Interaction of vortices in weakly viscous planar flows

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Abstract

We consider the inviscid limit for the two-dimensional incompressible Navier-Stokes equation in the particular case where the initial flow is a finite collection of point vortices. We suppose that the initial positions and the circulations of the vortices do not depend on the viscosity parameter ν, and we choose a time $T > 0$ such that the Helmholtz-Kirchhoff point vortex system is well-posed on the interval $[0, T]$. Under these assumptions, we prove that the solution of the Navier-Stokes equation converges, as $\nu \to 0$, to a superposition of Lamb-Oseen vortices whose centers evolve according to a viscous regularization of the point vortex system. Convergence holds uniformly in time, in a strong topology which allows us to give an accurate description of the asymptotic profile of each individual vortex. In particular, we compute to leading order the deformations of the vortices due to mutual interactions. This makes it possible to estimate the self-interactions, which play an important role in the convergence proof.

1 Introduction

It is a well established fact that coherent structures play a crucial role in the dynamics of two-dimensional turbulent flows. Experimental observations [11] and numerical simulations of decaying turbulence [31, 32] reveal that, in a two-dimensional flow with sufficiently high Reynolds number, isolated regions of concentrated vorticity appear after a short transient period, and persist over a very long time scale. These structures are nearly axisymmetric and behave like point vortices as long as they remain widely separated, but when two of them come sufficiently close to each other they get significantly deformed under the strain of the velocity field, and the interaction may even cause both vortices to merge into a single, larger structure [30, 43]. It thus appears that the long-time behavior of two-dimensional decaying turbulence is essentially governed by a few basic mechanisms, such as vortex interaction and, especially, vortex merging.

Although these phenomena are relatively well understood from a qualitative point of view, they remain largely beyond the scope of rigorous analysis. Vortex merging, in particular, is a genuinely nonperturbative process which seems extremely hard to describe mathematically, although it is certainly the key mechanism which explains the coarsening of vorticity structures in two-dimensional flows, in agreement with the inverse energy cascade. The situation is simpler for vortex interactions, which may be rigorously studied in the asymptotic regime where the distance between vortices is much larger than the typical core size, but complex phenomena can occur even in that case. Indeed, numerical calculations [30] and nonrigorous asymptotic expansions [52, 51] indicate that vortex interaction begins with a fast relaxation process, during which each vortex adapts its shape to the velocity field generated by the other vortices. This first step depends on the details of the initial data, and is characterized by temporal oscillations of the vortex cores which disappear on a non-viscous time scale. In a second step, the vortices relax to a Gaussian-like profile at a diffusive rate, and the system reaches a "metastable state" which is independent of the initial data, and will persist until two vortices get sufficiently close to start a merging process. In this metastable regime, the vortex centers move in the plane according to the Helmholtz-Kirchhoff dynamics, and the vortex profiles are uniquely determined, up to a scaling factor, by the relative positions of the centers.

From a mathematical point of view, a natural approach to study vortex interactions is to start with *point vortices* as initial data. After solving the Navier-Stokes equations, we obtain in this way a family of interacting vortices which, by construction, is directly in the metastable state that we have just described. In particular, as it will be proved below, we do not observe here the oscillatory and diffusive transient steps which take place in the general case. Point vortices can therefore be considered as well prepared initial data for the vortex interaction problem.

With this motivation in mind, we study in the present paper what we call the *viscous N-vortex solution*, namely the solution of the two-dimensional Navier-Stokes equations in the particular case where the initial vorticity is a superposition of N point vortices. For a given value of the viscosity parameter ν, this solution is entirely determined by the initial positions $x_1, \dots, x_N$ and the circulations $\alpha_1, \dots, \alpha_N$ of the vortices. It describes a family of interacting vortices of diameter $\mathcal{O}((\nu t)^{1/2})$, which are therefore widely separated if ν is sufficiently small. Our main goal is to obtain a rigorous asymptotic expansion of the N-vortex solution in the vanishing viscosity limit, assuming that vortex collisions do not occur. As was already explained, this problem is physically relevant, but it also has its own mathematical interest. Indeed, it is known that the two-dimensional Navier-Stokes equations have a unique solution, for any value of ν, when the initial vorticity is a finite measure [16], but computing the inviscid limit of rough solutions is a very difficult task in general, due to the underlying instabilites of the Euler flow. Surprisingly enough, although point vortices are perhaps the most singular initial data that can be considered for the Navier-Stokes equations, the inviscid limit appears to be tractable for the N-vortex solution, and provides a new rigorous derivation of the Helmholtz-Kirchhoff dynamics as well as a mathematical description of the metastable regime for interacting vortices.

In the rest of this introductory section, we recall a global well-posedness result for the two-dimensional Navier-Stokes equations which is adapted to our purposes, we introduce the Lamb-Oseen vortices which will play a crucial role in our analysis, and we briefly mention the difficulties related to the inviscid limit of rough solutions. Our main results concerning the N-vortex solution will be stated in Section 2, and proved in the subsequent sections.

The incompressible Navier-Stokes equations in the plane $\mathbb{R}^2$ have the following form:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u \;=\; \nu\Delta u - \nabla p\;, \qquad \operatorname{div} u = 0\;, \tag{1.1}$$

where $u(x,t) \in \mathbb{R}^2$ denotes the velocity of the fluid at point $x \in \mathbb{R}^2$ and time $t > 0$, and $p(x,t) \in \mathbb{R}$ is the pressure inside the fluid. The only physical parameter in (1.1) is the *kinematic viscosity* $\nu > 0$, which will play an important role in this work. For our purposes it will be convenient to consider the *vorticity field* $\omega(x,t) = \partial_1 u_2(x,t) - \partial_2 u_1(x,t)$, which evolves according to the remarkably simple equation

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega \;=\; \nu\Delta\omega\;. \tag{1.2}$$

Under mild assumptions, which will always be satisfied below, the velocity field $u(x,t)$ can be reconstructed from the vorticity $\omega(x,t)$ via the two-dimensional Biot-Savart law:

$$u(x,t) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y,t)\,\mathrm{d}y\;, \quad x\in\mathbb{R}^2\;, \tag{1.3}$$

where, for any $x=(x_1,x_2)\in\mathbb{R}^2$, we denote $x^\perp=(-x_2,x_1)$ and $|x|^2=x_1^2+x_2^2$.

Let $\mathcal{M}(\mathbb{R}^2)$ be the space of all real-valued finite measures on $\mathbb{R}^2$, equipped with the total variation norm

$$\|\mu\|_{\mathrm{tv}} \;=\; \sup\{\langle\mu,\phi\rangle\,;\,\phi\in C_0(\mathbb{R}^2)\,,\;\|\phi\|_{L^\infty}\le 1\}\;.$$

Here $\langle\mu,\phi\rangle = \int_{\mathbb{R}^2}\phi\,\mathrm{d}\mu$, and $C_0(\mathbb{R}^2)$ denotes the space of all continuous functions $\phi:\mathbb{R}^2\to\mathbb{R}$ which vanish at infinity. We say that a sequence $\{\mu_n\}$ in $\mathcal{M}(\mathbb{R}^2)$ converges weakly to $\mu\in\mathcal{M}(\mathbb{R}^2)$ if $\langle\mu_n,\phi\rangle\to\langle\mu,\phi\rangle$ as $n\to\infty$ for all $\phi\in C_0(\mathbb{R}^2)$. Weak convergence is denoted by $\mu_n\rightharpoonup\mu$.

Our starting point is the following result, which shows that the initial value problem for Eq. (1.2) is globally well-posed in the space $\mathcal{M}(\mathbb{R}^2)$:

Theorem 1.1 [16] *Fix $\nu>0$. For any initial measure $\mu\in\mathcal{M}(\mathbb{R}^2)$, Eq.* (1.2) *has a unique global solution*

$$\omega\in C^0((0,\infty),L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)) \tag{1.4}$$

such that $\|\omega(\cdot,t)\|_{L^1}\le\|\mu\|_{\mathrm{tv}}$ for all $t>0$, and $\omega(\cdot,t)\rightharpoonup\mu$ as $t\to 0+$.

Here and in what follows, it is understood that ω is a *mild* solution of (1.2), i.e. a solution of the associated integral equation

$$\omega(t) \;=\; e^{\nu t\Delta}\mu - \int_0^t \mathrm{div}\Big(e^{\nu(t-s)\Delta}u(s)\omega(s)\Big)\,\mathrm{d}s\;, \quad t>0\;, \tag{1.5}$$

where $e^{t\Delta}$ denotes the heat semigroup. The *existence* of a global solution to (1.2) for all initial data in $\mathcal{M}(\mathbb{R}^2)$ has been established more than 20 years ago by G.-H. Cottet [10], and independently by Y. Giga, T. Miyakawa and H. Osada [22]. In the same spirit, the later work by T. Kato [28] should also be mentioned. In addition to existence, it was shown in [22, 28] that the solution of (1.2) is *unique* if the atomic part of the initial measure is small compared to the viscosity. This smallness condition turns out to be necessary if one wants to obtain uniqueness by a standard application of Gronwall's lemma. On the other hand, in the particular case where the initial vorticity is a single Dirac mass (of arbitrary strength), uniqueness of the solution of (1.2) was proved recently by C.E. Wayne and the author [20], using a dynamical system approach. An alternative proof of the same result can also be found in [17]. Finally, in the general case, it is possible to obtain uniqueness of the solution of (1.2) by isolating the large Dirac masses in the initial measure and combining the approaches of [22] and [20]. This last step in the proof of Theorem 1.1 was achieved by I. Gallagher and the author in [16].

When the initial vorticity $\mu=\alpha\delta$ is a multiple of the Dirac mass (located at the origin), the unique solution of (1.2) is an explicit self-similar solution called the *Lamb-Oseen vortex:*

$$\omega(x,t) \;=\; \frac{\alpha}{\nu t}\,G\Big(\frac{x}{\sqrt{\nu t}}\Big)\;, \qquad u(x,t) \;=\; \frac{\alpha}{\sqrt{\nu t}}\,v^G\Big(\frac{x}{\sqrt{\nu t}}\Big)\;, \tag{1.6}$$

where

$$G(\xi) \;=\; \frac{1}{4\pi}\,e^{-|\xi|^2/4}\;, \qquad v^G(\xi) \;=\; \frac{1}{2\pi}\frac{\xi^\perp}{|\xi|^2}\Big(1-e^{-|\xi|^2/4}\Big)\;, \qquad \xi\in\mathbb{R}^2\;. \tag{1.7}$$

The circulation parameter $\alpha\in\mathbb{R}$ measures the intensity of the vortex. It coincides, in this particular case, with the integral of the vorticity ω over the whole plane $\mathbb{R}^2$, a quantity which is preserved under evolution. The dimensionless quantity $|\alpha|/\nu$ is usually called the *circulation Reynolds number*.

According to Theorem 1.1, the vorticity equation (1.2) has a unique global solution for any initial measure and any value of the viscosity parameter. It is very natural to investigate the behavior of that solution in the vanishing viscosity limit, especially if the initial data contain non-smooth structures such as point vortices, vortex sheets, or vortex patches. Indeed, if the viscosity is small, these structures will persist over a sufficiently long time scale to be observed and to influence the dynamics of the system. This question, however, is very difficult in its full generality, because Eq. (1.2) reduces formally, as $\nu \to 0$, to the Eulerian vorticity equation

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega \,=\, 0 \;, \tag{1.8}$$

which is not known to be well-posed in such a large space as $\mathcal{M}(\mathbb{R}^2)$. If the initial vorticity $\mu \in \mathcal{M}(\mathbb{R}^2)$ belongs to $H^{-1}(\mathbb{R}^2)$, and if the singular part of μ has a definite sign, then Eq. (1.8) has at least a global weak solution [14, 36], but this result does not cover the case of point vortices due to the assumption $\mu \in H^{-1}(\mathbb{R}^2)$. Furthermore, if we want to prove that the solution of (1.8) is unique, we have to assume that the initial vorticity is bounded [54] or almost bounded [55, 53].

From a more general point of view, it is relatively easy to show that solutions of the Navier-Stokes equations converge, in the vanishing viscosity limit, to solutions of the Euler equations if we restrict ourselves to *smooth* solutions in a domain *without boundary* [15, 50, 27, 2]. The situation is completely different in the presence of boundaries, especially if one adopts the classical non-slip boundary conditions for the solutions of the Navier-Stokes equations. In that case Prandtl boundary layers may form, and the inviscid limit becomes an extremely difficult question which has been rigorously treated so far only for analytic data [48, 23]. But even in the absence of boundaries, hard problems can occur in the inviscid limit if one considers non-smooth solutions. Classical examples are listed below, in increasing order of singularity:

1) *Vortex patches.* The simplest example in this category is the case where the vorticity is the characteristic function of a smooth bounded domain in $\mathbb{R}^2$. The corresponding velocity field is Lipschitz continuous, or almost Lipschitz if the boundary of the patch has singularities. The first convergence results, due to P. Constantin and J. Wu [8, 9], and to J.-Y. Chemin [6], hold for a general class of solutions including two-dimensional vortex patches. Several improvements have been subsequently obtained, especially by R. Danchin [12, 13], H. Abidi and R. Danchin [1], T. Hmidi [25, 26], and N. Masmoudi [42]. Closer to the spirit of the present work, we also quote a recent paper by F. Sueur [49], where viscous transition profiles at the boundary of a vortex patch are systematically constructed, and provide a complete asymptotic expansion of the solution in powers of $(\nu t)^{1/2}$.

2) *Vortex sheets.* In a two-dimensional setting, this is the case where the initial vorticity is concentrated on a piece of curve in $\mathbb{R}^2$. The tangential component of the velocity field is discontinuous along the curve, thus creating a shear flow which is responsible for the celebrated Kelvin-Helmholtz instability. Due to this underlying instability, the inviscid limit for vortex sheets is at least as difficult as for Prandtl boundary layers, and has been rigorously treated so far only in the case of analytic data [7, 4].

3) *Point vortices.* This is the most singular example in our list, since here the velocity field is not even bounded near the vortex centers. However, in the case of point vortices, the evolution of the inviscid solution is given, at least formally, by the Helmholtz-Kirchhoff system which does not exhibit any dynamical instability. Therefore, one can reasonably hope to control the vanishing viscosity limit in this particular situation. The first rigorous results in this direction were obtained by C. Marchioro [38, 39], and the aim of the present paper is to show how these results can be complemented to obtain an accurate description of the slightly viscous solution of (1.2) when the initial condition is a finite collection of point vortices.

After this general introduction, we now give a precise definition of the problem we want to study, and we state our main results.

2 The viscous N-vortex solution

Let N be a positive integer. We take $x_1, \ldots, x_N \in \mathbb{R}^2$ such that $x_i \neq x_j$ for $i \neq j$, and we also fix $\alpha_1, \ldots, \alpha_N \in \mathbb{R} \setminus \{0\}$. Given any $\nu > 0$, we denote by $\omega^\nu(x,t)$, $u^\nu(x,t)$ the unique solution of the vorticity equation (1.2) with initial data

$$\mu \;=\; \sum_{i=1}^N \alpha_i\, \delta(\cdot - x_i)\;. \tag{2.1}$$

This initial measure, which does not depend on the viscosity ν, describes a superposition of N point vortices of circulations $\alpha_1, \ldots, \alpha_N$ located at the points $x_1, \ldots, x_N$ in $\mathbb{R}^2$. Existence of solutions of (1.2) with singular initial data such as (2.1) has first been established by G. Benfatto, R. Esposito, and M. Pulvirenti in [3]. Uniqueness is guaranteed by Theorem 1.1, and our goal is to describe the behavior of the vorticity $\omega^\nu(x,t)$ in the vanishing viscosity limit.

The measure μ is very singular, and we do not know how to construct even a weak solution of Euler's equation (1.8) with such initial data. However, the inviscid motion of point vortices in the plane has been investigated by many authors, starting with H. von Helmholtz [24] and G. R. Kirchhoff [29] who derived a system of ordinary differential equations describing the motion of the vortex centers. It is therefore reasonable to expect, in our case, that

$$\omega^\nu(\cdot,t) \;\approx\; \sum_{i=1}^N \alpha_i\, \delta(\cdot - z_i(t))\;, \quad \text{as } \nu \to 0\;, \tag{2.2}$$

where $z(t) = (z_1(t), \ldots, z_N(t))$ denotes the solution of the Helmholtz-Kirchhoff system

$$z_i'(t) \;=\; \frac{1}{2\pi}\sum_{j\neq i} \alpha_j\, \frac{(z_i(t)-z_j(t))^\perp}{|z_i(t)-z_j(t)|^2}\;, \qquad z_i(0) \;=\; x_i\;. \tag{2.3}$$

As was already mentioned, the expression in the right-hand side of (2.2) is *not* a weak solution of Euler's equation, because in that case the self-interaction terms in the nonlinearity $u \cdot \nabla\omega$ are too singular to make sense even as distributions. However, as was shown by C. Marchioro and M. Pulvirenti [37, 40], it is possible to derive system (2.3) from Euler's equation by a rigorous procedure, which consists in approximating the Dirac masses in the initial data by small vortex patches, of diameter $\epsilon > 0$ and circulations $\alpha_1, \ldots, \alpha_N$, whose centers are located at the points $x_1, \ldots, x_N$. Then the corresponding solution of (1.8) converges weakly, as $\epsilon \to 0$, to the expression (2.2) where the vortex positions $z_1(t), \ldots, z_N(t)$ are solutions of (2.3). We also recall that system (2.3) is not globally well-posed for all initial data, because if $N \geq 3$ and if the circulations α_i do not have all the same sign, vortex collisions can occur in finite time for some exceptional initial configurations [41, 46].

Our first result shows that the solution $\omega^\nu(x,t)$ of Eq. (1.2) given by Theorem 1.1 converges weakly, in the vanishing viscosity limit, to a superposition of point vortices which evolve according to (2.3), provided that vortex collisions do not occur.

Theorem 2.1 *Assume that the point vortex system* (2.3) *is well-posed on the time interval* $[0,T]$. *Then the solution* $\omega^\nu(x,t)$ *of the Navier-Sokes equation* (1.2) *with initial data* (2.1) *satisfies*

$$\omega^\nu(\cdot,t) \xrightarrow[\nu\to 0]{} \sum_{i=1}^N \alpha_i\,\delta(\cdot - z_i(t))\ , \quad \textit{for all } t\in[0,T]\ , \tag{2.4}$$

where $z(t) = (z_1(t),\dots, z_N(t))$ *is the solution of* (2.3).

This theorem is closely related to a result by C. Marchioro [38, 39], which we now briefly describe. Instead of point vortices, Marchioro considers initial data of the form

$$\omega_0^\epsilon(x) \,=\, \sum_{i=1}^N \omega_i^\epsilon(x)\ , \qquad \epsilon > 0\ ,$$

where, for each $i\in\{1,\dots,N\}$, ω_i^ϵ is a smooth vortex patch with a definite sign, which is centered at point $x_i\in\mathbb{R}^2$, has compact support of size $\mathcal{O}(\epsilon)$, and satisfies

$$\int_{\mathbb{R}^2}\omega_i^\epsilon(x)\,\mathrm{d}x \,=\, \alpha_i^\epsilon \xrightarrow[\epsilon\to 0]{} \alpha_i\ .$$

Under these assumptions, it is proved that the solution $\omega^{\epsilon,\nu}(x,t)$ of (1.2) with initial data ω_0^ϵ converges to the expression in the right-hand side of (2.2) in the double limit $\nu\to 0$, $\epsilon\to 0$, provided

$$\nu \,\le\, \nu_0\,\epsilon^\beta\ , \quad \text{for some } \nu_0 > 0 \text{ and } \beta > 0\ . \tag{2.5}$$

Theorem 2.1 above corresponds to the limiting case $\epsilon = 0$, $\nu\to 0$, which is precisely excluded by hypothesis (2.5). It should be mentioned, however, that restriction (2.5) can be removed if the circulations $\alpha_1,\dots,\alpha_N$ all have the same sign, in which case Theorem 2.1 may probably be established using the techniques developped in [38].

Marchioro's proof is based on a decomposition of the solution $\omega^{\epsilon,\nu}(x,t)$ into a sum of N viscous vortex patches. The main idea is to control the spread of each patch by computing its moment of inertia with respect to a suitable point $z_i^{\epsilon,\nu}(t)$, which is an approximate solution of (2.3). This argument does not give any information on the actual shape of the vortex patches, and is therefore not sufficient to provide a qualitative description of the solution $\omega^{\epsilon,\nu}(x,t)$ for small ϵ,ν. Theorem 2.1 above suffers exactly from the same drawback. Its main interest is to provide a natural and rigorous derivation of the point vortex system (2.3), which differs from the classical approach of [37, 40].

The main goal of the present paper is to obtain a quantitative version of Theorem 2.1 which specifies the convergence rate in (2.4) and provides a precise asymptotic expansion of the N-vortex solution $\omega^\nu(x,t)$ in the vanishing viscosity limit. As in Marchioro's approach, our starting point is a decomposition of $\omega^\nu(x,t)$ into a sum of N viscous vortex patches, which correspond to the atoms of the initial measure (2.1).

Lemma 2.2 *The N-vortex solution can be decomposed as*

$$\omega^\nu(x,t) \,=\, \sum_{i=1}^N \omega_i^\nu(x,t)\ , \qquad u^\nu(x,t) \,=\, \sum_{i=1}^N u_i^\nu(x,t)\ , \tag{2.6}$$

where, for each $i\in\{1,\dots,N\}$, $\omega_i^\nu \in C^0((0,\infty), L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$ *is the (unique) solution of the convection-diffusion equation*

$$\frac{\partial\omega_i^\nu}{\partial t} + (u^\nu\cdot\nabla)\omega_i^\nu \,=\, \nu\Delta\omega_i^\nu\ , \quad \textit{with} \quad \omega_i^\nu(\cdot,t) \xrightarrow[t\to 0]{} \alpha_i\,\delta(\cdot - x_i)\ , \tag{2.7}$$

and the velocity field u_i^ν is obtained from ω_i^ν via the Biot-Savart law (1.3)*. Moreover $\omega_i^\nu(x,t)$ has the same sign as α_i for all $x \in \mathbb{R}^2$ and all $t > 0$, and $\int_{\mathbb{R}^2} \omega_i^\nu(x,t)\,\mathrm{d}x = \alpha_i$ for all $t > 0$. Finally, there exists $K_0 > 0$ (depending only on ν and $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$) such that*

$$|\omega_i^\nu(x,t)| \,\le\, K_0 \,\frac{|\alpha_i|}{\nu t}\, \exp\Bigl(-\frac{|x-x_i|^2}{5\nu t}\Bigr)~, \tag{2.8}$$

for all $i \in \{1,\dots,N\}$, all $x \in \mathbb{R}^2$, and all $t > 0$.

The proof of Lemma 2.2, which is borrowed from [20, 16], will be reproduced in Section 5 for the reader's convenience. For the time being, we just point out that estimate (2.8) gives a precise information on the N-vortex solution in the limit $t \to 0$ for any fixed $\nu > 0$, but cannot be used to control $\omega_i^\nu(x,t)$ in the limit $\nu \to 0$ for fixed $t > 0$, because the constant $K_0(\nu, |\alpha|)$ blows up rapidly as $\nu \to 0$.

In the very particular case where $N = 1$, we know from [20, 17] that $\omega^\nu(x,t)$ is just a suitable translate of Oseen's vortex (1.6). From now on, we assume that $N \ge 2$, and we suppose that the point vortex system (2.3) is well-posed on the time interval $[0,T]$. We denote

$$d \,=\, \min_{t\in[0,T]}\, \min_{i\neq j}\, |z_i(t) - z_j(t)| \,>\, 0~, \tag{2.9}$$

and we also introduce the turnover time

$$T_0 \,=\, \frac{d^2}{|\alpha|}~, \quad \text{where} \quad |\alpha| \,=\, |\alpha_1| + \cdots + |\alpha_N|~. \tag{2.10}$$

As is well known [46], T_0 is a natural time scale for the inviscid dynamics described by (2.3). For instance, for a pair of vortices with the same circulation α separated by a distance d, one can check that the rotation period of each vortex around the midpoint is $4\pi^2 T_0$. Our goal is to show that, if $\nu > 0$ is sufficiently small, the N-vortex solution $\omega^\nu(x,t)$ looks like a superposition of N Oseen vortices located at some points $z_1^\nu(t), \dots, z_N^\nu(t)$ which satisfy the following *viscous regularization* of the Helmholtz-Kirchhoff system:

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i^\nu(t) \,=\, \sum_{j=1}^N \frac{\alpha_j}{\sqrt{\nu t}}\, v^G\Bigl(\frac{z_i^\nu(t) - z_j^\nu(t)}{\sqrt{\nu t}}\Bigr)~, \qquad z_i^\nu(0) \,=\, x_i~, \tag{2.11}$$

where v^G is given by (1.7). The reason for using that system instead of (2.3) will be explained at the beginning of Section 3. For the moment, we just observe that system (2.11) is globally well-posed for positive times, and that the solutions $z_i^\nu(t)$ are exponentially close to the solutions $z_i(t)$ of (2.3) if the viscosity ν is sufficiently small.

Lemma 2.3 *Assuming pairwise distinct initial positions $x_1, \dots, x_N$, system* (2.11) *is globally well-posed for positive times, for any value of $\nu > 0$. Moreover, if the solution of* (2.3) *satisfies* (2.9)*, there exists $K_1 > 0$ (depending only on the ratio T/T_0) such that*

$$\frac{1}{d}\max_{i=1,\dots,N} |z_i^\nu(t) - z_i(t)| \,\le\, K_1 \,\exp\Bigl(-\frac{d^2}{5\nu t}\Bigr)~, \quad \textit{for all } t \in (0,T]~. \tag{2.12}$$

The proof of Lemma 2.3 is also postponed to Section 5. We refer to [45] for a discussion of the validity of system (2.11) as a model for the dynamics of interacting vortices.

To obtain a precise description of each vortex patch $\omega_i^\nu(x,t)$ in a neighborhood of $z_i^\nu(t)$, we introduce, for each $i \in \{1,\dots,N\}$, the self-similar variable

$$\xi \;=\; \frac{x - z_i^\nu(t)}{\sqrt{\nu t}}\;,$$

and we define rescaled functions $w_i^\nu(\xi,t) \in \mathbb{R}$ and $v_i^\nu(\xi,t) \in \mathbb{R}^2$ by setting

$$\omega_i^\nu(x,t) \;=\; \frac{\alpha_i}{\nu t}\, w_i^\nu\Big(\frac{x - z_i^\nu(t)}{\sqrt{\nu t}}\,,\, t\Big)\;, \qquad u_i^\nu(x,t) \;=\; \frac{\alpha_i}{\sqrt{\nu t}}\, v_i^\nu\Big(\frac{x - z_i^\nu(t)}{\sqrt{\nu t}}\,,\, t\Big)\;. \tag{2.13}$$

Given a small $\beta \in (0,1)$, which will be specified later, we also introduce a weighted L^2 space X defined by the following norm:

$$\|w\|_X \;=\; \Big(\int_{\mathbb{R}^2} |w(\xi)|^2\, e^{\beta|\xi|/4}\,\mathrm{d}\xi\Big)^{1/2}\;. \tag{2.14}$$

We already know from [16] that $w_i^\nu(\xi,t)$ converges to the Gaussian profile $G(\xi)$ as $t \to 0$ for any fixed $\nu > 0$. Our first main result shows that a similar result holds in the vanishing viscosity limit, uniformly in time on the interval $(0,T]$.

Theorem 2.4 *Assume that the point vortex system* (2.3) *is well-posed on the time interval* $[0,T]$*, and let* $\omega^\nu(x,t)$ *be the solution of* (1.2) *with initial data* (2.1)*. If* $\omega^\nu(x,t)$ *is decomposed as in* (2.6)*, then the rescaled profiles* $w_i^\nu(\xi,t)$ *defined by* (2.13) *satisfy*

$$\max_{i=1,\dots,N} \|w_i^\nu(\cdot,t) - G\|_X \;=\; \mathcal{O}\Big(\frac{\nu t}{d^2}\Big)\;, \quad \textit{as } \nu \to 0\;, \tag{2.15}$$

uniformly for $t \in (0,T]$*, where* d *is given by* (2.9)*.*

More precisely, the proof shows that there exist positive constants β and K_2, depending only on the ratio T/T_0, such that

$$\max_{i=1,\dots,N} \|w_i^\nu(\cdot,t) - G\|_X \;\le\; K_2\,\frac{\nu t}{d^2}\;, \tag{2.16}$$

for all $t \in (0,T]$, provided ν is small enough so that $\nu T/d^2 \le K_2^{-1}$. This result means that, when the viscosity ν is small, the N-vortex solution $\omega^\nu(x,t)$ looks like a superposition of N Oseen vortices located at the points $z_1^\nu(t),\dots,z_N^\nu(t)$, which evolve in time according to (2.11). Since $X \hookrightarrow L^1(\mathbb{R}^2)$ and since $z_i^\nu(t) \to z_i(t)$ as $\nu \to 0$ by Lemma 2.3, estimate (2.15) implies in particular (2.4), and Theorem 2.1 is thus a direct consequence of Theorem 2.4. Needless to say, the constant K_2 blows up when $T/T_0 \to \infty$, see the discussion at the end of this section.

Theorem 2.4 is already very satisfactory, but it does not seem possible to prove it directly without computing a higher order approximation of the N-vortex solution. This rather surprising claim will be justified in Section 3, but for the moment it can be roughly explained as follows. As is clear from (1.6), the velocity field of Oseen's vortex is very large near the center if the viscosity ν is small, with a maximal angular speed of the order of $|\alpha|/(\nu t)$. As long as the vortex stays isolated, it does not feel at all the effect of its own velocity field, because of radial symmetry. However, if a vortex is advected by a non-homogenous external field, which in our case is produced by the other $N-1$ vortices, it will get deformed and, consequently, will start feeling the influence of its own velocity field. If the ratio $|\alpha|/\nu$ is large, this self-interaction will have a very strong effect, even if the deformation is quite small. In particular, one may fear that the vortex gets further deformed, increasing in turn the self-interaction itself, and that the whole process results in a violent instability. In fact, this catastrophic scenario does not happen. Remarkably enough, a rapidly rotating Oseen vortex in an external field adapts its shape in such a way that the self-interaction *counterbalances* the strain of the external field [52, 51]. This fundamental observation will be the basis for our analysis in Section 3. It explains why one can observe, in turbulent two-dimensional flows, stable asymmetric vortices which in a first approximation are simply advected by the main stream. The same mechanism accounts for the existence of stable asymmetric Burgers vortices, which are stationary solutions of the three-dimensional Navier-Stokes equations in a linear strain field [44, 21, 34, 35].

To compute the self-interactions of the vortices, which play a crucial role in the convergence proof, we construct a higher order approximation of the N-vortex solution in the following way. For each $i \in \{1, \dots, N\}$ and all $t \in [0, T]$, we denote

$$w_i^{\mathrm{app}}(\xi, t) \;=\; G(\xi) + \Big(\frac{\nu t}{d^2}\Big)\Big\{\bar{F}_i(\xi, t) + F_i^\nu(\xi, t)\Big\}~, \qquad \xi \in \mathbb{R}^2~, \tag{2.17}$$

where $\bar{F}_i(\xi, t)$ is a radially symmetric function of ξ, whose precise expression is given in (3.42) below, and $F_i^\nu(\xi, t)$ is a nonsymmetric correction which satisfies

$$F_i^\nu(\xi, t) \;=\; \frac{d^2}{4\pi}\,\omega(|\xi|) \sum_{j \neq i} \frac{\alpha_j}{\alpha_i}\,\frac{1}{|z_{ij}(t)|^2}\Big(2\frac{|\xi \cdot z_{ij}(t)|^2}{|\xi|^2 |z_{ij}(t)|^2} - 1\Big) + \mathcal{O}\Big(\frac{\nu}{|\alpha|}\Big)~, \tag{2.18}$$

where $z_{ij}(t) = z_i^\nu(t) - z_j^\nu(t)$. Here $\omega : (0, \infty) \to \mathbb{R}$ is a smooth, positive function satisfying $\omega(r) \approx C_1 r^2$ as $r \to 0$ and $\omega(r) \approx C_2 r^4 e^{-r^2/4}$ as $r \to \infty$ for some $C_1, C_2 > 0$, see Eq. (3.28) below. The right-hand side of (2.17) is the beginning of an asymptotic expansion of the rescaled vortex patch $w_i^\nu(\xi, t)$ in powers of the non-dimensional parameter $(\nu t)/d^2$. Each term in this expansion can in turn be developped in powers of $\nu/|\alpha|$. The most important physical effect is due to the nonsymmetric term $F_i^\nu(\xi, t)$, which describes to leading order the deformation of the i^{th} vortex due to the influence of the other vortices. Keeping only that term and using polar coordinates $\xi = (r\cos\theta, r\sin\theta)$, we can rewrite (2.17) in the following simplified form

$$w_i^{\mathrm{app}}(\xi, t) \;=\; g(r) + \frac{\omega(r)}{4\pi} \sum_{j \neq i} \frac{\alpha_j}{\alpha_i}\,\frac{\nu t}{|z_{ij}(t)|^2}\cos\Big(2(\theta - \theta_{ij}(t))\Big) \;+\; \dots~, \tag{2.19}$$

where $g(|\xi|) = G(\xi)$ and $\theta_{ij}(t)$ is the argument of the planar vector $z_{ij}(t) = z_i^\nu(t) - z_j^\nu(t)$. This formula allows us to compute the principal axes and the eccentricities of the vorticity contours, which are elliptical at this level of approximation.

Using these notations, our final result can now be stated as follows:

Theorem 2.5 *Assume that the point vortex system* (2.3) *is well-posed on the time interval* $[0, T]$*, and let* $\omega^\nu(x, t)$ *be the solution of* (1.2) *with initial data* (2.1)*. If* $\omega^\nu(x, t)$ *is decomposed as in* (2.6)*, then the rescaled profiles* $w_i^\nu(\xi, t)$ *defined by* (2.13) *satisfy*

$$\max_{i=1,\dots,N} \|w_i^\nu(\cdot, t) - w_i^{\mathrm{app}}(\cdot, t)\|_X \;=\; \mathcal{O}\Big(\Big(\frac{\nu t}{d^2}\Big)^{3/2}\Big)~, \quad \textit{as } \nu \to 0~, \tag{2.20}$$

uniformly for $t \in (0, T]$*, where* w_i^{app} *is given by* (2.17).

As is clear from (2.15), (2.17), (2.20), Theorem 2.5 implies immediately Theorem 2.4, hence also Theorem 2.1. Note that the convergence result is now accurate enough so that the first order corrections to the Gaussian profile in (2.17) are much larger, for small viscosities, than the remainder terms which are summarized in the right-hand side of (2.20). This means in particular that the deformations of the interacting vortices are really given, to leading order, by (2.19). According to that formula, each vortex adapts its shape *instantaneously* to the relative positions of the other vortices, without oscillations or inertia. Indeed, for each $t \in (0,T]$, the angular factor $\cos(2(\theta-\theta_{ij}))$ in (2.19) which gives the leading order deformation (up to a time-dependent prefactor $\nu t/|z_{ij}|^2$) is entirely determined by the instantaneous positions of the vortex centers. In this sense, point vortices can be considered as well-prepared initial data, and the N-vortex solution is an example of the "metastable regime" described in the introduction. In contrast, one should mention that the first order radially symmetric corrections $\bar{F}_i(\xi,t)$ do not only depend on the instantaneous vortex positions, but on the whole history of the system, see (3.42).

The rest of this paper is devoted to the proof of Theorem 2.5, which is divided in two main steps. In Section 3, we construct an approximate solution of our system, with the property that the associated residuum is extremely small in the vanishing viscosity limit. This approximation differs from (2.17) by higher order corrections which are necessary to reach the desired accuracy, but will eventually be absorbed in the right-hand side of (2.20). As is explained in Section 3.3, a difficulty in this construction comes from the fact that the radially symmetric and the nonsymmetric terms in the approximate solution $w_i^{\rm app}(\xi,t)$ have a different origin, and play a different role. Once a suitable approximation has been constructed, our final task is to control the remainder $w_i(\xi,t)-w_i^{\rm app}(\xi,t)$ uniformly in time and in the viscosity parameter ν. This will be done in Section 4, using appropriate energy estimates. A major technical difficulty comes here from the fact that we do not want to assume that the ratio T/T_0 is small. If we did so, the proof would be considerably simpler, and we could replace the weight $e^{\beta|\xi|/4}$ by $e^{\beta|\xi|^2/4}$ in the definition (2.14) of our function space X, thus improving our convergence result. In the general case, however, we have to use a rather delicate energy estimate involving time-dependent weights $p_i(\xi,t)$, which will be constructed in Section 4.1. The price to pay is a slightly weaker control of the remainder, and the fact that all our constants, such as K_2 and β in (2.16), have a bad dependence on T if $T \gg T_0$. For instance, the proof shows that one can take $K_2 = C\exp(C(T/T_0)^2)$ for some $C>0$, but there is no reason to believe that this relation is optimal.

In conclusion, our results show that the vanishing viscosity limit can be rigorously controlled in the particular case of point vortices, due to the remarkable dynamic and structural stability properties of the Oseen vortices. These properties, which were established in [20, 21, 33], play a crucial role both in the construction of the approximate solution in Section 3, and in the energy estimates of Section 4.

3 Construction of an approximate solution

In this section, we show how to construct an asymptotic expansion of the N-vortex solution in the vanishing viscosity limit. Our starting point is the evolution system satisfied by the rescaled profiles $w_i^\nu(\xi,t)$, $v_i^\nu(\xi,t)$, which from now on will be denoted by $w_i(\xi,t)$, $v_i(\xi,t)$ for simplicity. Inserting (2.13) into (2.7), we obtain, for $i=1,\dots,N$,

$$t\partial_t w_i(\xi,t) + \left\{\sum_{j=1}^N \frac{\alpha_j}{\nu}\, v_j\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\Big) - \sqrt{\frac{t}{\nu}}\, z_i'(t)\right\} \cdot \nabla w_i(\xi,t) \;=\; (\mathcal{L}w_i)(\xi,t)\;, \tag{3.1}$$

where

$$\mathcal{L}w \;=\; \Delta w + \frac{1}{2}\xi\cdot\nabla w + w\;. \tag{3.2}$$

Here and in what follows, we denote the vortex positions by $z_i(t)$ instead of $z_i^\nu(t)$, to keep the formulas as simple as possible. We also recall that $z_{ij}(t) = z_i(t) - z_j(t)$.

The initial value problem for system (3.1) at time $t = 0$ is not well-posed, because the time derivative appears in the singular form $t\partial_t$. A convenient way to avoid this difficulty is to introduce a new variable $\tau = \log(t/T)$, so that $\partial_\tau = t\partial_t$. With this parametrization, the solution of (3.1) given by Lemma 2.2 is defined for all $\tau \in (-\infty, 0]$, and converges to the profile G of Oseen's vortex as $\tau \to -\infty$, see [16, Proposition 4.5]. For simplicity, we keep here the original time t, because this is the natural variable for the ODE system (2.3) or (2.11).

3.1 Residuum of the naive approximation

If $w(\xi, t) = (w_1(\xi, t), \dots, w_N(\xi, t))$ is an approximate solution of system (3.1), we define the *residuum* $R(\xi, t) = (R_1(\xi, t), \dots, R_N(\xi, t))$ of this approximation by

$$R_i(\xi,t) \;=\; t\partial_t w_i(\xi,t) + \biggl\{\sum_{j=1}^N \frac{\alpha_j}{\nu}\, v_j\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\Bigr) - \sqrt{\frac{t}{\nu}}\, z_i'(t)\biggr\} \cdot \nabla w_i(\xi,t) - (\mathcal{L}w_i)(\xi,t)\ ,$$

for all $i \in \{1, \dots, N\}$. Here and in the sequel, it is always understood that $v_i(\xi, t)$ is the velocity field corresponding to $w_i(\xi, t)$ via the Biot-Savart law (1.3), which also holds for the rescaled functions (2.13) due to scaling invariance.

In view of Theorem 2.4, the solution of (3.1) we are interested in satisfies $w_i(\xi, t) \approx G(\xi)$ and $v_i(\xi, t) \approx v^G(\xi)$ for all $t \in (0, T]$, if ν is sufficiently small. Since $\partial_t G = \mathcal{L}G = 0$, the residuum of this naive approximation (where $w_i(\xi, t) = G(\xi)$ for all $i \in \{1, \dots, N\}$) is

$$R_i^{(0)}(\xi,t) \;=\; \biggl\{\sum_{j=1}^N \frac{\alpha_j}{\nu}\, v^G\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr) - \sqrt{\frac{t}{\nu}}\, z_i'(t)\biggr\} \cdot \nabla G(\xi)\ . \tag{3.3}$$

This expression looks singular in the limit $\nu \to 0$, but the problem can be eliminated by an appropriate choice of the vortex positions $z_1(t), \dots, z_N(t)$. Indeed, in (3.3), the quantity inside the curly brackets $\{\cdot\}$ vanishes for $\xi = 0$ if we set

$$z_i'(t) \;=\; \sum_{j=1}^N \frac{\alpha_j}{\sqrt{\nu t}}\, v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\ , \qquad i = 1, \dots, N\ . \tag{3.4}$$

This is exactly the regularized point vortex system, which was already introduced in (2.11).

From now on we always assume, as in Theorems 2.4 and 2.5, that the original point vortex system (2.3) is well-posed on the time interval $[0, T]$. For any $\nu > 0$, we denote by $z_1(t), \dots, z_N(t)$ the solution of (3.4) with initial data $x_1, \dots, x_N$. In view of Lemma 2.3, we can assume that this solution satisfies (2.9) for some $d > 0$ (independent of ν), provided ν is sufficiently small. Inserting (3.4) into (3.3), we obtain the following expression of the residuum

$$R_i^{(0)}(\xi,t) \;=\; \sum_{j\neq i} \frac{\alpha_j}{\nu}\biggl\{v^G\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\biggr\} \cdot \nabla G(\xi)\ . \tag{3.5}$$

Remark that the sum now runs on the indices $j \neq i$, because the term corresponding to $j = i$ vanishes. Our first task is to compute an asymptotic expansion of the right-hand side of (3.5) in the vanishing viscosity limit.

Proposition 3.1 *For $i = 1, \dots, N$, we have*

$$R_i^{(0)}(\xi,t) \;=\; \frac{\alpha_i t}{d^2}\Big\{A_i(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)^{1/2} B_i(\xi,t) + \Big(\frac{\nu t}{d^2}\Big) C_i(\xi,t) + \tilde R_i^{(0)}(\xi,t)\Big\}\;, \tag{3.6}$$

for all $\xi \in \mathbb{R}^2$ and all $t \in (0,T]$, where

$$\begin{aligned}
A_i(\xi,t) &= \frac{d^2}{2\pi}\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{(\xi\cdot z_{ij}(t))(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^4}\,G(\xi)\;,\\
B_i(\xi,t) &= \frac{d^3}{4\pi}\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^6}\Big(|\xi|^2|z_{ij}(t)|^2 - 4(\xi\cdot z_{ij}(t))^2\Big)\,G(\xi)\;,\\
C_i(\xi,t) &= \frac{d^4}{\pi}\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{(\xi\cdot z_{ij}(t))(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^8}\Big(2(\xi\cdot z_{ij}(t))^2 - |\xi|^2|z_{ij}(t)|^2\Big)\,G(\xi)\;.
\end{aligned} \tag{3.7}$$

Moreover, for any $\gamma < 1$, there exists $C > 0$ such that

$$|\tilde R_i^{(0)}(\xi,t)| \;\le\; C\Big(\frac{\nu t}{d^2}\Big)^{3/2} e^{-\gamma|\xi|^2/4}\;, \qquad \xi \in \mathbb{R}^2\;, \quad 0 < t \le T\;. \tag{3.8}$$

Remarks. Proposition 3.1 provides an expansion of the residuum $R_i^{(0)}(\xi,t)$ in powers of the dimensionless parameter $(\nu t)^{1/2}/d$, where $(\nu t)^{1/2}$ is a diffusion length which gives the typical diameter of each vortex at time t, and d is the minimal distance between the vortex centers, see (2.9). As is clear from (3.7), the parameter d in (3.6) has been introduced rather artificially, to ensure that the quantities A_i, B_i, C_i are dimensionless, as is the residuum itself. The proof will show that an expansion of the form (3.6) can be performed to arbitrarily high orders, but for simplicity we keep only the terms which will be necessary to prove Theorems 2.4 and 2.5. Finally, we remark that the prefactor $\alpha_i t/d^2$ in (3.6) is bounded by t/T_0, where T_0 is the turnover time introduced in (2.10).

Proof. Fix $\gamma \in (0,1)$, and let $\gamma_1 = 1-\gamma$. Since $\nabla G(\xi) = -\frac{1}{2}\xi G(\xi)$ decreases rapidly as $|\xi| \to \infty$, it is clear that the residuum (3.5) is extremely small if $|\xi|$ is large. For instance, if $|\xi| \ge d/(2\sqrt{\nu t})$, we can bound

$$\begin{aligned}
|R_i^{(0)}(\xi,t)| &\le \frac{|\alpha|}{\nu}\,\|v^G\|_{L^\infty}|\xi|G(\xi) \;=\; \frac{1}{4\pi}\,\frac{|\alpha|}{\nu}\,\|v^G\|_{L^\infty}|\xi|\,e^{-\gamma_1|\xi|^2/4}\,e^{-\gamma|\xi|^2/4}\\
&\le C\,\frac{|\alpha| t}{d^2}\Big(\frac{d^2}{\nu t}\Big)^{3/2}\exp\Big(-\frac{\gamma_1 d^2}{16\nu t}\Big)\,e^{-\gamma|\xi|^2/4}\;,
\end{aligned}$$

where in the last inequality we have used the fact that $|\xi|\,e^{-\gamma_1|\xi|^2/4}$ is a decreasing function of $|\xi|$ when $|\xi| \gg 1$. A similar argument shows that

$$|A_i(\xi,t)| + |B_i(\xi,t)| + |C_i(\xi,t)| \;\le\; C\Big(\frac{d^2}{\nu t}\Big)^2\exp\Big(-\frac{\gamma_1 d^2}{16\nu t}\Big)\,e^{-\gamma|\xi|^2/4}\;,$$

if $|\xi| \ge d/(2\sqrt{\nu t})$. We conclude that expansion (3.6) holds in that region, with a remainder term satisfying a much better estimate than (3.8).

We now consider the case where $|\xi| \le d/(2\sqrt{\nu t})$. Since $|z_{ij}(t)| = |z_i(t) - z_j(t)| \ge d$ by (2.9) if $i \neq j$, we have

$$\Big|\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big| \;\ge\; \frac{d}{\sqrt{\nu t}} \;\ge\; 2|\xi|\;, \quad \text{and} \quad \Big|\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Big| \;\ge\; \frac{d}{2\sqrt{\nu t}} \;\ge\; |\xi|\;. \tag{3.9}$$

To estimate the right-hand side of (3.5), we have to compute the difference $v^G(\xi+\eta) - v^G(\eta)$, with $\eta = z_{ij}(t)/\sqrt{\nu t}$. Using definition (1.7), we obtain the identity

$$v^G(\xi+\eta) - v^G(\eta) \;=\; \frac{1}{2\pi}\Bigl(V_1(\xi,\eta) + V_2(\xi,\eta)\Bigr)\;, \qquad \xi,\eta \in \mathbb{R}^2\;, \tag{3.10}$$

where

$$V_1(\xi,\eta) \;=\; \frac{(\xi+\eta)^\perp}{|\xi+\eta|^2} - \frac{\eta^\perp}{|\eta|^2}\;, \qquad V_2(\xi,\eta) \;=\; \frac{\eta^\perp}{|\eta|^2}\,e^{-|\eta|^2/4} - \frac{(\xi+\eta)^\perp}{|\xi+\eta|^2}\,e^{-|\xi+\eta|^2/4}\;.$$

In particular, it follows from (3.9) that

$$\Bigl|V_2\Bigl(\xi,\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr| \;\le\; C\Bigl(\frac{\nu t}{d^2}\Bigr)^{1/2}\exp\Bigl(-\frac{d^2}{16\nu t}\Bigr)\;, \tag{3.11}$$

hence the contributions of the term V_2 to the residuum (3.5) are exponentially small and can be incorporated in the remainder term. To compute V_1, we use the following elementary lemma:

Lemma 3.2 *For all $\xi,\eta \in \mathbb{R}^2$ with $|\xi| < |\eta|$, we have*

$$\xi\cdot V_1(\xi,\eta) \;=\; \sum_{n=2}^{\infty}(-1)^{n-1}\,\frac{|\xi|^n}{|\eta|^n}\,\sin(n(\theta-\phi))\;, \tag{3.12}$$

where θ denotes the polar argument of ξ and ϕ the argument of η.

For completeness, the proof of Lemma 3.2 will be given in Section 5. Applying (3.12) with $\eta = z_{ij}(t)/\sqrt{\nu t}$ and keeping only the first three terms in the expansion, we obtain

$$\begin{aligned}\xi\cdot V_1\Bigl(\xi,\frac{z_{ij}}{\sqrt{\nu t}}\Bigr) \;&=\; -\,\frac{|\xi|^2\nu t}{|z_{ij}|^2}\,\sin(2(\theta-\phi)) + \frac{|\xi|^3(\nu t)^{3/2}}{|z_{ij}|^3}\,\sin(3(\theta-\phi))\\ &\quad -\,\frac{|\xi|^4(\nu t)^2}{|z_{ij}|^4}\,\sin(4(\theta-\phi)) + \mathcal{O}\Bigl(\frac{|\xi|^5(\nu t)^{5/2}}{|z_{ij}|^5}\Bigr)\;,\end{aligned} \tag{3.13}$$

where $\theta-\phi$ is the signed angle between ξ and z_{ij}. In particular, we have the relations

$$\sin(\theta-\phi) \;=\; \frac{\xi\cdot z_{ij}^\perp}{|\xi||z_{ij}|}\;, \qquad \cos(\theta-\phi) \;=\; \frac{\xi\cdot z_{ij}}{|\xi||z_{ij}|}\;,$$

in terms of which the higher order trigonometric expressions appearing in (3.13) can be computed using the well-known formulas $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$, $\sin(3\alpha) = \sin(\alpha)(4\cos^2(\alpha)-1)$, and $\sin(4\alpha) = 4\sin(\alpha)\cos(\alpha)(2\cos^2(\alpha)-1)$. Summarizing, we have shown that

$$\begin{aligned}\sum_{j\neq i}\frac{\alpha_j}{\nu}\,\frac{1}{2\pi}V_1\Bigl(\xi,\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\cdot\nabla G(\xi) \;&=\; -\frac{1}{4\pi}\sum_{j\neq i}\frac{\alpha_j}{\nu}\,\xi\cdot V_1\Bigl(\xi,\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)G(\xi)\\ &=\; \frac{\alpha_i t}{d^2}\Bigl\{A_i(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)^{1/2}B_i(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)C_i(\xi,t) + \hat{R}_i(\xi,t)\Bigr\}\;,\end{aligned}$$

for $|\xi| \le d/(2\sqrt{\nu t})$, where A_i, B_i, C_i are given by (3.7) and $\hat{R}_i(\xi,t)$ satisfies the bound (3.8). As was already observed, the same result holds if we replace V_1 with V_1+V_2, so we conclude that expansion (3.6) is valid in the region $|\xi| \le d/(2\sqrt{\nu t})$ too. The proof of Proposition 3.1 is thus complete. □

It is important to remark that the residuum $R_i^{(0)}(\xi,t)$ does not converge to zero as $\nu \to 0$, because of the leading order term $A_i(\xi,t)$. If we decompose the solution of (3.1) as $w_i(\xi,t) = G(\xi) + \tilde{w}_i(\xi,t)$, the equation for $\tilde{w}_i(\xi,t)$ will contain a source term of size $\mathcal{O}(1)$ as $\nu \to 0$, and we therefore expect that the remainder $\tilde{w}_i(\xi,t)$ itself will be of size $\mathcal{O}(1)$ after a short time. But, as is easily verified, the equation for $\tilde{w}_i(\xi,t)$ contains nonlinear terms with a prefactor of size $\mathcal{O}(\nu^{-1})$, and such terms cannot be controlled in the vanishing viscosity limit if $\tilde{w}_i(\xi,t)$ is $\mathcal{O}(1)$. This is the reason why it is necessary to construct a more precise approximate solution of (3.1), with a sufficiently small residuum, in order to desingularize the equation for the remainder.

3.2 First order approximation

We look for an approximate solution of (3.1) of the form

$$w_i^{\mathrm{app}}(\xi,t) \,=\, G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr)F_i(\xi,t)\;, \qquad v_i^{\mathrm{app}}(\xi,t) \,=\, v^G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr)v^{F_i}(\xi,t)\;, \tag{3.14}$$

where, for each $i \in \{1,\dots,N\}$, $F_i(\xi,t)$ is a smooth vorticity profile to be determined, and $v^{F_i}(\xi,t)$ is the velocity field obtained from $F_i(\xi,t)$ via the Biot-Savart law (1.3). In fact, we shall need later a more precise approximation of the N-vortex solution, but we prefer starting with (3.14) to describe the procedure in a relatively simple setting. Our goal is to chose the profile $F_i(\xi,t)$ so as to minimize the residuum of our approximation, which is

$$R_i^{(1)}(\xi,t) \,=\, (t\partial_t - \mathcal{L})w_i^{\mathrm{app}}(\xi,t) + \sum_{j=1}^N \frac{\alpha_j}{\nu}\Bigl\{v_j^{\mathrm{app}}\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,t\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\}\cdot\nabla w_i^{\mathrm{app}}(\xi,t)\;.$$

Using (3.14) and the definition (3.5) of $R_i^{(0)}(\xi,t)$, we find after some calculations

$$R_i^{(1)}(\xi,t) \,=\, R_i^{(0)}(\xi,t) + \frac{\alpha_i t}{d^2}\Bigl(v^G\cdot\nabla F_i + v^{F_i}\cdot\nabla G\Bigr)(\xi,t) + \tilde{R}_i^{(1)}(\xi,t)\;, \tag{3.15}$$

where

$$\begin{aligned}\tilde{R}_i^{(1)}(\xi,t) \,=\, &\Bigl(\frac{\nu t}{d^2}\Bigr)\Bigl(t\partial_t F_i + F_i - \mathcal{L}F_i\Bigr)(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)\sum_{j=1}^N \frac{\alpha_j t}{d^2}\,v^{F_j}\Bigl(\xi + \frac{z_{ij}}{\sqrt{\nu t}}\,,t\Bigr)\cdot\nabla F_i(\xi,t)\\ &+ \sum_{j\neq i}\frac{\alpha_j t}{d^2}\Bigl\{\Bigl(v^G\Bigl(\xi + \frac{z_{ij}}{\sqrt{\nu t}}\Bigr) - v^G\Bigl(\frac{z_{ij}}{\sqrt{\nu t}}\Bigr)\Bigr)\cdot\nabla F_i(\xi,t) + v^{F_j}\Bigl(\xi + \frac{z_{ij}}{\sqrt{\nu t}}\,,t\Bigr)\cdot\nabla G(\xi)\Bigr\}\;.\end{aligned}$$

It is easy to check, at least formally, that $\tilde{R}_i^{(1)}(\xi,t)$ is $\mathcal{O}(\nu t/d^2)$ as $\nu \to 0$ (this calculation will be done rigorously later, when the profile F_i will be determined). So it follows from (3.6) and (3.15) that

$$R_i^{(1)}(\xi,t) \,=\, \frac{\alpha_i t}{d^2}\Bigl(A_i + v^G\cdot\nabla F_i + v^{F_i}\cdot\nabla G\Bigr)(\xi,t) + \mathcal{O}\Bigl(\Bigl(\frac{\nu t}{d}\Bigr)^{\frac{1}{2}}\Bigr)\;. \tag{3.16}$$

To minimize the residuum, it is natural to impose $A_i + v^G\cdot\nabla F_i + v^{F_i}\cdot\nabla G = 0$. As we shall see, this "elliptic" equation has a solution, but does not completely determine the profile F_i.

To prove this claim, we first introduce some notation. Let Y denote the Hilbert space

$$Y \,=\, \Bigl\{w \in L^2(\mathbb{R}^2)\,\Big|\,\int_{\mathbb{R}^2}|w(\xi)|^2\,e^{|\xi|^2/4}\,\mathrm{d}\xi < \infty\Bigr\}\;, \tag{3.17}$$

equipped with the scalar product

$$(w_1,w_2)_Y \,=\, \int_{\mathbb{R}^2} w_1(\xi)w_2(\xi)\,e^{|\xi|^2/4}\,\mathrm{d}\xi\;. \tag{3.18}$$

We consider the linear operator $\Lambda : D(\Lambda) \to Y$ defined by $D(\Lambda) = \{w \in Y \,|\, v^G \cdot \nabla w \in Y\}$ and

$$\Lambda w \;=\; v^G \cdot \nabla w + v \cdot \nabla G \;, \qquad w \in D(\Lambda) \;, \tag{3.19}$$

where (as always) v denotes the velocity field obtained from w via the Biot-Savart law (1.3). With these notations, the equation we have to solve becomes $\Lambda F_i + A_i = 0$, and we are therefore interested in computing the (partial) inverse of Λ on appropriate subspaces. This operator has been extensively studied, because it plays a prominent role in the stability properties of the Oseen vortices and the construction of asymmetric Burgers vortices [20, 21, 33, 34, 35]. In particular, we have

Proposition 3.3 *[20, 33] The operator Λ is skew-adjoint in Y, so that $\Lambda^* = -\Lambda$. Moreover,*

$$\mathrm{Ker}(\Lambda) \;=\; Y_0 \oplus \{\beta_1\partial_1 G + \beta_2\partial_2 G \,|\, \beta_1, \beta_2 \in \mathbb{R}\} \;, \tag{3.20}$$

where $Y_0 \subset Y$ is the subspace of all radially symmetric functions.

If $w \in Y_0$, the corresponding velocity field v satisfies $\xi \cdot v(\xi) = 0$, hence (3.19) immediately implies that $\Lambda w = 0$. On the other hand, if we differentiate the identity $v^G \cdot \nabla G = 0$ with respect to ξ_1 and ξ_2, we obtain $\Lambda(\partial_1 G) = \Lambda(\partial_2 G) = 0$. So the right-hand side of (3.20) is certainly contained in the kernel of Λ, and the converse inclusion was proved in [33]. On the other hand, the fact that Λ is skew-adjoint in Y implies that $\mathrm{Ker}(\Lambda) = \mathrm{Ran}(\Lambda)^\perp$, so we know that the range of Λ is a dense subspace of $\mathrm{Ker}(\Lambda)^\perp$. We shall now prove that $\mathrm{Ran}(\Lambda)$ contains $\mathrm{Ker}(\Lambda)^\perp \cap Z$, where

$$Z \;=\; \Big\{ w : \mathbb{R}^2 \to \mathbb{R} \,\Big|\, e^{|\xi|^2/8} w \in \mathcal{S}(\mathbb{R}^2) \Big\} \;\subset\; Y \;.$$

Here $\mathcal{S}(\mathbb{R}^2)$ denotes the space of all smooth, rapidly decreasing functions on $\mathbb{R}^2$.

As was observed e.g. in [20], the operator Λ commutes with the group $SO(2)$ of all rotations about the origin. It is thus natural to decompose

$$Y \;=\; \bigoplus_{n=0}^{\infty} Y_n \;=\; \bigoplus_{n=0}^{\infty} P_n Y \;,$$

where P_n is the orthogonal projection defined in polar coordinates (r,θ) by the formula

$$(P_n w)(r\cos\theta, r\sin\theta) \;=\; \frac{2-\delta_{n,0}}{2\pi} \int_0^{2\pi} w(r\cos\theta', r\sin\theta') \cos(n(\theta-\theta'))\,\mathrm{d}\theta' \;, \quad n \in \mathbb{N} \;.$$

Then $Y_0 = P_0 Y$ is the subspace of all radially symmetric functions, and for $n \ge 1$ the subspace $Y_n = P_n Y$ contains all functions of the form $w(r\cos\theta, r\sin\theta) = a_1(r)\cos(n\theta) + a_2(r)\sin(n\theta)$. It is not difficult to verify that the projections P_n commute with Λ for all $n \in \mathbb{N}$, and explicit formulas for the restrictions $\Lambda_n = P_n \Lambda P_n$ are given in [20, Section 4.1.1]. To formulate the main technical result of this section, we shall use the following notations:

$$g(r) \;=\; \frac{1}{4\pi}\,e^{-r^2/4} \;, \quad \phi(r) \;=\; \frac{1}{2\pi r^2}(1-e^{-r^2/4}) \;, \quad h(r) \;=\; \frac{g(r)}{2\phi(r)} \;=\; \frac{r^2/4}{e^{r^2/4}-1} \;, \quad r > 0 \;. \tag{3.21}$$

Lemma 3.4 *If $z \in Y_n \cap Z$ for some $n \ge 2$, there exists a unique $w \in Y_n \cap Z$ such that $\Lambda w = z$. In particular, if $z = a(r)\sin(n\theta)$, then $w = -\omega(r)\cos(n\theta)$, where*

$$\omega(r) \;=\; h(r)\Omega(r) + \frac{a(r)}{n\phi(r)} \;, \qquad r > 0 \;, \tag{3.22}$$

and $\Omega : (0,\infty) \to \mathbb{R}$ is the unique solution of the differential equation

$$-\frac{1}{r}(r\Omega'(r))' + \Big(\frac{n^2}{r^2} - h(r)\Big)\Omega(r) \;=\; \frac{a(r)}{n\phi(r)}~, \qquad r > 0~, \tag{3.23}$$

such that $\Omega(r) = \mathcal{O}(r^n)$ as $r \to 0$ and $\Omega(r) = \mathcal{O}(r^{-n})$ as $r \to \infty$.

Remarks.
1. By rotation invariance, if $z = a(r)\cos(n\theta)$, then $w = \omega(r)\sin(n\theta)$, and the relation between ω and a is unchanged. The general case where $z = a_1(r)\cos(n\theta) + a_2(r)\sin(n\theta)$ follows by linearity.
2. The conclusion of Lemma 3.4 is wrong for $n = 1$. Indeed, since $\partial_k G = -\frac{1}{2}\xi_k G$ for $k = 1,2$, it is clear that $\partial_1 G, \partial_2 G \in Y_1 \cap Z$, but Proposition 3.3 asserts that these functions belong to $\mathrm{Ker}(\Lambda) = \mathrm{Ran}(\Lambda)^\perp$. However, if $z \in Y_1 \cap Z$ satisfies $(z, \partial_k G)_Y = 0$ for $k = 1,2$, one can show that there exists a unique $w \in Y_1 \cap Z \cap \mathrm{Ker}(\Lambda)^\perp$ such that $\Lambda w = z$. This result will not be needed in what follows, so we omit the proof.
3. If w is as in Lemma 3.4 and if v is the velocity field associated to w via the Biot-Savart law (1.3), the proof will show that v is smooth and satisfies

$$|v(\xi)| \;=\; \mathcal{O}(|\xi|^{n-1}) \quad \text{as } \xi \to 0~, \quad \text{and} \quad |v(\xi)| \;=\; \mathcal{O}(|\xi|^{-n-1}) \quad \text{as } |\xi| \to \infty~. \tag{3.24}$$

Proof of Lemma 3.4. A particular case of Lemma 3.4 was proved in [21, Proposition 3.1]. Since the general case is quite similar, we just indicate here the main steps and refer to [21] for further details.

Assume that $z \in Y_n \cap Z$ for some $n \ge 2$. By Proposition 3.3, we have $z \in \mathrm{Ker}(\Lambda)^\perp = \overline{\mathrm{Ran}(\Lambda)}$. Our task is to verify that $z \in \mathrm{Ran}(\Lambda)$, and that there exists a unique $w \in Y_n \cap Z$ such that $\Lambda w = z$. Without loss of generality, we assume that $z = a(r)\sin(n\theta)$. Then $a : \mathbb{R}_+ \to \mathbb{R}$ is a smooth function with the property that $e^{r^2/8}a(r)$ decays rapidly as $r \to \infty$. Furthermore, we can write $a(r) = r^n A(r^2)$, where $A : [0,\infty) \to \mathbb{R}$ is again a smooth function. In particular, we have $a(r) = \mathcal{O}(r^n)$ as $r \to 0$. We look for a solution w of the form $w = -\omega(r)\cos(n\theta)$. The corresponding stream function, which is defined by the relation $-\Delta\Psi = w$, satisfies $\Psi = -\Omega(r)\cos(n\theta)$, where Ω is the unique regular solution of the differential equation

$$-\frac{1}{r}(r\Omega'(r))' + \frac{n^2}{r^2}\,\Omega(r) \;=\; \omega(r)~, \qquad r > 0~. \tag{3.25}$$

Moreover, the velocity field $v = -\nabla^\perp\Psi$ has the following expression

$$v \;=\; \frac{n}{r}\,\Omega(r)\sin(n\theta)\,\mathbf{e}_r + \Omega'(r)\cos(n\theta)\,\mathbf{e}_\theta~,$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ are the unit vectors in the radial and azimuthal directions, respectively. Thus, using definitions (1.7) and (3.21), we obtain

$$v\cdot\nabla G \;=\; -\frac{n}{2}\,\Omega(r)g(r)\sin(n\theta)~, \quad \text{and} \quad v^G\cdot\nabla w \;=\; n\omega(r)\phi(r)\sin(n\theta)~.$$

In particular, we see that $\Lambda w = z$ if and only if $-\frac{n}{2}\Omega g + n\phi\omega = a$, which is (3.22). Furthermore, combining (3.22) and (3.25), we obtain the differential equation (3.23) which determines Ω.

It remains to verify that (3.23) has indeed a unique solution with the desired properties. We first consider the *homogeneous* equation obtained by setting $a(r) \equiv 0$ in (3.23). Since $h(r) \to 1$

as $r \to 0$ and $h(r)$ decays rapidly as $r \to \infty$, this linear equation has two particular solutions ψ_+, ψ_- which satisfy

$$\psi_-(r) \sim r^n \quad \text{as } r \to 0\ , \quad \text{and} \quad \psi_+(r) \sim r^{-n} \quad \text{as } r \to \infty\ . \tag{3.26}$$

These solutions are of course unique. Moreover, since $n \ge 2$, the coefficient $n^2/r^2 - h(r)$ in (3.23) is always positive, because $\sup_{r>0} r^2 h(r) \cong 2.59\ldots < n^2$. It then follows from the Maximum Principle that the functions ψ_+, ψ_- are strictly positive and satisfy

$$\psi_-(r) \sim \kappa_- r^n \quad \text{as } r \to \infty\ , \quad \text{and} \quad \psi_+(r) \sim \kappa_+ r^{-n} \quad \text{as } r \to 0\ ,$$

for some $\kappa_-, \kappa_+ > 0$. In particular, ψ_+, ψ_- are linearly independent. In addition, their Wronskian determinant satisfies $W = \psi_+\psi_-' - \psi_-\psi_+' = w_0/r$ for some $w_0 > 0$, and it follows that $\kappa_+ = \kappa_- = w_0/(2n)$. We now return to the full equation (3.23), and consider the particular solution given by the explicit formula

$$\Omega(r) = \psi_+(r)\int_0^r \frac{y}{w_0}\,\psi_-(y)\frac{a(y)}{n\phi(y)}\,\mathrm{d}y + \psi_-(r)\int_r^\infty \frac{y}{w_0}\,\psi_+(y)\frac{a(y)}{n\phi(y)}\,\mathrm{d}y\ , \quad r > 0\ . \tag{3.27}$$

As is easily verified, we have $\Omega(r) = \mathcal{O}(r^n)$ as $r \to 0$, $\Omega(r) = \mathcal{O}(r^{-n})$ as $r \to \infty$, and Ω is the unique solution of (3.23) with these properties. If $w = -\omega(r)\cos(n\theta)$, where ω is defined by (3.22), then $\Lambda w = z$ by construction, and it is not difficult to see that $w \in Y_n \cap Z$. Indeed, it is clear that w is smooth away from the origin, and the fact that $e^{|\xi|^2/8}w$ decays rapidly at infinity follows immediately from (3.22). To prove that w is smooth in a neighborhood of zero, we observe that any regular solution of (3.23) has the form $\Omega(r) = r^n\Phi(r^2)$, where $\Phi : [0,\infty) \to \mathbb{R}$ is a smooth function. Using (3.22), we conclude that

$$w(\xi) = -|\xi|^n \cos(n\theta)\Big(h(|\xi|)\Phi(|\xi|^2) + \frac{A(|\xi|^2)}{n\phi(|\xi|)}\Big)$$

is smooth also near the origin, because $|\xi|^n\cos(n\theta) = \mathrm{Re}((\xi_1 + i\xi_2)^n)$ is a homogeneous polynomial in ξ. The proof of Lemma 3.4 is thus complete. □

Remark. Of course, the argument above fails if $n = 1$, because the coefficient $1/r^2 - h(r)$ in (3.23) is no longer positive. In fact, it is easy to verify that the functions ψ_+, ψ_- defined by (3.26) are linearly dependent in that case.

Equipped with Lemma 3.4, we now go back to the determination of the vorticity profile $F_i(\xi,t)$ in (3.14). The equation we have to solve is $\Lambda F_i + A_i = 0$, where $A_i(\xi,t)$ is given by (3.7). Using polar coordinates (r,θ) as before, we can write

$$A_i(\cdot,t) = \frac{d^2}{4\pi}\,r^2 g(r)\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{1}{|z_{ij}(t)|^2}\,\sin(2(\theta - \theta_{ij}(t)))\ ,$$

where $\theta_{ij}(t)$ is the argument of the vector $z_{ij}(t) = z_i(t) - z_j(t)$. This expression shows that $A_i(\cdot,t) \in Y_2 \cap Z$ for any $t \in [0,T]$, hence by Lemma 3.4 there exists a unique $F_i^0(\cdot,t) \in Y_2 \cap Z$ such that $\Lambda F_i^0 + A_i = 0$. Explicitly,

$$F_i^0(\cdot,t) = \frac{d^2}{4\pi}\,\omega(r)\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{1}{|z_{ij}(t)|^2}\,\cos(2(\theta - \theta_{ij}(t)))\ , \tag{3.28}$$

where $\omega(r)$ is given by (3.22), (3.23) with $n = 2$ and $a(r) = r^2 g(r)$. It follows in particular from (3.22), (3.28) that

$$|F_i^0(\xi,t)| \le C|\xi|^2(1 + |\xi|^2)\,e^{-|\xi|^2/4}\ , \qquad \xi \in \mathbb{R}^2\ , \quad t \in (0,T]\ .$$

If we now return to (3.15) and choose $F_i(\xi,t) = F_i^0(\xi,t)$, it is easy to verify that

$$R_i^{(1)}(\xi,t) \;=\; R_i^{(0)}(\xi,t) - \frac{\alpha_i t}{d^2}\,A_i(\xi,t) + \tilde R_i^{(1)}(\xi,t) \;=\; \mathcal{O}\Bigl(\Bigl(\frac{\nu t}{d^2}\Bigr)^{\frac{1}{2}}\Bigr)\;,$$

so we succeded in constructing an approximate solution with a smaller residuum than $R_i^{(0)}$. As is explained in Section 2, the profile $F_i^0(\xi,t)$ describes to leading order the deformations of the vortices due to mutual interaction.

Before going further, we state and prove a variant of Lemma 3.4 which will be useful in the next section.

Lemma 3.5 *Assume that $z \in Y_n \cap Z$ for some $n \ge 2$, and let $w \in Y_n \cap Z$ be the solution of $\Lambda w = z$ given by Lemma 3.4. Then for all $\epsilon \neq 0$ the equation*

$$\epsilon(1-\mathcal{L})w^\epsilon + \Lambda w^\epsilon \;=\; z \tag{3.29}$$

has a unique solution $w^\epsilon \in Y_n \cap Z$. Moreover, there exists $C > 0$ (depending on z) such that

$$\|w^\epsilon - w\|_Y \;\le\; \frac{C|\epsilon|}{1+|\epsilon|}\;, \qquad \textit{for all } \epsilon \neq 0\;. \tag{3.30}$$

Proof. Again, a particular case of Lemma 3.5 has been proved in [21, Proposition 3.4]. As is well-known (see e.g. [19]), the operator $\mathcal{L}$ defined by (3.2) is self-adjoint in Y and its spectrum is given by $\sigma(\mathcal{L}) = \{-\frac{n}{2}\,|\, n = 0,1,2,\dots\}$. Since Λ is skew-symmetric and relatively compact with respect to $\mathcal{L}$, it follows that the operator $-\epsilon\mathcal{L} + \Lambda$ is maximal accretive for any $\epsilon > 0$. Thus, for any $z \in Y$, equation (3.29) has a unique solution $w^\epsilon \in Y$, which satisfies $\|w^\epsilon\|_Y \le \epsilon^{-1}\|z\|_Y$. A similar result holds of course for $\epsilon < 0$.

We now consider the particular case where $z \in Y_n \cap Z$ for some $n \ge 2$. Since $\mathcal{L}$ commutes with the projection P_n, it is clear that $P_n w^\epsilon = w^\epsilon$, hence $w^\epsilon \in Y_n$. In that subspace, Eq. (3.29) reduces to an ordinary differential equation which can be studied as in the proof of Lemma 3.4. In particular, it is straightforward to check that $e^{|\xi|^2/8} w^\epsilon$ decays rapidly as $|\xi| \to \infty$, so that $w^\epsilon \in Y_n \cap Z$. On the other hand, since

$$\epsilon(1-\mathcal{L})(w^\epsilon - w) + \Lambda(w^\epsilon - w) \;=\; -\epsilon(1-\mathcal{L})w\;, \tag{3.31}$$

the fact that $\|[\epsilon(1-\mathcal{L}) + \Lambda]^{-1}\| \le |\epsilon|^{-1}$ implies

$$\|w^\epsilon - w\|_Y \;\le\; \|[\epsilon(1-\mathcal{L}) + \Lambda]^{-1}\|\,\|\epsilon(1-\mathcal{L})w\|_Y \;\le\; \|(1-\mathcal{L})w\|_Y\;, \tag{3.32}$$

hence $\|w^\epsilon - w\|_Y$ is uniformly bounded for all $\epsilon \neq 0$.

Finally, since $w \in Y_n \cap Z$, we have $(1-\mathcal{L})w \in Y_n \cap Z$, hence by Lemma 3.4 there exists a unique $\hat w \in Y_n \cap Z$ such that $\Lambda\hat w = (1-\mathcal{L})w$. Then (3.31) takes the equivalent form

$$[\epsilon(1-\mathcal{L}) + \Lambda](w^\epsilon - w + \epsilon\hat w) \;=\; \epsilon^2(1-\mathcal{L})\hat w\;,$$

from which we deduce

$$w^\epsilon - w \;=\; -\epsilon\hat w + \epsilon^2[\epsilon(1-\mathcal{L}) + \Lambda]^{-1}(1-\mathcal{L})\hat w\;.$$

Since $\|[\epsilon(1-\mathcal{L}) + \Lambda]^{-1}\| \le |\epsilon|^{-1}$, we conclude that

$$\|w^\epsilon - w\|_Y \;\le\; |\epsilon|(\|\hat w\|_Y + \|(1-\mathcal{L})\hat w\|_Y)\;. \tag{3.33}$$

Combining (3.32) and (3.33), we obtain (3.30). This concludes the proof of Lemma 3.5. $\square$

Remark 3.6 *For later use, we also introduce the space $Z_* \subset Z$ defined by*

$$Z_* \;=\; \Big\{ w : \mathbb{R}^2 \to \mathbb{R} \,\Big|\, e^{|\xi|^2/4} w \in \mathcal{S}_*(\mathbb{R}^2) \Big\}~,$$

where $\mathcal{S}_*(\mathbb{R}^2)$ denotes the space of all smooth functions $w : \mathbb{R}^2 \to \mathbb{R}$ such that w and all its derivatives have at most a polynomial growth at infinity. As is easily verified, Lemmas 3.4 and 3.5 still hold if we replace everywhere Z by Z_*. In particular, if $z \in Y_n \cap Z_*$, the solution w^ϵ of (3.29) *belongs to $Y_n \cap Z_*$. Thus, for any $\gamma < 1$, there exists $C > 0$ such that*

$$|w^\epsilon(\xi)| + |\nabla w^\epsilon(\xi)| \;\le\; C\, e^{-\gamma|\xi|^2/4}~, \qquad \xi \in \mathbb{R}^2~,$$

and using for instance (3.31) *one can show that the constant C is independent of ϵ.*

3.3 Third order approximation

We now construct an approximate solution that will be accurate enough to prove Theorem 2.5. We set

$$\begin{aligned} w_i^{\rm app}(\xi,t) &= G(\xi) + \Big(\frac{\nu t}{d^2}\Big) F_i(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)^{3/2} H_i(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)^2 K_i(\xi,t)~, \\ v_i^{\rm app}(\xi,t) &= v^G(\xi) + \Big(\frac{\nu t}{d^2}\Big) v^{F_i}(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)^{3/2} v^{H_i}(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)^2 v^{K_i}(\xi,t)~, \end{aligned} \tag{3.34}$$

where the vorticity profiles F_i, H_i, K_i have to be determined, and the velocity fields v^{F_i}, v^{H_i}, v^{K_i} are obtained from F_i, H_i, K_i via the Biot-Savart law (1.3). As in Proposition 3.1, the main expansion parameter in (3.34) is $(\nu t/d^2)$. The profiles F_i, H_i, K_i still depend on the viscosity ν, but they all have a finite limit as $\nu \to 0$.

Our first task is to compute the residuum $R_i^{(3)}(\xi,t)$ of the third-order expansion (3.34), as an approximate solution of (3.1). By a direct calculation, we find

$$\begin{aligned} (t\partial_t - \mathcal{L}) w_i^{\rm app}(\xi,t) \;=\; &\Big(\frac{\nu t}{d^2}\Big)\Big(t\partial_t F_i + F_i - \mathcal{L}F_i\Big)(\xi,t) \\ &+ \Big(\frac{\nu t}{d^2}\Big)^{3/2}\Big(t\partial_t H_i + \frac{3}{2}H_i - \mathcal{L}H_i\Big)(\xi,t) \\ &+ \Big(\frac{\nu t}{d^2}\Big)^{2}\Big(t\partial_t K_i + 2K_i - \mathcal{L}K_i\Big)(\xi,t)~, \end{aligned} \tag{3.35}$$

and using (3.5), (3.19) we obtain

$$\begin{aligned} &\sum_{j=1}^N \frac{\alpha_j}{\nu}\Big\{ v_j^{\rm app}\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,t\Big) - v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\Big\} \cdot \nabla w_i^{\rm app}(\xi,t) \\ &\quad = R_i^{(0)}(\xi,t) + \frac{\alpha_i t}{d^2}\Big\{\Lambda F_i + \Big(\frac{\nu t}{d^2}\Big)^{\frac12}\Lambda H_i + \Big(\frac{\nu t}{d^2}\Big)\Lambda K_i\Big\}(\xi,t) \\ &\quad + \sum_{j\neq i}\frac{\alpha_j}{\nu}\Big\{\Big(\frac{\nu t}{d^2}\Big)v^{F_j} + \Big(\frac{\nu t}{d^2}\Big)^{\frac32} v^{H_j} + \Big(\frac{\nu t}{d^2}\Big)^{2} v^{K_j}\Big\}\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,t\Big)\cdot\nabla G(\xi) \\ &\quad + \sum_{j\neq i}\frac{\alpha_j}{\nu}\Big\{v^G\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Big) - v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\Big\}\cdot\nabla\Big\{\Big(\frac{\nu t}{d^2}\Big)F_i + \Big(\frac{\nu t}{d^2}\Big)^{\frac32} H_i + \Big(\frac{\nu t}{d^2}\Big)^{2} K_i\Big\}(\xi,t) \\ &\quad + \sum_{j=1}^N \frac{\alpha_j}{\nu}\Big\{\Big(\frac{\nu t}{d^2}\Big)v^{F_j} + \Big(\frac{\nu t}{d^2}\Big)^{\frac32} v^{H_j} + \Big(\frac{\nu t}{d^2}\Big)^{2} v^{K_j}\Big\}\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,t\Big)\cdot \\ &\qquad\quad \cdot\nabla\Big\{\Big(\frac{\nu t}{d^2}\Big)F_i + \Big(\frac{\nu t}{d^2}\Big)^{\frac32} H_i + \Big(\frac{\nu t}{d^2}\Big)^{2} K_i\Big\}(\xi,t)~. \end{aligned} \tag{3.36}$$

By definition, the residuum $R_i^{(3)}(\xi,t)$ is the sum of all terms in (3.35) and (3.36).

Next, we separate the lower order terms, which will have to be eliminated by an appropriate choice of F_i, H_i, K_i, from the higher order terms, which will be automatically negligible. We thus decompose

$$R_i^{(3)}(\xi,t) \;=\; R_i^\ell(\xi,t) + R_i^h(\xi,t)\;,$$

where R_i^ℓ collects the lower order terms in the residuum, namely

$$\begin{aligned} R_i^\ell(\xi,t) \;=\; & R_i^{(0)}(\xi,t) + \frac{\alpha_i t}{d^2}\Big\{\Lambda F_i + \Big(\frac{\nu t}{d^2}\Big)^{\frac12}\Lambda H_i + \Big(\frac{\nu t}{d^2}\Big)\Lambda K_i\Big\}(\xi,t) \\ & + \Big(\frac{\nu t}{d^2}\Big)\Big(t\partial_t F_i + F_i - \mathcal{L}F_i\Big)(\xi,t) + \frac{\alpha_i t}{d^2}\Big(\frac{\nu t}{d^2}\Big) v^{F_i}(\xi,t)\cdot\nabla F_i(\xi,t) \\ & + \Big(\frac{\nu t}{d^2}\Big)\sum_{j\neq i}\frac{\alpha_j}{\nu}\Big\{v^G\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Big) - v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\Big\}\cdot\nabla F_i(\xi,t)\;. \end{aligned} \tag{3.37}$$

Our goal is to choose the vorticity profiles F_i, H_i, K_i so as to minimize the quantity $R_i^\ell(\xi,t)$ in the vanishing viscosity limit. The contributions of the naive residuum (3.6) are easily eliminated by successive applications of Lemma 3.4: we first take $F_i(\xi,t)$ so that $\Lambda F_i + A_i = 0$, then $H_i(\xi,t)$ so that $\Lambda H_i + B_i = 0$, and so on. In this way, we obtain a residuum $R_i^\ell(\xi,t)$ of size $\mathcal{O}(\nu t/d^2)$, but unfortunately this is not sufficient to prove Theorems 2.4 and 2.5. To obtain a more precise estimate, we also need to eliminate all terms in the last two lines of (3.37). This requires a more careful choice of F_i, H_i, K_i, which will be done in three steps:

1) First, we take

$$F_i(\xi,t) \;=\; \bar F_i(\xi,t) + F_i^\nu(\xi,t)\;, \qquad \xi\in\mathbb{R}^2\;, \quad t\in(0,T]\;, \tag{3.38}$$

where $\bar F_i(\cdot,t)\in Y_0\cap Z$ and $F_i^\nu(\cdot,t)\in Y_2\cap Z$. More precisely:

a) The profile $F_i^\nu(\xi,t)$ is the unique solution of the elliptic equation

$$\frac{\nu}{\alpha_i}(F_i^\nu - \mathcal{L}F_i^\nu) + \Lambda F_i^\nu + A_i \;=\; 0\;, \tag{3.39}$$

as given by Lemma 3.5. Since $A_i(\cdot,t)\in Y_2\cap Z$, we have $F_i^\nu(\cdot,t)\in Y_2\cap Z$ for all $t\in(0,T]$. Moreover, by (3.30),

$$\|F_i^\nu(\cdot,t) - F_i^0(\cdot,t)\|_Y \;\le\; C\,\frac{\nu}{|\alpha_i|+\nu}\;, \qquad t\in(0,T]\;,$$

where $F_i^0(\xi,t)$ is given by (3.28).

b) The profile $\bar F_i(\xi,t)$ is the unique solution of the linear parabolic equation

$$t\partial_t\bar F_i + \bar F_i - \mathcal{L}\bar F_i + \frac{\alpha_i t}{d^2}\Big\{P_0(V^{F_i^\nu}\cdot\nabla F_i^\nu) + P_0(D_i\cdot\nabla F_i^\nu)\Big\} \;=\; 0\;, \tag{3.40}$$

with initial data $\bar F_i(\cdot,0) = 0$. Here P_0 is the orthogonal projection (in Y) onto the radially symmetric functions, and $D_i(\xi,t)$ is the divergence-free vector field given by

$$D_i(\xi,t) \;=\; \frac{1}{2\pi}\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{d^2}{|z_{ij}(t)|^4}\Big(\xi^\perp|z_{ij}(t)|^2 - 2(\xi\cdot z_{ij}(t))z_{ij}(t)^\perp\Big)\;. \tag{3.41}$$

It is clear that $\bar F_i(\xi,t)$ is well-defined. In fact, if $S(\tau) = \exp(\tau\mathcal{L})$ denotes the C_0-semigroup in Y generated by $\mathcal{L}$, we have the explicit formula

$$\bar F_i(\cdot,t) \;=\; -\frac{\alpha_i}{d^2}\int_0^t S\Bigl(\log\frac{t}{s}\Bigr)\frac{s}{t}\,P_0 Q_i(\cdot,s)\,\mathrm{d}s\ , \tag{3.42}$$

where $Q_i = V^{F_i^\nu}\cdot\nabla F_i^\nu + D_i\cdot\nabla F_i^\nu$.

2) Next, we determine the profile $H_i(\xi,t)$. From the proof of Proposition 3.1, it is clear that $B_i(\cdot,t) \in Y_3\cap Z$. Thus, by Lemma 3.4, there exists a unique solution $H_i(\cdot,t) \in Y_3\cap Z$ to the equation

$$\Lambda H_i(\cdot,t) + B_i(\cdot,t) \;=\; 0\ , \qquad t\in(0,T]\ . \tag{3.43}$$

3) Finally, we set

$$K_i(\xi,t) \;=\; K_{i2}(\xi,t) + K_{i4}(\xi,t)\ , \qquad \xi\in\mathbb{R}^2\ , \quad t\in(0,T]\ , \tag{3.44}$$

where $K_{i2}(\cdot,t)\in Y_2\cap Z$ and $K_{i4}(\cdot,t)\in Y_4\cap Z$ are chosen as follows:

a) The profile $K_{i2}(\xi,t)$ is the unique solution, given by Lemma 3.4, of the equation

$$\Lambda K_{i2} + P_2(V^{F_i}\cdot\nabla F_i) + P_2(D_i\cdot\nabla F_i) + \frac{d^2}{\alpha_i}\partial_t F_i^\nu \;=\; 0\ , \tag{3.45}$$

where P_2 is the orthogonal projection in Y onto Y_2.

b) The profile $K_{i4}(\xi,t)$ is the unique solution, given by Lemma 3.4, of the equation

$$\Lambda K_{i4} + P_4(V^{F_i}\cdot\nabla F_i) + P_4(D_i\cdot\nabla F_i) + C_i \;=\; 0\ , \tag{3.46}$$

where P_4 is the orthogonal projection onto Y_4, and C_i is as in (3.7).

The main result of this section is:

Proposition 3.7 *Fix $\frac12 < \gamma < 1$. There exists $C > 0$ (depending on T/T_0) such that, with the above choices of the vorticity profiles F_i, H_i, K_i, the residuum of the approximate solution* (3.34) *satisfies*

$$|R_i^{(3)}(\xi,t)| \;\le\; C\Bigl(\frac{\nu t}{d^2}\Bigr)^{3/2} e^{-\gamma|\xi|^2/4}\ , \tag{3.47}$$

for all $\xi\in\mathbb{R}^2$, all $t\in(0,T]$, and all $i\in\{1,\cdots,N\}$.

Proof. The proof is a long sequence of rather straightforward verifications, some of which will be left to the reader. We first summarize the informations we have on the vorticity profiles F_i, H_i, K_i, and on the associated velocity fields v^{F_i}, v^{H_i}, v^{K_i}. Given any $\gamma < 1$, we claim that there exists $C > 0$ such that

$$|F_i(\xi,t)| + |\nabla F_i(\xi,t)| \;\le\; C\,e^{-\gamma|\xi|^2/4}\ , \qquad |v^{F_i}(\xi,t)| \;\le\; \frac{C}{(1+|\xi|^2)^{3/2}}\ , \tag{3.48}$$

for all $\xi\in\mathbb{R}^2$, all $t\in(0,T]$, and all $i\in\{1,\cdots,N\}$. Indeed, we recall that $F_i = \bar F_i + F_i^\nu$, where F_i^ν is defined by (3.39) and $\bar F_i$ by (3.40). From (3.39) and Remark 3.6, we know that $F_i^\nu(\cdot,t)\in Y_2\cap Z_*$. Thus F_i^ν and ∇F_i^ν satisfy the Gaussian bound in (3.48), and it follows from [19, Proposition B.1] that $|v^{F_i}(\xi,t)| \le C(1+|\xi|^2)^{-3/2}$, see also (3.24). On the other hand, using (3.42) and the explicit expression of the integral kernel of the semigroup $S(\tau) = \exp(\tau\mathcal{L})$

[19, Appendix A], it is straightforward to verify that $|\bar{F}_i(\xi,t)| \le Ce^{-\gamma|\xi|^2/4}$. Since $\nabla S(\tau) = e^{\tau/2}S(\tau)\nabla$, we also have

$$\nabla \bar{F}_i(\cdot,t) \;=\; -\frac{\alpha_i}{d^2}\int_0^t S\Big(\log\frac{t}{s}\Big)\Big(\frac{s}{t}\Big)^{\frac{1}{2}}\nabla P_0 Q_i(\cdot,s)\,\mathrm{d}s \;,$$

from which we deduce that $|\nabla\bar{F}_i(\xi,t)| \le Ce^{-\gamma|\xi|^2/4}$. Finally, since $\bar{F}_i(\cdot,t)$ is a radially symmetric function with zero average, we have a simple formula for the associated velocity field

$$v^{\bar{F}_i}(\xi,t) \;=\; \frac{1}{2\pi}\,\frac{\xi^\perp}{|\xi|^2}\int_{|\xi'|\ge|\xi|}\bar{F}_i(\xi',t)\,\mathrm{d}\xi' \;,$$

which implies that $v^{\bar{F}_i}(\xi,t)$ has also a Gaussian decay as $|\xi|\to\infty$. This proves (3.48).

The corresponding estimates for $H_i(\xi,t)$ and $K_i(\xi,t)$ are easier to establish. Since $B_i(\cdot,t) \in Y_3\cap Z_*$, it follows from (3.43) and Remark 3.6 that $H_i(\cdot,t)\in Y_3\cap Z_*$ for all $t\in(0,T]$. Using the same arguments as before, we obtain

$$|H_i(\xi,t)| + |\nabla H_i(\xi,t)| \;\le\; C\,e^{-\gamma|\xi|^2/4}\;, \qquad |v^{H_i}(\xi,t)| \;\le\; \frac{C}{(1+|\xi|^2)^2}\;. \tag{3.49}$$

Similarly, it follows from (3.45), (3.46) that $K_{i2}(\cdot,t)\in Y_2\cap Z_*$ and $K_{i4}(\cdot,t)\in Y_4\cap Z_*$. We conclude that $K_i = K_{i2}+K_{i4}$ satisfies

$$|K_i(\xi,t)| + |\nabla K_i(\xi,t)| \;\le\; C\,e^{-\gamma|\xi|^2/4}\;, \qquad |v^{K_i}(\xi,t)| \;\le\; \frac{C}{(1+|\xi|^2)^{3/2}}\;. \tag{3.50}$$

Next, we make the following observations, which were implicitely used in the definitions of the profiles F_i and K_i. Since $F_i = \bar{F}_i + F_i^\nu \in Y_0+Y_2$, it is easy to verify that $v^{F_i}\cdot\nabla F_i \in Y_0+Y_2+Y_4$. Similarly, using the definition (3.41) of the vector field D_i, we find that $D_i\cdot\nabla F_i \in Y_0+Y_2+Y_4$. Thus we have the identities

$$v^{F_i}\cdot\nabla F_i \;=\; (P_0+P_2+P_4)(v^{F_i}\cdot\nabla F_i)\;, \qquad D_i\cdot\nabla F_i \;=\; (P_0+P_2+P_4)(D_i\cdot\nabla F_i)\;. \tag{3.51}$$

Moreover, it is straightforward to check that

$$P_0(v^{F_i}\cdot\nabla F_i) \;=\; P_0(v^{F_i^\nu}\cdot\nabla F_i^\nu)\;, \qquad P_0(D_i\cdot\nabla F_i) \;=\; P_0(D_i\cdot\nabla F_i^\nu)\;. \tag{3.52}$$

Remark that both expressions in (3.52) appear in the definition (3.40) of $\bar{F}_i$.

Now, we replace the definitions (3.38)–(3.40) and (3.43)–(3.46) into the expression (3.37) of the residuum $R_i^\ell(\xi,t)$. Using in addition (3.6), (3.51), (3.52), we obtain the simple formula

$$R_i^\ell(\xi,t) \;=\; \frac{\alpha_i t}{d^2}\,\tilde{R}_i^{(0)}(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)\,\Delta_i(\xi,t)\cdot\nabla F_i(\xi,t)\;, \tag{3.53}$$

where

$$\Delta_i(\xi,t) \;=\; \sum_{j\ne i}\frac{\alpha_j}{\nu}\Big(v^G\Big(\xi+\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big) - v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\Big) - \frac{\alpha_i t}{d^2}\,D_i(\xi,t)\;. \tag{3.54}$$

Our goal is to obtain an estimate of the form (3.47) for $R_i^\ell(\xi,t)$. Since $\tilde{R}_i^{(0)}(\xi,t)$ satisfies (3.8), it is sufficient to bound the second term in the right-hand side of (3.53). As in the proof of Proposition 3.1, we can assume that $|\xi|\le d/(2\sqrt{\nu t})$, because in the converse case the quantity

$|\nabla F_i(\xi,t)|$ is extremely small due to (3.48). Using the decomposition (3.10) and the definition (3.41) of the vector field D_i, we find

$$\begin{aligned}\Delta_i(\xi,t) \;&=\; \frac{1}{2\pi}\sum_{j\neq i}\frac{\alpha_j}{\nu}\Big(V_1\Big(\xi,\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)+V_2\Big(\xi,\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\Big)-\frac{\alpha_i t}{d^2}\,D_i(\xi,t)\\ &=\; \frac{1}{2\pi}\sum_{j\neq i}\frac{\alpha_j}{\nu}W\Big(\xi,\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)+\frac{1}{2\pi}\sum_{j\neq i}\frac{\alpha_j}{\nu}V_2\Big(\xi,\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\;,\end{aligned} \tag{3.55}$$

where

$$W(\xi,\eta) \;=\; \frac{(\xi+\eta)^\perp}{|\xi+\eta|^2}-\frac{(\xi+\eta)^\perp}{|\eta|^2}+2\frac{(\xi\cdot\eta)\eta^\perp}{|\eta|^4} \;=\; \mathcal{O}\Big(\frac{|\xi|^2}{|\eta|^3}\Big)\;, \quad \text{as } |\eta|\to\infty\;. \tag{3.56}$$

In view of (3.11), the contributions of V_2 are negligible, and using (3.48) we easily obtain

$$|\Delta_i(\xi,t)\cdot\nabla F_i(\xi,t)| \;\le\; C\frac{|\alpha_i|t}{d^2}\Big(\frac{\nu t}{d^2}\Big)^{1/2}e^{-\gamma|\xi|^2/4}\;, \qquad \xi\in\mathbb{R}^2\;,$$

which is the desired estimate.

To complete the proof of Proposition 3.7, it remains to verify that

$$|R_i^{(3)}(\xi,t)-R_i^\ell(\xi,t)| \;\le\; C\left(\frac{\nu t}{d^2}\right)^{3/2}e^{-\gamma|\xi|^2/4}\;.$$

This follows immediately from (3.36), (3.37) if one uses the bounds (3.48), (3.49), (3.50) on the vorticity profiles F_i, H_i, K_i and the associated velocities. In particular, it is straightforward to check that each term in the difference $R_i^{(3)}-R_i^\ell$ is of the order of $(\nu t/d^2)^n$ for some $n\ge 3/2$, either due to an explicit prefactor or as a consequence of the polynomial decay of the velocity fields v^G, v^{F_i}, v^{H_i}, or v^{K_i} as $|\xi|\to\infty$. This concludes the proof. □

Remark. Instead of (3.43), one can define the profile $H_i(\xi,t)$ as the (unique) solution of the elliptic equation

$$\frac{\nu}{\alpha_i}\Big(\frac{3}{2}H_i-\mathcal{L}H_i\Big)+\Lambda H_i+B_i \;=\; 0\;,$$

which is the analog of (3.39). In the same spirit, one can replace (3.45) by

$$\frac{\nu}{\alpha_i}(2K_{i2}-\mathcal{L}K_{i2})+\Lambda K_{i2}+P_2(V^{F_i}\cdot\nabla F_i)+P_2(D_i\cdot\nabla F_i)+\frac{d^2}{\alpha_i}\partial_t F_i^\nu \;=\; 0\;,$$

and proceed similarly with (3.46). After these modifications, it is easy to verify that the residuum satisfies the improved bound

$$|R_i^{(3)}(\xi,t)| \;\le\; C\,\frac{|\alpha|t}{d^2}\left(\frac{\nu t}{d^2}\right)^{3/2}e^{-\gamma|\xi|^2/4}\;,$$

which is sharper than (3.47) for small times. This refinement is not needed in the proof of Theorem 2.5, but it indicates the correct way to proceed if one wants to construct even more precise approximations of the N-vortex solution.

4 Control of the remainder

In the previous section, we constructed an approximate solution $w_i^{\mathrm{app}}(\xi,t)$, $v_i^{\mathrm{app}}(\xi,t)$ of equation (3.1), with a very small residuum. We now consider the exact solution $w_i(\xi,t)$ of (3.1) given by (2.13) and Lemma 2.2, and we try control the difference $w_i(\xi,t) - w_i^{\mathrm{app}}(\xi,t)$ for $\xi \in \mathbb{R}^2$ and $t \in (0,T]$, in the vanishing viscosity limit. To this end, it is convenient to write

$$w_i(\xi,t) \,=\, w_i^{\mathrm{app}}(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)\,\tilde w_i(\xi,t)~, \quad v_i(\xi,t) \,=\, v_i^{\mathrm{app}}(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr)\,\tilde v_i(\xi,t)~, \tag{4.1}$$

and to study the evolution system satisfied by the remainder $\tilde w_i(\xi,t)$, $\tilde v_i(\xi,t)$.

Inserting (4.1) into (3.1), and using the definition (3.4), we find

$$t\partial_t \tilde w_i(\xi,t) - (\mathcal{L}\tilde w_i)(\xi,t) + \tilde w_i(\xi,t) \tag{4.2}$$

$$+ \frac{\alpha_i}{\nu}\Bigl(v_i^{\mathrm{app}}(\xi,t)\cdot\nabla\tilde w_i(\xi,t) + \tilde v_i(\xi,t)\cdot\nabla w_i^{\mathrm{app}}(\xi,t)\Bigr) \tag{4.3}$$

$$+ \sum_{j\neq i}\frac{\alpha_j}{\nu}\Bigl\{v_j^{\mathrm{app}}\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\}\cdot\nabla\tilde w_i(\xi,t) \tag{4.4}$$

$$+ \sum_{j\neq i}\frac{\alpha_j}{\nu}\,\tilde v_j\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Bigr)\cdot\nabla w_i^{\mathrm{app}}(\xi,t) \tag{4.5}$$

$$+ \sum_{j=1}^{N}\frac{\alpha_j t}{d^2}\,\tilde v_j\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Bigr)\cdot\nabla\tilde w_i(\xi,t) + \tilde R_i(\xi,t) \,=\, 0~, \tag{4.6}$$

where $\tilde R_i(\xi,t) = (\nu t/d^2)^{-1}R_i^{(3)}(\xi,t)$. From Proposition 3.7, we know that

$$|\tilde R_i(\xi,t)| \,\le\, C\,\Bigl(\frac{\nu t}{d^2}\Bigr)^{1/2}\,e^{-\gamma|\xi|^2/4}~, \tag{4.7}$$

for all $\xi \in \mathbb{R}^2$, all $t \in (0,T]$, and all $i \in \{1,\dots,N\}$. Also, since $\int_{\mathbb{R}^2} w_i(\xi,t)\,\mathrm{d}\xi = 1$ by (2.13) and Lemma 2.2, it is clear that $\int_{\mathbb{R}^2}\tilde w_i(\xi,t)\,\mathrm{d}\xi = 0$ for all $t \in (0,T]$.

To prove Theorem 2.5, our strategy is to consider the (unique) solution $\tilde w_i(\xi,t)$ of system (4.2)–(4.6) with zero initial data, and to control it on the time interval $(0,T]$ using an energy functional of the form

$$E(t) \,=\, \frac{1}{2}\sum_{i=1}^{N}\int_{\mathbb{R}^2} p_i(\xi,t)|\tilde w_i(\xi,t)|^2\,\mathrm{d}\xi~, \qquad t \in (0,T]~, \tag{4.8}$$

where the weight functions $p_i(\xi,t)$, $i \in \{1,\dots,N\}$, will be carefully constructed below. In particular, we shall require that $p_i(\xi,t) \ge Ce^{\beta|\xi|/4}$ for some $\beta > 0$. Using (4.2)–(4.7), we shall derive a differential inequality for $E(t)$ which will imply that $E(t) = \mathcal{O}(\nu t/d^2)$ as $\nu \to 0$. This will show that

$$\sum_{i=1}^{N}\int_{\mathbb{R}^2} e^{\beta|\xi|/4}|\tilde w_i(\xi,t)|^2\,\mathrm{d}\xi \,\le\, C\Bigl(\frac{\nu t}{d^2}\Bigr)~, \qquad t \in (0,T]~, \tag{4.9}$$

for some $C > 0$, if ν is sufficiently small. In view of (4.1), this estimate is equivalent to (2.20), which is the desired result.

4.1 Construction of the weight functions

Since the construction of the weights $p_i(\xi,t)$ is rather delicate, we first explain the main ideas in a heuristic way. Ideally, we would like to use for each $i \in \{1,\dots,N\}$ the time-independent weight $p(\xi) = e^{|\xi|^2/4}$, in order to control the remainder $\tilde{w}_i(\cdot,t)$ in the function space Y defined in (3.17). This is a natural choice for at least two reasons. First, the linear operator $\mathcal{L}$ defined in (3.2) is self-adjoint in Y, and a straightforward calculation (which will be reproduced below) shows that

$$\int_{\mathbb{R}^2} e^{|\xi|^2/4}\tilde{w}_i(\mathcal{L}\tilde{w}_i - \tilde{w}_i)\,\mathrm{d}\xi \,\le\, -\int_{\mathbb{R}^2} e^{|\xi|^2/4}\Big(\frac{1}{4}|\nabla\tilde{w}_i|^2 + \frac{|\xi|^2}{24}|\tilde{w}_i|^2 + \frac{1}{2}|\tilde{w}_i|^2\Big)\,\mathrm{d}\xi\;. \qquad (4.10)$$

Next, using (3.34), we observe that the self-interaction terms (4.3) have the form

$$\frac{\alpha_i}{\nu}\Big(v_i^{\mathrm{app}}(\xi,t)\cdot\nabla\tilde{w}_i(\xi,t) + \tilde{v}_i(\xi,t)\cdot\nabla w_i^{\mathrm{app}}(\xi,t)\Big) \;=\; \frac{\alpha_i}{\nu}\,\Lambda\tilde{w}_i(\xi,t) \,+\, \text{regular terms}\;,$$

where Λ is the linear operator defined in (3.19). Here and below, we call "regular" all terms which have a finite limit as $\nu \to 0$. Since Λ is skew-symmetric in the space Y by Proposition 3.3, we see that the singular term $(\alpha_i/\nu)\Lambda\tilde{w}_i$ will not contribute at all to the variation of the energy if we use the Gaussian weight $e^{|\xi|^2/4}$.

Unfortunately, this naive choice is not appropriate to treat the advection terms (4.4), which describe how the perturbation $\tilde{w}_i$ of the i^{th} vortex is transported by the velocity field of the other vortices. Indeed, using again (3.34), we can write (4.4) as

$$\sum_{j\neq i}\frac{\alpha_j}{\nu}\left\{v^G\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Big) - v^G\Big(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Big)\right\}\cdot\nabla\tilde{w}_i(\xi,t) \,+\, \text{regular terms}\;. \qquad (4.11)$$

Since $v^G(\xi)$ is given by (1.7), and since $|z_{ij}(t)| \ge d > 0$ when $i \neq j$, it is easy to verify (as in the proof of Proposition 3.1) that the first term in (4.11) has a finite limit as $\nu \to 0$, provided $|\xi| \ll d/\sqrt{\nu t}$. Although this is not obvious a priori, we shall see below that the same term is also harmless if $|\xi| \gg D/\sqrt{\nu t}$, where

$$D \,=\, \max_{t\in[0,T]}\,\max_{i\neq j}\,|z_i(t) - z_j(t)| \,<\, \infty\;. \qquad (4.12)$$

In the intermediate region, however, the first term in (4.11) can be of size $\mathcal{O}(|\alpha|/\nu)$, and there is no hope to obtain a better bound. But we should keep in mind that the whole term (4.4) describes the advection of the perturbation $\tilde{w}_i$ by a divergence-free velocity field, and therefore does not contribute to the variation of the energy in the regions where the weight function is constant. The idea is thus to modify the Gaussian weight to obtain a large plateau in the intermediate region where the advection term (4.4) is singular.

Our improved try is therefore a time-dependent weight of the form

$$p(\xi,t) \,=\, \begin{cases} e^{|\xi|^2/4} & \text{if } \;|\xi| \le \rho(t)\;, \\ e^{\rho(t)^2/4} & \text{if } \;\rho(t) \le |\xi| \le K\rho(t)\;, \\ e^{|\xi|^2/(4K^2)} & \text{if } \;|\xi| \ge K\rho(t)\;, \end{cases} \qquad (4.13)$$

where

$$\rho(t) \,=\, \frac{d}{2\sqrt{\nu t}}\;, \quad \text{and} \quad K \,=\, \frac{4D}{d}\;.$$

By construction, the function $\xi \mapsto p(\xi,t)$ coincides with $e^{|\xi|^2/4}$ in a large disk near the origin, is identically constant in the intermediate region where the advection terms (4.4) are dangerous, and becomes Gaussian again when $|\xi|$ is very large. Moreover, this weight is continuous, radially symmetric, and satisfies $e^{|\xi|^2/(4K^2)} \le p(\xi,t) \le e^{|\xi|^2/4}$ for all $\xi \in \mathbb{R}^2$ and all $t \in (0,T]$. Of course, the operator $\mathcal{L}$ is no longer self-adjoint in the function space defined by the modified weight $p(\xi,t)$, but an estimate of the form (4.10) can nevertheless be established by a direct calculation. Similarly, the operator Λ is no longer skew-symmetric, but we shall see below that it remains approximately skew-symmetric with the modified weight, and this will be sufficient to treat the self-interaction terms (4.3).

We now consider the contributions of the advection terms (4.4) to the variation of the energy (4.8), when $p_i(\xi,t) = p(\xi,t)$ for all $i \in \{1,\dots,N\}$. Integrating by parts, we obtain

$$\int_{\mathbb{R}^2} p(\xi,t)\,\tilde{w}_i(\xi,t) \sum_{j\neq i} \frac{\alpha_j}{\nu} \Bigl\{ v_j^{\rm app}\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\} \cdot \nabla \tilde{w}_i(\xi,t)\,{\rm d}\xi$$
$$= -\frac{1}{2}\int_{\mathbb{R}^2} |\tilde{w}_i(\xi,t)|^2 \sum_{j\neq i} \frac{\alpha_j}{\nu} \Bigl\{ v_j^{\rm app}\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\, t\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\} \cdot \nabla p(\xi,t)\,{\rm d}\xi~. \qquad (4.14)$$

If $\rho(t) \le |\xi| \le K\rho(t)$, then $\nabla p(\xi,t) \equiv 0$ by construction, hence it remains to consider the regions where $|\xi| \le \rho(t)$ or $|\xi| \ge K\rho(t)$. Unfortunately, although the integrand in (4.14) is regular when $|\xi| \le \rho(t)$, the contributions from that region are still difficult to control if t is large. To see this, we first replace $v_j^{\rm app}$ by v^G as in (4.11), because the difference $v_j^{\rm app} - v^G$ is negligible at this level of analysis. Assuming that $|\xi| \le \rho(t)$, we find as in (3.54)

$$\sum_{j\neq i} \frac{\alpha_j}{\nu} \Bigl\{ v^G\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\} ~=~ \frac{\alpha_i t}{d^2}\,\Bigl(D_i(\xi,t) + \mathcal{O}\Bigl(|\xi|^2 \frac{\sqrt{\nu t}}{d}\Bigr)\Bigr)~,$$

where $D_i(\xi,t)$ is defined in (3.41). As $\nabla p(\xi,t) = \frac{\xi}{2}p(\xi,t)$ for $|\xi| \le \rho(t)$, we conclude that the main contribution to (4.14) has the form

$$\frac{1}{4\pi}\,\frac{\alpha_i t}{d^2} \int_{|\xi|\le\rho(t)} |\tilde{w}_i(\xi,t)|^2 \sum_{j\neq i} \frac{\alpha_j}{\alpha_i}\,\frac{d^2(\xi\cdot z_{ij}(t))(\xi\cdot z_{ij}(t)^\perp)}{|z_{ij}(t)|^4}\,p(\xi,t)\,{\rm d}\xi~, \qquad (4.15)$$

and is therefore bounded by

$$C\,\frac{|\alpha_i| t}{d^2} \int_{|\xi|\le\rho(t)} |\xi|^2\, e^{|\xi|^2/4}\,|\tilde{w}_i(\xi,t)|^2\,{\rm d}\xi~. \qquad (4.16)$$

If t is small with respect to the turnover time T_0 defined in (2.10), then $|\alpha_i| t \ll d^2$ and the quantity (4.16) can be controlled by the negative terms originating from the diffusion operator $\mathcal{L}$, see (4.10). In that case, one can show that the expression (4.14) is harmless also in the outer region where $|\xi| \ge K\rho(t)$, and it is not difficult to verify that the linear terms (4.5) and the nonlinear terms (4.6) can be controlled in a similar way. Thus, if $T \ll T_0$, it is possible to carry out the whole proof of Theorem 2.5 using the energy functional (4.8) with $p_i(\xi,t) = p(\xi,t)$ for all $i \in \{1,\dots,N\}$.

The main difficulty is of course to get rid of the condition $T \ll T_0$, which is obviously too restrictive. We follow here the same strategy as in the construction of the approximate solution $w_i^{\rm app}(\xi,t)$ in Sections 3.2 and 3.3. So far, we have used radially symmetric weights to eliminate, or at least to minimize, the influence of the singular self-interaction terms (4.3). The idea is now to add small, nonsymmetric corrections of size $\mathcal{O}(\nu t/d^2)$ which, by interacting with the singular expression (4.3), will produce counter-terms of size $\mathcal{O}(1)$ that will exactly compensate for (4.15). Unfortunately, the construction of these corrections is quite technical, and requires a non-trivial modification of the underlying radially symmetric weight.

We now give the precise definition of the weights $p_i(\xi,t)$ that will be used in the definition (4.8) of the energy. We start from the radially symmetric weight

$$p_0(\xi,t) \;=\; \begin{cases} e^{|\xi|^2/4}\Bigl(1-\psi(|\xi|^2-a(t)^2)\Bigr) & \text{if} \quad 0 \le |\xi|^2 \le a(t)^2{+}1\;, \\ e^{a(t)^2/4} & \text{if} \quad a(t)^2{+}1 \le |\xi|^2 \le b(t)^2\;, \\ e^{\beta|\xi|/4} & \text{if} \quad |\xi|^2 \ge b(t)^2\;, \end{cases} \tag{4.17}$$

where

$$a(t) \;=\; a_0\Bigl(\frac{d^2}{\nu t}\Bigr)^{1/4}\;, \quad b(t) \;=\; b_0\Bigl(\frac{d^2}{\nu t}\Bigr)^{1/2}\;, \quad \text{and} \quad \beta \;=\; \frac{a_0^2}{b_0} \ll 1\;. \tag{4.18}$$

Here $a_0 \ll 1$ and $b_0 \gg 1$ are positive constants which will be chosen later. In (4.17), we use a cut-off function $\psi : (-\infty,1] \to \mathbb{R}$ which satisfies $\psi(y) = 0$ for $y < -1$, $\psi(-1) = \psi'(-1) = 0$, $0 < \psi'(y) < \frac{1}{4}(1-\psi(y))$ for $|y| < 1$, $\psi(1) = 1 - e^{-1/4}$, and $\psi'(1) = \frac{1}{4}e^{-1/4}$. For definiteness, we can take

$$\psi(y) \;=\; \zeta_2(y+1)^2 - \zeta_3(y+1)^3 \quad \text{for} \quad |y| \le 1\;,$$

where $\zeta_2 = \frac{3}{4} - \frac{7}{8}e^{-1/4} \approx 0.068$ and $\zeta_3 = \frac{1}{4} - \frac{5}{16}e^{-1/4} \approx 0.0066$. It follows from these definitions that the function $\xi \mapsto p_0(\xi,t)$ is piecewise smooth and nondecreasing along rays. Moreover, we have

$$\nabla p_0(\xi,t) \;=\; \begin{cases} \frac{\xi}{2}\, e^{|\xi|^2/4}\Bigl(1-\tilde\psi(|\xi|^2-a(t)^2)\Bigr) & \text{if} \quad 0 \le |\xi|^2 \le a(t)^2{+}1\;, \\ 0 & \text{if} \quad a(t)^2{+}1 \le |\xi|^2 \le b(t)^2\;, \\ \frac{\beta}{4}\frac{\xi}{|\xi|}\, e^{\beta|\xi|/4} & \text{if} \quad |\xi|^2 \ge b(t)^2\;, \end{cases} \tag{4.19}$$

where $\tilde\psi(y) = \psi(y) + 4\psi'(y)$. The graph of the function $p_0(\xi,t)$ is depicted in Fig. 1.

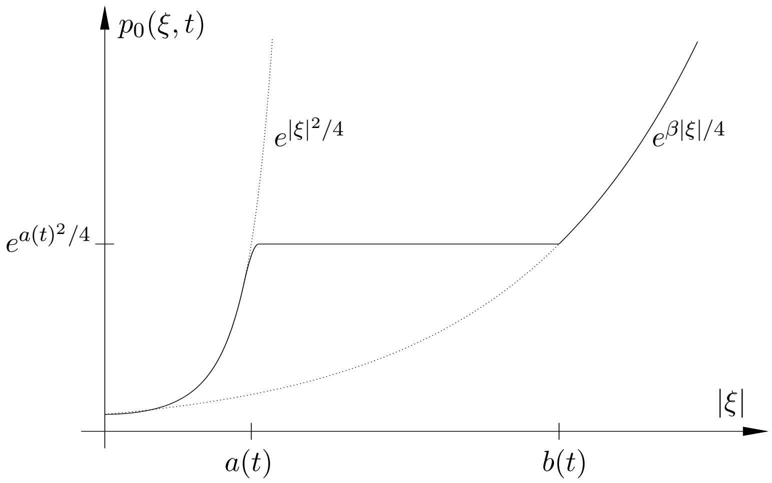

Fig. 1: The radially symmetric, time-dependent weight $p_0(\xi,t)$ is represented as a function of $|\xi|$. The gradient of $p_0(\xi,t)$ has a jump discontinuity at $|\xi| = b(t)$, but is Lipschitz continuous near $|\xi| = a(t)$.

Next, for each $i \in \{1,\dots,N\}$, we define

$$p_i(\xi,t) \;=\; p_0(\xi,t) + \Bigl(\frac{\nu t}{d^2}\Bigr) q_i(\xi,t)\;, \tag{4.20}$$

where the correction $q_i(\xi,t)$ vanishes identically when $|\xi|^2 \ge a(t)^2 + 1$ and satisfies

$$v^G(\xi)\cdot\nabla q_i(\xi,t) \;=\; \frac{1}{\pi}\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\frac{d^2}{|z_{ij}(t)|^4}(\xi\cdot z_{ij}(t))(\nabla p_0(\xi,t)\cdot z_{ij}(t)^\perp)\;, \tag{4.21}$$

for $|\xi|^2 \le a(t)^2 + 1$. Proceeding as in Section 3, it is easy to obtain the explicit expression

$$q_i(\xi,t) \;=\; -\frac{1}{2}\,\frac{\xi\cdot\nabla p_0(\xi,t)}{1-e^{-|\xi|^2/4}}\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{d^2}{|z_{ij}(t)|^4}\Bigl((\xi\cdot z_{ij}(t))^2-(\xi\cdot z_{ij}(t)^\perp)^2\Bigr)\;, \tag{4.22}$$

which is in fact valid for $|\xi| \le b(t)$, because $\nabla p_0(\xi,t)$ vanishes when $a(t)^2+1 \le |\xi|^2 \le b(t)^2$. It follows from (4.22) that

$$(1+|\xi|^2)|q_i(\xi,t)| + |\xi||\nabla q_i(\xi,t)| + |t\partial_t q_i(\xi,t)| \;\le\; C|\xi|^2(1+|\xi|^2)^2 p_0(\xi,t)\;, \tag{4.23}$$

whenever $|\xi|^2 \le a(t)^2+1$. Using (4.20) and the definition (4.18) of $a(t)$, we deduce from (4.23) that $|p_i(\xi,t)-p_0(\xi,t)| \le Ca_0^4\,p_0(\xi,t)$ for some $C>0$. Thus, if we choose the constant $a_0>0$ sufficiently small, we see that the weight $p_i(\xi,t)$ is a very small perturbation of $p_0(\xi,t)$ for all $\xi\in\mathbb{R}^2$ and all $t\in(0,T]$. In particular, for $i=0,\dots,N$, we have the uniform bounds

$$\frac{1}{2}\,e^{\beta|\xi|/4} \;\le\; p_i(\xi,t) \;\le\; 2\,e^{|\xi|^2/4}\;, \qquad \xi\in\mathbb{R}^2\;, \quad t\in(0,T]\;. \tag{4.24}$$

4.2 Energy estimates

Now that we have defined appropriate weights $p_1(\xi,t),\dots,p_N(\xi,t)$, it remains to control the evolution of the energy functional $E(t)$ introduced in (4.8). Here and in what follows, we always assume that the parameters in (4.18) satisfy $a_0 \ll 1$, $b_0 \gg 1$, and that the viscosity $\nu>0$ is small enough so that $\nu T \ll d^2$.

Proposition 4.1 *There exist positive constants ϵ_0, ϵ_1 and κ_1, κ_2, κ_3, depending only on the ratio T/T_0, such that, if $a_0 = b_0^{-1} = \epsilon_0$ and $\nu T/d^2 \le \epsilon_1$, and if $\tilde w = (\tilde w_1,\dots,\tilde w_N) \in C^0((0,T],Y^N)$ is any solution of system* (4.2)–(4.6)*, then the energy $E(t)$ defined in* (4.8) *satisfies*

$$tE'(t) \;\le\; -\kappa_1 E(t) + \kappa_2\frac{t}{T_0}\Bigl(E(t)+E(t)^3\Bigr) + \kappa_3\Bigl(\frac{\nu t}{d^2}\Bigr)\;, \qquad 0<t\le T\;. \tag{4.25}$$

Proof. Differentiating (4.8) with respect to time and using (4.2)–(4.6), we find

$$tE'(t) \;=\; \sum_{i=1}^N \int_{\mathbb{R}^2}\Bigl\{\frac{1}{2}\,t\partial_t p_i(\xi,t)|\tilde w_i(\xi,t)|^2 + p_i(\xi,t)\tilde w_i(\xi,t)\,t\partial_t\tilde w_i(\xi,t)\Bigr\}\,\mathrm{d}\xi \;=\; \sum_{k=1}^6 \mathcal{E}_k(t)\;, \tag{4.26}$$

where

$$\mathcal{E}_1(t) \;=\; \sum_{i=1}^N\int_{\mathbb{R}^2}\Bigl\{\frac{1}{2}\,t\partial_t p_i(\xi,t)|\tilde w_i(\xi,t)|^2 + p_i(\xi,t)\tilde w_i(\xi,t)(\mathcal{L}\tilde w_i(\xi,t)-\tilde w_i(\xi,t))\Bigr\}\,\mathrm{d}\xi\;,$$

$$\mathcal{E}_2(t) \;=\; -\sum_{i=1}^N\int_{\mathbb{R}^2} p_i(\xi,t)\tilde w_i(\xi,t)\frac{\alpha_i}{\nu}\Bigl(v_i^{\rm app}(\xi,t)\cdot\nabla\tilde w_i(\xi,t) + \tilde v_i(\xi,t)\cdot\nabla w_i^{\rm app}(\xi,t)\Bigr)\,\mathrm{d}\xi\;,$$

$$\mathcal{E}_3(t) \;=\; -\sum_{i=1}^N\int_{\mathbb{R}^2} p_i(\xi,t)\tilde w_i(\xi,t)\sum_{j\neq i}\frac{\alpha_j}{\nu}\Bigl\{v_j^{\rm app}\Bigl(\xi+\frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\Bigr\}\cdot\nabla\tilde w_i(\xi,t)\,\mathrm{d}\xi\;,$$

$$\mathcal{E}_4(t) \;=\; -\sum_{i=1}^N\int_{\mathbb{R}^2} p_i(\xi,t)\tilde w_i(\xi,t)\sum_{j\neq i}\frac{\alpha_j}{\nu}\,\tilde v_j\Bigl(\xi+\frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Bigr)\cdot\nabla w_i^{\rm app}(\xi,t)\,\mathrm{d}\xi\;,$$

$$\mathcal{E}_5(t) \;=\; -\sum_{i=1}^N \int_{\mathbb{R}^2} p_i(\xi,t)\tilde{w}_i(\xi,t)\sum_{j=1}^N \frac{\alpha_j t}{d^2}\,\tilde{v}_j\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Bigr)\cdot\nabla\tilde{w}_i(\xi,t)\,\mathrm{d}\xi\;,$$

$$\mathcal{E}_6(t) \;=\; -\sum_{i=1}^N \int_{\mathbb{R}^2} p_i(\xi,t)\tilde{w}_i(\xi,t)\tilde{R}_i(\xi,t)\,\mathrm{d}\xi\;.$$

The general strategy is to control the contributions of $\mathcal{E}_2(t),\dots,\mathcal{E}_6(t)$ using the negative terms contained in $\mathcal{E}_1(t)$. We shall proceed in six steps, each one being devoted to the detailed analysis of a specific term. Very often, we will have to consider separately the three cases $|\xi|^2 \le a(t)^2+1$, $a(t)^2+1 \le |\xi|^2 \le b(t)^2$, and $|\xi| \ge b(t)$, see (4.17). For simplicity, these three domains in $\mathbb{R}^2$ will be denoted by "region I", "region II", and "region III", respectively.

Step 1: *Diffusive terms*
Our goal is to show that there exists a constant $\kappa > 0$ such that

$$\mathcal{E}_1(t) \;\le\; -\kappa\sum_{i=1}^N \Bigl(\mathcal{I}_i(t) + \mathcal{J}_i(t) + \mathcal{K}_i(t)\Bigr)\;, \tag{4.27}$$

where

$$\mathcal{I}_i(t) \;=\; \int_{\mathbb{R}^2} p_0(\xi,t)|\nabla\tilde{w}_i(\xi,t)|^2\,\mathrm{d}\xi\;,\qquad \mathcal{J}_i(t) \;=\; \int_{\mathbb{R}^2} p_0(\xi,t)\,\chi(\xi,t)|\tilde{w}_i(\xi,t)|^2\,\mathrm{d}\xi\;,$$
$$\mathcal{K}_i(t) \;=\; \int_{\mathbb{R}^2} p_0(\xi,t)|\tilde{w}_i(\xi,t)|^2\,\mathrm{d}\xi\;. \tag{4.28}$$

Here $\chi(\xi,t)$ is the continuous, radially symmetric function defined by

$$\chi(\xi,t) \;=\; \begin{cases} |\xi|^2 & \text{if} \quad 0 \le |\xi| \le a(t)\;, \\ a(t)^2 & \text{if} \quad a(t) \le |\xi| \le b(t)\;, \\ \beta|\xi| & \text{if} \quad |\xi| \ge b(t)\;. \end{cases}$$

To prove (4.27), we start from the identity

$$\int_{\mathbb{R}^2} p_i\tilde{w}_i(\mathcal{L}\tilde{w}_i - \tilde{w}_i)\,\mathrm{d}\xi \;=\; -\int_{\mathbb{R}^2}\Bigl(p_i|\nabla\tilde{w}_i|^2 + \tilde{w}_i(\nabla p_i\cdot\nabla\tilde{w}_i) + \frac{1}{4}(\xi\cdot\nabla p_i)|\tilde{w}_i|^2 + \frac{1}{2}p_i|\tilde{w}_i|^2\Bigr)\,\mathrm{d}\xi\;,$$

which is easily obtained using (3.2) and integrating by parts. Since

$$\Bigl|\int_{\mathbb{R}^2}\tilde{w}_i(\nabla p_i\cdot\nabla\tilde{w}_i)\,\mathrm{d}\xi\Bigr| \;\le\; \frac{3}{4}\int_{\mathbb{R}^2} p_i|\nabla\tilde{w}_i|^2\,\mathrm{d}\xi + \frac{1}{3}\int_{\mathbb{R}^2}\frac{|\nabla p_i|^2}{p_i}\,|\tilde{w}_i|^2\,\mathrm{d}\xi\;,$$

we see that $\mathcal{E}_1(t) \le \tilde{\mathcal{E}}_1(t)$, where

$$\tilde{\mathcal{E}}_1(t) \;=\; -\sum_{i=1}^N \int_{\mathbb{R}^2}\Bigl(\frac{1}{4}p_i(\xi,t)|\nabla\tilde{w}_i(\xi,t)|^2 + p_i(\xi,t)\tilde{\chi}(\xi,t)|\tilde{w}_i(\xi,t)|^2 + \frac{1}{2}p_i(\xi,t)|\tilde{w}_i(\xi,t)|^2\Bigr)\,\mathrm{d}\xi\;,$$

and

$$\tilde{\chi}(\xi,t) \;=\; \frac{\xi\cdot\nabla p_i(\xi,t)}{4p_i(\xi,t)} - \frac{|\nabla p_i(\xi,t)|^2}{3p_i(\xi,t)^2} - \frac{t\partial_t p_i(\xi,t)}{2p_i(\xi,t)}\;.$$

We already observed that $p_i(\xi,t) \ge \frac{1}{2}p_0(\xi,t)$ if a_0 is sufficiently small. So, to prove (4.27), it remains to show that $\tilde{\chi}(\xi,t) \ge C\chi(\xi,t)$ for some $C > 0$. This is easily verified in regions II and III, because $p_i(\xi,t) = p_0(\xi,t)$ in these regions. From (4.17), we find by a direct calculation

$$\tilde{\chi}(\xi,t) \;=\; \begin{cases} \dfrac{a(t)^2}{16} & \text{if} \quad a(t)^2+1 \le |\xi|^2 \le b(t)^2\;, \\ \dfrac{\beta}{16}|\xi| - \dfrac{\beta^2}{48} & \text{if} \quad |\xi| \ge b(t)\;. \end{cases}$$

In region I, we first compute the contributions of the radially symmetric weight $p_0(\xi,t)$ to the function $\tilde\chi(\xi,t)$. Using (4.19), we obtain

$$\tilde\chi_0(\xi,t) \;=\; \frac{|\xi|^2}{8}\frac{1-\tilde\psi(|\xi|^2{-}a^2)}{1-\psi(|\xi|^2{-}a^2)} - \frac{|\xi|^2}{12}\Bigl(\frac{1-\tilde\psi(|\xi|^2{-}a^2)}{1-\psi(|\xi|^2{-}a^2)}\Bigr)^2 + \frac{a(t)^2}{4}\frac{\psi'(|\xi|^2{-}a^2)}{1-\psi(|\xi|^2{-}a^2)}~.$$

In particular, we see that $\tilde\chi_0(\xi,t) = |\xi|^2/(24)$ if $|\xi|^2 \le a(t)^2-1$. If $y = |\xi|^2 - a(t)^2 \in [-1,1]$, we use the elementary bounds

$$0 \le \frac{1-\tilde\psi(y)}{1-\psi(y)} \le 1~, \quad \text{and} \quad \frac{\frac{1}{24}(1-\tilde\psi(y)) + \frac14\psi'(y)}{1-\psi(y)} \ge \delta > 0~,$$

which follow from the definition of ψ, and we deduce that $\tilde\chi_0(\xi,t) \ge \delta\min\{|\xi|^2, a(t)^2\}$ for some $\delta > 0$. Summarizing, we have shown that there exists $C_0 > 0$ such that $\tilde\chi_0(\xi,t) \ge C_0|\xi|^2$ in region I. On the other hand, using (4.23), it is straightforward to verify that $\tilde\chi(\xi,t) \ge \tilde\chi_0(\xi,t) - C_1 a_0^4|\xi|^2$ when $|\xi|^2 \le a(t)^2+1$. Thus, if the constant a_0 in (4.18) is sufficiently small, there exists $C_2 > 0$ such that $\tilde\chi(\xi,t) \ge C_2\chi(\xi,t)$ for all $\xi \in \mathbb{R}^2$ and all $t \in (0,T]$. This concludes the proof of (4.27).

Step 2: *Self-interaction terms*
We next consider the term $\mathcal{E}_2(t)$ in (4.26). To simplify the notations, we rewrite (3.34) in the form

$$w_i^{\mathrm{app}}(\xi,t) \;=\; G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr)\mathcal{F}_i(\xi,t)~, \qquad v_i^{\mathrm{app}}(\xi,t) \;=\; v^G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr)v^{\mathcal{F}_i}(\xi,t)~, \tag{4.29}$$

where $\mathcal{F}_i(\xi,t) = F_i(\xi,t) + (\nu t/d^2)^{1/2}H_i(\xi,t) + (\nu t/d^2)K_i(\xi,t)$, and $v^{\mathcal{F}_i}$ is the velocity field obtained from $\mathcal{F}_i$ via the Biot-Savart law (1.3). We thus have $\mathcal{E}_2(t) = \Omega_1(t) + \cdots + \Omega_4(t)$, where

$$\begin{aligned}
\Omega_1(t) \;&=\; -\sum_{i=1}^N \frac{\alpha_i}{\nu}\int_{\mathbb{R}^2} p_i(\xi,t)\tilde w_i(\xi,t)(v^G(\xi)\cdot\nabla\tilde w_i(\xi,t))\,\mathrm{d}\xi~,\\
\Omega_2(t) \;&=\; -\sum_{i=1}^N \frac{\alpha_i}{\nu}\int_{\mathbb{R}^2} p_0(\xi,t)\tilde w_i(\xi,t)(\tilde v_i(\xi,t)\cdot\nabla G(\xi))\,\mathrm{d}\xi~,\\
\Omega_3(t) \;&=\; -\sum_{i=1}^N \frac{\alpha_i t}{d^2}\int_{\mathbb{R}^2} q_i(\xi,t)\tilde w_i(\xi,t)(\tilde v_i(\xi,t)\cdot\nabla G(\xi))\,\mathrm{d}\xi~,\\
\Omega_4(t) \;&=\; -\sum_{i=1}^N \frac{\alpha_i t}{d^2}\int_{\mathbb{R}^2} p_i(\xi,t)\tilde w_i(\xi,t)\Bigl(v^{\mathcal{F}_i}(\xi,t)\cdot\nabla\tilde w_i(\xi,t) + \tilde v_i(\xi,t)\cdot\nabla\mathcal{F}_i(\xi,t)\Bigr)\,\mathrm{d}\xi~.
\end{aligned}$$

To prove that $\Omega_1(t)$ has a finite limit as $\nu \to 0$, we integrate by parts and use the fact that $v^G\cdot\nabla p_0 = 0$ because the weight p_0 is radially symmetric. In view of (4.20), we find

$$\Omega_1(t) \;=\; \sum_{i=1}^N \frac{\alpha_i}{2\nu}\int_{\mathbb{R}^2}|\tilde w_i(\xi,t)|^2(v^G(\xi)\cdot\nabla p_i(\xi,t))\,\mathrm{d}\xi \;=\; \sum_{i=1}^N \frac{\alpha_i t}{2d^2}\int_{\mathbb{R}^2}|\tilde w_i(\xi,t)|^2(v^G(\xi)\cdot\nabla q_i(\xi,t))\,\mathrm{d}\xi~.$$

This term cannot be controlled by (4.27), unless $t \ll T_0$, but it will exactly compensate for another term coming from $\mathcal{E}_3(t)$. As was already explained, this is precisely the reason why the correction $(\nu t/d^2)q_i(\xi,t)$ was added to the weight $p_0(\xi,t)$. To treat Ω_2, we use the fact that the linear operator $\tilde w_i \mapsto \tilde v_i\cdot\nabla G$ (where $\tilde v_i$ is obtained from $\tilde w_i$ via the Biot-Savart law) is skew-symmetric in the function space Y defined by (3.17), see [20]. We thus have

$$
\begin{aligned}
|\Omega_2(t)| &= \left|\sum_{i=1}^N \frac{\alpha_i}{\nu}\int_{\mathbb{R}^2}\Bigl(e^{|\xi|^2/4} - p_0(\xi,t)\Bigr)\tilde{w}_i(\xi,t)(\tilde{v}_i(\xi,t)\cdot\nabla G(\xi))\,\mathrm{d}\xi\right| \\
&\le \sum_{i=1}^N \frac{|\alpha_i|}{\nu}\int_{|\xi|^2\ge a(t)^2-1} e^{|\xi|^2/4}|\tilde{w}_i(\xi,t)|\,|\tilde{v}_i(\xi,t)|\,|\nabla G(\xi)|\,\mathrm{d}\xi \\
&\le C\sum_{i=1}^N \frac{|\alpha_i|}{\nu}\int_{|\xi|^2\ge a(t)^2-1} e^{\beta|\xi|/8}|\tilde{w}_i(\xi,t)|\,|\tilde{v}_i(\xi,t)|\,|\xi|e^{-\beta|\xi|/8}\,\mathrm{d}\xi\ .
\end{aligned}
$$

In the last line, we have used the definition (1.7) of G. To estimate the last integral, we apply the trilinear Hölder inequality with exponents 2, 4, 4. We have $\|e^{\beta|\xi|/8}\tilde{w}_i\|_{L^2} \le C\mathcal{K}_i^{1/2}$ by (4.24), (4.28), and $\|\tilde{v}_i\|_{L^4} \le C\|\tilde{w}_i\|_{L^{4/3}} \le C\mathcal{K}_i^{1/2}$ (see e.g. [19]). Moreover, by direct calculation,

$$
\Bigl(\int_{|\xi|^2\ge a(t)^2-1}|\xi|^4\, e^{-\beta|\xi|/2}\,\mathrm{d}\xi\Bigr)^{1/4} \le C\, a(t)^{3/2}\,\frac{e^{-\beta a(t)/8}}{(\beta a(t))^{1/4}}\ ,
$$

provided that $\beta a(t) \ge 1$. Thus, if the constant ϵ_1 in Proposition 4.1 is sufficiently small (depending on ϵ_0 and T/T_0), we find

$$
|\Omega_2(t)| \le C\sum_{i=1}^N \frac{|\alpha_i|}{\nu}\,\mathcal{K}_i(t)\, a(t)^{3/2}\, e^{-\beta a(t)/8} \le \epsilon\sum_{i=1}^N \mathcal{K}_i(t)\ .
$$

Here and in what follows, ϵ denotes a positive constant which can be made arbitrarily small by an appropriate choice of ϵ_0 and ϵ_1. Using similar estimates, it is also easy to bound the regular terms Ω_3 and Ω_4. We find

$$
\begin{aligned}
|\Omega_3(t)| &\le C\sum_{i=1}^N \frac{|\alpha_i|t}{d^2}\int_{\mathbb{R}^2} p_0(\xi,t)|\xi|^2(1+|\xi|^2)|\tilde{w}_i(\xi,t)|\,|\tilde{v}_i(\xi,t)|\,|\nabla G(\xi)|\,\mathrm{d}\xi \le C\frac{t}{T_0}\sum_{i=1}^N \mathcal{K}_i(t)\ , \\
|\Omega_4(t)| &\le C\sum_{i=1}^N \frac{|\alpha_i|t}{d^2}\int_{\mathbb{R}^2} p_0(\xi,t)|\tilde{w}_i(\xi,t)|\Bigl(|v^{\mathcal{F}_i}(\xi,t)|\,|\nabla\tilde{w}_i(\xi,t)| + |\tilde{v}_i(\xi,t)|\,|\nabla\mathcal{F}_i(\xi,t)|\Bigr)\,\mathrm{d}\xi \\
&\le C\sum_{i=1}^N \frac{|\alpha_i|t}{d^2}\Bigl(\mathcal{K}_i(t)^{1/2}\mathcal{I}_i(t)^{1/2} + \mathcal{K}_i(t)\Bigr) \le \epsilon\sum_{i=1}^N \mathcal{I}_i(t) + C\frac{t}{T_0}\sum_{i=1}^N \mathcal{K}_i(t)\ .
\end{aligned}
$$

Step 3: *Advection terms*
Using (4.29) and integrating by parts, we write $\mathcal{E}_3(t) = \Psi_1(t) + \Psi_2(t) + \Psi_3(t)$, where

$$
\begin{aligned}
\Psi_1(t) &= \sum_{i=1}^N \int_{\mathbb{R}^2}|\tilde{w}_i(\xi,t)|^2\,\nabla p_0(\xi,t)\cdot\sum_{j\ne i}\frac{\alpha_j}{2\nu}\left\{v^G\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\right\}\mathrm{d}\xi\ , \\
\Psi_2(t) &= \sum_{i=1}^N \int_{\mathbb{R}^2}|\tilde{w}_i(\xi,t)|^2\,\nabla q_i(\xi,t)\cdot\sum_{j\ne i}\frac{\alpha_j t}{2d^2}\left\{v^G\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr) - v^G\Bigl(\frac{z_{ij}(t)}{\sqrt{\nu t}}\Bigr)\right\}\mathrm{d}\xi\ , \\
\Psi_3(t) &= \sum_{i=1}^N \int_{\mathbb{R}^2}|\tilde{w}_i(\xi,t)|^2\,\nabla p_i(\xi,t)\cdot\sum_{j\ne i}\frac{\alpha_j t}{2d^2}\,v^{\mathcal{F}_j}\Bigl(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\ ,\,t\Bigr)\,\mathrm{d}\xi\ .
\end{aligned}
$$

By construction, all three integrands vanish for $a(t)^2+1 \le |\xi|^2 \le b(t)^2$, so we need only consider regions I and III. The most important term is the contribution of region I to $\Psi_1(t)$, which we denote by $\Psi_1^I(t)$. Using (3.54) together with the identity

$$v^G(\xi)\cdot\nabla q_i(\xi,t) \;=\; -\nabla p_0(\xi,t)\cdot D_i(\xi,t)\;, \qquad |\xi|^2 \le a(t)^2+1\;,$$

which follows immediately from the definitions (3.41), (4.21), we find

$$\Psi_1^I(t) \;=\; -\Omega_1(t) + \frac{1}{2}\sum_{i=1}^N \int_{|\xi|^2\le a(t)^2+1} |\tilde{w}_i(\xi,t)|^2\,\nabla p_0(\xi,t)\cdot\Delta_i(\xi,t)\,\mathrm{d}\xi\;.$$

Thus, using the estimates on $\Delta_i(\xi,t)$ which follow from (3.55), (3.56), we obtain

$$\begin{aligned}|\Psi_1^I(t)+\Omega_1(t)| \;&\le\; C\sum_{i=1}^N\sum_{j\ne i}\frac{|\alpha_j|}{\nu}\int_{|\xi|^2\le a(t)^2+1} p_0(\xi,t)|\tilde{w}_i(\xi,t)|^2|\xi|^3\Bigl(\frac{\nu t}{d^2}\Bigr)^{3/2}\mathrm{d}\xi\\ &\le\; C\frac{|\alpha|t}{d^2}\,a(t)\Bigl(\frac{\nu t}{d^2}\Bigr)^{1/2}\sum_{i=1}^N \mathcal{J}_i(t) \;\le\; \epsilon\sum_{i=1}^N\mathcal{J}_i(t)\;.\end{aligned}$$

On the other hand, the contribution of region III to $\Psi_1(t)$ can be estimated as follows

$$\begin{aligned}|\Psi_1^{III}(t)| \;&\le\; C\sum_{i=1}^N\sum_{j\ne i}\frac{|\alpha_j|}{\nu}\int_{|\xi|\ge b(t)} \beta p_0(\xi,t)|\tilde{w}_i(\xi,t)|^2\Bigl(\frac{\nu t}{d^2}\Bigr)^{1/2}\mathrm{d}\xi\\ &\le\; C\frac{|\alpha|t}{d^2}\,\frac{1}{b_0}\sum_{i=1}^N\int_{|\xi|\ge b(t)}\beta|\xi|p_0(\xi,t)|\tilde{w}_i(\xi,t)|^2\,\mathrm{d}\xi \;\le\; \epsilon\sum_{i=1}^N\mathcal{J}_i(t)\;,\end{aligned}$$

if the parameter $b_0>0$ is chosen sufficiently large. Similarly, using (4.23), we can bound $\Psi_2(t)$ in the following way

$$\begin{aligned}|\Psi_2(t)| \;&\le\; C\sum_{i=1}^N\sum_{j\ne i}\frac{|\alpha_j|t}{d^2}\int_{|\xi|^2\le a(t)^2+1} p_0(\xi,t)|\xi|^2(1+|\xi|^2)^2|\tilde{w}_i(\xi,t)|^2\Bigl(\frac{\nu t}{d^2}\Bigr)\mathrm{d}\xi\\ &\le\; C\frac{|\alpha|t}{d^2}\,a_0^4\sum_{i=1}^N\int_{|\xi|^2\le a(t)^2+1} p_0(\xi,t)|\xi|^2|\tilde{w}_i(\xi,t)|^2\,\mathrm{d}\xi \;\le\; \epsilon\sum_{i=1}^N\mathcal{J}_i(t)\;,\end{aligned}$$

if $a_0>0$ is sufficiently small. Finally, to bound $\Psi_3(t)$, we recall that $|v^{\mathcal{F}_i}(\xi,t)| \le C(1+|\xi|^2)^{-3/2}$, see (3.48), (3.49), (3.50). Since $|\xi+\frac{z_{ij}}{\sqrt{\nu t}}| \ge \frac{d}{2\sqrt{\nu t}}$ in regions I and III when $i\ne j$, we find

$$|\Psi_3(t)| \;\le\; C\sum_{i=1}^N\sum_{j\ne i}\frac{|\alpha_j|t}{d^2}\Bigl(\frac{\nu t}{d^2}\Bigr)^{3/2}\int_{\mathbb{R}^2}|\nabla p_0(\xi,t)||\tilde{w}_i(\xi,t)|^2\,\mathrm{d}\xi \;\le\; \epsilon\sum_{i=1}^N\mathcal{K}_i(t)\;.$$

Step 4: *Cross-interaction terms*
To bound $\mathcal{E}_4(t)$, we need a good estimate on the velocity field $\tilde{v}_i(\xi,t)$. We claim that there exists a constant $C>0$ such that

$$\|(1+|\xi|^2)\tilde{v}_i(\xi,t)\|_{L^\infty}^2 \;\le\; C\Bigl(\mathcal{I}_i(t)+\mathcal{K}_i(t)\Bigr)\;, \qquad i\in\{1,\dots,N\}\;. \tag{4.30}$$

Indeed, since $\int_{\mathbb{R}^2}\tilde{w}_i(\xi,t)\,\mathrm{d}\xi \;=\; 0$ for all $t\in(0,T]$, it follows from [19, Proposition B.1] that

$$\|(1+|\xi|^2)\tilde{v}_i\|_{L^\infty}^2 \;\le\; C\Bigl(\|(1+|\xi|^2)\tilde{w}_i\|_{L^4}+\|(1+|\xi|^2)\tilde{w}_i\|_{L^{4/3}}\Bigr)\;.$$

As $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ and $(1+|\xi|^2) \le Cp_0(\xi,t)$, we have $\|(1+|\xi|^2)\tilde{w}_i\|_{L^4} \le C(\mathcal{I}_i^{1/2} + \mathcal{K}_i^{1/2})$. In addition, using Hölder's inequality, we find $\|(1+|\xi|^2)\tilde{w}_i\|_{L^{4/3}} \le C\mathcal{K}_i^{1/2}$. This proves (4.30).

The main contribution to $\mathcal{E}_4(t)$ comes from region I. Since $|\xi + \frac{z_{ij}}{\sqrt{\nu t}}| \ge \frac{d}{2\sqrt{\nu t}}$ when $i \neq j$ and $|\xi|^2 \le a(t)^2 + 1$, we obtain, using (4.30),

$$\begin{aligned} |\mathcal{E}_4^I(t)| \;&\le\; C\sum_{i=1}^N \sum_{j\neq i} \frac{|\alpha_j|t}{d^2}\Big(\mathcal{I}_j(t) + \mathcal{K}_j(t)\Big)^{1/2} \int_{|\xi|^2 \le a(t)^2+1} p_0(\xi,t)|\tilde{w}_i(\xi,t)||\nabla w_i^{\rm app}(\xi,t)| \,\mathrm{d}\xi \\ &\le\; C\frac{|\alpha|t}{d^2}\sum_{j=1}^N \Big(\mathcal{I}_j(t) + \mathcal{K}_j(t)\Big)^{1/2} \sum_{i=1}^N \mathcal{K}_i(t)^{1/2} \;\le\; \epsilon\sum_{i=1}^N \mathcal{I}_i(t) + C\frac{t}{T_0}\sum_{i=1}^N \mathcal{K}_i(t)\ . \end{aligned}$$

In regions II and III, the quantity $|\nabla w_i^{\rm app}(\xi,t)|$ is bounded by $Ce^{-\gamma|\xi|^2/4}$ for any $\gamma < 1$. Choosing $\gamma > 1/2$ and proceeding as in the second step, we easily find

$$|\mathcal{E}_4^{II}(t)| + |\mathcal{E}_4^{III}(t)| \;\le\; C\sum_{i=1}^N \sum_{j\neq i} \frac{|\alpha_j|}{\nu}\,\mathcal{K}_i(t)^{1/2}\mathcal{K}_j(t)^{1/2} e^{-(\gamma-\frac{1}{2})a(t)^2/4} \;\le\; \epsilon\sum_{i=1}^N \mathcal{K}_i(t)\ .$$

Step 5: *Nonlinear terms*
Instead of (4.30), we use here the simpler inequality

$$\|\tilde{v}_i\|_{L^\infty}^2 \;\le\; C\|\tilde{w}_i\|_{L^4}\|\tilde{w}_i\|_{L^{4/3}} \;\le\; C\Big(\mathcal{I}_i + \mathcal{K}_i\Big)^{1/2}\mathcal{K}_i^{1/2}\ ,$$

which follows from [19, Lemma 2.1]. Applying Hölder's inequality, we thus obtain

$$\begin{aligned} |\mathcal{E}_5(t)| \;&\le\; C\sum_{i,j=1}^N \frac{|\alpha_j|t}{d^2}\mathcal{K}_i(t)^{1/2}\Big(\mathcal{I}_j(t) + \mathcal{K}_j(t)\Big)^{1/4}\mathcal{K}_j(t)^{1/4}\mathcal{I}_i(t)^{1/2} \\ &\le\; \epsilon\sum_{i=1}^N \mathcal{I}_i(t) + C\frac{t}{T_0}\sum_{i=1}^N \Big(\mathcal{K}_i(t)^2 + \mathcal{K}_i(t)^3\Big)\ . \end{aligned}$$

Step 6: *Remainder terms*
Finally, using Hölder's inequality and estimate (4.7), we find

$$|\mathcal{E}_6(t)| \;\le\; C\sum_{i=1}^N \Big(\frac{\nu t}{d^2}\Big)^{1/2}\mathcal{K}_i(t)^{1/2} \;\le\; \epsilon\sum_{i=1}^N \mathcal{K}_i(t) + C\Big(\frac{\nu t}{d^2}\Big)\ .$$

Collecting the estimates established in Steps 1–6 and using the fact that $E(t) \approx \frac{1}{2}\sum_{i=1}^N \mathcal{K}_i(t)$, we easily obtain (4.25) provided that $\epsilon > 0$ is sufficiently small. As was already explained, this last condition is easy to fulfill if we choose the constants ϵ_0, ϵ_1 appropriately. This concludes the proof of Proposition 4.1. □

4.3 End of the proof of Theorem 2.5

It is now quite easy to conclude the proof of Theorem 2.5. If the solution $\omega^\nu(x,t)$ of (1.2) with initial data (2.1) is decomposed as in (2.6), we know from Lemma 2.2 that the rescaled vorticity profiles $w_i(\xi,t) \equiv w_i^\nu(\xi,t)$ defined by (2.13) satisfy $w_i(\cdot,t) \in Y$ for all $t \in (0,T]$, see (2.8). Standard parabolic estimates then imply that $w = (w_1,\dots,w_N) \in C^0((0,T],Y^N)$ is a solution of system (3.1), where the vortex positions $z_i(t) \equiv z_i^\nu(t)$ are given by (3.4). Moreover, we know from [16, Proposition 4.5] that $w_i(\cdot,t) \to G$ in Y as $t \to 0$, for all $i \in \{1,\dots,N\}$. In fact, in Section 6.3 of [16], this convergence is established in a polynomially weighted space only, but the proof also works (and is in fact simpler) in the Gaussian space Y. Using the approximate solution $w_i^{\mathrm{app}}(\xi,t)$ of (3.1) constructed in Section 3, we can even obtain the following improved estimate for short times:

Lemma 4.2 *For any fixed $\nu > 0$, one has $\|w_i(\cdot,t) - w_i^{\mathrm{app}}(\cdot,t)\|_Y = \mathcal{O}(t^{3/2})$ as $t \to 0$, for all $i \in \{1,\dots,N\}$.*

Proof. Let $\hat w_i(\xi,t) = w_i(\xi,t) - w_i^{\mathrm{app}}(\xi,t)$. Then $\hat w_i$ satisfies a system which is similar to (4.2)–(4.6), except that the first line (4.2) is replaced by $t\partial_t \hat w_i(\xi,t) - (\mathcal{L}\hat w_i)(\xi,t)$, and in the last line (4.6) the left-hand side becomes

$$\sum_{j=1}^N \frac{\alpha_j}{\nu}\,\hat v_j\Big(\xi + \frac{z_{ij}(t)}{\sqrt{\nu t}}\,,\,t\Big)\cdot\nabla\hat w_i(\xi,t) \,+\, R_i^{(3)}(\xi,t)\;, \tag{4.31}$$

where $R_i^{(3)}(\xi,t)$ satisfies (3.47). Note that the nonlinear terms in (4.31) are singular as $\nu \to 0$, but this is not a problem here because $\nu > 0$ is fixed. To estimate $\hat w_i(\cdot,t)$ in the space Y, we use the energy functional

$$\hat E(t) \,=\, \frac12\int_{\mathbb{R}^2} e^{|\xi|^2/4}\Big(|\hat w_1(\xi,t)|^2 + \dots + |\hat w_N(\xi,t)|^2\Big)\,\mathrm{d}\xi\;.$$

Repeating the proof of Proposition 4.1, with substantial simplifications, we obtain for small times a differential inequality of the form

$$t\hat E'(t) \,\le\, -\eta_1\hat E(t) + \eta_2\hat E(t)^3 + \eta_3\Big(\frac{\nu t}{d^2}\Big)^3\;, \tag{4.32}$$

where the positive constants η_1,η_2,η_3 may depend on ν. In the derivation of (4.32), the only new ingredient is the estimate

$$\int_{\mathbb{R}^2} e^{|\xi|^2/4}\,\hat w_i\mathcal{L}\hat w_i\,\mathrm{d}\xi \,\le\, -\kappa\int_{\mathbb{R}^2} e^{|\xi|^2/4}\Big(|\nabla\hat w_i|^2 + |\xi|^2|\hat w_i|^2 + |\hat w_i|^2\Big)\,\mathrm{d}\xi\;,$$

which holds for some $\kappa > 0$ because the self-adjoint operator $\mathcal{L}$ is strictly negative in the subspace of functions with zero mean, see [19]. Since we already know that $\hat E(t) \to 0$ as $t \to 0$, inequality (4.32) implies that $\hat E'(t) \le Ct^2$ for $t > 0$ sufficiently small, hence $\hat E(t) = \mathcal{O}(t^3)$ as $t \to 0$. This is the desired result. $\square$

We now consider the energy functional $E(t)$ defined by (4.8). Since $\hat w_i(\xi,t) = (\nu t/d^2)\tilde w_i(\xi,t)$, and since the weights $p_i(\xi,t)$ satisfy (4.24), it follows from Lemma 4.2 that $E(t) \to 0$ as $t \to 0$, for any fixed $\nu > 0$. As long as $E(t) \le 1$, we have by Proposition 4.1

$$tE'(t) \,\le\, 2\kappa_2\,\frac{t}{T_0}E(t) + \kappa_3\frac{\nu t}{d^2}\;,$$

hence

$$E(t) \,\le\, \kappa_3\int_0^t e^{2\kappa_2 s/T_0}\,\frac{\nu}{d^2}\,\mathrm{d}s \,\le\, \kappa_3\,e^{2\kappa_2 t/T_0}\,\frac{\nu t}{d^2}\;. \tag{4.33}$$

If we choose $\nu > 0$ sufficiently small so that

$$\kappa_3\,e^{2\kappa_2 T/T_0}\,\frac{\nu T}{d^2} \,\le\, 1\;,$$

we see that (4.33) holds for all $t \in (0,T]$. Since $p_i(\xi,t) \ge Ce^{\beta|\xi|/4}$ for all $\xi \in \mathbb{R}^2$ and all $t \in (0,T]$, we obtain (4.9), and (2.20) follows. The proof of Theorem 2.5 is thus complete. $\square$

5 Appendix

Proof of Lemma 2.2. Let $U : \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}^2$ be a smooth, divergence-free vector field, and fix $\nu > 0$. We assume that

$$\frac{1}{\nu}\sup_{t>0}\Bigl((\nu t)^{1/2}\|U(\cdot,t)\|_{L^\infty} + \|\Omega(\cdot,t)\|_{L^1}\Bigr) \,=\, C_0 \,<\, \infty\;, \tag{5.1}$$

where $\Omega = \partial_1 U_2 - \partial_2 U_1$. Then any solution of the linear equation $\partial_t\omega + (U\cdot\nabla)\omega = \nu\Delta\omega$ can be represented as

$$\omega(x,t) \,=\, \int_{\mathbb{R}^2}\Gamma_U^\nu(x,t;y,s)\omega(y,s)\,\mathrm{d}y\;, \quad x\in\mathbb{R}^2\;, \quad t>s>0\;, \tag{5.2}$$

where the *fundamental solution* $\Gamma_U^\nu(x,t;y,s)$ has the following properties:
1. For any $\beta\in(0,1)$, there exists $C_1 = C_1(\beta,C_0) > 0$ such that

$$0 \,<\, \Gamma_U^\nu(x,t;y,s) \,\le\, \frac{C_1}{\nu(t-s)}\exp\Bigl(-\beta\,\frac{|x-y|^2}{4\nu(t-s)}\Bigr)\;, \tag{5.3}$$

for all $x,y\in\mathbb{R}^2$ and all $t>s>0$. This very precise upper bound is due to E. Carlen and M. Loss [5].
2. There exists $\gamma = \gamma(C_0) > 0$ and, for any $\delta > 0$, there exists $C_2 = C_2(\delta,C_0) > 0$ such that

$$|\Gamma_U^\nu(x,t;y,s) - \Gamma_U^\nu(x',t';y',s')| \,\le\, C_2\Bigl(|x-x'|^\gamma + |t-t'|^{\gamma/2} + |y-y'|^\gamma + |s-s'|^{\gamma/2}\Bigr)\;, \tag{5.4}$$

whenever $t-s\ge\delta$ and $t'-s'\ge\delta$. This Hölder continuity property, which is due to H. Osada [47], implies in particular that $\Gamma_U^\nu(x,t;y,s)$ can be continuously extended up to $s=0$, and that this extension (which is still denoted by Γ_U^ν) satisfies (5.3) and (5.4) with $s=0$.
3. For all $x,y\in\mathbb{R}^2$ and all $t>s>0$, we have

$$\int_{\mathbb{R}^2}\Gamma_U^\nu(x,t;y,s)\,\mathrm{d}x \,=\, 1\;, \quad \text{and} \quad \int_{\mathbb{R}^2}\Gamma_U^\nu(x,t;y,s)\,\mathrm{d}y \,=\, 1\;. \tag{5.5}$$

Note that the first equality uses the fact that U is divergence-free.

We now consider the particular case where $\omega(x,t) = \omega^\nu(x,t)$ and $U(x,t) = u^\nu(x,t)$. Then $\partial_t\omega + (U\cdot\nabla)\omega = \nu\Delta\omega$ by construction, and the results established in [22] show that assumption (5.1) is satisfied with $C_0 = C|\alpha|/\nu$, where $C>0$ is a universal constant and $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$. Using the Hölder continuity (5.4) and the fact that $\omega^\nu(\cdot,t) \rightharpoonup \mu$ as $t\to 0$, we can take the limit $s\to 0$ in the representation (5.2) and obtain, for any $\nu>0$, the following expression

$$\omega^\nu(x,t) \,=\, \int_{\mathbb{R}^2}\Gamma_{u^\nu}^\nu(x,t;y,0)\,\mathrm{d}\mu_y \,=\, \sum_{i=1}^N \alpha_i\Gamma_{u^\nu}^\nu(x,t;x_i,0)\;. \tag{5.6}$$

Setting $\omega_i^\nu(x,t) = \alpha_i\Gamma_{u^\nu}^\nu(x,t;x_i,0)$, we obtain the desired decomposition (2.6), and the various properties of $\omega_i^\nu(x,t)$ follow directly from (5.3) and (5.5). □

Proof of Lemma 2.3. For simplicity, we set $z^\nu(t) = (z_1^\nu(t),\dots,z_N^\nu(t)) \in (\mathbb{R}^2)^N$ and we rewrite system (2.11) as $\dot z^\nu(t) = F(z^\nu(t),\nu t)$, where $F : (\mathbb{R}^2)^N\times(0,\infty)\to(\mathbb{R}^2)^N$ is defined by

$$F_i(z,\eta) \,=\, \sum_{j\neq i}\frac{\alpha_j}{\sqrt{\eta}}\,v^G\Bigl(\frac{z_i-z_j}{\sqrt{\eta}}\Bigr)\;, \qquad i\in\{1,\dots,N\}\;. \tag{5.7}$$

For any $\delta \ge 0$, we denote $\Omega_\delta = \{z \in (\mathbb{R}^2)^N\,;\, |z_i - z_j| > \delta \text{ for all } i \neq j\}$. Then F extends to a smooth map from $\Omega_0 \times [0,\infty)$ to $(\mathbb{R}^2)^N$, with

$$F_i(z,0) \;=\; \sum_{j\neq i} \frac{\alpha_j}{2\pi}\,\frac{(z_i - z_j)^\perp}{|z_i - z_j|^2}\;, \qquad i \in \{1,\dots,N\}\;, \tag{5.8}$$

This remark already implies that system (2.11) is locally well-posed for all initial data in Ω_0. Global well-posedness easily follows, because as soon as η is bounded away from zero, the vector field $z \mapsto F(z,\eta)$ is smooth and uniformly bounded.

We now compare the solutions of (2.11) in Ω_0 with those of system (2.3), which can be written as $\dot z(t) = F(z(t),0)$. If $z \in \Omega_\delta$ for some $\delta > 0$, then using (5.7), (5.8) and the definition (1.7) of v^G, we easily find

$$|F_i(z,\eta) - F_i(z,0)| \;\le\; \sum_{j\neq i} \frac{|\alpha_j|}{2\pi}\,\frac{1}{|z_i - z_j|}\,e^{-|z_i-z_j|^2/(4\eta)} \;\le\; \frac{|\alpha|}{2\pi\delta}\,e^{-\delta^2/(4\eta)}\;,$$

for any $\eta > 0$. Similarly, if $z, \tilde z \in \Omega_\delta$, then

$$|F_i(z,0) - F_i(\tilde z,0)| \;\le\; \sum_{j\neq i} \frac{|\alpha_j|}{\pi\delta^2}\,\max\{|z_1 - \tilde z_1|,\dots,|z_N - \tilde z_N|\} \;\le\; \frac{|\alpha|}{\pi\delta^2}\,\|z - \tilde z\|\;,$$

where $\|z - \tilde z\| = \max\{|z_1 - \tilde z_1|,\dots,|z_N - \tilde z_N|\}$.

Assume now that $z \in C^0([0,T],(\mathbb{R}^2)^N)$ is a solution of (2.3) satisfying (2.9) for some $d > 0$, and take $\delta \in (0,d)$. For any $\nu > 0$, let $z^\nu(t)$ denote the unique solution of (2.11) with initial data $z^\nu(0) = z(0)$. As long as $z^\nu(t)$ stays in Ω_δ, we have

$$\begin{aligned}\|\dot z^\nu(t) - \dot z(t)\| \;&\le\; \|F(z^\nu(t),\nu t) - F(z^\nu(t),0)\| + \|F(z^\nu(t),0) - F(z(t),0)\| \\ &\le\; \frac{|\alpha|}{2\pi\delta}\,e^{-\delta^2/(4\nu t)} + \frac{|\alpha|}{\pi\delta^2}\,\|z^\nu(t) - z(t)\|\;,\end{aligned}$$

hence

$$\|z^\nu(t) - z(t)\| \;\le\; \frac{|\alpha|}{2\pi\delta}\int_0^t e^{|\alpha|(t-s)/(\pi\delta^2)}\,e^{-\delta^2/(4\nu s)}\,\mathrm{d}s \;\le\; \frac{\delta}{2}\,e^{|\alpha|t/(\pi\delta^2)}\,e^{-\delta^2/(4\nu t)}\;. \tag{5.9}$$

If $\nu > 0$ is sufficiently small, this implies that $z^\nu(t) \in \Omega_\delta$ for all $t \in [0,T]$, hence (5.9) holds for $t \in [0,T]$. Choosing for instance $\delta = d\sqrt{4/5}$, we obtain (2.12) with $K_1 = \exp(CT/T_0)$. For larger values of ν, the solution $z^\nu(t)$ may leave Ω_δ, but in that case the bound (2.12) still holds if we take the constant K_1 large enough. $\square$

Proof of Lemma 3.2. Let $r = |\xi|/|\eta| < 1$, and $\psi = \theta - \phi$. We have

$$|\xi + \eta|^2 \;=\; |\eta|^2(1 + 2r\cos(\psi) + r^2) \;=\; |\eta|^2\,|1 + z|^2\;,$$

where $z = r\,e^{i\psi} \in \mathbb{C}$. Now

$$\begin{aligned}\frac{1}{|1+z|^2} \;&=\; \Bigl(1 - z + z^2 - z^3 + \dots\Bigr)\Bigl(1 - \bar z + \bar z^2 - \bar z^3 + \dots\Bigr) \\ &=\; 1 - (z + \bar z) + (z^2 + z\bar z + \bar z^2) - (z^3 + z^2\bar z + z\bar z^2 + \bar z^3) + \dots\;.\end{aligned}$$

But, for each $n \in \mathbb{N}$,

$$z^n + z^{n-1}\bar z + \dots + z\bar z^{n-1} + \bar z^n \;=\; \frac{z^{n+1} - \bar z^{n+1}}{z - \bar z} \;=\; r^n\,\frac{\sin((n+1)\psi)}{\sin(\psi)}\;,$$

if $\sin(\psi) \neq 0$. Thus

$$\frac{1}{|\xi+\eta|^2} - \frac{1}{|\eta|^2} \;=\; \frac{1}{|\eta|^2}\Big(\frac{1}{|1+z|^2} - 1\Big) \;=\; \frac{1}{|\eta|^2}\sum_{n=1}^{\infty}(-1)^n \frac{|\xi|^n}{|\eta|^n}\,\frac{\sin((n+1)\psi)}{\sin(\psi)}\;. \tag{5.10}$$

Multiplying the first and the last member of (5.10) by $\xi \cdot \eta^\perp = |\xi||\eta|\sin(\psi)$, we obtain (3.12). This concludes the proof. □

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References

[1] H. Abidi and R. Danchin. Optimal bounds for the inviscid limit of Navier-Stokes equations. *Asymptot. Anal.* **38** (2004), 35–46.

[2] Th. Beale and A. Majda. Rates of convergence for viscous splitting of the Navier-Stokes equations. *Math. Comp.* **37** (1981), 243–259.

[3] G. Benfatto, R. Esposito, and M. Pulvirenti. Planar Navier-Stokes flow for singular initial data. *Nonlinear Anal.* **9** (1985), 533–545.

[4] R. Caflisch and M. Sammartino. Vortex layers in the small viscosity limit. "WASCOM 2005"—13th Conference on Waves and Stability in Continuous Media, 59–70, World Sci. Publ., Hackensack, NJ, 2006.

[5] E. A. Carlen and M. Loss. Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier-Stokes equation. *Duke Math. J.*, **81**, 135–157 (1996), 1995.

[6] J.-Y. Chemin. A remark on the inviscid limit for two-dimensional incompressible fluids. *Comm. Partial Diff. Equations* **21** (1996), 1771–1779.

[7] P.-H. Chen and W.-L. Wang. Roll-up of a viscous vortex sheet. *J. Chinese Inst. Engrs.* **14** (1991), 507–517.

[8] P. Constantin and J. Wu. Inviscid limit for vortex patches. *Nonlinearity* **8** (1995), 735–742.

[9] P. Constantin and J. Wu. The inviscid limit for non-smooth vorticity. *Indiana Univ. Math. J.* **45** (1996), 67–81.

[10] G.-H. Cottet. Équations de Navier-Stokes dans le plan avec tourbillon initial mesure. *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 105–108.

[11] Y. Couder. Observation expérimentale de la turbulence bidimensionnelle dans un film liquide mince. *C. R. Acad. Sci. Paris* II **297** (1983), 641–645.

[12] R. Danchin. Poches de tourbillon visqueuses. *J. Math. Pures Appl.* **76** (1997), 609–647.

[13] R. Danchin. Persistance de structures géométriques et limite non visqueuse pour les fluides incompressibles en dimension quelconque. *Bull. Soc. Math. France* **127** (1999), 179–227.

[14] J.-M. Delort. Existence de nappes de tourbillon en dimension deux. *J. Amer. Math. Soc.* **4** (1991), 553–586.

[15] D. Ebin and J. Marsden. Groups of diffeomorphisms and the notion of an incompressible fluid. *Ann. of Math.* **92** (1970), 102–163.

[16] I. Gallagher and Th. Gallay. Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity. *Math. Ann.* **332** (2005), 287–327.

[17] I. Gallagher, Th. Gallay, and P.-L. Lions. On the uniqueness of the solution of the two-dimensional Navier-Stokes equation with a Dirac mass as initial vorticity. *Math. Nachr.* **278** (2005), 1665–1672.

[18] Th. Gallay. Equations de Navier-Stokes dans le plan avec tourbillon initial mesure. Séminaire EDP de l'Ecole Polytechnique 2003-2004, exposé n°XIV.

[19] Th. Gallay and C. E. Wayne. Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^2$. *Arch. Ration. Mech. Anal.* **163** (2002), 209–258.

[20] Th. Gallay and C.E. Wayne. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.* **255** (2005), 97–129.

[21] Th. Gallay and C.E. Wayne. Existence and stability of asymmetric Burgers vortices. *J. Math. Fluid Mech.* **9** (2007), 243–261.

[22] Y. Giga, T. Miyakawa, and H. Osada. Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.* **104** (1988), 223–250.

[23] E. Grenier. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53** (2000), 1067–1091.

[24] H. von Helmholtz. Über Integrale des hydrodynamischen Gleichungen, welche die Wirbelbewegungen entsprechen. *J. reine angew. Math.* **55** (1858), 25-55.

[25] T. Hmidi. Régularité höldérienne des poches de tourbillon visqueuses. *J. Math. Pures Appl.* **84** (2005), 1455–1495.

[26] T. Hmidi. Poches de tourbillon singulières dans un fluide faiblement visqueux. *Rev. Mat. Iberoamericana* **22** (2006), 489–543.

[27] T. Kato. Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. *J. Functional Analysis* **9** (1972), 296–305.

[28] T. Kato. The Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as the initial vorticity. *Differential Integral Equations* **7** (1994), 949–966.

[29] G. R. Kirchhoff. Vorlesungen über Mathematische Physik. Mekanik. Teubner, Leipzig, 1876.

[30] S. Le Dizès and A. Verga. Viscous interactions of two co-rotating vortices before merging. *J. Fluid Mech.* **467** (2002), 389–410.

[31] J.C. Mc Williams. The emergence of isolated coherent vortices in turbulent flow. *J. Fluid. Mech.* **146** (1984), 21–43.

[32] J.C. Mc Williams. The vortices of two-dimensional turbulence. *J. Fluid. Mech.* **219** (1990), 361–385.

[33] Y. Maekawa. Spectral properties of the linearization at the Burgers vortex in the high rotation limit. *J. Math. Fluid Mech.*, to appear.

[34] Y. Maekawa. On the existence of Burgers vortices for high Reynolds numbers. *J. Math. Analysis and Applications* **349** (2009), 181–200.

[35] Y. Maekawa. Existence of asymmetric Burgers vortices and their asymptotic behavior at large circulations. *Math. Models Methods Appl. Sci.* **19** (2009), 669–705.

[36] A. Majda. Remarks on weak solutions for vortex sheets with a distinguished sign. *Indiana Univ. Math. J.* **42** (1993), 921–939.

[37] C. Marchioro. Euler evolution for singular initial data and vortex theory: a global solution. *Comm. Math. Phys.* **116** (1988), 45–55.

[38] C. Marchioro. On the vanishing viscosity limit for two-dimensional Navier-Stokes equations with singular initial data. *Math. Methods Appl. Sci.* **12** (1990), 463–470.

[39] C. Marchioro. On the inviscid limit for a fluid with a concentrated vorticity. *Comm. Math. Phys.* **196** (1998), 53–65.

[40] C. Marchioro and M. Pulvirenti. Vortices and localization in Euler flows. *Comm. Math. Phys.* **154** (1993), 49–61.

[41] C. Marchioro and M. Pulvirenti. Mathematical theory of incompressible nonviscous fluids. Applied Mathematical Sciences **96**, Springer-Verlag, New York, 1994.

[42] N. Masmoudi. Remarks about the inviscid limit of the Navier-Stokes system. *Comm. Math. Phys.* **270** (2007), 777–788.

[43] P. Meunier, S. Le Dizès, and T. Leweke. Physics of vortex merging. *Comptes Rendus Physique* **6** (2005), 431–450.

[44] H. K. Moffatt, S. Kida, and K. Ohkitani. Stretched vortices—the sinews of turbulence; large-Reynolds-number asymptotics. *J. Fluid Mech.* **259** (1994), 241–264.

[45] R. Nagem, G. Sandri, D. Uminsky, and C.E. Wayne. Generalized Helmholtz-Kirchhoff model for two-dimensional distributed vortex motion. *SIAM J. Appl. Dyn. Syst.* **8** (2009), 160–179.

[46] P. Newton. The N-vortex problem. Analytical techniques. Applied Mathematical Sciences **145**, Springer-Verlag, New York, 2001.

[47] H. Osada. Diffusion processes with generators of generalized divergence form. *J. Math. Kyoto Univ.* **27** (1987), 597–619.

[48] M. Sammartino and R. Caflisch. Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* **192** (1998), 433–461. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.* **192** (1998), 463–491.

[49] F. Sueur. Vorticity internal transition layers for the Navier-Stokes equations, preprint (2008).

[50] H. Swann. The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in $\mathbb{R}^3$. *Trans. Amer. Math. Soc.* **157** (1971), 373–397.

[51] L. Ting and R. Klein. Viscous vortical flows. Lecture Notes in Physics **374**. Springer-Verlag, Berlin, 1991.

[52] L. Ting and C. Tung. Motion and decay of a vortex in a nonuniform stream. *Phys. Fluids* **8** (1965), 1039–1051.

[53] M. Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. Sci. Ecole Norm. Sup.* **32** (1999), 769–812.

[54] V. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.* **3** (1963), 1032–1066.

[55] V. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.* **2** (1995), 27–38.