Energy bounds for the two-dimensional Navier-Stokes equations in an infinite cylinder

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Abstract

We consider the incompressible Navier-Stokes equations in the cylinder $\mathbb{R} \times \mathbb{T}$, with no exterior forcing, and we investigate the long-time behavior of solutions arising from merely bounded initial data. Although we do not prove that such solutions stay uniformly bounded for all times, we show that they converge in an appropriate sense to the family of spatially homogeneous equilibria as $t \to \infty$. Convergence is uniform on compact subdomains, and holds for all times except on a sparse subset of the positive real axis. We also improve the known upper bound on the L^∞ norm of the solutions, although our results in this direction are not optimal. Our approach is based on a detailed study of the local energy dissipation in the system, in the spirit of a recent work devoted to a class of dissipative partial differential equations with a formal gradient structure [5].

Keywords: Navier-Stokes equations, localized energy estimates, extended dissipative system.
MSC(2010): 35Q30, 35B40, 76D05

1 Introduction

The aim of this paper is to give some insight into the intrinsic dynamics of the two-dimensional incompressible Navier-Stokes equations in an unbounded domain. We consider the situation where a viscous fluid evolves freely without being driven by any external force, so that the motion originates entirely from the initial data, and we aim at obtaining general informations on the long-time behavior of the system. If the fluid fills a bounded domain $\Omega \subset \mathbb{R}^2$ and satisfies no-slip boundary conditions, it is well-known that the velocity converges exponentially fast to zero as $t \to \infty$, and the long-time asymptotics can be accurately described [3]. In an unbounded domain such as the whole plane $\Omega = \mathbb{R}^2$, solutions with finite kinetic energy also converge to the uniform rest state [10], and the (algebraic) decay rate of the velocity can be specified under appropriate localization assumptions on the initial data [12, 15]. Similar results can be obtained for infinite-energy solutions if the vorticity of the fluid is integrable [7], in which case the velocity field decays to zero roughly like $|x|^{-1}$ as $|x| \to \infty$.

The problem is far more complicated if we consider the situation where the velocity field is merely bounded, or decays very slowly to zero at infinity. In that case the Cauchy problem for the Navier-Stokes equations is still globally well-posed [8, 11], but essentially nothing is known about the long-time behavior of the solutions. In fact, it is even unclear whether the L^∞ norm of the velocity field $u(\cdot, t)$ stays bounded for all times. For instance, if the fluid fills the whole plane $\mathbb{R}^2$, the best estimate we are aware of is due to Zelik [16], and takes the form $\|u(\cdot,t)\|_{L^\infty} \le C(1+t)^2$, but there is no reason to believe that such a bound is sharp. Although the question may look rather academic at first sight, we believe that it is important to understand the behavior of solutions that are just bounded, because they may exhibit a nontrivial dynamics that is created by the equation itself, through the initial data, and does not result from an artificial exterior forcing. As an intimately related problem, we would like to understand the dynamics of the Navier-Stokes equations in large but bounded domains, and to derive *uniformly local* energy estimates that are independent of the size of the domain, or of the total kinetic energy.

As a first step in this direction, following Afendikov and Mielke [1], we study in the present paper the situation where the fluid velocity $u(x,t) \in \mathbb{R}^2$ and the pressure $p(x,t) \in \mathbb{R}$ are *periodic* in one space direction. As is discussed in [1], this assumption considerably simplifies the dynamics of the Navier-Stokes equations, but the periodic setting still includes interesting and nontrivial examples, such as Kolmogorov flows. We thus consider the system

$$\partial_t u + (u\cdot\nabla)u \,=\, \Delta u - \nabla p\;, \qquad \operatorname{div} u \,=\, 0\;, \tag{1.1}$$

in the two-dimensional cylinder $\mathbb{O} = \mathbb{R}\times\mathbb{T}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $x = (x_1, x_2) \in \mathbb{O}$, the unbounded coordinate $x_1 \in \mathbb{R}$ will be referred to as the "horizontal variable" and $x_2 \in \mathbb{T}$ will be the "vertical variable", although no gravity is included in our model. For simplicity, all physical parameters, such as the kinematic viscosity and the fluid density, have been eliminated by rescaling. Eq. (1.1) is the system studied by Afendikov and Mielke in the particular case where no exterior force drives the fluid. Our goal is to sharpen the conclusions of [1] and to obtain more precise information on the long-time behavior of the solutions in that particular situation.

Since we want to consider solutions of (1.1) that are merely bounded, we suppose that the velocity field $u(\cdot,t)$ belongs for each $t \ge 0$ to the Banach space $\mathrm{BUC}(\mathbb{O})$ of all bounded and uniformly continuous functions $u : \mathbb{O} \to \mathbb{R}^2$ equipped with the supremum norm

$$||u||_{L^\infty(\mathbb{O})} \,=\, \sup_{x\in\mathbb{O}} |u(x)|\;, \qquad \text{where} \quad |u| = (u_1^2 + u_2^2)^{1/2}\;.$$

Given a velocity field $u \in \mathrm{BUC}(\mathbb{O})$, the associated pressure p is determined by solving the elliptic equation $-\Delta p = \operatorname{div}((u\cdot\nabla)u)$ with appropriate conditions at infinity. As is explained in [1], it is very important to specify the behavior of the pressure, because different choices lead to genuinely different dynamics. For flows that are not driven by a pressure gradient at infinity, the standard choice is

$$p \,=\, \sum_{k,\ell=1}^{2} R_k R_\ell u_k u_\ell\;, \tag{1.2}$$

see [8, 9, 1], where R_1, R_2 are the Riesz transforms in the cylinder $\mathbb{O} = \mathbb{R}\times\mathbb{T}$. For instance, if $u \in \mathrm{BUC}(\mathbb{O})$, the pressure p defined by (1.2) belongs to $\mathrm{BMO}(\mathbb{O})$, the space of all functions with bounded mean oscillation on $\mathbb{O}$. Taking (1.2) into account, the Navier-Stokes equations (1.1) can be written in the compact form $\partial_t u + \mathbb{P}(u\cdot\nabla)u = \Delta u$, where $\mathbb{P}$ is the Leray-Hopf projection defined by its matrix elements $\mathbb{P}_{jk} = \delta_{jk} + R_j R_k$, $1 \le j,k \le 2$.

The following result summarizes the main conclusions of [1] for flows without exterior forcing:

Theorem 1.1. **[1]** *For any initial data $u_0 \in \mathrm{BUC}(\mathbb{O})$ with $\operatorname{div} u_0 = 0$, the Navier-Stokes equations (1.1), (1.2) have a unique global solution $u \in C^0([0,+\infty), \mathrm{BUC}(\mathbb{O}))$ such that $u(0) = u_0$. Moreover, there exists $C > 0$ (depending on u_0) such that*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{O})} \,\le\, C(1+t)^{1/2}\;, \qquad \text{for all} \;\; t \ge 0\;. \tag{1.3}$$

An interesting open question is whether the solutions of (1.1), (1.2) given by Theorem 1.1 stay uniformly bounded for all times. In this direction, we just mention the following improvement of estimate (1.3), which comes as a byproduct of our analysis.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exists a constant $C > 0$ (depending on the initial data u_0) such that*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{O})} \,\le\, C(1+t)^{1/6}\;, \qquad \textit{for all} \;\; t \ge 0\;. \tag{1.4}$$

As is explained in Section 7 below, there are good reasons to believe that the bound (1.4) is not sharp either for large times, but this question will not be addressed here and we plan to come back to it in a future work [6]. For the time being, our main purpose is to show that it is possible to obtain rather precise information on the long-time dynamics of equation (1.1) even if uniform bounds on the solutions are not known a priori. To formulate our results, it is convenient to assume that the *mean horizontal flow* vanishes identically :

$$\langle u_1\rangle(x_1,t) \,:=\, \int_{\mathbb{T}} u_1(x_1,x_2,t)\,\mathrm{d}x_2 \,=\, 0\;, \qquad \text{for all} \;\; x_1 \in \mathbb{R}\,,\; t \ge 0\;. \tag{1.5}$$

This, however, does not restrict the generality, as can be seen by the following argument. Given a solution of (1.1), we define $m_1 = \langle u_1\rangle$ and $a = \langle u_1^2 + p\rangle$, where the brackets $\langle\cdot\rangle$ denote the vertical average, as in (1.5). Using the divergence-free condition and integrating by parts, it is easy to verify that $\partial_{x_1} m_1 = 0$, so that m_1 is a function of t only. On the other hand, using the first equation in (1.1) we find $\partial_t m_1 + \partial_{x_1} a = 0$, thus $\partial_{x_1} a$ is also a function of t only. If we assume that $u \in \mathrm{BUC}(\mathbb{O})$ and $p \in \mathrm{BMO}(\mathbb{O})$, as in Theorem 1.1, this implies that $\partial_{x_1} a = 0$, hence $m_1 = \langle u_1\rangle$ is a constant that can be computed from the initial data. We now define

$$\begin{pmatrix}\tilde u_1(x_1,x_2,t)\\ \tilde u_2(x_1,x_2,t)\end{pmatrix} \,=\, \begin{pmatrix}u_1(x_1+m_1t,x_2,t)-m_1\\ u_2(x_1+m_1t,x_2,t)\end{pmatrix}\;, \qquad \tilde p(x_1,x_2,t) \,=\, p(x_1+m_1t,x_2,t)\;,$$

for all $x \in \mathbb{O}$ and $t \ge 0$. By Galilean invariance, it is clear that $\tilde u, \tilde p$ solve the Navier-Stokes equation (1.1), and by construction the mean horizontal flow $\langle\tilde u_1\rangle$ vanishes identically.

The following theorem collects a few typical consequences of our main results, which will be formulated more precisely and in a greater generality in the subsequent sections.

Theorem 1.3. *Let u be a solution of the Navier-Stokes equations given by Theorem 1.1 and satisfying* (1.5)*. Then the following estimates hold :*

1) There exists $C > 0$ (depending only on u_0) such that, for all $T > 0$,

$$\sup_{x_1\in\mathbb{R}} \int_0^T\!\!\int_{\mathbb{T}} |u(x_1,x_2,t)|^2\,\mathrm{d}x_2\,\mathrm{d}t \,\le\, CT\;. \tag{1.6}$$

2) There exists $C > 0$ (depending only on u_0) such that, for all $T > 0$ and all $R > 0$,

$$\int_{-R}^R\!\int_{\mathbb{T}} |u(x_1,x_2,T)|^2\,\mathrm{d}x_2\,\mathrm{d}x_1 + \int_0^T\!\!\int_{-R}^R\!\int_{\mathbb{T}} |\nabla u(x_1,x_2,t)|^2\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t \,\le\, C(R+T^{1/2})\;. \tag{1.7}$$

3) For all $\epsilon > 0$ and all $R > 0$, we have

$$\frac{1}{T}\,\mathrm{meas}\Big\{t\in[0,T]\,\Big|\,\inf_{m\in\mathbb{R}}\,\sup_{|x_1|\le R}\,\sup_{x_2\in\mathbb{T}}|u(x_1,x_2,t)-(0,m)^{\mathrm{t}}| \ge \epsilon\Big\} \xrightarrow[T\to\infty]{} 0\;. \tag{1.8}$$

Before giving an idea of our general strategy, we briefly comment on the results summarized in Theorem 1.3. Estimate (1.6) shows that the "kinetic energy" $\langle\frac{1}{2}|u|^2\rangle$, computed at each point $x_1 \in \mathbb{R}$, behaves like a bounded quantity when averaged over time. This already indicates that a bound like (1.3) or (1.4) is necessarily pessimistic: if the quantity $\langle|u|^2\rangle$ is not uniformly bounded in time, it can reach large values only on a relatively small subset of the time interval, otherwise (1.6) would give a contradiction. In particular $\langle|u|^2\rangle$ cannot increase to infinity as $t \to \infty$ at any fixed point $x_1 \in \mathbb{R}$. Estimate (1.7) contains even more information, and for simplicity we only comment on the particular case where $R = T^{1/2}$, which is especially instructive. We first learn from (1.7) that the energy $\frac{1}{2}\langle|u|^2\rangle$ computed at any time $T > 0$ behaves like a bounded quantity when averaged over an interval of size $T^{1/2}$ in the horizontal variable. As before, this indicates that, for any fixed $T > 0$, the quantity $\langle|u|^2\rangle$ can reach large values only on relatively small spatial subdomains. Next, Eq. (1.7) shows that the energy dissipation $\langle|\nabla u|^2\rangle$ converges to zero as $T \to \infty$ when averaged over a horizontal interval of size $T^{1/2}$ and a time interval of size T. This information is new and valuable, even for solutions for which a uniform upper bound is known a priori. As a consequence of (1.7), we immediately see that the only time-independent or time-periodic solutions of the Navier-Stokes equations (1.1), (1.2) in $\mathrm{BUC}(\mathbb{O})$ are spatially homogeneous equilibria of the form $u = (m_1, m_2)^{\mathrm{t}}$, where $m_1, m_2 \in \mathbb{R}$ (of course, under assumption (1.5), we have $m_1 = 0$). Finally, it follows from (1.7) that any solution of (1.1), (1.2) in $\mathrm{BUC}(\mathbb{O})$ converges uniformly on compact subdomains to this family of equilibria, except perhaps on a sparse subset of the time axis. A simple version of this last statement is given in (1.8), and we refer to Corollary 8.4 below for a more precise and quantitative estimate.

Our analysis of the dynamics of the two-dimensional Navier-Stokes equations (1.1) is based on the following simple ideas. If $u(x,t), p(x,t)$ is a solution of (1.1), (1.2) given by Theorem 1.1, we introduce the *energy density* $e = \langle\frac{1}{2}|u|^2\rangle + 1$, the *inviscid flux* $h = \langle(\frac{1}{2}|u|^2 + p)u_1\rangle$, and the *energy dissipation rate* $d = \langle|\nabla u|^2\rangle$. More explicitly, for all $x_1 \in \mathbb{R}$ and all $t > 0$, we define

$$e(x_1,t) \,=\, \frac{1}{2}\int_{\mathbb{T}} |u(x_1,x_2,t)|^2 \,\mathrm{d}x_2 + 1 \;, \tag{1.9}$$

$$h(x_1,t) \,=\, \int_{\mathbb{T}} \Big(\frac{1}{2}|u(x_1,x_2,t)|^2 + p(x_1,x_2,t)\Big) u_1(x_1,x_2,t) \,\mathrm{d}x_2 \;, \tag{1.10}$$

$$d(x_1,t) \,=\, \int_{\mathbb{T}} |\nabla u(x_1,x_2,t)|^2 \,\mathrm{d}x_2 \;, \tag{1.11}$$

where $|\nabla u|^2 = |\partial_1 u_1|^2 + |\partial_2 u_1|^2 + |\partial_1 u_2|^2 + |\partial_2 u_2|^2$. Here and in the sequel, we denote $\partial_1 = \partial_{x_1}$ and $\partial_2 = \partial_{x_2}$ for simplicity. A straightforward calculation then shows that the energy density satisfies $\partial_t e + \partial_1 h \,=\, \partial_1^2 e - d$ for all $x_1 \in \mathbb{R}$ and $t > 0$. In particular, if we introduce the *total energy flux* $f = \partial_1 e - h$, we arrive at the energy balance equation

$$\partial_t e(x_1,t) \,=\, \partial_1 f(x_1,t) - d(x_1,t) \;, \qquad x_1 \in \mathbb{R} \;, \quad t > 0 \;, \tag{1.12}$$

which is the starting point of our approach.

At this point, we would like to mention that (1.11) is *not* the usual definition of the energy dissipation rate that can be found in textbooks of Fluid Mechanics. Indeed, energy is dissipated in viscous fluids due to internal friction, and the rate of dissipation is therefore proportional to $|D(u)|^2$ instead of $|\nabla u|^2$, where $D(u) = \nabla u + (\nabla u)^{\mathrm{t}}$ is the strain rate tensor. Albeit less natural from a physical point of view, the definition (1.11) is nevertheless more convenient for our purposes, because the energy dissipation rates then controls all first-order derivatives of u.

To exploit (1.12), it is necessary to bound the energy flux f in terms of e and d. In Section 3 below we show that, for any $t_0 > 0$, there exists a constant $C > 0$ such that

$$h(x_1,t)^2 \,\le\, C e(x_1,t) d(x_1,t) \;, \qquad x_1 \in \mathbb{R} \;, \quad t \ge t_0 \;. \tag{1.13}$$

This simple bound is obtained under the assumption that $e \geq 1$, and this is why we added a constant to the kinetic energy in the definition (1.9). On the other hand, using (1.9), (1.11) and the Cauchy-Schwarz inequality, we easily obtain $(\partial_1 e)^2 \leq 2ed$. Summarizing, there exists $\beta > 0$ such that

$$f(x_1,t)^2 \;\leq\; \beta e(x_1,t)d(x_1,t)\;, \qquad x_1 \in \mathbb{R}\;, \quad t \geq t_0\;. \tag{1.14}$$

This inequality is very important, because it shows that energy is necessarily dissipated in the system as soon as the flux f is nonzero. More precisely, if we have an upper bound on the energy density e, then (1.14) allows to quantify how much energy is dissipated during transport.

In a recent paper [5], we introduced the notion of an *extended dissipative system* in a rather general framework. Roughly speaking, this is a system in which one can define an energy density e, an energy flux f, and an energy dissipation rate d satisfying (1.14) and such that the energy balance (1.12) holds for all solutions, see Section 3 below for more details. Under these assumptions, we established in [5] a few general results which impose rather severe constraints to the dynamics of the system. For instance, nontrivial time-periodic orbits cannot exist, and any global solution with uniformly bounded energy density converges, in a suitable localized and averaged sense, to the set of equilibria. Unfortunately, the results of [5] do not apply directly to the Navier-Stokes equations (1.1), (1.2), because we do not know a priori if the energy density (1.9) stays uniformly bounded for all times, see the discussion near Theorem 1.2 above. The purpose of the present paper is precisely to extend the techniques developed in [5] so as to cover the important case of the Navier-Stokes equations. The main new ingredient is the estimate $(\partial_1 e)^2 \leq 2ed$, which holds in the present case but was not included in our abstract definition of an extended dissipative system (because it is not satisfied in some other important examples). When combined with (1.14), this estimate allows to obtain convergence results that are exactly as accurate as those derived in [5] for uniformly bounded solutions of general extended dissipative systems.

The rest of this paper is organized as follows. In Section 2, we briefly recall what is known about the Cauchy problem for the Navier-Stokes equations (1.1), (1.2) in the space $\mathrm{BUC}(\mathbb{O})$, and we give a short proof of Theorem 1.1. In Section 3, we show that the Navier-Stokes equations define an extended dissipative system in the sense of [5], and we establish the crucial estimate (1.14) using a uniform bound on the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. The main part of our analysis begins in Sections 4 and 5, where we obtain (in an abstract framework) accurate estimates on the integrated energy flux and the integrated energy density. Unlike in [5], we do not have to assume here that the energy density is uniformly bounded for all times; nevertheless, we can draw similar conclusions concerning the long-time behavior of the solutions, some of which are presented in Section 6. The (rather delicate) question of obtaining pointwise estimates on the energy density is briefly discussed in Section 7, which contains in particular a proof of Theorem 1.2. Finally, we show in Section 8 that the solutions of the Navier-Stokes equations (1.1), (1.2) converge (in an appropriate sense) to spatially homogeneous equilibria as $t \to \infty$, and we give a proof of Theorem 1.3 which includes a much more precise version of estimate (1.8). The last section is an appendix which collects the proofs of some auxiliary results stated in Section 2.

Acknowledgements. The research of the second named author was partially supported by the grant No 037-0372791-2803 of the Croatian Ministry of Science. We also thank the anonymous referees for useful comments.

2 The Navier-Stokes and vorticity equations in $\mathbb{O} = \mathbb{R} \times \mathbb{T}$

In this section, we study the Cauchy problem for the Navier-Stokes equations (1.1), (1.2) in the cylinder $\mathbb{O} = \mathbb{R} \times \mathbb{T}$, and we establish a few general properties of the solutions that will be used later on. We do not claim for much originality at this stage, because the results collected here are essentially taken from [1, 8, 5].

As was observed in the introduction, the Navier-Stokes equation can be written in the form $\partial_t u + \mathbb{P}(u \cdot \nabla)u = \Delta u$, where $\mathbb{P}$ is the Leray-Hopf projection. Given initial data u_0, the corresponding integral equation reads

$$u(t) \;=\; e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbb{P}(u(s) \otimes u(s))\,\mathrm{d}s\;, \qquad t \ge 0\;, \tag{2.1}$$

where $u(t) = u(\cdot, t)$ and $\nabla \cdot e^{t\Delta}\mathbb{P}(u \otimes v)$ is a shorthand notation for the vector with j^{th} component

$$\Bigl(\nabla \cdot e^{t\Delta}\mathbb{P}(u \otimes v)\Bigr)_j \;=\; \sum_{k,\ell=1}^{2} \partial_\ell\, e^{t\Delta}\mathbb{P}_{jk} u_\ell v_k\;, \qquad j = 1, 2\;. \tag{2.2}$$

It is well known that the heat kernel $e^{t\Delta}$ defines a strongly continuous semigroup of contractions in the space $\mathrm{BUC}(\mathbb{O})$, see e.g. [2]. Moreover, the following estimate allows to control the nonlinear term in (2.1) :

Lemma 2.1. *There exists a constant $C_0 > 0$ such that, for all $t > 0$ and all $u, v \in \mathrm{BUC}(\mathbb{O})$, one has $\nabla \cdot e^{t\Delta}\mathbb{P}(u \otimes v) \in \mathrm{BUC}(\mathbb{O})$ and*

$$\|\nabla \cdot e^{t\Delta}\mathbb{P}(u \otimes v)\|_{L^\infty(\mathbb{O})} \;\le\; \frac{C_0}{\sqrt{t}}\, \|u\|_{L^\infty(\mathbb{O})} \|v\|_{L^\infty(\mathbb{O})}\;. \tag{2.3}$$

For the reader's convenience, we give a short proof of estimate (2.3) in the Appendix. Using Lemma 2.1 and a standard fixed point argument, one easily obtains the following local existence result.

Proposition 2.2. **[1, 8]** *For any initial data $u_0 \in \mathrm{BUC}(\mathbb{O})$ with $\operatorname{div} u_0 = 0$, there exists a time $T > 0$ such that the integral equation (2.1) has a unique solution $u \in C^0([0, T], \mathrm{BUC}(\mathbb{O}))$, which satisfies $t^{1/2}\partial_j u \in C^0_b((0, T], \mathrm{BUC}(\mathbb{O}))$ for $j = 1, 2$.*

As in [8], one can also show that the solutions of (2.1) are smooth and satisfy (1.1), (1.2) for positive times. The proof of Theorem 2.2, which is reproduced in the Appendix, gives a local existence time of the form $T = \mathcal{O}(\|u_0\|_{L^\infty}^{-2})$, so that any upper bound on $\|u_0\|_{L^\infty}$ provides a lower bound on T. In particular, all solutions either blow up in finite time in L^∞ norm, or can be extended to the whole time axis $[0, +\infty)$. To rule out the first scenario, the most efficient way is to consider the vorticity $\omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$, which is well defined for positive times by Proposition 2.2 and evolves according to the advection-diffusion equation

$$\partial_t \omega + u \cdot \nabla \omega \;=\; \Delta \omega\;. \tag{2.4}$$

The parabolic maximum principle applies to (2.4), hence $\|\omega(\cdot, t)\|_{L^\infty}$ is a nonincreasing function of $t > 0$. This, however, does not imply that the velocity field $u(\cdot, t)$ stays uniformly bounded for all times, because the Biot-Savart law does not allow to control the vertical average of u in terms of ω, as we now explain.

Given any divergence-free velocity field $u \in \mathrm{BUC}(\mathbb{O})$, we decompose $u = \langle u \rangle + \widehat{u}$, where $\langle u \rangle = \fint_{\mathbb{T}} u \,\mathrm{d}x_2$ denotes the vertical average of u. More explicitly,

$$u(x_1, x_2) \;=\; \begin{pmatrix} m_1(x_1) \\ m_2(x_1) \end{pmatrix} + \begin{pmatrix} \widehat{u}_1(x_1, x_2) \\ \widehat{u}_2(x_1, x_2) \end{pmatrix} , \qquad (x_1, x_2) \in \mathbb{R} \times \mathbb{T} , \tag{2.5}$$

where $m_j = \langle u_j \rangle$ for $j = 1, 2$. The divergence-free condition implies that $\mathrm{div}\langle u \rangle = \partial_1 m_1 = 0$ and $\mathrm{div}\,\widehat{u} = 0$. In particular, the mean horizontal flow m_1 is a constant which (according to the discussion in the previous section) can be set to zero without loss of generality. We thus assume that $(m_1, m_2) = (0, m)$, where $m = \langle u_2 \rangle$ is the mean vertical flow. Taking the curl of (2.5), we also obtain $\omega = \langle \omega \rangle + \widehat{\omega}$, where $\langle \omega \rangle = \partial_1 m$ and $\widehat{\omega} = \partial_1 \widehat{u}_2 - \partial_2 \widehat{u}_1$. Now, a direct calculation which can be found in [1] shows that the oscillating part $\widehat{u}$ of the velocity field is entirely determined by the associated vorticity $\widehat{\omega}$. More precisely, we have the Biot-Savart formula:

$$\widehat{u}(x_1, x_2) \;=\; \int_{\mathbb{R}} \int_{\mathbb{T}} \nabla^{\perp} K(x_1 - y_1, x_2 - y_2)\, \widehat{\omega}(y_1, y_2) \,\mathrm{d}y_2 \,\mathrm{d}y_1 , \tag{2.6}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)^{\mathrm{t}}$ and K is the fundamental solution of the Laplace operator in $\mathbb{O} = \mathbb{R} \times \mathbb{T}$:

$$K(x_1, x_2) \;=\; \frac{1}{4\pi} \log\Bigl(2\cosh(2\pi x_1) - 2\cos(2\pi x_2)\Bigr) . \tag{2.7}$$

In contrast, the mean vertical flow $m = \langle u_2 \rangle$ cannot be completely expressed in terms of the vorticity, and we only know that $\partial_1 m = \langle \omega \rangle$.

For later use, we also give an explicit formula for the pressure. Note that (1.2) only defines p up to a constant if $u \in \mathrm{BUC}(\mathbb{O})$, and it is necessary to fix that constant if we want to control the pressure in a space like $L^{\infty}(\mathbb{O})$. In the Appendix, we show that p can be taken as

$$p \;=\; -u_1^2 - 2\partial_2 K * (\omega\, u_1) , \tag{2.8}$$

where $*$ denotes the convolution product in $\mathbb{O}$, see (2.6). We then have the following result, whose proof is also postponed to the Appendix.

Lemma 2.3. *Assume that $u \in \mathrm{BUC}(\mathbb{O})$ satisfies $\mathrm{div}\, u = 0$ and $\langle u_1 \rangle = 0$, and let p be defined by (2.8) where $\omega = \partial_1 u_2 - \partial_2 u_1$. If $\omega \in L^{\infty}(\mathbb{O})$, we have*

$$\|\widehat{u}\|_{L^{\infty}(\mathbb{O})} \;\le\; C_1 \|\omega\|_{L^{\infty}(\mathbb{O})} , \qquad \|p\|_{L^{\infty}(\mathbb{O})} \;\le\; C_2 \|\omega\|_{L^{\infty}(\mathbb{O})}^2 , \tag{2.9}$$

where C_1, C_2 are positive constants which do not depend on u. Moreover, there exists $C_3 > 0$ such that

$$\|\nabla u\|_{\mathrm{BMO}(\mathbb{O})} \;\le\; C_3 \|\omega\|_{L^{\infty}(\mathbb{O})} . \tag{2.10}$$

Here $\mathrm{BMO}(\mathbb{O})$ denotes the space of functions with bounded mean oscillation on $\mathbb{O}$, which can be identified with the space of all $f \in \mathrm{BMO}(\mathbb{R}^2)$ that are 1-periodic in the vertical direction, see [14, Chapter IV] for precise definitions. If $f \in \mathrm{BMO}(\mathbb{O})$, then $\|f\|_{\mathrm{BMO}(\mathbb{O})} = \|\tilde{f}\|_{\mathrm{BMO}(\mathbb{R}^2)}$ where $\tilde{f}$ denotes the periodic extension of f to the whole plane $\mathbb{R}^2$.

We now return to the solutions of (1.1) given by Proposition 2.2 and show that they cannot blow up in finite time. Take $u_0 \in \mathrm{BUC}(\mathbb{O})$ such that $\mathrm{div}\, u_0 = 0$ and $\langle (u_0)_1 \rangle = 0$, and let $T_* > 0$ be the maximal existence time of the solution of (1.1), (1.2) with initial data u_0. As was already observed, the vorticity $\omega(\cdot, t)$ is uniformly bounded for all $t \in [t_0, T_*)$, if $t_0 > 0$. Lemma 2.3 then shows that the horizontal speed $u_1 = \widehat{u}_1$ and the oscillating part $\widehat{u}_2$ of the vertical speed are uniformly bounded for all $t \in [0, T_*)$. Thus we only need to estimate the mean vertical flow $m(x_1, t)$, which satisfies the simple equation

$$\partial_1 m + \partial_1 \langle \widehat{u}_1 \widehat{u}_2 \rangle \;=\; \partial_1^2 m \;, \qquad x_1 \in \mathbb{R} \;, \quad 0 < t < T_* \;. \tag{2.11}$$

Indeed, using the identity $(u \cdot \nabla) u = \frac{1}{2} \nabla |u|^2 + u^{\perp} \omega$ and averaging over $x_2 \in \mathbb{T}$ the evolution equation for u_2 in (1.1), we easily obtain the equation $\partial_t m + \langle u_1 \omega \rangle = \partial_1^2 m$, which is equivalent to (2.11) since $\langle u_1 \omega \rangle = \langle \widehat{u}_1 \widehat{\omega} \rangle = \langle \widehat{u}_1 \partial_1 \widehat{u}_2 \rangle - \langle \widehat{u}_1 \partial_2 \widehat{u}_1 \rangle = \partial_1 \langle \widehat{u}_1 \widehat{u}_2 \rangle$. Now, the integral equation corresponding to (2.11) is

$$m(t) \;=\; e^{t\partial_1^2} m(0) - \int_0^t \partial_1 e^{(t-s)\partial_1^2} \langle \widehat{u}_1(s) \widehat{u}_2(s) \rangle \,\mathrm{d}s \;, \qquad 0 < t < T_* \;,$$

where $e^{t\partial_1^2}$ denotes the heat semigroup on $\mathbb{R}$. Since there exists $M > 0$ such that $\|\widehat{u}(t)\|_{L^\infty} \le M$ for all $t \in [0, T_*)$, we have

$$\|m(t)\|_{L^\infty} \;\le\; \|m(0)\|_{L^\infty} + \int_0^t \frac{M^2}{\sqrt{\pi(t-s)}} \,\mathrm{d}s \;=\; \|m(0)\|_{L^\infty} + \frac{2M^2\, t^{1/2}}{\sqrt{\pi}} \;, \tag{2.12}$$

for all $t \in [0, T_*)$. Thus $\|m(t)\|_{L^\infty}$ cannot blow up in finite time, hence $T_* = +\infty$ and estimate (2.12) holds for all times. The proof of Theorem 1.1 is thus complete. $\square$

Remark 2.4. Although our motivation for studying solutions of the Navier-Stokes equations that are periodic in one space direction is mainly mathematical, it is worth noting that there is a physical situation which naturally leads to periodic boundary conditions. Indeed, consider a viscous fluid that fills the two-dimensional strip $\Omega = \mathbb{R} \times [0, 1]$ and satisfies *perfect slip* boundary conditions, namely the fluid does not exert any tangential force on the boundary. If $u = (u_1, u_2)^{\mathrm{t}}$ denotes the fluid velocity, this means that u_1 (resp. u_2) satisfies homogeneous Neumann (resp. Dirichlet) boundary conditions on $\partial\Omega$. If we now extend u to the larger strip $\tilde{\Omega} = \mathbb{R} \times [-1, 1]$ in such a way that the horizontal (resp. vertical) velocity is an even (resp. odd) function of x_2, then the extension $\tilde{u} : \tilde{\Omega} \to \mathbb{R}^2$ satisfies periodic boundary conditions on $\tilde{\Omega}$, and can therefore be studied using the techniques developed in the present paper. Note, however, that the mean vertical flow vanishes identically in that situation, because the extension $\tilde{u}$ is (by construction) an odd function of x_2. If we eliminate the mean horizontal flow by a Galilean transformation, this implies that the velocity field is entirely controlled by the vorticity, hence remains uniformly bounded for all times. In fact, since ω satisfies (2.4) with homogeneous Dirichlet boundary conditions on $\partial\Omega$, it is even possible to show that $\|\omega(t)\|_{L^\infty} \to 0$ as $t \to \infty$, which in turns implies that the velocity converges uniformly to zero, see [6] for more details. Thus the general conclusions of Theorem 1.3 are clearly suboptimal in the case of perfect slip boundary conditions.

3 The Navier-Stokes equation as an extended dissipative system

In a previous work [5], we introduced the notion of an *extended dissipative system*, which is an abstract framework describing the essential properties of an important class of dissipative partial differential equations on unbounded domains. In this section, we show that the Navier-Stokes equation (1.1) belongs to that class, so that interesting conclusions can be drawn at least for solutions that stay uniformly bounded for all times.

We first recall the main definitions. If X is a metrizable topological space, we say that a family $(\Phi(t))_{t \ge 0}$ of continuous maps $\Phi(t) : X \to X$ is a *continuous semiflow* on X if

- $\Phi(0) = \mathbf{1}$ (the identity map);

- $\Phi(t_1 + t_2) = \Phi(t_1) \circ \Phi(t_2)$ for all $t_1, t_2 \geq 0$;
- For any $T > 0$, the map $(t, u) \mapsto \Phi(t)u$ is continuous from $[0, T] \times X$ to X.

In particular, given initial data $u_0 \in X$, the *trajectory* $u : \mathbb{R}_+ \to X$ defined by $u(t) = \Phi(t)u_0$ is a continuous function of the time $t \geq 0$, and the solution $u(t)$ depends continuously on u_0, uniformly in time on compact intervals. If $\Phi(t)u_0 = u_0$ for all $t \geq 0$, we say that u_0 is an *equilibrium*.

Definition 3.1. *We say that a continuous semiflow $(\Phi(t))_{t\geq 0}$ on a metrizable space X is an* extended dissipative system *on $\mathbb{R}$ if one can associate to each element $u \in X$ a triple (e, f, d) with $e, d \in C^0(\mathbb{R}, \mathbb{R}_+)$ and $f \in C^0(\mathbb{R}, \mathbb{R})$ such that*

(A1) *The functions e, f, d depend continuously on $u \in X$, uniformly on compact sets of $\mathbb{R}$;*

(A2) *$|f|^2 \leq b(e)d$ for some nondecreasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$;*

and such that, under the evolution defined by the semiflow $(\Phi(t))_{t\geq 0}$, the time-dependent quantities e, f, d have the following properties :

(A3) *If $d(x, t) = 0$ for all $(x, t) \in \mathbb{R} \times [0, t_0]$, where $t_0 > 0$, then u is an equilibrium;*

(A4) *The energy balance $\partial_t e = \partial_x f - d$ holds in the sense of distributions on $\mathbb{R} \times \mathbb{R}_+$.*

Remark 3.2. As in [5] there is a slight abuse of notation in the definition above. To any state of the system, namely to any point $u \in X$, we associate an energy density $e(x) \geq 0$, an energy flux $f(x) \in \mathbb{R}$, and an energy dissipation rate $d(x) \geq 0$, which are continuous functions of $x \in \mathbb{R}$ and satisfy properties (A1), (A2). In addition, given any $t \geq 0$, we associate to the evolved state $\Phi(t)u \in X$ an energy density $e(x, t) \geq 0$, an energy flux $f(x, t) \in \mathbb{R}$, and an energy dissipation rate $d(x, t) \geq 0$, and these are the time-dependent quantities that satisfy (A3), (A4). For simplicity, we will use the same notation e, f, d in both cases, although the quantities that evolve according to the semiflow $\Phi(t)$ depend on an additional variable $t \geq 0$.

Remark 3.3. Unlike in [5], where definitions are given in full generality, we only consider here an extended dissipative system on the real line $\mathbb{R}$. This is because we want to study the Navier-Stokes equation in the cylinder $\mathbb{O}$, which has only one unbounded direction, so that we indeed obtain a one-dimensional extended dissipative system if we consider the energy flow through vertical sections of the cylinder, as in (1.9)–(1.11).

We next introduce a functional-analytic framework that is appropriate for the Navier-Stokes equation (1.1). Let X denote the Banach space

$$X = \Big\{ u \in \mathrm{BUC}(\mathbb{O}) \,\Big|\, \partial_j u \in \mathrm{BUC}(\mathbb{O}) \text{ for } j = 1, 2\,, \ \operatorname{div} u = 0\,, \ \langle u_1 \rangle = 0 \Big\}\,, \tag{3.1}$$

equipped with the norm $\|u\|_X = \max(\|u\|_{L^\infty(\mathbb{O})}, \|\nabla u\|_{L^\infty(\mathbb{O})})$. We recall that $\langle u_1 \rangle$ is the vertical average of the horizontal velocity, see (1.5). Proceeding as in the proof of Theorem 1.1, it is straightforward to verify that the Cauchy problem for Eq. (1.1) is globally well-posed in X. More precisely, for any $u_0 \in X$, the integral equation (2.1) has a unique global solution $u \in C^0([0, \infty), X)$, which depends continuously on the initial data u_0 in the topology of X, uniformly in time on compact intervals. In other words, the Navier-Stokes equation defines a continuous semiflow $(\Phi(t))_{t\geq 0}$ on X. Given a constant $M > 0$, we also consider the subset

$$X_M = \Big\{ u \in X \,\Big|\, |\operatorname{curl} u| = |\partial_1 u_2 - \partial_2 u_1| \leq M \Big\} \subset X\,, \tag{3.2}$$

which is invariant under the semiflow $\Phi(t)$ since the vorticity $\omega = \operatorname{curl} u$ obeys the parabolic maximum principle. Note that X_M is an *unbounded* subset of X, because (as was discussed in the previous section) a bound on the vorticity allows to control the oscillating part $\widehat{u}$ of the velocity field, but not the average vertical velocity $m = \langle u_2 \rangle$.

For any $u \in X_M$, we define as in (1.9)–(1.11):

$$\begin{aligned} e(x_1) &= \frac{1}{2}\int_{\mathbb{T}} |u(x_1,x_2)|^2 \,\mathrm{d}x_2 + 1 \;, \\ h(x_1) &= \int_{\mathbb{T}} \Bigl(\frac{1}{2}|u(x_1,x_2)|^2 + p(x_1,x_2)\Bigr) u_1(x_1,x_2)\,\mathrm{d}x_2 \;, \\ d(x_1) &= \int_{\mathbb{T}} |\nabla u(x_1,x_2)|^2 \,\mathrm{d}x_2 \;, \end{aligned} \tag{3.3}$$

where p is given by (2.8). We also denote $f(x_1) = \partial_1 e(x_1) - h(x_1)$.

Lemma 3.4. *Assume that $u \in X_M$ for some $M > 0$, and let e, h, d be defined by* (3.3)*. Then there exists a constant $C_4 > 0$, depending only on M, such that*

$$h(x_1)^2 \le C_4 e(x_1) d(x_1) \;, \qquad |\partial_1 e(x_1)|^2 \le 2e(x_1)d(x_1) \;, \tag{3.4}$$

for all $x_1 \in \mathbb{R}$.

Proof. If $u \in X_M$, then $\langle u_1 \rangle = \int_{\mathbb{T}} u_1(x_1,x_2)\,\mathrm{d}x_2 = 0$, and the Poincaré-Wirtinger inequality implies

$$\int_{\mathbb{T}} |u_1(x_1,x_2)|^2 \,\mathrm{d}x_2 \le \frac{1}{4\pi^2}\int_{\mathbb{T}} |\partial_2 u_1(x_1,x_2)|^2 \,\mathrm{d}x_2 \le \frac{d(x_1)}{4\pi^2} \;. \tag{3.5}$$

To prove (3.4), we write $h(x_1) = h_1(x_1) + h_2(x_1)$, where

$$h_1(x_1) = \frac{1}{2}\int_{\mathbb{T}} |u(x_1,x_2)|^2 u_1(x_1,x_2)\,\mathrm{d}x_2 \;, \qquad h_2(x_1) = \int_{\mathbb{T}} p(x_1,x_2) u_1(x_1,x_2)\,\mathrm{d}x_2 \;.$$

Since $u_2 = m + \widehat{u}_2$, where $m = \langle u_2 \rangle$, we have $|u|^2 = u_1^2 + m^2 + 2m\widehat{u}_2 + \widehat{u}_2^2$, hence

$$h_1 = \frac{1}{2}\int_{\mathbb{T}} \Bigl(u_1^2 + 2m\widehat{u}_2 + \widehat{u}_2^2\Bigr) u_1 \,\mathrm{d}x_2 = \frac{1}{2}\int_{\mathbb{T}} \Bigl(u_1^2 - \widehat{u}_2^2 + 2u_2\widehat{u}_2\Bigr) u_1 \,\mathrm{d}x_2 \;.$$

Using Lemma 2.3, Hölder's inequality, and Poincaré's inequality (3.5), we thus obtain

$$\begin{aligned} |h_1| &\le CM \int_{\mathbb{T}} (|u_1| + |u_2|)|u_1|\,\mathrm{d}x_2 \le CM\Bigl(\int_{\mathbb{T}} |u|^2 \,\mathrm{d}x_2\Bigr)^{1/2}\Bigl(\int_{\mathbb{T}} u_1^2 \,\mathrm{d}x_2\Bigr)^{1/2} \le CM(ed)^{1/2} \;, \\ |h_2| &\le CM^2 \int_{\mathbb{T}} |u_1|\,\mathrm{d}x_2 \le CM^2\Bigl(\int_{\mathbb{T}} u_1^2 \,\mathrm{d}x_2\Bigr)^{1/2} \le CM^2 d^{1/2} \le CM^2(ed)^{1/2} \;. \end{aligned}$$

In the last inequality, we used the fact that $e(x_1) \ge 1$ for all $x_1 \in \mathbb{R}$. Finally,

$$|\partial_1 e| \le \int_{\mathbb{T}} |u_1\partial_1 u_1 + u_2\partial_1 u_2|\,\mathrm{d}x_2 \le \Bigl(\int_{\mathbb{T}} |u|^2 \,\mathrm{d}x_2\Bigr)^{1/2}\Bigl(\int_{\mathbb{T}} |\nabla u|^2 \,\mathrm{d}x_2\Bigr)^{1/2} \le (2ed)^{1/2} \;.$$

□

As a direct consequence of Lemma 3.4, we obtain

Proposition 3.5. *The Navier-Stokes equations* (1.1), (1.2) *define an extended dissipative system in the space X_M for any $M > 0$. More precisely, the triple (e, f, d) defined by* (3.3) *and by the relation $f = \partial_1 e - h$ satisfies assumptions (A1), (A3), (A4) of Definition 3.1, as well as*

- **(A2')** $|f|^2 \le \beta ed$ *for some positive constant β depending only on M;*
- **(A5)** $|\partial_1 e|^2 \le \gamma ed$ *for some positive constant γ.*

We are now in position to apply the results of [5]. If $u(x_1,x_2,t)$ is a solution of (1.1), (1.2) with initial data $u_0 \in X_M$, we define for all $T \ge 0$:

$$e_*(T) \;=\; \sup_{x_1\in\mathbb{R}} e(x_1,T)\;, \qquad \bar e_*(T) \;=\; \sup_{0\le t\le T} e_*(t)\;, \tag{3.6}$$

where $e(x_1,t)$ is the energy density (1.9). With these notations we have

Corollary 3.6. [5] *If $u_0 \in X_M$ for some $M > 0$, the solution of* (1.1), (1.2) *with initial data u_0 satisfies, for all $x_1 \in \mathbb{R}$ and all $T > 0$,*

$$\Bigl|\int_0^T f(x_1,t)\,\mathrm{d}t\Bigr| \;\le\; \sqrt{\beta T \bar e_*(T) e_*(0)}\;, \tag{3.7}$$

where $\beta > 0$ is as in Proposition 3.5. Moreover, for all $R > 0$,

$$\int_0^T\int_{|x_1|\le R} d(x_1,t)\,\mathrm{d}x_1\,\mathrm{d}t \;\le\; 2\sqrt{\beta T \bar e_*(T) e_*(0)} + 2R e_*(0)\;. \tag{3.8}$$

In particular, if $e_*(t)$ is uniformly bounded for all times, we have

$$\limsup_{T\to\infty} \frac{1}{\sqrt{\beta T}}\int_0^T\int_{|x_1|\le\sqrt{\beta T}} d(x_1,t)\,\mathrm{d}x_1\,\mathrm{d}t \;\le\; 4\sqrt{e_*(0)\bar e_*(\infty)}\;. \tag{3.9}$$

The weakness of Corollary 3.6 lies in the fact that the right-hand side of inequalities (3.7)–(3.9) involves the quantity $\bar e_*(T)$, which depends on T in an unknown way. If we restrict ourselves to solutions for which $\bar e_*(\infty) < \infty$, then Corollary 3.6 allows to draw interesting consequences on the long-time behavior. For instance, inequality (3.7) shows that the energy flux $f(x_1,t)$ through any fixed point $x_1 \in \mathbb{R}$ is, on average, very small when t is large. It follows that the total energy that is dissipated in a spatial domain of size $\mathcal{O}(T^{1/2})$ over the time interval $[0,T]$ grows at most like $T^{1/2}$ as $T \to \infty$, as indicated by (3.8), (3.9). Since, by assumption (A3), the energy dissipation $d(x_1,t)$ vanishes only on the set of equilibria, one can deduce, as in [5], that any solution of (1.1) with uniformly bounded energy density converges, in a suitable localized and averaged sense, to the set of equilibria as $t \to \infty$. In particular, it follows from (3.8) or (3.9) that the Navier-Stokes equation (1.1) has no solutions in $\mathrm{BUC}(\mathbb{O})$ that are periodic in time, except for spatially homogeneous equilibria.

The main goal of the present paper is to reproduce the results of [5] for the Navier-Stokes equation without assuming that the energy density stays uniformly bounded. As we shall show in the subsequent sections, this can be achieved by using the additional assumption (A5) in Proposition 3.5, which holds in our case but not for some of the systems considered in [5]. We shall thus obtain estimates which are similar to (3.7)–(3.9) but do not contain the quantity $\bar e_*(T)$ in the right-hand side.

Remark 3.7. If we equip our function space X with a topology that is weak enough so that all solutions with uniformly bounded energy density are compact, then Corollary 3.6 also gives some information on the omega-limit set of such solutions. A natural choice is the "localized" topology $\mathcal{T}_{\mathrm{loc}}$, which is the topology of uniform convergence on compact sets for the velocity field $u(x_1,x_2)$. Indeed, by Ascoli's theorem, any bounded subset of X is relatively compact with respect to $\mathcal{T}_{\mathrm{loc}}$. Moreover, although the Navier-Stokes equation is nonlocal, it is straightforward to verify that the solutions of (1.1), (1.2) depend continuously on the initial data in the topology $\mathcal{T}_{\mathrm{loc}}$, so that (1.1) defines a continuous semiflow in $X_{\mathrm{loc}} = (X,\mathcal{T}_{\mathrm{loc}})$. If we restrict ourselves to the subset X_M for some $M > 0$, then (1.1) nearly defines an extended dissipative system in the sense of Definition 3.1. The only caveat concerns assumption (A1): the flux f and the dissipation d do not depend continuously on u in the localized topology $\mathcal{T}_{\rm loc}$, but this is a minor point and most of the results of [5] can be derived without that property. For instance, if $u(t)$ is a solution of (1.1) that stays uniformly bounded in X, then the trajectory $\{u(t)\}_{t\ge0}$ is relatively compact in $X_{\rm loc}$, and it follows from [5] that the omega-limit set $\Omega(u)$ contains at least an equilibrium $\bar u$ with $\nabla\bar u \equiv 0$. Moreover the solution $u(t)$ stays most of the time, in a sense that can be quantified precisely, within an arbitrary neighborhood (in $\mathcal{T}_{\rm loc}$) of the set of spatially homogeneous equilibria.

4 Integrated flux bounds

From now on, we do not consider specifically the Navier-Stokes equation anymore, but we study a general extended dissipative system on $\mathbb{R}$ in the sense of Definition 3.1. Keeping in mind the applications to (1.1), we strengthen assumption (A2) as follows:

(A2') $|f|^2 \le \beta e d$ for some positive constant β.

Moreover, we add another hypothesis, which will be crucial in obtaining results without a priori bounds on the solutions :

(A5) $|\partial_x e|^2 \le \gamma e d$ for some positive constant γ.

Here and in what follows, we denote the space variable by $x \in \mathbb{R}$ (instead of x_1). Assumption (A5) means that the energy gradient generates dissipation, and in combination with (A2') this will drive the whole theory.

Given a solution $u(t) = \Phi(t)u_0$ of our system, we consider the (time-dependent) energy density $e(x,t) \ge 0$, the energy flux $f(x,t) \in \mathbb{R}$, and the energy dissipation rate $d(x,t) \ge 0$, which are continuous functions on $\mathbb{R}\times\mathbb{R}_+$. In view of (A4), the local energy dissipation law $\partial_t e = \partial_x f - d$ holds in the sense of distributions on $\mathbb{R}\times\mathbb{R}_+$. As a consequence, given $T > 0$ and $a, b \in \mathbb{R}$ with $a < b$, we have the integrated energy balance equation

$$\int_a^b \Big(e(x,T) - e(x,0)\Big)\,\mathrm{d}x \;=\; \int_0^T \Big(f(b,t) - f(a,t)\Big)\,\mathrm{d}t - \int_0^T\int_a^b d(x,t)\,\mathrm{d}x\,\mathrm{d}t\;. \tag{4.1}$$

The left-hand side is the variation of energy in the segment $[a,b]$ from initial time $t = 0$ to final time $t = T$. The first term in the right-hand side represents the energy entering (or leaving) the segment $[a,b]$ through the endpoints over the time interval $[0,T]$, and the last term is the energy that is dissipated in $[a,b]$ for $t \in [0,T]$. This relation will be used so often that we now introduce shorthand notations for the various quantities in (4.1).

We use capital letters E, F, D to denote the integrals of e, f, d with respect to time, namely

$$E(x,T) \;=\; \int_0^T e(x,t)\,\mathrm{d}t\;,\qquad F(x,T) \;=\; \int_0^T f(x,t)\,\mathrm{d}t\;,\qquad D(x,T) \;=\; \int_0^T d(x,t)\,\mathrm{d}t\;, \tag{4.2}$$

for all $x \in \mathbb{R}$ and all $T \ge 0$. Thus, if $a < b$, the total energy dissipated in the segment $[a,b]$ over the time interval $[0,T]$ is

$$D([a,b],T) \;=\; \int_a^b D(x,T)\,\mathrm{d}x \;=\; \int_a^b\int_0^T d(x,t)\,\mathrm{d}t\,\mathrm{d}x\;. \tag{4.3}$$

Another important quantity is the "available" energy in the segment $[a,b]$ at time T, which we define as

$$A([a,b],T) \;=\; \int_a^b e(x,0)\,\mathrm{d}x + F(b,T) - F(a,T)\;. \tag{4.4}$$

This is the energy that would be present in the segment $[a,b]$ at time T, due to the initial data and to the flux through the endpoints, if no dissipation was included in our model. Indeed, using this notation, the integrated energy balance (4.1) reads

$$A([a,b],T) \;=\; \int_a^b e(x,T)\,\mathrm{d}x + D([a,b],T)\ . \tag{4.5}$$

Finally, one of our main goals is investigating the energy growth, so it is convenient to introduce the following notations for the supremum of the energy density:

$$\begin{aligned} e_*(t) \;&=\; \sup_{x\in\mathbb{R}} e(x,t)\ , \qquad & e_*([a,b],t) \;&=\; \sup_{x\in[a,b]} e(x,t)\ ,\\ E_*(t) \;&=\; \sup_{x\in\mathbb{R}} E(x,t)\ , \qquad & E_*([a,b],t) \;&=\; \sup_{x\in[a,b]} E(x,t)\ . \end{aligned} \tag{4.6}$$

As a first application, we use the energy balance equation and assumption (A2') to derive useful bounds on the integrated energy flux $F(x,T)$, which will serve as a basis for the analysis in the subsequent sections. We begin with a local version of the integrated flux bound.

Proposition 4.1. *Let $u(t) = \Phi(t)u_0$ be any solution of an extended dissipative system on $\mathbb{R}$ satisfying (A2'). Then, for all $a,b\in\mathbb{R}$ with $a<b$ and all $T>0$, we have*

$$F(a,T) \;\le\; \sqrt{\beta e_*([a,b],0)E_*([a,b],T)} + \frac{\beta E_*([a,b],T)}{b-a}\ , \tag{4.7}$$

$$F(b,T) \;\ge\; -\sqrt{\beta e_*([a,b],0)E_*([a,b],T)} - \frac{\beta E_*([a,b],T)}{b-a}\ . \tag{4.8}$$

Proof. Let $\widetilde{e} = e_*([a,b],0)$ and $\widetilde{E} = E_*([a,b],T)$. If $\widetilde{E}=0$, then by (A2') we have $F(x,T)=0$ for all $x\in[a,b]$, and (4.7), (4.8) trivially hold. So we assume that $\widetilde{E}>0$ and prove (4.7), the proof of (4.8) being analogous. Using assumption (A2') and Hölder's inequality, we find for any $x\in[a,b]$:

$$\begin{aligned} F(x,T)^2 \;&\le\; \Bigl(\int_0^T |f(x,t)|\,\mathrm{d}t\Bigr)^2 \;\le\; \Bigl(\int_0^T \beta^{1/2}e(x,t)^{1/2}d(x,t)^{1/2}\,\mathrm{d}t\Bigr)^2\\ &\le\; \beta\Bigl(\int_0^T e(x,t)\,\mathrm{d}t\Bigr)\Bigl(\int_0^T d(x,t)\,\mathrm{d}t\Bigr) \;\le\; \beta\widetilde{E}\,D(x,T)\ . \end{aligned} \tag{4.9}$$

On the other hand, if we integrate in time the energy dissipation law (A4) and use the fact that $e(x,T)\ge 0$ and $e(x,0)\le\widetilde{e}$, we obtain for all $x\in[a,b]$:

$$\partial_x F(x,T) \;=\; e(x,T) - e(x,0) + D(x,T) \;\ge\; -\widetilde{e} + D(x,T)\ . \tag{4.10}$$

Thus, combining (4.9) and (4.10), we see that the integrated flux $F(x,t)$ satisfies the differential inequality

$$\partial_x F(x,T) \;\ge\; -\widetilde{e} + \frac{1}{\beta\widetilde{E}}\,F(x,T)^2\ , \qquad x\in[a,b]\ . \tag{4.11}$$

Let $\rho = (\beta\widetilde{e}\widetilde{E})^{1/2}$. If $F(a,T)\le\rho$, then (4.7) is proved. If $F(a,T)>\rho$, then $\partial_x F(a,T)>0$, and it follows from (4.11) that $F(x,T)>\rho$ for all $x\in[a,b]$, so that

$$\frac{\partial_x F(x,T)}{F(x,T)^2-\rho^2} \;\ge\; \frac{1}{\beta\widetilde{E}}\ , \qquad x\in[a,b]\ .$$

Integrating both sides over $x \in [a,b]$ we deduce

$$\frac{b-a}{\beta\widetilde{E}} \;\le\; \frac{1}{2\rho}\ln\left(\frac{F(a,T)+\rho}{F(a,T)-\rho}\cdot\frac{F(b,T)-\rho}{F(b,T)+\rho}\right) \;\le\; \frac{1}{2\rho}\ln\left(\frac{F(a,T)+\rho}{F(a,T)-\rho}\right)\;.$$

Thus, if we denote $Y = \rho(b-a)/(\beta\widetilde{E})$, we arrive at

$$F(a,T) \;\le\; \rho\,\frac{\exp(2Y)+1}{\exp(2Y)-1} \;\le\; \rho\Big(1+\frac{1}{Y}\Big) \;=\; \sqrt{\beta\widetilde{e}\widetilde{E}} + \frac{\beta\widetilde{E}}{b-a}\;,$$

which is the desired result. □

A remarkable feature of inequalities (4.7), (4.8) is that they give estimates on the integrated flux $F(x_0,T)$ in terms of the energy density $e(x,t)$ for $t \in [0,T]$ and x in a *neighborhood* of x_0. Simpler estimates involving the energy over the whole line $\mathbb{R}$ easily follow from Proposition 4.1 and are often sufficient in the applications.

Corollary 4.2. *Under the assumptions of Proposition 4.1, one has for all $x \in \mathbb{R}$ and all $T > 0$*

$$|F(x,T)| \;\le\; \sqrt{\beta e_*(0)E_*(T)}\;, \tag{4.12}$$

where $e_*(0)$ and $E_*(T)$ are defined in (4.6).

Proof. Since $e_*([a,b],0) \le e_*(0)$ and $E_*([a,b],T) \le E_*(T)$, it follows from (4.7), (4.8) that

$$F(a,T) \;\le\; \sqrt{\beta e_*(0)E_*(T)} + \frac{\beta E_*(T)}{b-a}\;, \qquad F(b,T) \;\ge\; -\sqrt{\beta e_*(0)E_*(T)} - \frac{\beta E_*(T)}{b-a}\;.$$

If we now take the limit $b \to +\infty$ in the first inequality and $a \to -\infty$ in the second one, we obtain (4.12). □

The proof of Proposition 4.1 also shows that, in a one-dimensional extended dissipative system, the energy density $e(x,t)$ cannot be everywhere an increasing function of time. More precisely, we have

Corollary 4.3. *Let $u(t) = \Phi(t)u_0$ be a solution of an extended dissipative system on $\mathbb{R}$ satisfying (A2'), and assume that there exists $T > 0$ such that $e(x,T) \ge e(x,0)$ for all $x \in \mathbb{R}$. Then $f(x,t) = d(x,t) = 0$ and $e(x,t) = e(x,0)$ for all $x \in \mathbb{R}$ and all $t \in [0,T]$.*

Proof. If $e(x,T) \ge e(x,0)$ for all $x \in \mathbb{R}$, it is clear that inequality (4.10) holds for all $x \in \mathbb{R}$ with $\widetilde{e} = 0$. Proceeding as in the proof of Proposition 4.1 and Corollary 4.2, we deduce that $F(x,T) \le 0$ for all $x \in \mathbb{R}$, and finally that $F(\cdot,T) \equiv 0$. Since $e(x,T) \ge e(x,0)$, the integrated energy balance (4.1) then implies that the energy dissipation $d(x,t)$ vanishes identically for $t \in [0,T]$, and so does the energy flux $f(x,t)$ by (A2'). Now, using the local energy dissipation law (A4), we conclude that $e(x,t) = e(x,0)$ for all $x \in \mathbb{R}$ and all $t \in [0,T]$. □

In view of assumption (A3), Corollary 4.3 implies that equilibria are the only possible solutions for which $e(x,t) \ge e(x,0)$ for all $x \in \mathbb{R}$ and all $t \ge 0$. In particular, a state $u \in X$ for which the energy density $e(x)$ vanishes identically (or is equal to its minimal value) is necessary an equilibrium. For instance, it follows from Corollary 4.3 that the Navier-Stokes equation (1.1) has no nontrivial solution $u \in C_b^0((0,T],X)$ in the space (3.1) such that $u(x,t)$ converges to zero uniformly on compact sets of $\mathbb{R}$ as $t \to 0$.

5 Integrated energy bounds

We have seen in Corollary 4.2 that the integrated energy flux $F(x,T)$ can be bounded by an expression depending only on the initial data and the integrated energy density $E_*(T)$. Using the additional assumption (A5), we now prove that $E_*(T)$ can in turn be estimated in terms of the initial data and the observation time T. We begin with an auxiliary result :

Lemma 5.1. *Let $u(t) = \Phi(t)u_0$ be any solution of an extended dissipative system on $\mathbb{R}$ satisfying (A5). Given $T > 0$ and $a, b \in \mathbb{R}$ with $a < b$, we have for all $x \in [a,b]$:*

$$\Big(\int_0^T e(x,t)^2 \, \mathrm{d}t\Big)^{1/2} \;\le\; \Big(\frac{\sqrt{T}}{b-a} + \sqrt{\gamma}\Big) \sup_{0\le t\le T} A([a,b],t) \;, \tag{5.1}$$

where the available energy $A([a,b],t)$ is defined in (4.4) *and satisfies* (4.5).

Proof. For any $x_0 \in [a,b]$ and any $t \in [0,T]$, we have

$$\Big| e(x_0,t) - \frac{1}{b-a}\int_a^b e(x,t)\,\mathrm{d}x \Big| \;\le\; \int_a^b |\partial_x e(x,t)|\,\mathrm{d}x \;.$$

Applying (A5) and Hölder's inequality, we obtain

$$\int_a^b |\partial_x e(x,t)|\,\mathrm{d}x \;\le\; \sqrt{\gamma}\int_a^b \Big(e(x,t)d(x,t)\Big)^{1/2}\mathrm{d}x \;\le\; \sqrt{\gamma}\Big(\int_a^b e(x,t)\,\mathrm{d}x\Big)^{1/2}\Big(\int_a^b d(x,t)\,\mathrm{d}x\Big)^{1/2} ,$$

hence

$$e(x_0,t) \;\le\; \frac{1}{b-a}\int_a^b e(x,t)\,\mathrm{d}x + \sqrt{\gamma}\Big(\int_a^b e(x,t)\,\mathrm{d}x\Big)^{1/2}\Big(\int_a^b d(x,t)\,\mathrm{d}x\Big)^{1/2} . \tag{5.2}$$

Using (5.2) we now estimate the L^2 norm in time of $e(x_0,t)$. Since $\int_a^b e(x,t)\,\mathrm{d}x \le A([a,b],t)$ by (4.5), we have

$$\int_0^T \Big(\int_a^b e(x,t)\,\mathrm{d}x\Big)^2 \mathrm{d}t \;\le\; \int_0^T A([a,b],t)^2\,\mathrm{d}t \;\le\; TA_*^2 \;, \tag{5.3}$$

where we introduced the shorthand notation $A_* = \sup\{A([a,b],t)\,|\,t \in [0,T]\}$. Similarly, in view of (4.3) and (4.5), we obtain

$$\int_0^T \Big(\int_a^b e(x,t)\,\mathrm{d}x\Big)\Big(\int_a^b d(x,t)\,\mathrm{d}x\Big)\,\mathrm{d}t \le \sup_{0\le t\le T}\Big(\int_a^b e(x,t)\,\mathrm{d}x\Big)D([a,b],T) \;\le\; A_*^2 \;. \tag{5.4}$$

Applying Minkowski's inequality to the right-hand side of (5.2) and using (5.3), (5.4), we thus find

$$\Big(\int_0^T e(x_0,t)^2\,\mathrm{d}t\Big)^{1/2} \;\le\; \frac{\sqrt{T}A_*}{b-a} + \sqrt{\gamma}A_* \;,$$

which is the desired result. ☐

Remark 5.2. It is clear from inequality (5.2) that we can estimate the L^p norm in time of the energy density $e(x_0,t)$ for $p \le 2$ only. Indeed, the only control we have on the energy dissipation $d(x,t)$ is the bound $D([a,b],T) \le A([a,b],T)$, which comes from (4.5), and this corresponds to the limiting case $p = 2$.

In view of (5.1), it is natural to introduce the quantity

$$\mathcal{E}_*(T) \;=\; \sup_{x\in\mathbb{R}} \left(\int_0^T e(x,t)^2\,\mathrm{d}t\right)^{1/2}\;, \qquad T>0\;, \tag{5.5}$$

which controls $E_*(T)$ since $E_*(T) = \sup_{x\in\mathbb{R}}\int_0^T e(x,t)\,\mathrm{d}t \le \sqrt{T}\mathcal{E}_*(T)$. Combining Corollary 4.2 and Lemma 5.1, we obtain the main result of this section:

Proposition 5.3. *Let $u(t) = \Phi(t)u_0$ be any solution of an extended dissipative system on $\mathbb{R}$ satisfying (A2') and (A5). There exists a constant $\kappa > 1$, depending only on the product $\beta\gamma$, such that, for all $T > 0$,*

$$\mathcal{E}_*(T) \;\le\; \kappa e_*(0)\sqrt{T}\;, \qquad \textit{and} \qquad E_*(T) \;\le\; \kappa e_*(0) T\;. \tag{5.6}$$

Proof. We need only prove the first inequality in (5.6), as it implies the second one. Fix $T > 0$, and take $a, b \in \mathbb{R}$ with $a < b$. By (4.4) and (4.12), the available energy in the interval $[a, b]$ at any time $t \in [0, T]$ satisfies

$$A([a,b],t) \;\le\; e_*(0)(b-a) + |F(b,t)| + |F(a,t)| \;\le\; e_*(0)(b-a) + 2\sqrt{\beta e_*(0)E_*(t)}\;.$$

If $x \in [a, b]$, it thus follows from (5.1) that

$$\left(\int_0^T e(x,t)^2\,\mathrm{d}t\right)^{1/2} \;\le\; \left(\frac{\sqrt{T}}{b-a} + \sqrt{\gamma}\right)\left(e_*(0)(b-a) + 2\sqrt{\beta e_*(0)E_*(T)}\right)\;, \tag{5.7}$$

because $E_*(t) \le E_*(T)$ if $t \in [0, T]$. We now assume that $b - a = \epsilon\sqrt{T}$, where $\epsilon = (\beta/\gamma)^{1/4}$. Inserting this relation into the right-hand side of (5.7), and then taking the supremum over $x \in \mathbb{R}$ in the left-hand side, we obtain the inequality

$$\mathcal{E}_*(T) \;\le\; (1+\sigma)\left(e_*(0)\sqrt{T} + 2\sigma\sqrt{e_*(0)E_*(T)}\right)\;, \tag{5.8}$$

where $\sigma = (\beta\gamma)^{1/4}$. If $e_*(0) = 0$, then $\mathcal{E}_*(T) = 0$ in agreement with (5.6). In the converse case, we define $Z > 0$ such that

$$Z^2 \;=\; \frac{\mathcal{E}_*(T)}{e_*(0)\sqrt{T}}\;,$$

and since $E_*(T) \le \sqrt{T}\mathcal{E}_*(T)$ we deduce from (5.8) that $Z^2 \le (1+\sigma)(1+2\sigma Z)$. This quadratic inequality implies that $Z^2 \le \kappa$, where

$$\sqrt{\kappa} \;=\; \sigma(1+\sigma) + \sqrt{\sigma^2(1+\sigma)^2 + (1+\sigma)}\;, \tag{5.9}$$

and (5.6) follows. This concludes the proof. $\square$

As an immediate consequence of Corollary 4.2 and Proposition 5.3, we obtain our final estimate on the integrated energy flux:

Corollary 5.4. *Under the assumptions of Proposition 5.3 we have, for all $x \in \mathbb{R}$ and all $T > 0$,*

$$|F(x,T)| \;\le\; e_*(0)\sqrt{\kappa\beta T}\;. \tag{5.10}$$

6 Some dynamical implications

In this section, we draw a few consequences of the previous results, in the spirit of what was done in [5] for *bounded* solutions of extended dissipative systems. As in Proposition 5.3, we always assume that $u(t) = \Phi(t)u_0$ is a solution of an extended dissipative system on $\mathbb{R}$ satisfying (A2') and (A5). We first observe that our bound on the integrated energy flux implies a useful estimate on the energy dissipation.

Proposition 6.1. *Under the assumptions of Proposition 5.3 we have, for all $T > 0$ and all $R > 0$,*

$$\int_{-R}^{R} e(x,T)\,\mathrm{d}x + \int_0^T \int_{-R}^{R} d(x,t)\,\mathrm{d}x\,\mathrm{d}t \;\le\; 2e_*(0)\Big(R + \sqrt{\kappa\beta T}\Big) \;, \tag{6.1}$$

where κ is defined in (5.9)*.*

Proof. By (4.5) the left-hand side of (6.1) is equal to the available energy $A([-R,R],T)$. Now, using definition (4.4) and Corollary 5.4, we see that $A([-R,R],T) \le 2Re_*(0) + 2e_*(0)\sqrt{\kappa\beta T}$, and (6.1) follows. □

Inequality (6.1) shows that the dissipated energy $D([-R,R],T)$ grows at most like $\sqrt{T}$ as $T \to \infty$. In particular, all equilibria of our extended dissipative system are non-dissipative (i.e., they satisfy $d \equiv 0$), and there exist no other time-periodic solutions. Moreover, since by assumption (A3) only equilibria satisfy $d \equiv 0$, Proposition 6.1 can be used to prove that all trajectories converge, in a suitable sense, to the set of equilibria as $t \to \infty$. For instance, arguing as in [4] or [5, Proposition 5.1], we obtain

Corollary 6.2. *Consider an extended dissipative system on $\mathbb{R}$ satisfying (A2') and (A5). If $\bar{u} \in X$ is not an equilibrium, then $\bar{u}$ has a neighborhood $\mathcal{V}$ in X such that, for any solution $u(t) = \Phi(t)u_0$, one has*

$$\limsup_{T\to\infty} \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{\mathcal{V}}(u(t))\,\mathrm{d}t \;<\; \infty \;,$$

where $\mathbf{1}_{\mathcal{V}}$ denotes the characteristic function of $\mathcal{V}$.

Corollary 6.2 shows that any trajectory $u(t)$ spends a very small fraction of its lifetime in a sufficiently small neighborhood of any nonequilibrium point. If we assume in addition that our configuration space X is *compact* (see Remark 3.7), then using a finite covering argument we can deduce that any trajectory spends most of its time near the set of equilibria. More precisely, proceeding as in [4] or [5, Proposition 5.4], we find

Corollary 6.3. *Consider a compact extended dissipative system on $\mathbb{R}$ satisfying (A2') and (A5). If $\mathcal{V}$ is a neighborhood of the set of equilibria, then any solution $u(t) = \Phi(t)u_0$ satisfies*

$$\limsup_{T\to\infty} \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{\mathcal{V}^c}(u(t))\,\mathrm{d}t \;<\; \infty \;,$$

where $\mathbf{1}_{\mathcal{V}^c}$ denotes the characteristic function of $X \setminus \mathcal{V}$.

Remark 6.4. Corollary 6.3. has several ergodic-theoretical implications for compact extended dissipative systems satisfying (A2') and (A5). For instance, one can show by applying Birkhoff's ergodic theorem that all invariant measures are supported on the set of equilibria. Furthermore, using the variational principle for topological and metric entropy, one can conclude that the topological entropy of the system is necessarily zero, see [13, Section 4] for a related discussion.

We now drop the compactness assumption and return to the general case considered in Proposition 5.3. We already observed in Corollary 4.3 that only trivial solutions (namely, equilibria) have the property that the energy density $e(x,t)$ does not decrease anywhere in space when times varies. We now derive a more precise result which strongly constraints the set of points where energy can increase. Given a solution $u(t) = \Phi(t)u_0$ of an extended dissipative system on $\mathbb{R}$, we define, for any $T > 0$,

$$J_T \;=\; \Bigl\{R > 0 \Bigm| \int_{-R}^{R} e(x,T)\,\mathrm{d}x \;\ge\; \int_{-R}^{R} e(x,0)\,\mathrm{d}x\Bigr\} \;\subset\; (0,\infty)\ . \tag{6.2}$$

Proposition 6.5. *Let $u(t) = \Phi(t)u_0$ be any solution of an extended dissipative system on $\mathbb{R}$ satisfying (A2') and (A5), and assume that $u_0 \in X$ is not an equilibrium. Then, for any $T > 0$, the set J_T defined by* (6.2) *has a finite Lebesgue measure.*

Proof. Given $T > 0$, we define for any $R > 0$

$$\partial E(R,T) \;=\; \int_{-R}^{R} e(x,T)\,\mathrm{d}x - \int_{-R}^{R} e(x,0)\,\mathrm{d}x\ .$$

The energy balance (4.1) then implies

$$F(R,T) - F(-R,T) \;=\; \partial E(R,T) + D([-R,R],T)\ , \qquad R > 0\ . \tag{6.3}$$

On the other hand, proceeding as in (4.9) and using (5.6), we find for all $x \in \mathbb{R}$

$$F(x,T)^2 \;\le\; \beta\Bigl(\int_0^T e(x,t)\,\mathrm{d}t\Bigr)\Bigl(\int_0^T d(x,t)\,\mathrm{d}t\Bigr) \;\le\; \beta\kappa e_*(0)T\,D(x,T)\ ,$$

so that $\int_{-R}^{R} F(x,T)^2\,\mathrm{d}x \le \beta\kappa e_*(0)T\,D([-R,R],T)$. In view of Corollary 4.3, the assumption that u_0 is not an equilibrium implies that $e_*(0) > 0$, hence we deduce from (6.3) that

$$F(R,T) - F(-R,T) \;\ge\; \partial E(R,T) + \frac{1}{\beta\kappa e_*(0)T}\int_{-R}^{R} F(x,T)^2\,\mathrm{d}x\ . \tag{6.4}$$

Since $u_0 \in X$ is not an equilibrium, assumption (A3) implies that $D([-R,R],T) > 0$ for all sufficiently large $R > 0$, say for all $R \ge R_0 > 0$. On the other hand, by definition, we have $\partial E(R,T) \ge 0$ for all $R \in J_T$. Thus, using (6.3), we conclude that $F(R,T) - F(-R,T) > 0$ for all $R \in J_T \cap [R_0,\infty)$.

If $J_T \cap [R_0,\infty)$ is empty, the claim is proved. Otherwise, we choose $R_1 \in J_T \cap [R_0,\infty)$, and we define

$$\mathcal{F}(R) \;=\; \frac{1}{2(\beta\kappa e_*(0)T)^2}\int_{-R}^{R} F(x,T)^2\,\mathrm{d}x\ , \qquad R > 0\ .$$

The function $\mathcal{F} : (0,\infty) \to \mathbb{R}_+$ is nondecreasing and $\mathcal{F}(R) > 0$ for all $R \ge R_1$. Using (6.4) and the definition of J_T, we obtain

$$\begin{aligned}\mathcal{F}'(R) \;&=\; \frac{F(R,T)^2 + F(-R,T)^2}{2(\beta\kappa e_*(0)T)^2} \;\ge\; \frac{|F(R,T) - F(-R,T)|^2}{4(\beta\kappa e_*(0)T)^2}\\ &\ge\; \frac{\mathbf{1}_{J_T}(R)}{4(\beta\kappa e_*(0)T)^4}\Bigl(\int_{-R}^{R} F(x,T)^2\,\mathrm{d}x\Bigr)^2 \;=\; \mathbf{1}_{J_T}(R)\mathcal{F}(R)^2\ .\end{aligned}$$

Thus, for all $R > R_1$, we have

$$\int_{R_1}^{R} \mathbf{1}_{J_T}(x)\,\mathrm{d}x \;\le\; \int_{R_1}^{R} \frac{\mathcal{F}'(R)}{\mathcal{F}(R)^2}\,\mathrm{d}R \;\le\; \frac{1}{\mathcal{F}(R_1)} - \frac{1}{\mathcal{F}(R)} \;\le\; \frac{1}{\mathcal{F}(R_1)}\ .$$

This proves that $J_T \cap [R_1,\infty)$ has finite Lebesgue measure. $\square$

7 Pointwise estimates on the energy density

In Sections 4 and 5 we have shown that, under assumptions (A2') and (A5), the energy density associated to any solution of an extended dissipative system on $\mathbb{R}$ satisfies nice integral bounds, which are summarized in Proposition 5.3. A more difficult question is whether our hypotheses also imply a uniform estimate in time on the energy density. Before giving a partial result in that direction, we observe that some naive blow-up scenarios are already excluded by Propositions 5.3 and 6.1. For instance, if for some $x \in \mathbb{R}$ the energy density $e(x,t)$ is a nondecreasing function of time, then (5.6) implies that $e(x,t) \le \kappa e_*(0)$ for all $t \ge 0$. Indeed, for any $T > t$ we have

$$e(x,t) \;\le\; \frac{1}{T-t}\int_t^T e(x,\tau)\,\mathrm{d}\tau \;\le\; \frac{E_*(T)}{T-t} \;\le\; \frac{\kappa e_*(0)T}{T-t}\;,$$

and the claim follows by taking $T \to \infty$. Thus a standard scenario where the maximum of the energy density is reached at a fixed point $x \in \mathbb{R}$ and increases with time cannot lead to any unbounded growth. On the other hand, applying (6.1) with $R = \sqrt{\beta T}$, we obtain

$$\frac{1}{\sqrt{\beta T}}\int_{-\sqrt{\beta T}}^{\sqrt{\beta T}} e(x,T)\,\mathrm{d}x \;\le\; 2(1+\sqrt{\kappa})e_*(0)\;.$$

Thus, if for some $T > 0$ the energy density $e(x,T)$ is comparable to $e_*(T)$ over an interval of size $2\sqrt{\beta T}$, then $e_*(T)$ is in turn comparable to $e_*(0)$. This indicates that strong spatial inhomogeneities necessarily occur in unbounded solutions, if they exist.

To obtain a pointwise bound on the energy density in the abstract framework of extended dissipative systems, it appears necessary to introduce an additional assumption, which allows to control the spatial derivative of $e(x,t)$ at a given time. A reasonable possibility is:

(A6) $(\partial_x e)^2 \le \delta e$ for some $\delta > 0$.

Of course, (A6) follows from (A5) if the energy dissipation rate $d(x,t)$ is uniformly bounded, which is indeed the case in many applications. Under this hypothesis, we have the following result.

Proposition 7.1. *Let $u(t) = \Phi(t)u_0$ be any solution of an extended dissipative system on $\mathbb{R}$ satisfying (A2'), (A5), and (A6). There exists a constant $C > 0$, depending only on the product $\beta\gamma$, such that, for all $T > 0$,*

$$e_*(T) \;\le\; C\Big(e_*(0) + e_*(0)^{2/3}(\delta\beta T)^{1/3}\Big)\;. \tag{7.1}$$

Proof. We fix $T > 0$ and assume that $e_*(T) > 0$ (otherwise there is nothing to prove). Given $x_0 \in \mathbb{R}$, we have either $e(x_0,T) \le 4e_*(0)$, or $e(x_0,T) > 4e_*(0)$. In the latter case, we define $a = e(x_0,T)/\sqrt{\delta e_*(T)}$. Since $|\partial_x e(x,T)| \le \sqrt{\delta e_*(T)}$ by (A6), we have for all $x \in \mathbb{R}$

$$e(x,T) \;\ge\; e(x_0,T) - \sqrt{\delta e_*(T)}\,|x-x_0| \;=\; e(x_0,T)\Big(1 - \frac{|x-x_0|}{a}\Big)\;.$$

Thus, by (4.5), the available energy in the interval $[x_0-a, x_0+a]$ at time T satisfies

$$A([x_0-a,x_0+a],T) \;\ge\; \int_{x_0-a}^{x_0+a} e(x,T)\,\mathrm{d}x \;\ge\; ae(x_0,T)\;. \tag{7.2}$$

On the other hand, in view of (4.4) and Corollary 5.4, we also have

$$A([x_0-a,x_0+a],T) \;\le\; 2ae_*(0) + 2e_*(0)\sqrt{\kappa\beta T}\;. \tag{7.3}$$

Combining (7.2), (7.3) and recalling that $e(x_0,T) > 4e_*(0)$, we thus find

$$ae(x_0,T) \equiv \frac{e(x_0,T)^2}{\sqrt{\delta e_*(T)}} \le 4e_*(0)\sqrt{\kappa\beta T}\ . \tag{7.4}$$

Summarizing, given $x_0 \in \mathbb{R}$, we have shown that (7.4) holds whenever $e(x_0,T) > 4e_*(0)$. Since $e_*(T) = \sup_{x_0\in\mathbb{R}} e(x_0,T)$, we conclude that

$$e_*(T) \le \max\Bigl(4e_*(0)\,,\, (4e_*(0))^{2/3}(\delta\kappa\beta T)^{1/3}\Bigr)\ ,$$

and (7.1) follows. $\square$

In the particular case of the Navier-Stokes equation (1.1), it is possible to use Proposition 7.1 to prove Theorem 1.2, but this approach requires a uniform bound on the energy dissipation rate d which is not obvious a priori. In fact, if u is any solution of (1.1) in the space X defined by (3.1), we know from (2.10) that ∇u is uniformly bounded in the space $\mathrm{BMO}(\mathbb{O})$ for all $t \ge 0$, because the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ is bounded in $L^\infty(\mathbb{O})$, but this is not sufficient to control the energy dissipation (1.11) in $L^\infty(\mathbb{R})$. However, using the vorticity equation (2.4) and the fact that the only possibly unbounded component of the velocity field is the vertical average $m = \langle u_2\rangle$, it is possible to prove that the vorticity ω is uniformly bounded in some Hölder space $C^\alpha(\mathbb{O})$ for all $t \ge t_0 > 0$, where $\alpha > 0$, and using the Biot-Savart formula we can deduce that ∇u is uniformly bounded in $L^\infty(\mathbb{O})$ for all $t \ge 0$. This implies that the energy dissipation $d(x_1,t)$ is bounded in $L^\infty(\mathbb{R})$, so that (A6) follows from (A2'), and Proposition 7.1 allows us to conclude that the energy density $e(x_1,t)$ cannot grow faster than $t^{1/3}$ as $t \to \infty$, which shows (1.4).

Alternatively, Theorem 1.2 can be established by the following direct argument, which does not rely on Proposition 7.1.

Proof of Theorem 1.2. Let $u(x,t)$ be a solution of the Navier-Stokes equations (1.1), (1.2) given by Theorem 1.1. Since we are interested in the long-time behavior, we can assume without loss of generality that the initial data u_0 belong to the set X_M for some $M > 0$, see definition (3.2) and Proposition 2.2. Also, we suppose that u is decomposed as in (2.5) with $(m_1,m_2) = (0,m)$. We already know that, for all $t \ge 0$,

$$\|\omega(\cdot,t)\|_{L^\infty(\mathbb{O})} \le M\ , \qquad \text{and} \qquad \|\widehat{u}(\cdot,t)\|_{L^\infty(\mathbb{O})} \le C_1 M\ , \tag{7.5}$$

see Lemma 2.3. Since $\partial_1 m = \langle\omega\rangle$, it follows that $|\partial_1 m(x_1,t)| \le M$ for all $x_1 \in \mathbb{R}$ and all $t \ge 0$.

In the subsequent calculations, we fix a time $t > 0$ and, for simplicity, we denote the space variable by x instead of x_1. Given $a > 0$, we have for all $x \in \mathbb{R}$:

$$|m(x,t)| \le \frac{Ma}{2} + \frac{1}{2a}\int_{x-a}^{x+a} |m(y,t)|\,\mathrm{d}y\ , \tag{7.6}$$

because

$$\frac{1}{2a}\int_{x-a}^{x+a}\Bigl(|m(x,t)| - |m(y,t)|\Bigr)\,\mathrm{d}y \le \frac{1}{2a}\int_{x-a}^{x+a} M|x-y|\,\mathrm{d}y = \frac{Ma}{2}\ .$$

To bound the last term in (7.6), we observe that $|u|^2 = \widehat{u}_1^2 + (m+\widehat{u}_2)^2 \ge \frac{1}{2}m^2 - \widehat{u}_2^2$. Integrating that inequality with respect to the vertical variable and using (1.9), (7.5), we easily obtain

$$m(x,t)^2 \le 4e(x,t) + C\ , \qquad x \in \mathbb{R}\ , \tag{7.7}$$

where $C = 2C_1^2 M^2$. Thus, if we apply Hölder's inequality to the integral in (7.6) and use (7.7), we arrive at

$$|m(x,t)| \le \frac{Ma}{2} + \Bigl(\frac{2}{a}\int_{x-a}^{x+a} e(y,t)\,\mathrm{d}y + C\Bigr)^{1/2}, \qquad x \in \mathbb{R}. \tag{7.8}$$

Finally, we know from Lemma 3.4 and Proposition 3.5 that the Navier-Stokes equation in X_M defines an extended dissipative system satisfying (A2') for some $\beta > 0$ (depending on M) and (A5) with $\gamma = 2$. Thus, proceeding as in Proposition 6.1, we find

$$\frac{2}{a}\int_{x-a}^{x+a} e(y,t)\,\mathrm{d}y \le \frac{2}{a}\Bigl(2ae_*(0) + 2e_*(0)\sqrt{\kappa\beta t}\Bigr) = 4e_*(0)\Bigl(1 + \frac{\sqrt{\kappa\beta t}}{a}\Bigr),$$

for all $x \in \mathbb{R}$. After replacing this inequality in the right-hand side of (7.8) and taking the supremum over $x \in \mathbb{R}$, we conclude that

$$\sup_{x\in\mathbb{R}} |m(x,t)| \le \frac{Ma}{2} + \left(C + 4e_*(0)\Bigl(1 + \frac{\sqrt{\kappa\beta t}}{a}\Bigr)\right)^{1/2}. \tag{7.9}$$

If we now take $a = t^{1/6}$, we see from (7.9) that $\|m(\cdot,t)\|_{L^\infty} \le C'(1+t)^{1/6}$ for some $C' > 0$. Since $\widehat{u}$ is uniformly bounded, this proves (1.4). □

8 Convergence results for the Navier-Stokes equations

This final section is entirely devoted to the particular example of the Navier-Stokes equations in the cylinder $\mathbb{O} = \mathbb{R} \times \mathbb{T}$. Our goal is to use the results of Sections 4 to 6 to obtain qualitative informations on the long-time behavior of the solutions. Without loss of generality, we fix $M > 0$ and work in the function space X_M defined by (3.2), where equations (1.1), (1.2) define an extended dissipative system satisfying assumptions (A2'), (A5) for some constants β, γ. In particular, applying Proposition 5.3 and using the explicit formula (1.9) for the energy density, we obtain the estimate

$$\sup_{x_1\in\mathbb{R}} \int_0^T \int_{\mathbb{T}} \frac{1}{2}|u(x_1,x_2,t)|^2\,\mathrm{d}x_2\,\mathrm{d}t \le \kappa e_*(0)T,$$

which proves (1.6). Similarly, if we denote $B_R = [-R,R] \times \mathbb{T}$, then Proposition 6.1 implies that

$$\int_{B_R} \frac{1}{2}|u(x,T)|^2\,\mathrm{d}x + \int_0^T \int_{B_R} |\nabla u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \le 2e_*(0)(R + \sqrt{\kappa\beta T}), \tag{8.1}$$

which is (1.7). Thus, to complete the proof of Theorem 1.3, it remains to establish (1.8).

Let $\mathcal{E} \subset X$ denote the set of equilibria of the Navier-Stokes equation (1.1) in X, namely the set of all constant velocity fields of the form $u = (0,m)^{\mathrm{t}}$, with $m \in \mathbb{R}$. Given $u \in X$ and $R > 0$, we define the distance from u to $\mathcal{E}$ on the finite cylinder $B_R = [-R,R] \times \mathbb{T}$ as

$$d_R(u,\mathcal{E}) = \inf_{m\in\mathbb{R}} \sup_{x\in B_R} |u(x) - (0,m)^{\mathrm{t}}|. \tag{8.2}$$

The following estimate will be useful:

Lemma 8.1. *Fix $\theta \in (0,1)$. There exists $C_5 > 0$ such that, for any $u \in X_M$ and any $R \ge 1$, one has*

$$d_R(u,\mathcal{E}) \le C_5 M^\theta R^{\frac{1+\theta}{2}} \|\nabla u\|_{L^2(B_R)}^{1-\theta}. \tag{8.3}$$

Proof. We decompose $u = (0,m)^{\mathrm{t}} + \widehat{u}$, where $m = \langle u_2 \rangle$. Since $\partial_1 m = \langle \omega \rangle$ and $|\omega| \le M$, we have using Hölder's inequality

$$\sup_{|x_1|\le R} |m(x_1) - m(0)| \;\le\; \int_{B_R} |\omega| \,\mathrm{d}x \;\le\; M^\theta \int_{B_R} |\omega|^{1-\theta} \,\mathrm{d}x \;\le\; C M^\theta R^{\frac{1+\theta}{2}} \|\omega\|_{L^2(B_R)}^{1-\theta} \;. \tag{8.4}$$

On the other hand, since $\widehat{u}$ has zero mean over B_R, the Sobolev embedding theorem and the Poincaré-Wirtinger inequality imply that, if $2 < p < \infty$,

$$\|\widehat{u}\|_{L^\infty(B_R)} \;\le\; C_p R^{1-\frac{1}{p}} \|\nabla \widehat{u}\|_{L^p(B_R)} \;,$$

where the constant depends only on p (here we use the assumption that $R \ge 1$). Moreover, interpolating between L^2 and BMO and using (2.10), we obtain

$$\|\nabla \widehat{u}\|_{L^p(B_R)} \;\le\; C_p \|\nabla \widehat{u}\|_{L^2(B_R)}^{\frac{2}{p}} \|\nabla \widehat{u}\|_{\mathrm{BMO}(\mathbb{O})}^{1-\frac{2}{p}} \;\le\; C_p M^{1-\frac{2}{p}} \|\nabla \widehat{u}\|_{L^2(B_R)}^{\frac{2}{p}} \;.$$

Choosing $p = 2/(1-\theta)$, we thus find

$$\|\widehat{u}\|_{L^\infty(B_R)} \;\le\; C M^\theta R^{\frac{1+\theta}{2}} \|\nabla \widehat{u}\|_{L^2(B_R)}^{1-\theta} \;. \tag{8.5}$$

If we now combine (8.4) and (8.5), we obtain

$$d_R(u,\mathcal{E}) \;\le\; \sup_{x\in B_R} |u(x) - (0,m(0))^{\mathrm{t}}| \;\le\; C_5 M^\theta R^{\frac{1+\theta}{2}} \|\nabla u\|_{L^2(B_R)}^{1-\theta} \;,$$

where $C_5 > 0$ depends only on θ. This is the desired estimate. $\square$

The distance (8.2) allows us to introduce the following family of neighborhoods of the set of equilibria. Given $\epsilon > 0$ and $R > 0$, we denote

$$\mathcal{U}_{\epsilon,R} \;=\; \{u \in X \,|\, d_R(u,\mathcal{E}) < \epsilon\} \;.$$

Using estimate (8.1) and Lemma 8.1, we now show that any solution of the Navier-Stokes equation in X_M spends a relatively small fraction of its lifetime outside $\mathcal{U}_{\epsilon,R}$, even if $\epsilon > 0$ is very small and $R > 0$ very large. More precisely, we have

Proposition 8.2. *Fix $\theta \in (0,1)$ and $M > 0$. There exists $C_6 > 0$ such that, if $u \in C^0([0,\infty), X)$ is any solution of the Navier-Stokes equations (1.1), (1.2) with initial data in X_M, the following estimate holds for any $\epsilon > 0$, any $R \ge 1$, and any $T > 0$:*

$$\int_0^T \mathbf{1}_{\mathcal{U}^c_{\epsilon,R}}(u(t)) \,\mathrm{d}t \;\le\; \frac{C_6}{\epsilon} \,(RT)^{\frac{1+\theta}{2}} \Bigl(e_*(0)(R + \sqrt{\kappa\beta T})\Bigr)^{\frac{1-\theta}{2}} \;, \tag{8.6}$$

where $\mathbf{1}_{\mathcal{U}^c_{\epsilon,R}}$ is the characteristic function of the complement of $\mathcal{U}_{\epsilon,R}$.

Proof. Using the definition of the set $\mathcal{U}_{\epsilon,R}$ and estimate (8.3), we easily find

$$\int_0^T \mathbf{1}_{\mathcal{U}^c_{\epsilon,R}}(u(t)) \,\mathrm{d}t \;\le\; \frac{1}{\epsilon} \int_0^T d_R(u(t),\mathcal{E}) \,\mathrm{d}t \;\le\; \frac{C_5}{\epsilon} \,M^\theta R^{\frac{1+\theta}{2}} \int_0^T \|\nabla u(t)\|_{L^2(B_R)}^{1-\theta} \,\mathrm{d}t \;.$$

Moreover, Hölder's inequality and estimate (8.1) imply

$$\int_0^T \|\nabla u(t)\|_{L^2(B_R)}^{1-\theta} \,\mathrm{d}t \;\le\; \biggl(\int_0^T \|\nabla u(t)\|_{L^2(B_R)}^2 \,\mathrm{d}t\biggr)^{\frac{1-\theta}{2}} T^{\frac{1+\theta}{2}} \;\le\; \Bigl(2e_*(0)(R + \sqrt{\kappa\beta T})\Bigr)^{\frac{1-\theta}{2}} T^{\frac{1+\theta}{2}} \;.$$

Combining both inequalities, we arrive at (8.6). $\square$

There are several ways to exploit the conclusion of Proposition 8.2. If we fix ϵ, R and take θ sufficiently small, we obtain the following result which already implies estimate (1.8) in Theorem 1.3.

Corollary 8.3. *Any solution $u \in C^0([0,\infty), X)$ of the Navier-Stokes equations* (1.1), (1.2) *with initial data in X_M satisfies*

$$\limsup_{T\to\infty} \frac{1}{T^{\frac{3+\theta}{4}}} \int_0^T \mathbf{1}_{\mathcal{U}^c_{\epsilon,R}}(u(t))\,\mathrm{d}t \;<\; \infty\ ,$$

for all $\epsilon > 0$, all $R \ge 1$, and all $\theta > 0$.

It is also interesting to consider a time-dependent domain $B_{R(T)}$ whose size increases (sufficiently slowly) as $T \to \infty$. In that case, we can still show that any solution of (1.1) converges to the set of equilibria inside $B_{R(T)}$, except perhaps on a sparse subset of the time axis.

Corollary 8.4. *Fix $a, b, c > 0$ such that $a/2 + b < c < 1/4$. If $u \in C^0([0,\infty), X)$ is any solution of the Navier-Stokes equations* (1.1), (1.2) *with initial data in X_M, there exists $C_7 > 0$ such that, for all $T \ge 1$,*

$$\mathrm{meas}\left\{ t \in [0,T] \;\middle|\; \inf_{m\in\mathbf{R}} \sup_{|x_1|\le T^a} \sup_{x_2\in\mathbb{T}} |u(x_1,x_2) - (0,m)^{\mathrm{t}}| \ge \frac{1}{T^b} \right\} \;\le\; C_7 T^{\frac{3}{4}+c}\ . \tag{8.7}$$

Proof. If we set $R = T^a$ and $\epsilon = T^{-b}$, the quantity in the left-hand side of (8.7) is exactly the integral $\int_0^T \mathbf{1}_{\mathcal{U}^c_{\epsilon,R}}(u(t))\,\mathrm{d}t$. Using (8.6) and the fact that $R = T^a \le \sqrt{T}$ since $a < 1/2$ and $T \ge 1$, we easily obtain

$$\int_0^T \mathbf{1}_{\mathcal{U}^c_{\epsilon,R}}(u(t))\,\mathrm{d}t \;\le\; C(e_*(0))^{\frac{1-\theta}{2}} T^{\frac{3}{4}+\frac{a}{2}+b+\frac{\theta}{4}(1+2a)}\ .$$

The conclusion now follows if we take $\theta > 0$ small enough. □

9 Appendix

9.1 Proof of Lemma 2.1

We start from (2.2) and recall that $\mathbb{P}_{jk} = \delta_{jk} + R_j R_k$, where R_1, R_2 are the Riesz transforms. It follows that $\nabla \cdot e^{t\Delta}\mathbb{P}(u \otimes v) = \nabla \cdot e^{t\Delta}(u \otimes v) + W(t,u,v)$, where

$$W_j(t,u,v) \;=\; \sum_{k,\ell=1}^{2} \partial_\ell\, e^{t\Delta} R_j R_k u_\ell v_k \;=\; \sum_{k,\ell=1}^{2} \int_t^\infty \partial_j \partial_k \partial_\ell\, e^{s\Delta} u_\ell v_k \,\mathrm{d}s\ , \qquad j = 1,2\ .$$

In the last equality, we used the fact that $R_j R_k \Delta = -\partial_j \partial_k$ for $j,k = 1,2$. Now, for any $g \in L^\infty(\mathbb{O})$ and any $t > 0$, we know that $e^{t\Delta} g \in C^\infty(\mathbb{O})$ and $\|\partial^\alpha e^{t\Delta} g\|_{L^\infty} \le C_\alpha t^{-|\alpha|/2} \|g\|_{L^\infty}$ for all $\alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^2$ with $|\alpha| = |\alpha_1| + |\alpha_2| > 0$. If $u, v \in \mathrm{BUC}(\mathbb{O})$, we thus have

$$\|\nabla \cdot e^{t\Delta}\mathbb{P}(u\otimes v)\|_{L^\infty} \;\le\; \frac{C}{\sqrt{t}}\|u\|_{L^\infty}\|v\|_{L^\infty} + \int_t^\infty \frac{C}{s^{3/2}}\|u\|_{L^\infty}\|v\|_{L^\infty}\,\mathrm{d}s \;\le\; \frac{C}{\sqrt{t}}\|u\|_{L^\infty}\|v\|_{L^\infty}\ ,$$

which proves (2.3). The same argument shows that $\nabla \cdot e^{t\Delta}\mathbb{P}(u\otimes v) \in \mathrm{BUC}(\mathbb{O})$. □

9.2 Proof of Proposition 2.2

Given $u_0 \in \mathrm{BUC}(\mathbb{O})$ with $\operatorname{div} u_0 = 0$, we take $R > 0$ and $T > 0$ such that $2\|u_0\|_{L^\infty} \le R$ and $4C_0RT^{1/2} < 1$, where $C_0 > 0$ is the constant in Lemma 2.1. We introduce the Banach space $X = C^0([0,T], \mathrm{BUC}(\mathbb{O}))$ equipped with the norm

$$\|u\|_X \;=\; \sup_{0\le t\le T} \|u(t)\|_{L^\infty(\mathbb{O})}\;,$$

and we set $B_R = \{u \in X \,|\, \|u\|_X \le R\}$. For all $u \in X$ and all $t \in [0,T]$, we denote by $(Fu)(t)$ the expression in the right-hand side of (2.1).

If $u \in B_R$, then by Lemma 2.1 we have for all $t \in [0,T]$:

$$\|(Fu)(t)\|_{L^\infty} \;\le\; \|u_0\|_{L^\infty} + \int_0^t \frac{C_0}{\sqrt{t-s}}\,\|u(s)\|_{L^\infty}^2 \,\mathrm{d}s \;\le\; \|u_0\|_{L^\infty} + 2C_0T^{1/2}R^2 \;\le\; R\;.$$

Thus F maps B_R into itself, and a similar calculation shows that $\|Fu - Fv\|_X \le \kappa\|u-v\|_X$ for all $u, v \in B_R$, where $\kappa = 4C_0RT^{1/2} < 1$. Thus Eq. (2.1) has a unique solution in B_R, and applying Gronwall's lemma it is easy to verify that u is also the unique solution of (2.1) in the whole space X. Finally, proceeding as in [8], one can prove that $t^{1/2}\nabla u \in C_b^0((0,T], \mathrm{BUC}(\mathbb{O}))$. □

9.3 Proof of Lemma 2.3

We first observe that, since $\widehat{\omega}$ has zero average in the vertical variable, the Biot-Savart formula (2.6) can be written in the equivalent form $\widehat{u} = \nabla^\perp \overline{K} * \widehat{\omega}$, where

$$\overline{K}(x_1,x_2) \;=\; K(x_1,x_2) - \frac{|x_1|}{2}\;, \qquad (x_1,x_2) \in \mathbb{O}\;.$$

Now it is easy to verify that $\overline{K} \in L^1(\mathbb{O})$ and $\partial_j\overline{K} \in L^1(\mathbb{O})$ for $j = 1,2$, see [1]. Using Young's inequality, we deduce

$$\|\widehat{u}_1\|_{L^\infty} \;\le\; \|\partial_2 K\|_{L^1}\|\widehat{\omega}\|_{L^\infty}\;, \qquad \|\widehat{u}_2\|_{L^\infty} \;\le\; \|\partial_1\overline{K}\|_{L^1}\|\widehat{\omega}\|_{L^\infty}\;, \tag{9.1}$$

and the first inequality in (2.9) follows since $\|\widehat{\omega}\|_{L^\infty} \le 2\|\omega\|_{L^\infty}$.

The next step is to establish the formula (2.8) for the pressure. The easiest way is to use the identity

$$\operatorname{div}((u\cdot\nabla)u) \;=\; \Delta(u_1^2) + 2\partial_2(\omega u_1)\;, \tag{9.2}$$

which holds for any divergence-free vector field $u = (u_1,u_2)^{\mathrm{t}}$ with vorticity $\omega = \partial_1u_2 - \partial_2u_1$. If we define $p = -u_1^2 - 2\partial_2K * (\omega u_1)$, where K is the fundamental solution of the Laplace operator, it follows immediately from (9.2) that $-\Delta p = \operatorname{div}((u\cdot\nabla)u)$, which is the desired result. Alternatively, it is possible to derive (2.8) directly from the formal expression (1.2). Now, using (2.8), (9.1), and the fact that $u_1 = \widehat{u}_1$, we find

$$\|p\|_{L^\infty} \;\le\; \|\widehat{u}_1\|_{L^\infty}^2 + 2\|\partial_2K\|_{L^1}\|\omega\|_{L^\infty}\|\widehat{u}_1\|_{L^\infty} \;\le\; C\|\partial_2K\|_{L^1}^2\|\omega\|_{L^\infty}^2\;.$$

This proves the second inequality in (2.9).

Finally, to estimate ∇u, we observe that

$$\partial_1u_1 \;=\; -\partial_2u_2 \;=\; R_1R_2\omega\;, \quad \partial_1u_2 = -R_1^2\omega\;, \quad \partial_2u_1 = R_2^2\omega\;,$$

and we use the well-known fact that the Riesz operators are bounded from $L^\infty(\mathbb{O})$ to $\mathrm{BMO}(\mathbb{O})$, see [14, Chapter IV]. This concludes the proof of Lemma 2.3. □

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