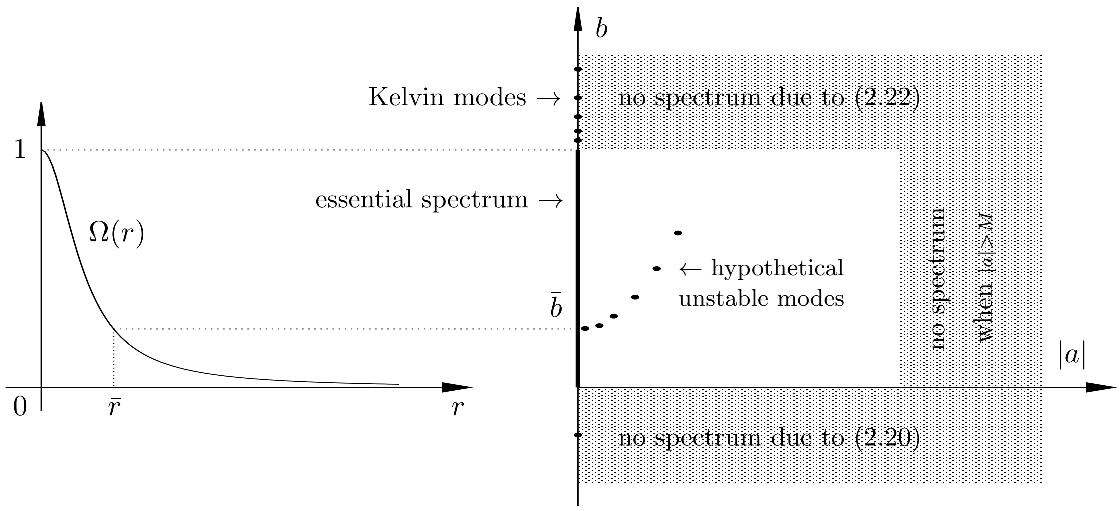

Fig. 1: The information obtained so far on the spectrum of the linearized operator using the spectral parametrization $s = m(a - ib)$. Kelvin modes are located on the imaginary axis $a = 0$, and accumulate only at the upper edge of the essential spectrum, which fills the segment $a = 0$, $b \in [0, 1]$. The rest of the spectrum, if any, consists of isolated eigenvalues which are contained in the region $|a| \leq M$, $b \in [0, 1]$, and can possibly accumulate only on the essential spectrum.

Next, we consider the function $v_r = u_r/\gamma_\star(r)^{1/2}$ which satisfies

$$-\partial_r\Big(\gamma_\star(r)\mathcal{A}(r)\partial_r^* v_r\Big) + \mathcal{E}(r)v_r \,=\, 0\,, \qquad r > 0\,, \tag{2.23}$$

where

$$\mathcal{E}(r) \;=\; \gamma_\star(r) + \frac{r}{2}\partial_r\Big(\frac{W(r)+2\Omega(r)}{m^2+k^2r^2}\Big) + \frac{1}{4}\frac{\Omega'(r)^2}{\gamma_\star(r)}\,\mathcal{A}(r) - \frac{k^2}{m^2}\,\frac{\mathcal{A}(r)\Phi(r)}{\gamma_\star(r)}\,.$$

If we multiply (2.23) by $r\bar v_r$, integrate the result over $\mathbb{R}_+$, and take the imaginary part, we obtain the relation

$$-a\int_0^\infty\Big\{\mathcal{A}(r)|\partial_r^* v_r|^2 + |v_r|^2 + \frac{\mathcal{A}(r)}{a^2+(\Omega-b)^2}\Big(\frac{k^2\Phi(r)}{m^2} - \frac{\Omega'(r)^2}{4}\Big)|v_r|^2\Big\}r\,\mathrm{d}r \;=\; 0\,, \qquad (2.24)$$

which is analogous to identity (1.18). Introducing the "Richardson number"

$$\mathrm{Ri}(r) \;=\; \frac{k^2}{m^2}\,\frac{\Phi(r)}{\Omega'(r)^2}\,, \qquad (2.25)$$

we deduce from (2.24) that equation (2.23), hence also equation (2.17), has no nontrivial solution satisfying the boundary conditions if $a \neq 0$ and $\mathrm{Ri}(r) \geq 1/4$ for all $r > 0$. Unfortunately, unlike for the Taylor-Goldstein equation, the Richardson number (2.25) depends on the Fourier parameters m, k, and it is obvious that the inequality $\mathrm{Ri}(r) \geq 1/4$ cannot hold for all values of m and k. So the above approach fails to give any stability criterion that would hold for arbitrary perturbations. The situation is plainly summarized by Howard and Gupta in [21] :

> "The overall conclusion of this consideration of the non-axisymmetric case is thus essentially negative : the methods used to derive the Richardson number and semi-circle results in the axisymmetric case reproduce the known results of Rayleigh for two-dimensional perturbations and pure axial flow, but seem to give very little more. In fact the present situation with regard to non-axisymmetric perturbations seems to be very unsatisfactory from a theoretical point of view."

Remark 2.3. In the spirit of Howard's semi-circle law for shear flows [10], it is possible in the case of columnar vortices to locate the (hypothetical) unstable modes in a slightly more precise way than what is depicted in Fig. 1, see e.g. [12]. We do not comment further on that, because in the next section we give conditions on the vorticity profile which entirely preclude the existence of unstable eigenvalues.

3 Spectral Stability of Inviscid Columnar Vortices

In this section, we present the main results that were obtained recently in collaboration with D. Smets [17, 18]. We first state our precise assumptions on the unperturbed columnar vortex.

Assumption H1: *The vorticity profile $W : \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ is a $\mathcal{C}^2$ function satisfying $W'(0) = 0$, $W'(r) < 0$ for all $r > 0$, $rW'(r) \to 0$ as $r \to \infty$, and*

$$\Gamma \;:=\; \int_0^\infty W(r)r\,\mathrm{d}r \;<\; \infty\,. \qquad (3.1)$$

The crucial point here is the monotonicity of the vorticity distribution W, which implies stability with respect to two-dimensional perturbations, see Section 2.3. We also suppose that $W(r) \to 0$ as $r \to \infty$ fast enough so that the integral in (3.1) converges; in other words, the *total circulation* of the vortex is finite. It follows in particular that $W(r) > 0$ for all $r > 0$, and the expression (2.15) of the angular velocity shows that $\Omega(r) > 0$ and $\Omega'(r) < 0$ for all $r > 0$. As a consequence, the Rayleigh function $\Phi = 2\Omega W$ is positive everywhere, which implies stability with respect to axisymmetric perturbations too.

Assumption H2: *The "Richardson function" $J : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$J(r) = \frac{\Phi(r)}{\Omega'(r)^2}\,, \qquad r > 0\,, \tag{3.2}$$

satisfies $J'(r) < 0$ for all $r > 0$ and $rJ'(r) \to 0$ as $r \to \infty$.

This second assumption is less natural, and probably only technical in nature. The quantity $J(r)$ appears in the definition of the "Richardson number" (2.25), which plays an important role in the stability analysis of columnar vortices. If, for some given value of the ratio $k^2/m^2 > 0$, the Richardson number (2.25) is not everywhere larger than 1/4, assumption H2 implies the existence of a unique $r_* > 0$ such that $\mathrm{Ri}(r) > 1/4$ if $r < r_*$ (stable region) and $\mathrm{Ri}(r) < 1/4$ if $r > r_*$ (possibly unstable region). If we do not suppose that the function J is monotone, more regions have to be considered, which greatly complicates the analysis. The monotonicity of J is also essential to construct simple subsolutions of equation (2.11) for large r, see [17, Section 4.6]. On the positive side, we emphasize that assumptions H1 and H2 are satisfied in all classical examples, such as the Lamb-Oseen vortex or the Kaufmann-Scully vortex.

The following statement is our first main result.

Theorem 3.1. **[17]** *Under assumptions H1, H2, the columnar vortex with vorticity profile W is spectrally stable in the following sense. Given any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$ with $(m,k) \neq (0,0)$, the stability equation* (2.10) *has no nontrivial solution $u_r \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ if the spectral parameter s has a nonzero real part.*

Theorem 3.1 asserts that, under assumptions H1, H2, the linearized operator in (2.5) has no unstable eigenmode of the form (2.7) with $\mathrm{Re}(s) \neq 0$ and $u \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)^3$. In some sense, this answers a long-standing question dating back to the pioneering contributions of Kelvin and Rayleigh. This rather optimistic view has to be tempered for at least two reasons: first, the status of assumption H2 is unclear, and it is conceivable that the conclusion of Theorem 3.1 holds under the sole hypothesis that the vorticity profile is monotone, although we do not know how to prove that. Next, the proof of Theorem 3.1 given in [17] is very indirect, and does not give much insight into the physical mechanisms leading to stability. Therefore, it is not clear if our approach can be applied to more complicated problems, such as the stability analysis of columnar vortices with nonzero axial flow.

As is explained in Section 2.4, if the angular Fourier mode m and the vertical wavenumber k are both nonzero, the historical approach to hydrodynamic stability based on integral identities such as (2.19) does not seem sufficient to preclude the existence of unstable eigenvalues in all regions of the complex plane, see Fig. 1. However, it is easy to verify that all unstable eigenvalues (if any) are simple, isolated, and depend continuously on the vortex profile W, which can be considered as an infinite-dimensional parameter in the differential equation (2.10). In addition, for the rescaled Kaufmann-Scully vortex

$$W_\epsilon(r) = \frac{2}{(1+\epsilon r^2)^2}\,, \qquad \text{where} \quad 0 < \epsilon \le \frac{4k^2}{m^2}\,, \tag{3.3}$$

a direct calculation shows that the Richardson number (2.25) satisfies $\mathrm{Ri}_\epsilon(r) \ge 1/4$ for all $r > 0$. By Howard and Gupta's result [21], it follows that the associated linearized operator has no unstable eigenvalue in the Fourier subspace indexed by m, k.

These observations suggest the following contradiction argument to prove Theorem 3.1. Assume that, for some vorticity profile W satisfying assumptions H1 and H2, the linearized operator in (2.5) has an unstable eigenmode of the form (2.7) for some $s \in \mathbb{C}\backslash i\mathbb{R}$ and some Fourier parameters $m \in \mathbb{N}$, $k \in \mathbb{R}$. We know from the results of Section 2.3 that both m and k are necessarily nonzero. The idea is now to perform a *continuous homotopy* $(W_t)_{t\in[0,1]}$ between the original profile $W_0 := W$ and the reference profile $W_1 := W_\epsilon$, where W_ϵ is defined in (3.3). For small t, the linearized operator associated with W_t has an unstable eigenvalue $s(t)$ which depends continuously on t and satisfies $s(0) = s$. But we also know that, for $t = 1$, the linearized operator associated with the reference profile W_ϵ has no unstable eigenvalue at all. Thus we logically conclude that there exists some $t_* \in (0, 1]$ such that the unstable eigenvalue $s(t)$ merges into the continuous spectrum on the imaginary axis at $t = t_*$. The core of our contradiction argument is the claim that, under assumptions H1 and H2, such a merger is actually impossible.

The way we actually arrive at a contradiction is not easily described in a few lines, and the interested reader is referred to [17, Section 4] for full details. If t_n is an increasing sequence converging to t_*, we denote $s_n = s(t_n) = m(a_n - ib_n)$, so that $a_n \to 0$ as $n \to \infty$ by construction. Also, extracting a subsequence if needed, we can assume that $b_n \to \bar{b} \in [0, 1]$ as $n \to \infty$, see Fig. 1. For simplicity, we suppose here that $0 < \bar{b} < 1$, but of course the limiting cases $\bar{b} = 0$ and $\bar{b} = 1$ are also treated in [17]. If u_r^n denotes the (suitably normalized) eigenfunction associated with the eigenvalue s_n and the vorticity profile W_{t_n}, it is straightforward to verify that u_r^n converges as $n \to \infty$ to a solution u_r of the limiting equation (2.11), where $b = \bar{b}$ and Ω, W, Φ denote the angular velocity, vorticity, and Rayleigh function of the vortex profile at the bifurcation point $t = t_*$. That equation has a singularity at the point $\bar{r} := \Omega^{-1}(\bar{b})$, and it is crucial to study the behavior of u_r in the vicinity of $\bar{r}$ (this is what is referred to as a *critical layer analysis* in the physical literature). If $\mathrm{Ri}(\bar{r}) > 1/4$, it is relatively easy to obtain a contradiction from identity (2.24), because all main terms in the integrand are positive in that case. If $\mathrm{Ri}(\bar{r}) < 1/4$, a contradiction can be obtained by a careful study of the solutions of (2.11) near the singularity, and by the construction of appropriate subsolutions in the region where $r > \bar{r}$, see [17].

Remark 3.2. The argument we have just sketched requires that assumption H2 be satisfied by the interpolated profile W_t for all $t \in [0, t_*]$. For that reason, we cannot use a linear interpolation of the form $W_t = (1 - t)W + tW_\epsilon$, because the class of vorticity profiles satisfying H2 is not a linear space nor even a convex set. Thus an additional technical difficulty in our proof is the necessity of constructing *ad hoc* interpolation and approximation schemes in the nonlinear class of profiles satisfying assumption H2, see [17, Section 6.4].

To state our second main result, we return to the linearized system (2.5) which we write in condensed form $\partial_t \tilde{u} = L\tilde{u}$. The linearized operator L is given by

$$L\tilde{u} \,=\, \begin{pmatrix} -\Omega\partial_\theta \tilde{u}_r + 2\Omega\tilde{u}_\theta - \partial_r P[\tilde{u}] \\ -\Omega\partial_\theta \tilde{u}_\theta - W\tilde{u}_r - \frac{1}{r}\partial_\theta P[\tilde{u}] \\ -\Omega\partial_\theta \tilde{u}_z - \partial_z P[\tilde{u}] \end{pmatrix}, \tag{3.4}$$

where $P[\tilde{u}]$ denotes the solution $\tilde{p}$ of elliptic equation (2.6). Our goal is to solve the linearized system in the Hilbert space

$$X \,=\, \Big\{ u = (u_r, u_\theta, u_z) \in L^2(\mathbb{R}^3)^3 \,\Big|\, \partial_r^* u_r + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z = 0 \Big\},$$

equipped with the standard L^2 norm.

Theorem 3.3. [18] *Assume that the vorticity profile W satisfies assumptions H1, H2. Then the linear operator L defined in (3.4) is the generator of a strongly continuous group $(e^{tL})_{t\in\mathbb{R}}$ of bounded linear operators in the energy space X. Moreover, for any $\epsilon > 0$, there exists a constant $C_\epsilon \geq 1$ such that*

$$\|e^{tL}\|_{X\to X} \,\le\, C_\epsilon \, e^{\epsilon|t|}, \qquad \text{for all } t \in \mathbb{R}. \tag{3.5}$$

Estimate (3.5) exactly means that the spectrum of the evolution operator e^{tL} in X is contained in the unit circle of the complex plane for all $t \in \mathbb{R}$. In that sense, Theorem 3.3 is arguably the strongest way of asserting that the columnar vortex with vorticity profile W is *spectrally stable*. In view of the Hille-Yosida theorem [13], it follows the spectrum of the generator L is entirely contained in the imaginary axis of the complex plane, and we have the following resolvent bound for any $a > 0$:

$$\sup\Big\{\|(z-L)^{-1}\|_{X\to X}\,\Big|\, z\in\mathbb{C}\,,\ |\mathrm{Re}(z)|\ge a\Big\} \,<\, \infty\,. \tag{3.6}$$

In fact, since X is a Hilbert space, the Gearhart-Prüss theorem [13, Section V.1] asserts that the resolvent bound (3.6) is *equivalent* to the group estimate (3.5).

Let $L_{m,k}$ denote the restriction of the linearized operator L to the Fourier subspace indexed by the angular mode $m \in \mathbb{Z}$ and the vertical wave number $k \in \mathbb{R}$. To prove Theorem 3.3, we fix some spectral parameter $s \in \mathbb{C}$ with $\mathrm{Re}(s) = a \neq 0$ and we consider the resolvent equation $(s - L_{m,k})u = f$, which is equivalent to the system

$$\begin{aligned} \gamma(r)u_r - 2\Omega(r)u_\theta \,&=\, -\partial_r p + f_r\,,\\ \gamma(r)u_\theta + W(r)u_r \,&=\, -\tfrac{im}{r}p + f_\theta\,,\\ \gamma(r)u_z \,&=\, -ikp + f_z\,, \end{aligned} \tag{3.7}$$

where $\gamma(r) = s + im\Omega(r)$ and the pressure $p = P_{m,k}[u]$ is chosen so as to preserve the incompressibility condition (2.9). Our goal is to show that the solution of (3.7) satisfies $\|u\| \le C(a)\|f\|$, where $C(a)$ is a positive constant depending only on the spectral abscissa a; in particular, the resolvent estimate is uniform in the Fourier parameters m, k and in the spectral parameter s on the vertical line $\mathrm{Re}(s) = a$. Such a uniform bound is essentially equivalent to (3.6), hence also to (3.5) by the Gearhart-Prüss theorem.

If $(m, k) \neq (0, 0)$, the resolvent system (3.7) can be reduced to a scalar equation for the radial velocity u_r, which can then be studied using the same techniques as in Section 2.4. This provides resolvent estimates with *explicit constant* $C(a)$ is some regions of the parameter space, but that approach fails in other regions where we have to invoke a contradiction argument that relies on the conclusion of Theorem 3.1. Thus our proof is again non-constructive, and does not provide any explicit expression for the constant $C(a)$ in general. In particular, we do not know if $C(a) = \mathcal{O}(|a|^{-N})$ as $a \to 0$ for some $N \in \mathbb{N}$. Such an improved estimate would indicate that the norm of the group e^{tL} grows at most polynomially as $|t| \to \infty$.

4 Conclusion and Perspectives

The results of the previous section apply to a large family of columnar vortices, including all classical models in atmospheric flows and engineering applications [1, 34]. They provide the first rigorous proof of spectral stability allowing for general perturbations, without any particular symmetry. In this sense, they solve an important problem that was formulated as early as 1880 by Lord Kelvin in the pioneering work [37]. However, many interesting questions remain open:

- Is assumption H2 really necessary for the conclusion of Theorem 3.1 to hold? Can one find a different proof, that does not rely on a non-constructive contradiction argument?
- Can one strengthen the conclusion of Theorem 3.3 and show that the group norm $\|e^{tL}\|$ grows at most polynomially as $|t| \to \infty$?

- Is it possible to prove some spectral stability results for more general equilibria of the form $u = V(r)e_\theta + W(r)e_z$, which include a nonzero axial flow ?
- Do our result give any useful information on the stability of columnar vortices in the slightly viscous case ?

In a broader perspective, a long-term project is the stability analysis of columnar vortices beyond the linear approximation, which is a completely open problem in the absence of useful variational characterization of such equilibria. In any case, we hope that our contribution will serve as a starting point for new developments in the stability analysis of concentrated vortices.

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