Global Existence and Long-Time Asymptotics for Rotating Fluids in a 3D layer

Thierry Gallay
Institut Fourier (UMR CNRS 5582)
Université de Grenoble I
B.P. 74
38402 Saint-Martin-d'Hères, France
thierry.gallay@ujf-grenoble.fr

Violaine Roussier-Michon
Institut de Mathématiques de Toulouse (UMR CNRS 5219)
INSA Toulouse
135 av Rangueil
31077 Toulouse cedex 4, France
roussier@math.univ-toulouse.fr

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Abstract

The Navier-Stokes-Coriolis system is a simple model for rotating fluids, which allows to study the influence of the Coriolis force on the dynamics of three-dimensional flows. In this paper, we consider the NSC system in an infinite three-dimensional layer delimited by two horizontal planes, with periodic boundary conditions in the vertical direction. If the angular velocity parameter is sufficiently large, depending on the initial data, we prove the existence of global, infinite-energy solutions with nonzero circulation number. We also show that these solutions converge toward two-dimensional Lamb-Oseen vortices as $t \to \infty$.

1 Introduction

In recent years a lot of activity has been devoted to the mathematical study of geophysical flows, and in particular to various models of rotating fluids. Taking advantage of the stratification effect due to the Coriolis force, significant results have been obtained which are still out of reach for the usual Navier-Stokes system, such as global existence of solutions for large initial data [1, 3] and stability of boundary layers for small viscosities [10, 15]. We refer the interested reader to the recent monograph [4] which contains a general introduction to geophysical flows, an overview of the mathematical theory, and an extensive bibliography.

In this article we study the so-called Navier-Stokes-Coriolis (NSC) system in a three-dimensional layer delimited by two infinite horizontal planes, assuming as usual that the rotation vector is constant and aligned with the vertical axis. This is a reasonably simple model for the motion of the ocean in a small geographic zone at mid-latitude, where the variation of the Coriolis force due to the curvature of Earth can be neglected. More realistic systems exist which take into account the variations of temperature and salinity inside the ocean, and include boundary effects modelling the influence of coasts, the topography of the bottom, or the action of the wind at the free surface, see [9, 17]. Nevertheless, keeping only the Coriolis force is meaningful in a first approximation, because its effect is very important on the ocean's motion at a global scale due to the fast rotation of Earth compared to typical velocities in the ocean.

Our main goal is to investigate the long-time behavior of the solutions to the NSC system for a fixed, but typically large, value of the rotation speed. As in [1, 3] we shall use the effect of the Coriolis force to prove global existence of solutions for large initial data, but the long-time asymptotics of those solutions turn out to be essentially two-dimensional and are therefore not affected by the rotation. Thus we shall recover as a leading term in our expansion the Lamb-Oseen vortex which plays a similar role for the usual Navier-Stokes system in the plane $\mathbb{R}^2$ [8] or the three-dimensional layer $\mathbb{R}^2 \times (0,1)$ [19]. To avoid all problems related to boundary layers, we shall always assume that the fluid motion is *periodic* in the vertical direction. This hypothesis has no physical justification and is only a convenient mathematical way to disregard the influence of the boundaries. Although boundary conditions do play an important role in the problem we study and will have to be considered ultimately, in this paper we chose to focus on the motion of the fluid in the bulk.

We thus consider the Navier-Stokes-Coriolis system in the three-dimensional layer $\mathbb{D} = \mathbb{R}^2 \times \mathbb{T}^1$, where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus. The points of $\mathbb{D}$ will be denoted by (x, z), where $x = (x_1, x_2) \in \mathbb{R}^2$ is the horizontal variable and $z \in \mathbb{T}^1$ is the vertical coordinate. The system reads

$$\partial_t u + (u \cdot \nabla)u + \Omega e_3 \wedge u \;=\; \Delta u - \nabla p \;, \qquad \operatorname{div} u \;=\; 0 \;, \tag{1}$$

where $u = u(t, x, z) \in \mathbb{R}^3$ is the velocity field of the fluid, and $p = p(t, x, z) \in \mathbb{R}$ is the pressure field. Here and in what follows, it is understood that differential operators such as ∇ or Δ act on all spatial variables (x, z), unless otherwise indicated. System (1) differs from the usual incompressible Navier-Stokes equations by the presence of the Coriolis term $\Omega e_3 \wedge u$, where $\Omega \in \mathbb{R}$ is a parameter and $e_3 = (0, 0, 1)^t$ is the unit vector in the vertical direction. This term is due to the fact that our reference frame rotates with constant angular velocity $\Omega/2$ around the vertical axis. Note that (1) does not contain any centrifugal force, because this effect can be included in the pressure term $-\nabla p$. For simplicity, the kinematic viscosity of the fluid has been rescaled to 1, and the fluid density has been incorporated in the definition of the pressure p.

As in the ordinary Navier-Stokes system, the role of the pressure in (1) is to enforce the incompressibility condition $\operatorname{div} u = 0$. To eliminate the pressure, one can apply to both sides the Leray projector $\mathbb{P}$, which is just the orthogonal projector in $L^2(\mathbb{D})^3$ onto the space of divergence-free vector fields. This operator has a rather simple expression in Fourier variables, which will be given in Appendix A. The projected equation then reads:

$$\partial_t u + \mathbb{P}((u\cdot\nabla)u) + \Omega\mathbb{P}(e_3\wedge u) \,=\, \Delta u\ , \qquad \operatorname{div} u \,=\, 0\ . \tag{2}$$

Another possibility is to consider the *vorticity field* $\omega = \operatorname{curl} u$, which satisfies the following evolution equation:

$$\partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u - \Omega\partial_z u \,=\, \Delta\omega\ . \tag{3}$$

Due to the incompressibility condition, the velocity field u can be reconstructed from the vorticity ω using the Biot-Savart law, which in the domain $\mathbb{D}$ has also a simple expression, see Appendix A.

As is clear from (3), the vertical coordinate z plays a distinguished role in our problem because the rotation acts trivially on z-independent velocity fields. As a matter of fact, even if rotation is absent, the linear evolution $\partial_t u = \Delta u$ leads to an exponential decay of the fluctuations of u in the vertical direction, due to the Poincaré inequality. For these reasons, it is appropriate to decompose the velocity field as $u(t,x,z) = \bar u(t,x) + \tilde u(t,x,z)$, where

$$\bar u(t,x) \,=\, (Qu)(t,x) \,\equiv\, \int_{\mathbb{T}^1} u(t,x,z)\,\mathrm{d}z \tag{4}$$

is the average of u with respect to the vertical variable, and the remainder $\tilde u = (1-Q)u$ has zero vertical average. We shall say that $\bar u$ is a *two-dimensional* vector field in the sense that it depends only on the spatial variable $x\in\mathbb{R}^2$, not on z, but one should keep in mind that $\bar u$ is not necessarily *horizontal* because its third component $\bar u_3$ is usually nonzero. A similar decomposition holds for the vorticity, and it is easy to verify that $\bar\omega = \operatorname{curl}\bar u$ and $\tilde\omega = \operatorname{curl}\tilde u$. In particular, since $\partial_1\bar u_1 + \partial_2\bar u_2 = 0$ and $\partial_1\bar u_2 - \partial_2\bar u_1 = \bar\omega_3$, the horizontal part of the two-dimensional velocity field $\bar u$ can be reconstructed from the third component of the vorticity $\bar\omega$ via the two-dimensional Biot-Savart law, see Appendix A. This means that the averaged velocity field $\bar u(t,x)$ can be represented by two scalar quantities, namely $\bar u_3(t,x)$ and $\bar\omega_3(t,x)$.

We shall solve the Cauchy problem for equation (2) in the Banach space X defined by

$$X \,=\, \Big\{u\in H^1_{\mathrm{loc}}(\mathbb{D})^3\ \Big|\ \operatorname{div} u = 0\,,\ \tilde u\in H^1(\mathbb{D})^3\,,\ \bar u_3\in H^1(\mathbb{R}^2)\,,\ \bar\omega_3\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)\Big\}\ , \tag{5}$$

equipped with the norm

$$\|u\|_X \,=\, \|\tilde u\|_{H^1(\mathbb{D})} + \|\bar u_3\|_{H^1(\mathbb{R}^2)} + \|\bar\omega_3\|_{L^1(\mathbb{R}^2)} + \|\bar\omega_3\|_{L^2(\mathbb{R}^2)}\ .$$

Observe that $X\not\subset H^1(\mathbb{D})^3$, because the two-dimensional horizontal velocity field $\bar u_h = (\bar u_1,\bar u_2)$ is not assumed to be square integrable. This slightly unusual choice is motivated by our desire to include *infinite-energy* solutions, which play a crucial role in the long-time asymptotics of the Navier-Stokes equations [7, 8]. The most important example of such a solution is the *Lamb-Oseen* vortex, whose velocity and vorticity fields are given by the following expressions:

$$u^G(t,x) \,=\, \frac{1}{\sqrt{1+t}}\,U^G\Big(\frac{x}{\sqrt{1+t}}\Big)\ ,\quad \text{where}\quad U^G(\xi) \,=\, \frac{1-e^{-|\xi|^2/4}}{2\pi|\xi|^2}\begin{pmatrix}-\xi_2\\ \xi_1\\ 0\end{pmatrix}\ , \tag{6}$$

$$\omega^G(t,x) \,=\, \frac{1}{1+t}\,G\Big(\frac{x}{\sqrt{1+t}}\Big)\ ,\qquad \text{where}\quad G(\xi) \,=\, \frac{1}{4\pi}\,e^{-|\xi|^2/4}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\ . \tag{7}$$

As is easily verified, for any $\alpha\in\mathbb{R}$ and any $\Omega\in\mathbb{R}$, the vortex $u(t,x,z) \,=\, \alpha u^G(t,x)$ is an exact solution of the NSC system (2). In fact, one has $\mathbb{P}(u^G\cdot\nabla)u^G = 0$ and $\mathbb{P}(e_3\wedge u^G) = 0$, so that u^G solves the linear heat equation $\partial_t u = \Delta u$.

We are now in position to formulate our main result:

Theorem 1.1 *For any initial data $u_0 \in X$, there exists $\Omega_0 \geq 0$ such that, for all $\Omega \in \mathbb{R}$ with $|\Omega| \geq \Omega_0$, the NSC system (2) has a unique global (mild) solution $u \in C^0([0,\infty), X)$ satisfying $u(0) = u_0$. Moreover $\|u(t,\cdot) - \alpha u^G(t,\cdot)\|_X \to 0$ as $t \to \infty$, where*

$$\alpha = \int_{\mathbb{D}} (\operatorname{curl} u_0)_3 \,\mathrm{d}x\,\mathrm{d}z \,. \tag{8}$$

This theorem contains in fact two different statements. The first one is the existence of global strong solutions to the NSC system (2) for arbitrarily large initial data in X, provided that the rotation speed $|\Omega|$ is sufficiently large (depending on the data). To prove this, we closely follow the existence results that have been established for rotating fluids in the whole space $\mathbb{R}^3$, see [4, Chapter 5]. In particular, if the three-dimensional part $\tilde{u}$ of the solution is not small at initial time, we assume that the rotation speed $|\Omega|$ is large enough so that $\tilde{u}$ is rapidly damped by the dispersive effect of the linearized equation

$$\partial_t \tilde{u} + \Omega \mathbb{P}(e_3 \wedge \tilde{u}) \,=\, \Delta \tilde{u} \,, \quad \operatorname{div} \tilde{u} \,=\, 0 \,. \tag{9}$$

For the reader's convenience, we briefly recall in Section 2.2 and Appendix B the Strichartz estimates satisfied by the solutions of (9) with compact support in Fourier space. Except for the choice of the spatial domain, the main difference of our approach with respect to [4] is that we do not assume that the whole velocity field u belongs to $L^2(\mathbb{D})^3$. As a consequence, we cannot use the energy inequality which plays an important role in the classical approach. To guarantee that the two-dimensional Navier-Stokes system has uniformly bounded solutions, the hypothesis $\bar{u}_h = (\bar{u}_1, \bar{u}_2)^t \in L^2(\mathbb{R}^2)^2$ is replaced by $\bar{\omega}_3 \in L^1(\mathbb{R}^2)$, a condition which allows for solutions with nonzero total circulation such as the Oseen vortex (6), (7).

The second part of Theorem 1.1, which concerns the long-time behavior of the solutions, is more in the spirit of the previous works [8, 19]. When stated more explicitly, our result shows that the solution $u(t,x,z)$ satisfies

$$\|\tilde{u}(t)\|_{H^1(\mathbb{D})} + \|\bar{u}_3(t)\|_{H^1(\mathbb{R}^2)} + \|\bar{\omega}_3(t)\|_{L^2(\mathbb{R}^2)} \xrightarrow[t\to\infty]{} 0 \,,$$

and

$$\Big\|\bar{\omega}_3(t) - \frac{\alpha}{1+t}\, g\Big(\frac{\cdot}{\sqrt{1+t}}\Big)\Big\|_{L^1(\mathbb{R}^2)} \xrightarrow[t\to\infty]{} 0 \,, \tag{10}$$

where $g(\xi) \,=\, (4\pi)^{-1}\, e^{-|\xi|^2/4}$. In particular, if the total circulation α is nonzero, we see that $\bar{\omega}_3(t)$ does not converge to zero in the (scale invariant) space $L^1(\mathbb{R}^2)$, but to the Oseen vortex with circulation α, which is thus the leading term in the asymptotic expansion of the solution as $t \to \infty$. This is in contrast with the case of finite-energy solutions, which always converge to zero in the energy norm.

We conclude this introduction with a few additional remarks on the scope of Theorem 1.1:

1) As is well-known, it is possible to prove the existence of solutions to the NSC system (2) under weaker assumptions on the initial data. For instance, it is sufficient to suppose that $\tilde{u}(0) \in H^{1/2}(\mathbb{D})^3$, $\bar{u}_3(0) \in L^2(\mathbb{R}^2)$, and $\bar{\omega}_3(0) \in L^1(\mathbb{R}^2)$, in which case the solution $u(t)$ will belong to X for any positive time. Since we are mainly interested in the long-time behavior of the solutions, we disregard these technical details and prefer working directly in the (noncritical) space X.

2) Theorem 1.1 does not give any information on the convergence rate towards Oseen's vortex. The proof shows that $\|\nabla \bar{u}_3(t)\|_{L^2(\mathbb{R}^2)} + \|\bar{\omega}_3(t)\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(t^{-1/2})$ and $\|\tilde{u}(t)\|_{H^1(\mathbb{D})} = \mathcal{O}(e^{-\nu t})$ for all $\nu < 4\pi^2$ as $t \to \infty$, but without additional assumptions on the data it is impossible to specify the decay rate of $\|\bar{u}_3(t)\|_{L^2(\mathbb{R}^2)}$ or the convergence rate in (10). However, algebraic convergence rates can be obtained if we assume that the initial data $\bar{u}_0(x)$ decay sufficiently fast as $|x| \to \infty$, see [8, 19].

3) In the proof of Theorem 1.1 we need a large rotation speed Ω only to prove the existence of a global solution, in the case where $\tilde{u}_0 = (1-Q)u_0$ is not small. Once existence has been established, the convergence to Oseen's vortex holds for any value of Ω and does not rely on the Coriolis force at all. Since our domain $\mathbb{D}$ has finite extension in the vertical direction, we can use the Poincaré inequality to show that $\tilde{u}(t)$ converges exponentially to zero as $t \to \infty$, but this point is not crucial: Our proof can be adapted to cover the case of the whole space $\mathbb{R}^3$, if we assume as in [4] that $u = \bar{u} + \tilde{u}$ with $\tilde{u} \in H^1(\mathbb{R}^3)$, or even $\tilde{u} \in \dot{H}^{1/2}(\mathbb{R}^3)$. In this situation the decay of $\tilde{u}(t)$ will not be exponential.

4) As is explained in [19], we can prove the analog of Theorem 1.1 in the layer $\mathbb{R}^2 \times (0,1)$ with different bounday conditions, for instance stress-free conditions. The case of no-slip (Dirichlet) boundary conditions is very different, because the solutions will converge exponentially to zero as $t \to \infty$, and the Oseen vortices can only appear as long-time transients.

5) A careful examination of the proof shows that the angular velocity Ω_0 in Theorem 1.1 can be chosen in the following way:

$$\Omega_0 \,=\, \max\Bigl(K_0^2 \|\nabla \tilde{u}_0\|_{L^2} - K_0\,,\, 0\Bigr)\,, \quad \text{with} \quad K_0 \,=\, C e^{C\|u_0\|_X^8}\,,$$

where $\tilde{u}_0 = (1-Q)u_0$ and $C > 0$ is a universal constant. In particular, one can take $\Omega_0 = 0$ if $\tilde{u}_0$ is sufficiently small, depending on $\bar{u}_0$. Of course, there is no reason to believe that this result is sharp.

The rest of this paper is organized as follows. In Section 2 we prove the existence part of Theorem 1.1 using energy estimates for the full system (2) and dispersive (Strichartz) estimates for the Rossby equation (9). Section 3 is devoted to the convergence proof, which relies on a compactness argument and a transformation into self-similar variables. In Appendix A we collect a few basic results concerning the Biot-Savart law in the domain $\mathbb{D}$, and in Appendix B we give a proof of the dispersive estimates for equation (9) which are used in the global existence proof.

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2 The Cauchy problem for the Navier-Stokes-Coriolis equation

In this section we prove that the Navier-Stokes-Coriolis system (2) is globally well-posed in the function space X defined by (5), provided that the rotation speed Ω is sufficiently large depending on the initial data. The precise statement is:

Theorem 2.1 *For any initial data $u_0 \in X$, there exists $\Omega_0 \ge 0$ such that, for all $\Omega \in \mathbb{R}$ with $|\Omega| \ge \Omega_0$, the NSC system (2) has a unique global solution $u \in C^0([0,\infty), X)$ satisfying $u(0) = u_0$. Moreover, there exists $C > 0$ (depending on u_0) such that $\|u(t)\|_X \le C$ for all $t \ge 0$.*

As is clear from the proof, one can take $\Omega_0 = 0$ in Theorem 2.1 (hence also in Theorem 1.1) if the three-dimensional part $\tilde{u}_0 = (1-Q)u_0$ of the initial velocity field is sufficiently small in X, see Remark 2.10 below. For large data, however, nobody knows how to prove global existence without assuming that the rotation speed Ω is large too.

2.1 Reformulation of the problem

If $u(t,x,z)$ is any solution of the NSC system (2), we decompose

$$u(t,x,z) \;=\; \bar{u}(t,x) + \tilde{u}(t,x,z)\;, \tag{11}$$

where $\bar{u} = Qu$, $\tilde{u} = (1-Q)u$, and Q is the vertical average operator defined in (4). Our first task is to derive evolution equations for $\bar{u}$ and $\tilde{u}$. Integrating (2) over the vertical variable $z \in \mathbb{T}^1$, and using the fact that $\mathbb{P}$ and Q commute with each other (see Appendix A), we obtain

$$\partial_t \bar{u} + \mathbb{P}[(\bar{u}\cdot\nabla)\bar{u} + Q(\tilde{u}\cdot\nabla)\tilde{u}] \;=\; \Delta\bar{u}\;, \quad \operatorname{div}\bar{u} \;=\; 0\;. \tag{12}$$

This is a two-dimensional Navier-Stokes equation for the three-component velocity field $\bar{u}(t,x)$, with a quadratic "source term" depending on $\tilde{u}$. Remark that the Coriolis force disappeared from (12), because $\operatorname{curl}(e_3\wedge\bar{u}) = -\partial_z\bar{u} = 0$, so that $\mathbb{P}(e_3\wedge\bar{u}) = 0$. On the other hand, subtracting (12) from (2), we find

$$\partial_t\tilde{u} + \mathbb{P}[(\bar{u}\cdot\nabla)\tilde{u} + (\tilde{u}\cdot\nabla)\bar{u} + (1-Q)(\tilde{u}\cdot\nabla)\tilde{u}] + \Omega\mathbb{P}(e_3\wedge\tilde{u}) \;=\; \Delta\tilde{u}\;, \quad \operatorname{div}\tilde{u} \;=\; 0\;. \tag{13}$$

Thus $\tilde{u}(t,x,z)$ satisfies a three-dimensional Navier-Stokes-Coriolis system, which is linearly coupled to (12) through the transport term $\mathbb{P}(\bar{u}\cdot\nabla)\tilde{u}$ and the stretching term $\mathbb{P}(\tilde{u}\cdot\nabla)\bar{u}$.

As is explained in the introdution, the averaged velocity field $\bar{u}(t,x)$ can be represented by two scalar quantities, namely its vertical component $\bar{u}_3(t,x)$ and the third component $\bar{\omega}_3(t,x)$ of the averaged vorticity field. Taking the third component of (12) and using the fact that $(\mathbb{P}\bar{u})_3 = \bar{u}_3$ (see Appendix A), we obtain the following evolution equation:

$$\partial_t\bar{u}_3 + (\bar{u}_h\cdot\nabla)\bar{u}_3 + N_1 \;=\; \Delta\bar{u}_3\;, \quad x\in\mathbb{R}^2\;, \quad t>0\;, \tag{14}$$

where $\bar{u}_h = (\bar{u}_1,\bar{u}_2)^t$ and $N_1 = Q(\tilde{u}\cdot\nabla)\tilde{u}_3$. Similarly, if we take the third component of (3) and integrate the resulting equation over the vertical variable z, we find

$$\partial_t\bar{\omega}_3 + (\bar{u}_h\cdot\nabla)\bar{\omega}_3 + N_2 \;=\; \Delta\bar{\omega}_3\;, \quad x\in\mathbb{R}^2\;, \quad t>0\;, \tag{15}$$

where $N_2 = Q((\tilde{u}\cdot\nabla)\tilde{\omega}_3 - (\tilde{\omega}\cdot\nabla)\tilde{u}_3)$. Here we have used the fact that $(\bar{\omega}\cdot\nabla)\bar{u}_3 = 0$, see (79) below.

By construction, the original NSC equation (2) is completely equivalent to the coupled system (13), (14), (15). To prove local existence of solutions, we consider the integral equations associated to these three PDE's (via Duhamel's formula), and we apply a standard fixed point argument in the function space $C^0([0,T],X)$. The result is:

Proposition 2.2 *For any $r>0$, there exists $T = T(r) > 0$ such that, for any $\Omega\in\mathbb{R}$ and all initial data $u_0 \in X$ with $\|u_0\|_X \le r$, the Navier-Stokes-Coriolis system* (2) *has a unique local solution $u \in C^0([0,T],X)$ satisfying $u(0) = u_0$.*

The proof of this statement uses classical arguments, which can be found in [6], [11], [12], and will therefore be omitted here. The fact that the local existence time T depends on u_0 only through (an upper bound of) the norm $\|u_0\|_X$ is not surprising, because we work in a function space X which is not critical with respect to the scaling of the Navier-Stokes equation. However, it is worth noticing that T is independent of the rotation speed Ω. This is because the rotation does not act at all on the two-dimensional part (14), (15) of our system, whereas in (13) it appears only in the term $\Omega\mathbb{P}(e_3\wedge\tilde{u})$, which is *skew-symmetric* in the space $H^1(\mathbb{D})^3$ and therefore does not affect the estimates.

To prove global existence and conclude the proof of Theorem 2.1, it remains to show that any solution $u \in C^0([0,T],X)$ of (2) is bounded for all $t \in [0,T]$ by a constant depending only on the initial data $u_0 = u(0)$. As is well-known, this is relatively easy to do if the three-dimensional part $\tilde{u}_0$ of the initial data is small in $H^1(\mathbb{D})$, see [6], [13]. In the general case, we shall use the dispersive properties of the Rossby equation (9) to prove that the solution $\tilde{u}(t,x,z)$ of (13) is rapidly damped for positive times if the rotation speed $|\Omega|$ is sufficiently large.

2.2 Dispersive properties

Since our spatial domain $\mathbb{D} = \mathbb{R}^2 \times \mathbb{T}^1$ is bounded in the vertical direction, the Poincaré inequality implies that the solutions of the linear equation (9) decay exponentially to zero as $t \to \infty$. More precisely, for any $s \geq 0$ and all divergence-free initial data $\tilde{u}_0 \in (1-Q)H^s(\mathbb{D})^3$, the solution $\tilde{u}(t,x,z)$ of (9) satisfies

$$\|\tilde{u}(t)\|_{H^s(\mathbb{D})} \;\leq\; \|\tilde{u}_0\|_{H^s(\mathbb{D})}\, e^{-4\pi^2 t} \;, \quad t \geq 0 \;. \tag{16}$$

This estimate is straightforward to establish by computing the time-derivative of $\|\tilde{u}(t)\|_{H^s}^2$ and using the Poincaré inequality $\|\nabla\tilde{u}\|_{H^s}^2 \geq 4\pi^2\|\tilde{u}\|_{H^s}^2$ together with the fact that the Coriolis operator $\tilde{u} \mapsto \mathbb{P}(e_3 \wedge \tilde{u})$ is skew-symmetric in $H^s(\mathbb{D})^3$ for divergence-free vector fields. Note in particular that (16) is independent ot the rotation speed Ω. However, as is shown e.g. in [4], additional information can be obtained for large $|\Omega|$ if we exploit the dispersive effect of the skew-symmetric term $\Omega\mathbb{P}(e_3 \wedge \tilde{u})$. The corresponding Strichartz-type estimates are most conveniently derived if we restrict ourselves to solutions with compact support in Fourier space.

Throughout this paper, we use the following conventions for Fourier transforms. If $f \in L^2(\mathbb{D})$ or $L^2(\mathbb{D})^3$, we set

$$f(x,z) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}} f_n(k)\, e^{i(k\cdot x + 2\pi n z)}\,\mathrm{d}k \;, \quad x \in \mathbb{R}^2 \;, \quad z \in \mathbb{T}^1 \;, \tag{17}$$

where

$$f_n(k) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{T}^1} f(x,z)\, e^{-i(k\cdot x + 2\pi n z)}\,\mathrm{d}z\,\mathrm{d}x \;, \quad k \in \mathbb{R}^2 \;, \quad n \in \mathbb{Z} \;. \tag{18}$$

With these notations, the norm of f in the Sobolev space $H^s(\mathbb{D})$ can be defined as

$$\|f\|_{H^s} \;=\; \Bigl(\int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}}(1+|k|^2+4\pi^2 n^2)^s |f_n(k)|^2\,\mathrm{d}k\Bigr)^{1/2} \;, \tag{19}$$

where $|k|^2 = k_1^2 + k_2^2$. Given any $R > 0$, we denote by $\mathcal{B}_R$ the ball

$$\mathcal{B}_R \;=\; \Bigl\{(k,n) \in \mathbb{R}^2 \times \mathbb{Z} \Bigm| \sqrt{|k|^2 + 4\pi^2 n^2} \leq R\Bigr\} \;. \tag{20}$$

Following closely the approach of [4, Chap. 5], we obtain our main dispersion estimate:

Proposition 2.3 *For any $R > 0$, there exists $C_R > 0$ such that, for all $\tilde{u}_0 \in (1-Q)L^2(\mathbb{D})^3$ with $\operatorname{div}\tilde{u}_0 = 0$ and $\operatorname{supp}(\tilde{u}_0)_n(k) \subset \mathcal{B}_R$, the solution $\tilde{u}$ of (9) with initial data $\tilde{u}_0$ satisfies*

$$\|\tilde{u}\|_{L^1(\mathbb{R}_+,L^\infty(\mathbb{D}))} \;\leq\; C_R|\Omega|^{-\frac{1}{4}}\|\tilde{u}_0\|_{L^2(\mathbb{D})} \;. \tag{21}$$

For completeness, the proof of this proposition will be given in Appendix B. Estimate (21) clearly demonstrates the dispersive effect of the Coriolis term in (9): If the initial data $\tilde{u}_0$ are compactly supported in Fourier space, the L^∞ norm of the solution $\tilde{u}(t,\cdot)$ will be very small (for most values of time) if the rotation speed $|\Omega|$ is large enough. This is in sharp contrast with what happens for Sobolev norms, for which the best we can have is estimate (16). As a side remark, if we consider initial data $\tilde{u}_0$ whose Fourier transform is supported *outside* the ball $\mathcal{B}_R$, then we clearly have $\|\tilde{u}(t)\|_{H^s} \le \|\tilde{u}_0\|_{H^s}\, e^{-R^2 t}$ for all $t \ge 0$.

Combining Proposition 2.3 with estimate (16), we deduce the following useful corollary:

Corollary 2.4 *Under the assumptions of Proposition 2.3, the solution $\tilde{u}$ of (9) satisfies, for any $p \in [1,+\infty]$ and any $q \in [2,+\infty]$ such that $\frac{1}{p} + \frac{2}{q} \le 1$,*

$$\|\tilde{u}\|_{L^p(\mathbb{R}_+,L^q(\mathbb{D}))} \;\le\; C_R \langle\Omega\rangle^{-\frac{1}{4p}} \|\tilde{u}_0\|_{L^2(\mathbb{D})} \;, \tag{22}$$

where $\langle\Omega\rangle = (1+|\Omega|^2)^{1/2}$.

Proof. Fix $s > 3/2$. Using Sobolev's embedding and our assumptions on $\tilde{u}_0$, we obtain from (16)

$$\|\tilde{u}(t)\|_{L^\infty} \;\le\; C\|\tilde{u}(t)\|_{H^s} \;\le\; C\|\tilde{u}_0\|_{H^s}\, e^{-4\pi^2 t} \;\le\; C_R\|\tilde{u}_0\|_{L^2}\, e^{-4\pi^2 t} \;, \quad t \ge 0 \;, \tag{23}$$

where C_R denotes a generic positive constant depending only on R. In particular, we have the estimate $\|\tilde{u}\|_{L^1(\mathbb{R}_+,L^\infty)} \le C_R\|\tilde{u}_0\|_{L^2}$ for all $\Omega \in \mathbb{R}$, so that (21) holds with $|\Omega|$ replaced by $\langle\Omega\rangle$. This gives (22) for $(p,q) = (1,\infty)$, and since the case $(p,q) = (\infty,\infty)$ is immediate from (23), we see that (22) holds for all $p \in [1,\infty]$ if $q = \infty$. Finally, as $\|\tilde{u}(t)\|_{L^2} \le \|\tilde{u}_0\|_{L^2}$ for all $t \ge 0$, the general case follows by a simple interpolation argument. ■

To exploit the dispersive properties of the linear equation (9) in the analysis of the nonlinear problem (13), we use the following decomposition, which is again borrowed from [4]. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function satisfying $0 \le \chi(x) \le 1$ for all $x \in \mathbb{R}$, $\chi(x) = 1$ for $|x| \le 1/2$ and $\chi(x) = 0$ for $|x| \ge 1$. Given any $R > 0$, we define the Fourier multiplyer $\mathcal{P}_R = \chi(|\nabla|/R)$ by the formula

$$(\mathcal{P}_R f)_n(k) \;=\; \chi\Big(\frac{\sqrt{|k|^2 + 4\pi^2 n^2}}{R}\Big) f_n(k) \;, \quad k \in \mathbb{R}^2 \;, \quad n \in \mathbb{Z} \;. \tag{24}$$

If $\tilde{u}(t,x,z)$ is a solution of (13) with initial data $\tilde{u}_0(x,z)$, we decompose

$$\tilde{u}(t,x,z) \;=\; \lambda(t,x,z) + r(t,x,z) \;, \tag{25}$$

where $\lambda(t,x,z)$ satisfies the linear Rossby equation

$$\partial_t \lambda + \Omega\mathbb{P}(e_3 \wedge \lambda) \;=\; \Delta\lambda \;, \quad \operatorname{div}\lambda \;=\; 0 \;, \tag{26}$$

with initial data $\lambda_0 = \mathcal{P}_R \tilde{u}_0$. By construction, the remainder $r(t,x,z)$ is a solution of the nonlinear equation

$$\partial_t r + \Omega\mathbb{P}(e_3 \wedge r) + N_3 \;=\; \Delta r \;, \quad \operatorname{div} r \;=\; 0 \;, \tag{27}$$

with initial data $r_0 = (1-\mathcal{P}_R)\tilde{u}_0$, where $N_3 = \mathbb{P}[(\bar{u}\cdot\nabla)\tilde{u} + (\tilde{u}\cdot\nabla)\bar{u} + (1-Q)(\tilde{u}\cdot\nabla)\tilde{u}]$.

In the rest of this section, we consider equations (26), (27) instead of (13), so that our final evolution system consists of (14), (15), (26), (27). Given $u_0 = \bar{u}_0 + \tilde{u}_0 \in X$, we will choose the parameter $R > 0$ large enough so that the initial data $r_0 = (1-\mathcal{P}_R)\tilde{u}_0$ for equation (27) are small in $H^1(\mathbb{D})$. Then the rotation speed $|\Omega|$ will be taken large enough so that we can exploit the dispersive estimates for $\lambda(t,x,z)$ given by Corollary 2.4.

2.3 Energy estimates

We now derive the energy estimates which will be used to control the solutions of the nonlinear equations (14), (15), (27).

Proposition 2.5 *There exist positive constants C_0, C_1 such that, if $u \in C^0([0,T], X)$ is a solution of* (2) *for some $\Omega \in \mathbb{R}$, and if u is decomposed as in* (11), (25) *for some $R > 0$, then the corresponding solutions of* (14), (15), (27) *satisfy, for any $t \in (0,T]$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{u}_3(t)\|^2_{L^2(\mathbb{R}^2)} \le -\|\nabla\bar{u}_3(t)\|^2_{L^2(\mathbb{R}^2)} + \|\tilde{u}(t)\|^4_{L^4(\mathbb{D})}\ , \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{u}_3(t)\|^2_{L^2(\mathbb{R}^2)} \le -\|\Delta\bar{u}_3(t)\|^2_{L^2(\mathbb{R}^2)} + C_0\big(\|\nabla\bar{u}_3(t)\|^2_{L^2}\|\bar{\omega}_3(t)\|^2_{L^2} + \||\tilde{u}(t)|\,|\nabla\tilde{u}(t)|\|^2_{L^2}\big)\ , \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\omega}_3(t)\|^2_{L^2(\mathbb{R}^2)} \le -\|\nabla\bar{\omega}_3(t)\|^2_{L^2(\mathbb{R}^2)} + 8\||\tilde{u}(t)||\nabla\tilde{u}(t)|\|^2_{L^2(\mathbb{D})}\ , \tag{30}$$

$$\|\bar{\omega}_3(t)\|_{L^1(\mathbb{R}^2)} \le \|\bar{\omega}_3(0)\|_{L^1(\mathbb{R}^2)} + 2\int_0^t \|\tilde{u}(s)\|_{L^2(\mathbb{D})}\|\Delta\tilde{u}(s)\|_{L^2(\mathbb{D})}\,\mathrm{d}s\ , \tag{31}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla r(t)\|^2_{L^2(\mathbb{D})} \le{}& -\|\Delta r(t)\|^2_{L^2(\mathbb{D})} + C_1\|\nabla r(t)\|^2_{L^2}\|\nabla\bar{u}(t)\|^2_{L^2}\|\Delta\bar{u}(t)\|^2_{L^2}\\ &+ C_1\big(\|\bar{u}(t)\|^2_{L^4}\|\nabla\lambda(t)\|^2_{L^4} + \|\nabla\bar{u}(t)\|^2_{L^2}\|\lambda(t)\|^2_{L^\infty} + \||\tilde{u}(t)|\,|\nabla\tilde{u}(t)|\|^2_{L^2}\big)\ .\end{aligned} \tag{32}$$

Remark 2.6 *Here and in what follows, if f is a vector valued or matrix valued function, we denote by $|f|$ the scalar function obtained by taking the Euclidean norm of the entries of f. Given any $p \in [1,\infty]$, we define $\|f\|_{L^p}$ as $\||f|\|_{L^p}$. With these conventions, if $\omega = \mathrm{curl}\, u$, we have for instance $|\omega| \le \sqrt{2}|\nabla u|$ and $\|\omega\|_{L^2} = \|\nabla u\|_{L^2}$.*

Proof. To prove (28), we multiply both sides of (14) by $\bar{u}_3$ and integrate over $\mathbb{R}^2$. The transport term $(\bar{u}_h\cdot\nabla)\bar{u}_3$ gives no contribution, because $\bar{u}$ is divergence-free, and the diffusion term $\Delta\bar{u}_3$ produces the negative contribution $-\|\nabla\bar{u}_3\|^2_{L^2}$ after integrating by parts. Since

$$-\int_{\mathbb{R}^2}\bar{u}_3 N_1\,\mathrm{d}x = -\int_{\mathbb{D}}\bar{u}_3(\tilde{u}\cdot\nabla)\tilde{u}_3\,\mathrm{d}x\,\mathrm{d}z = \int_{\mathbb{D}}\tilde{u}_3(\tilde{u}\cdot\nabla\bar{u}_3)\,\mathrm{d}x\,\mathrm{d}z \le \frac{1}{2}\|\tilde{u}\|^4_{L^4} + \frac{1}{2}\|\nabla\bar{u}_3\|^2_{L^2}\ ,$$

we obtain the desired estimate. In a similar way, to prove (29), we multiply (14) by $-\Delta\bar{u}_3$ and integrate over $\mathbb{R}^2$. The transport term gives here a nontrivial contribution which, after integrating by parts, can be bounded as follows:

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}(\Delta\bar{u}_3)(\bar{u}_h\cdot\nabla)\bar{u}_3\,\mathrm{d}x\Big| &\le \int_{\mathbb{R}^2}|\nabla\bar{u}_3||\nabla\bar{u}_h||\nabla\bar{u}_3|\,\mathrm{d}x \le \|\nabla\bar{u}_3\|^2_{L^4}\|\nabla\bar{u}_h\|_{L^2}\\ &\le C\|\Delta\bar{u}_3\|_{L^2}\|\nabla\bar{u}_3\|_{L^2}\|\bar{\omega}_3\|_{L^2} \le \frac{1}{4}\|\Delta\bar{u}_3\|^2_{L^2} + C\|\nabla\bar{u}_3\|^2_{L^2}\|\bar{\omega}_3\|^2_{L^2}\ .\end{aligned}$$

Here, to get from the first to the second line, we have used an interpolation inequality and the fact that $\bar{u}_h$ is obtained from $\bar{\omega}_3$ via the Biot-Savart law (81), see Appendix A. Since we also have

$$\Big|\int_{\mathbb{R}^2}\Delta\bar{u}_3\,N_1\,\mathrm{d}x\Big| \le \int_{\mathbb{D}}|\Delta\bar{u}_3||\tilde{u}||\nabla\tilde{u}|\,\mathrm{d}x\,\mathrm{d}z \le \frac{1}{4}\|\Delta\bar{u}_3\|^2_{L^2} + \||\tilde{u}||\nabla\tilde{u}|\|^2_{L^2}\ ,$$

we obtain again the desired inequality.

On the other hand, multiplying (15) by $\bar{\omega}_3$ and integrating over $\mathbb{R}^2$, we easily obtain (30), because

$$-\int_{\mathbb{R}^2}\bar{\omega}_3 N_2\,\mathrm{d}x = \int_{\mathbb{D}}\Big(\tilde{\omega}_3(\tilde{u}\cdot\nabla)\bar{\omega}_3 - \tilde{u}_3(\tilde{\omega}\cdot\nabla)\bar{\omega}_3\Big)\,\mathrm{d}x\,\mathrm{d}z \le \frac{1}{2}\|\nabla\bar{\omega}_3\|^2_{L^2} + 2\||\tilde{u}||\tilde{\omega}|\|^2_{L^2}\ ,$$

and $|\tilde{\omega}|^2 \le 2|\nabla\tilde{u}|^2$. The prove (31) we observe that, since the vector field $\bar{u}_h$ is divergence-free, any solution of (15) in $L^1(\mathbb{R}^2)$ satisfies

$$\|\bar{\omega}_3(t)\|_{L^1} \;\le\; \|\bar{\omega}_3(0)\|_{L^1} + \int_0^t \|N_2(s)\|_{L^1}\,\mathrm{d}s\;, \quad t \ge 0\;.$$

This bound can be established using the properties of the fundamental solution of the linear convection-diffusion equation $\partial_t f + (\bar{u}_h\cdot\nabla)f = \Delta f$, which will be recalled in Section 3.2 below. Since

$$\|N_2\|_{L^1} \;\le\; \|\tilde{u}\|_{L^2}\|\nabla\tilde{\omega}\|_{L^2} + \|\tilde{\omega}\|_{L^2}\|\nabla\tilde{u}\|_{L^2} \;\le\; \|\tilde{u}\|_{L^2}\|\Delta\tilde{u}\|_{L^2} + \|\nabla\tilde{u}\|_{L^2}^2 \;\le\; 2\|\tilde{u}\|_{L^2}\|\Delta\tilde{u}\|_{L^2}\;,$$

we obtain (31).

Finally, to prove (32), we multiply (27) with $-\Delta r$ and integrate over $\mathbb{D}$. As was already explained, the Coriolis term $\Omega\mathbb{P}(e_3\wedge r)$ gives no contribution, because it is skew-symmetric in any Sobolev space. So we just have to bound the contributions of the nonlinear term N_3, which are threefold. Since $\tilde{u} = \lambda + r$, the transport part $\mathbb{P}(\bar{u}\cdot\nabla)\tilde{u}$ in N_3 produces two terms, which can be estimated as follows:

$$\begin{aligned}
\Big|\int_{\mathbb{D}} \Delta r\cdot(\bar{u}\cdot\nabla)\lambda\,\mathrm{d}x\,\mathrm{d}z\Big| \;&\le\; \frac{1}{10}\|\Delta r\|_{L^2}^2 + C\|\bar{u}\|_{L^4}^2\|\nabla\lambda\|_{L^4}^2\;,\\
\Big|\int_{\mathbb{D}} \Delta r\cdot(\bar{u}\cdot\nabla)r\,\mathrm{d}x\,\mathrm{d}z\Big| \;&\le\; \int_{\mathbb{D}} |\nabla r||\nabla\bar{u}||\nabla r|\,\mathrm{d}x\,\mathrm{d}z \;\le\; \|\nabla r\|_{L^{\frac{8}{3}}}^2\|\nabla\bar{u}\|_{L^4}\\
&\le\; C\|\nabla r\|_{L^2}^{\frac{5}{4}}\|\Delta r\|_{L^2}^{\frac{3}{4}}\|\nabla\bar{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{u}\|_{L^2}^{\frac{1}{2}} \;\le\; C\|\nabla r\|_{L^2}^{\frac{1}{2}}\|\Delta r\|_{L^2}^{\frac{3}{2}}\|\nabla\bar{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{u}\|_{L^2}^{\frac{1}{2}}\\
&\le\; \frac{1}{10}\|\Delta r\|_{L^2}^2 + C\|\nabla r\|_{L^2}^2\|\nabla\bar{u}\|_{L^2}^2\|\Delta\bar{u}\|_{L^2}^2\;.
\end{aligned}$$

Here we have used interpolation inequalities, Sobolev embeddings, and the Poincaré inequality $\|\nabla r\|_{L^2} \le C\|\Delta r\|_{L^2}$. The two terms produced by the stretching part $\mathbb{P}(\tilde{u}\cdot\nabla)\bar{u}$ in N_3 can be estimated in a similar way:

$$\begin{aligned}
\Big|\int_{\mathbb{D}} \Delta r\cdot(\lambda\cdot\nabla)\bar{u}\,\mathrm{d}x\,\mathrm{d}z\Big| \;&\le\; \frac{1}{10}\|\Delta r\|_{L^2}^2 + C\|\lambda\|_{L^\infty}^2\|\nabla\bar{u}\|_{L^2}^2\;,\\
\Big|\int_{\mathbb{D}} \Delta r\cdot(r\cdot\nabla)\bar{u}\,\mathrm{d}x\,\mathrm{d}z\Big| \;&\le\; \|\Delta r\|_{L^2}\|r\|_{L^4}\|\nabla\bar{u}\|_{L^4} \;\le\; C\|\Delta r\|_{L^2}\|\nabla r\|_{L^2}\|\nabla\bar{u}\|_{L^2}^{\frac{1}{2}}\|\Delta\bar{u}\|_{L^2}^{\frac{1}{2}}\\
&\le\; \frac{1}{10}\|\Delta r\|_{L^2}^2 + C\|\nabla r\|_{L^2}^2\|\nabla\bar{u}\|_{L^2}^2\|\Delta\bar{u}\|_{L^2}^2\;.
\end{aligned}$$

Finally, the contribution of the quadratic term $\mathbb{P}(1-Q)(\tilde{u}\cdot\nabla)\tilde{u}$ in N_3 satisfies

$$\Big|\int_{\mathbb{D}} \Delta r\cdot(\tilde{u}\cdot\nabla)\tilde{u}\,\mathrm{d}x\,\mathrm{d}z\Big| \;\le\; \frac{1}{10}\|\Delta r\|_{L^2}^2 + C\||\tilde{u}|\,|\nabla\tilde{u}|\|_{L^2}^2\;.$$

Collecting all these estimates, we obtain (32). This concludes the proof. ■

2.4 Global existence

In this section, we combine the dispersive properties of Section 2.2 and the energy estimates of Section 2.3 to complete the proof of Theorem 2.1. We start with a preliminary result, which summarizes in a convenient way four of the five inequalities established in Proposition 2.5.

Lemma 2.7 *There exist positive constants C_2, C_3, and C_4 such that the following holds. Let $u \in C^0([0,T],X)$ be a solution of* (2) *for some $\Omega \in \mathbb{R}$, which is decomposed as in* (11), (25) *for some $R > 0$. Assume moreover that there exist $K \geq 1$ and $\varepsilon \in (0,1]$ such that the corresponding solutions of* (14), (15), (27) *satisfy*

$$\|\nabla \bar{u}_3(t)\|_{L^2(\mathbb{R}^2)} \leq K^2 \;, \quad \|\bar{\omega}_3(t)\|_{L^2(\mathbb{R}^2)} \leq K \;, \quad \|\nabla r(t)\|_{L^2(\mathbb{D})} \leq \varepsilon \;, \tag{33}$$

for all $t \in [0,T]$. If we define

$$\Phi(t) = \|\bar{u}_3(t)\|^2_{L^2(\mathbb{R}^2)} + \|\bar{\omega}_3(t)\|^2_{L^2(\mathbb{R}^2)} + \delta\|\nabla \bar{u}_3\|^2_{L^2(\mathbb{R}^2)} + \|\nabla r(t)\|^2_{L^2(\mathbb{D})} \;, \tag{34}$$

for some $\delta \in (0,1]$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq & -(\|\nabla \bar{u}_3(t)\|^2_{L^2} + \|\nabla \bar{\omega}_3(t)\|^2_{L^2} + \delta\|\Delta \bar{u}_3(t)\|^2_{L^2} + \|\Delta r(t)\|^2_{L^2}) \\ & + C_0\delta K^2\|\nabla \bar{u}_3(t)\|^2_{L^2} + C_2\varepsilon^2 K^4\|\Delta \bar{u}(t)\|^2_{L^2} + C_3\varepsilon^2\|\Delta r(t)\|^2_{L^2} \\ & + (\delta^{-1}\Phi(t) + K\|\bar{\omega}_3(t)\|_{L^1})G(t) + F(t) + \varepsilon^2 G(t) \;,\end{aligned} \tag{35}$$

for all $t \in (0,T]$, where

$$\begin{aligned} F(t) &= C_4(\|\lambda(t)\|^4_{L^4} + \|\lambda(t)\|^2_{L^\infty}\|\nabla\lambda(t)\|^2_{L^2}) \;, \\ G(t) &= C_4(\|\nabla\lambda(t)\|^2_{L^\infty} + \|\lambda(t)\|^2_{L^\infty} + \|\nabla\lambda(t)\|^2_{L^4}) \;. \end{aligned} \tag{36}$$

Proof. If Φ is defined by (34), it follows immediately from Proposition 2.5 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) \leq & -(\|\nabla \bar{u}_3(t)\|^2_{L^2} + \|\nabla \bar{\omega}_3(t)\|^2_{L^2} + \delta\|\Delta \bar{u}_3(t)\|^2_{L^2} + \|\Delta r(t)\|^2_{L^2}) \\ & + \|\tilde{u}(t)\|^4_{L^4} + C\||\tilde{u}(t)||\nabla\tilde{u}(t)|\|^2_{L^2} + C_0\delta\|\nabla \bar{u}_3(t)\|^2_{L^2}\|\bar{\omega}_3(t)\|^2_{L^2} \\ & + C(\|\nabla r(t)\|^2_{L^2}\|\nabla \bar{u}(t)\|^2_{L^2}\|\Delta \bar{u}(t)\|^2_{L^2} + \|\bar{u}(t)\|^2_{L^4}\|\nabla\lambda(t)\|^2_{L^4} + \|\nabla\bar{u}(t)\|^2_{L^2}\|\lambda(t)\|^2_{L^\infty}) \;.\end{aligned} \tag{37}$$

Using interpolation inequalities, Sobolev embeddings, and the a priori bounds (33), we first get

$$\begin{aligned}\|\tilde{u}(t)\|^4_{L^4} &\leq C(\|r(t)\|^4_{L^4} + \|\lambda(t)\|^4_{L^4}) \leq C(\|r(t)\|^3_{L^6}\|r(t)\|_{L^2} + \|\lambda(t)\|^4_{L^4}) \\ &\leq C(\|\nabla r(t)\|^3_{L^2}\|r(t)\|_{L^2} + \|\lambda(t)\|^4_{L^4}) \leq C\varepsilon^2\|\nabla r(t)\|_{L^2}\|r(t)\|_{L^2} + F_1(t) \;,\end{aligned}$$

where $F_1(t) = C\|\lambda(t)\|^4_{L^4}$. Proceeding in the same way, we also obtain

$$\begin{aligned}\||\tilde{u}||\nabla\tilde{u}|\|^2_{L^2} &\leq C(\||r||\nabla r|\|^2_{L^2} + \||r||\nabla\lambda|\|^2_{L^2} + \||\lambda||\nabla r|\|^2_{L^2} + \||\lambda||\nabla\lambda|\|^2_{L^2}) \\ &\leq C(\|r\|^2_{L^6}\|\nabla r\|^2_{L^3} + \|r\|^2_{L^2}\|\nabla\lambda\|^2_{L^\infty} + \|\nabla r\|^2_{L^2}\|\lambda\|^2_{L^\infty} + \|\lambda\|^2_{L^\infty}\|\nabla\lambda\|^2_{L^2}) \;,\end{aligned}$$

so that $\||\tilde{u}(t)||\nabla\tilde{u}(t)|\|^2_{L^2} \leq C\varepsilon^2\|\nabla r(t)\|_{L^2}\|\Delta r(t)\|_{L^2} + \varepsilon^2 G_1(t) + F_2(t)$, where

$$G_1(t) = C(\|\nabla\lambda(t)\|^2_{L^\infty} + \|\lambda(t)\|^2_{L^\infty}) \;, \quad F_2(t) = C\|\lambda(t)\|^2_{L^\infty}\|\nabla\lambda(t)\|^2_{L^2} \;.$$

It remains to estimate the last four terms in the right-hand side of (37). The first two in this group are independent of λ, and are simply bounded using assumption (33) and the fact that $\|\nabla\bar{u}\|^2_{L^2} = \|\nabla\bar{u}_3\|^2_{L^2} + \|\bar{\omega}_3\|^2_{L^2}$. On the other hand, in view of Proposition A.1, we have

$$\|\bar{u}\|^2_{L^4} \leq C(\|\bar{u}_3\|^2_{L^4} + \|\bar{\omega}_3\|^2_{L^{\frac{4}{3}}}) \leq C(\|\bar{u}_3\|_{L^2}\|\nabla\bar{u}_3\|_{L^2} + \|\bar{\omega}_3\|_{L^1}\|\bar{\omega}_3\|_{L^2}) \;,$$

hence

$$\|\bar{u}(t)\|^2_{L^4}\|\nabla\lambda(t)\|^2_{L^4} \leq (\delta^{-1/2}\Phi(t) + K\|\bar{\omega}_3(t)\|_{L^1})G_2(t) \;,$$

where $G_2(t) = C\|\nabla\lambda(t)\|^2_{L^4}$. Similarly, we find $\|\nabla\bar{u}(t)\|^2_{L^2}\|\lambda(t)\|^2_{L^\infty} \leq \delta^{-1}\Phi(t)G_1(t)$. Thus, using the Poincaré inequality $\|r\|_{L^2} \leq \|\nabla r\|_{L^2} \leq \|\Delta r\|_{L^2}$, we see that (35) holds with $F(t) = F_1(t) + F_2(t)$ and $G(t) = G_1(t) + G_2(t)$. ■

Remark 2.8 *In view of Corollary 2.4, there exists a constant $C_R > 0$ (depending only on R) such that*

$$\int_0^\infty F(t)\,\mathrm{d}t \;\le\; C_R\langle\Omega\rangle^{-\frac{1}{4}}\|\tilde u_0\|_{L^2}^4\;, \quad \textit{and} \quad \int_0^\infty G(t)\,\mathrm{d}t \;\le\; C_R\langle\Omega\rangle^{-\frac{1}{4}}\|\tilde u_0\|_{L^2}^2\;. \tag{38}$$

Remark 2.9 *Without loss of generality, we shall assume henceforth that the constants which appear in Proposition 2.5 and Lemma 2.7 satisfy $C_i \ge 1$, $i = 0,\dots,4$.*

Proof of theorem 2.1. Given $u_0 \in X$, we define $\bar u(0) = Qu_0$, $\tilde u_0 = (1-Q)u_0$, and $\bar\omega(0) = \mathrm{curl}(Qu_0)$, where Q is the vertical average operator (4). We first choose $K \ge 1$ such that

$$\|\bar u_3(0)\|_{H^1(\mathbb{R}^2)}^2 + \|\bar\omega_3(0)\|_{L^2(\mathbb{R}^2)}^2 + \|\bar\omega_3(0)\|_{L^1(\mathbb{R}^2)} + 2\|\tilde u_0\|_{H^1(\mathbb{D})}^2 \;\le\; \frac{K^2}{16C_0}\;, \tag{39}$$

where $C_0 \ge 1$ is as in Proposition 2.5. Next, we take $\varepsilon \in (0,1]$ sufficiently small so that

$$\varepsilon^2 \;\le\; \min\Big\{\frac{1}{2C_3}\,,\,\frac{\delta}{2C_2K^4}\Big\}\;, \quad \text{where} \quad \delta \;=\; \frac{1}{2C_0K^2} \;\in\; (0,1]\;, \tag{40}$$

and $C_2 \ge 1$, $C_3 \ge 1$ are as in Lemma 2.7. Once this is done, we set $\lambda_0 = \mathcal{P}_R\tilde u_0$ and $r_0 = (1-\mathcal{P}_R)\tilde u_0$, where $\mathcal{P}_R$ is the Fourier localization operator defined by (24). We assume that the parameter $R > 0$ is sufficiently large so that

$$4\,e^{2C_1K^8}\|\nabla r_0\|_{L^2}^2 \;\le\; \varepsilon^2\;, \tag{41}$$

and we denote by $\lambda(t,x,z)$ the solution of (26) with initial data λ_0. Finally, using Remark 2.8, we choose $\Omega_0 \ge 0$ sufficiently large so that, if $|\Omega| \ge \Omega_0$,

$$\int_0^\infty G(t)\,\mathrm{d}t \;\le\; \delta\log(2)\;, \quad \int_0^\infty (F(t) + \varepsilon^2 G(t))\,\mathrm{d}t \;\le\; \frac{K^2}{16C_0}\;, \tag{42}$$

and

$$4\,e^{2C_1K^8}\int_0^\infty \Big(F(t) + (K^4+\varepsilon^2)G(t)\Big)\,\mathrm{d}t \;\le\; \varepsilon^2\;. \tag{43}$$

Remark 2.10 *If $\tilde u_0$ is small enough so that $4\,e^{C_1K^8}\|\nabla\tilde u_0\|_{L^2}^2 \le \varepsilon^2$, then we can take formally $R = 0$, so that $r_0 = \tilde u_0$ and $\lambda_0 = 0$. In that case, one has $F(t) = G(t) \equiv 0$, and (42), (43) are of course satisfied for any $\Omega \in \mathbb{R}$.*

By Proposition 2.2, equation (2) has a unique maximal solution $u \in C^0([0,T_*),X)$ with initial data u_0, where $T_* \in (0,+\infty]$ denotes the maximal existence time. If we decompose $u(t) = \bar u(t) + \lambda(t) + r(t)$ as in (11), (25), then $\bar u_3(t)$, $\bar\omega_3(t)$, $r(t)$ are solutions of (14), (15), (27), respectively, and we know from (39) and (41) that

$$\|\nabla\bar u_3(0)\|_{L^2(\mathbb{R}^2)} \;\le\; \frac{K}{4}\;, \quad \|\bar\omega_3(0)\|_{L^2(\mathbb{R}^2)} \;\le\; \frac{K}{4}\;, \quad \|\nabla r_0\|_{L^2(\mathbb{D})} \;\le\; \frac{\varepsilon}{2}\;.$$

Thus, by continuity, the bounds (33) will be satisfied at least for $t > 0$ sufficiently small. Let

$$T \;=\; \sup\Big\{\tilde T \in [0,T_*)\;\Big|\;\text{The bounds (33) hold for all } t \in [0,\tilde T]\Big\} \;\in\; (0,T_*]\;. \tag{44}$$

We shall prove that $T = T_*$. This implies of course that $T = T_* = +\infty$, and that the solution $u(t)$ of (2) stays bounded in X for all $t \ge 0$, as is claimed in Theorem 2.1.

Assume on the contrary that $0 < T < T_*$, and let $\Psi(t) = \Phi(t) + \|\bar\omega_3(t)\|_{L^1}$, where Φ is defined in (34). Using (35) and (40), we find

$$\begin{aligned} \Phi(t) \;+\; \frac{1}{2}\int_0^t \Big(\|\nabla \bar u_3(s)\|_{L^2}^2 + \|\nabla \bar\omega_3(s)\|_{L^2}^2 + \delta\|\Delta \bar u_3(s)\|_{L^2}^2 + \|\Delta r(s)\|_{L^2}^2\Big)\,\mathrm{d}s \\ \le\; \Phi(0) + \delta^{-1}\int_0^t \Psi(s)G(s)\,\mathrm{d}s + \int_0^t (F(s) + \varepsilon^2 G(s))\,\mathrm{d}s\;, \quad t \in [0,T]\;. \end{aligned} \tag{45}$$

On the other hand, since

$$2\|\tilde u(t)\|_{L^2}\|\Delta \tilde u(t)\|_{L^2} \;\le\; \frac{1}{2\pi^2}\|\Delta\tilde u(t)\|_{L^2}^2 \;\le\; \frac{1}{\pi^2}\Big(\|\Delta r(t)\|_{L^2}^2 + \|\Delta\lambda(t)\|_{L^2}^2\Big)\;,$$

it follows from (31) that

$$\|\bar\omega_3(t)\|_{L^1} \;\le\; \|\bar\omega_3(0)\|_{L^1} + \frac{1}{4}\int_0^t \|\Delta r(s)\|_{L^2}^2\,\mathrm{d}s + \|\nabla\tilde u_0\|_{L^2}^2\;, \quad t \in [0,T]\;. \tag{46}$$

Here we have used the fact that $2\int_0^\infty \|\Delta\lambda(t)\|_{L^2}^2\,\mathrm{d}t = \|\nabla\lambda_0\|_{L^2}^2 \le \|\nabla\tilde u_0\|_{L^2}^2$ by (26). Summing up (45) and (46), we obtain for $t \in [0,T]$:

$$\begin{aligned} \Psi(t) \;+\; \frac{1}{2}\int_0^t \Big(\|\nabla \bar u_3(s)\|_{L^2}^2 + \|\nabla \bar\omega_3(s)\|_{L^2}^2 + \delta\|\Delta \bar u_3(s)\|_{L^2}^2 + \frac{1}{2}\|\Delta r(s)\|_{L^2}^2\Big)\,\mathrm{d}s \\ \le\; \Psi(0) + \|\nabla\tilde u_0\|_{L^2}^2 + \delta^{-1}\int_0^t \Psi(s)G(s)\,\mathrm{d}s + \int_0^t (F(s) + \varepsilon^2 G(s))\,\mathrm{d}s\;. \end{aligned} \tag{47}$$

This integral inequality for $\Psi(t)$ can be integrated using Gronwall's lemma. In view of (39), (41) and (42), we easily obtain

$$\begin{aligned} \Psi(t) \;+\; \frac{1}{2}\int_0^t \Big(\|\nabla \bar u_3(s)\|_{L^2}^2 + \|\nabla \bar\omega_3(s)\|_{L^2}^2 + \delta\|\Delta \bar u_3(s)\|_{L^2}^2 + \frac{1}{2}\|\Delta r(s)\|_{L^2}^2\Big)\,\mathrm{d}s \\ \le\; 2\Big(\Psi(0) + \|\nabla\tilde u_0\|_{L^2}^2 + \int_0^t (F(s) + \varepsilon^2 G(s))\,\mathrm{d}s\Big) \;\le\; \frac{K^2}{4C_0}\;, \end{aligned} \tag{48}$$

for all $t \in [0,T]$. In a similar way, using (32), (33) and proceeding as in the proof of Lemma 2.7, we find

$$\begin{aligned} \|\nabla r(t)\|_{L^2}^2 \;+\; \frac{1}{2}\int_0^t \|\Delta r(s)\|_{L^2}^2\,\mathrm{d}s \;\le\; \|\nabla r_0\|_{L^2}^2 + 2C_1K^4\int_0^t \|\nabla r(s)\|_{L^2}^2\|\Delta\bar u(s)\|_{L^2}^2\,\mathrm{d}s \\ +\; \delta^{-1}\int_0^t \Psi(s)G(s)\,\mathrm{d}s + \int_0^t (F(s) + \varepsilon^2 G(s))\,\mathrm{d}s\;, \quad t \in [0,T]\;. \end{aligned} \tag{49}$$

From (48) we know that $\int_0^t \|\Delta\bar u(s)\|_{L^2}^2\,\mathrm{d}s \le 2K^2/(4C_0\delta) = K^4$. Thus we can apply Gronwall's lemma to (49) and, using in addition (41) and (43), we obtain

$$\begin{aligned} \|\nabla r(t)\|_{L^2}^2 \;+\; \frac{1}{2}\int_0^t \|\Delta r(s)\|_{L^2}^2\,\mathrm{d}s \\ \le\; e^{2C_1K^8}\Big(\|\nabla r_0\|_{L^2}^2 + K^4\int_0^t G(s)\,\mathrm{d}s + \int_0^t (F(s) + \varepsilon^2 G(s))\,\mathrm{d}s\Big) \;\le\; \frac{\varepsilon^2}{2}\;, \end{aligned} \tag{50}$$

for all $t \in [0,T]$.

Now, it follows immediately from (48), (50) that

$$\|\nabla\bar u_3(t)\|_{L^2}^2 \;\le\; \frac{K^4}{2}\;, \quad \|\bar\omega_3(t)\|_{L^2}^2 \;\le\; \frac{K^2}{4C_0}\;, \quad \|\nabla r(t)\|_{L^2}^2 \;\le\; \frac{\varepsilon^2}{2}\;,$$

for all $t \in [0,T]$, which obviously contradicts the definition (44) of T. Thus $T = T_* = +\infty$, and estimates (33), (48), (50) hold for all $t \ge 0$. This concludes the proof of Theorem 2.1. ■

3 Convergence to Oseen Vortices

To complete the proof of Theorem 1.1, it remains to show that the global solution $u(t,x,z)$ of the Navier-Stokes-Coriolis system (2) constructed in Section 2 converges to Oseen's vortex as $t \to \infty$. To do that, we decompose $u(t,x,z) = \bar{u}(t,x) + \tilde{u}(t,x,z)$ as in (11), and we first show that the three-dimensional part $\tilde{u}(t)$ converges exponentially to zero in $H^1(\mathbb{D})^3$, due to Poincaré's inequality. We next turn our attention to the two-dimensional part $\bar{u}$, and prove that the third component $\bar{u}_3(t)$ decays to zero in $H^1(\mathbb{R}^2)$. Finally, the most delicate point is to show that $\bar{\omega}_3(t)$ converges to Oseen's vortex in $L^1(\mathbb{R}^2)$ as $t \to \infty$. Here the main ingredients are a transformation into self-similar variables, a compactness estimate for the rescaled solution, and a characterization of the complete trajectories of the two-dimensional Navier-Stokes equation which was obtained in [8].

3.1 Exponential decay of $\tilde{u}$

We recall from (25) that $\tilde{u}(t,x,z) = r(t,x,z) + \lambda(t,x,z)$, where λ satisfies the linear equation (26) and r is a solution of (27). We already know that $\|\lambda(t)\|_{H^s} \le C\,e^{-4\pi^2 t}$ for all $t \ge 0$ and any $s \ge 0$, see (16), so it remains to estimate $r(t,x,z)$. We start from equation (32) which, in view of the global bound obtained in Theorem 2.1 and the estimate above for λ, implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla r(t)\|_{L^2}^2 + \frac{1}{2}\|\Delta r(t)\|_{L^2}^2 \;\le\; C_1\|\nabla r(t)\|_{L^2}^2\|\Delta \bar{u}(t)\|_{L^2}^2 + C_2\,e^{-8\pi^2 t}\;, \tag{51}$$

for some constants $C_1, C_2 > 0$ (depending on the initial data). Fix $0 < \mu \le 2\pi^2$ and let $f(t) = e^{\mu t}\|\nabla r(t)\|_{L^2}^2$. Using (51) and the Poincaré inequality $\|\Delta r\|_{L^2} \ge 2\pi\|\nabla r\|_{L^2}$, we find

$$\begin{aligned} f'(t) \;&\le\; e^{\mu t}\Big(\mu\|\nabla r(t)\|_{L^2}^2 - \frac{1}{2}\|\Delta r(t)\|_{L^2}^2 + C_1\|\nabla r(t)\|_{L^2}^2\|\Delta\bar{u}(t)\|_{L^2}^2 + C_2\,e^{-8\pi^2 t}\Big) \\ &\le\; C_1 f(t)\|\Delta\bar{u}(t)\|_{L^2}^2 + C_2\,e^{-(8\pi^2-\mu)t}\;. \end{aligned} \tag{52}$$

Since $\int_0^\infty \|\Delta\bar{u}(t)\|_{L^2}^2\,\mathrm{d}t < \infty$ by (48), it follows from (52) that $f(t) \le C_3$ for all $t \ge 0$, hence $\|\nabla r(t)\|_{L^2} \le C_3\,e^{-\mu t/2}$ for some $C_3 > 0$. As $\|\tilde{u}\|_{H^1} \approx \|\nabla\tilde{u}\|_{L^2} \le \|\nabla r\|_{L^2} + \|\nabla\lambda\|_{L^2}$, this proves that $\tilde{u}(t)$ converges exponentially to zero in $H^1(\mathbb{D})^3$ as $t \to \infty$. The decay rate we have obtained so far is not optimal, but it is sufficient to conclude the proof of Theorem 1.1.

To get the optimal decay rate, the simplest solution is to go back to equation (13) satisfied by $\tilde{u}$. Using straightforward estimates to bound the nonlinear terms, we arrive at the differential inequality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tilde{u}(t)\|_{L^2}^2 \;&\le\; -2\|\Delta\tilde{u}(t)\|_{L^2}^2 + \int_{\mathbb{D}}\Delta\tilde{u}(t)\cdot N_3(t)\,\mathrm{d}x\,\mathrm{d}z \\ &\le\; -2\|\Delta\tilde{u}(t)\|_{L^2}^2 + C\|\Delta\tilde{u}(t)\|_{L^2}\|\nabla\tilde{u}(t)\|_{L^2}(\|\nabla\tilde{u}(t)\|_{L^3} + \|\nabla\bar{u}(t)\|_{L^3})\;, \end{aligned} \tag{53}$$

where $C > 0$ is a universal constant. Now we observe that

$$\int_0^\infty(\|\nabla\tilde{u}(t)\|_{L^3}^2 + \|\nabla\bar{u}(t)\|_{L^3}^2)\,\mathrm{d}t \;\le\; C\int_0^\infty(\|\nabla\tilde{u}(t)\|_{L^3}^2 + \|\nabla\bar{u}_3(t)\|_{L^3}^2 + \|\bar{\omega}_3(t)\|_{L^3}^2)\,\mathrm{d}t \;<\; \infty\;. \tag{54}$$

For $\tilde{u}$ and $\bar{u}_3$, this claim follows (48), (49), because $\|\nabla\tilde{u}\|_{L^3}^2 \le C\|\Delta\tilde{u}\|_{L^2}^2 \le C(\|\Delta r\|_{L^2}^2 + \|\Delta\lambda\|_{L^2}^2)$, and $\|\nabla\bar{u}_3\|_{L^3}^2 \le C\|\nabla\bar{u}_3\|_{L^2}^{4/3}\|\Delta\bar{u}_3\|_{L^2}^{2/3} \le C(\|\nabla\bar{u}_3\|_{L^2}^2 + \|\Delta\bar{u}_3\|_{L^2}^2)$. On the other hand, the decay rates established in Section 3.3 below will show that $\|\bar{\omega}_3(t)\|_{L^3} = \mathcal{O}(t^{-2/3})$ as $t \to \infty$, so that

(54) holds. Combining (53), (54), and using the Poincaré inequality $\|\Delta\tilde{u}\|_{L^2} \ge 2\pi\|\nabla\tilde{u}\|_{L^2}$, we easily obtain

$$\sup_{t\ge 0} e^{\mu t}\|\nabla\tilde{u}(t)\|_{L^2} \,<\, \infty \;, \quad \text{for any } \mu < 4\pi^2 \;. \tag{55}$$

Note, however, that the linear decay rate $\mu = 4\pi^2$ cannot be reached by this argument, because $\int_0^\infty \|\nabla\bar{u}(t)\|_{L^3}\,\mathrm{d}t = +\infty$ in general.

For later use, we mention that similar decay estimates can also be obtained for $\|\Delta\tilde{u}\|_{L^2}$, by differentiating (13) and repeating the same arguments. We thus obtain

$$\sup_{t\ge 1} e^{\mu t}\|\Delta\tilde{u}(t)\|_{L^2} \,<\, \infty \;, \quad \text{for any } \mu < 4\pi^2 \;. \tag{56}$$

3.2 Evanescence of $\bar{u}_3$

We next consider the third component of the two-dimensional velocity $\bar{u}$, which according to (14) satisfies the evolution equation

$$\partial_t\bar{u}_3 + (\bar{u}_h\cdot\nabla)\bar{u}_3 + N_1 \,=\, \Delta\bar{u}_3 \;, \tag{57}$$

where $\bar{u}_h = (\bar{u}_1,\bar{u}_2)^t$. The inhomogeneous term N_1 in (57) is clearly negligible for large times, because $\|N_1\|_{L^2} \le \||\tilde{u}||\nabla\tilde{u}|\|_{L^2} \le C\|\Delta\tilde{u}\|_{L^2}^2$ so that $\int_0^\infty \|N_1(t)\|_{L^2}\,\mathrm{d}t < \infty$. By Duhamel's formula, the solution of (57) can be represented as

$$\bar{u}_3(t) \,=\, S_{\bar{u}}(t,t_0)\bar{u}_3(t_0) - \int_{t_0}^t S_{\bar{u}}(t,s)N_1(s)\,\mathrm{d}s \;, \quad t\ge t_0\ge 0 \;, \tag{58}$$

where $S_{\bar{u}}(t,t_0)$ is the two-parameter evolution operator associated to the linear convection-diffusion equation $\partial_t f + (\bar{u}_h\cdot\nabla)f = \Delta f$ in $\mathbb{R}^2$. As is well-known [16, 2], the operator $S_{\bar{u}}$ can be expressed by an integral formula

$$(S_{\bar{u}}(t,t_0)f)(x) \,=\, \int_{\mathbb{R}^2}\Gamma_{\bar{u}}(t,x;t_0,x_0)f(x_0)\,\mathrm{d}x_0 \;, \quad t > t_0 \ge 0 \;,$$

where the kernel $\Gamma_{\bar{u}}(t,x;t_0,x_0)$ has the following properties:

i) For any $\beta\in(0,1)$ there exists $C_\beta > 0$ such that

$$0 \,<\, \Gamma_{\bar{u}}(t,x;t_0,x_0) \,\le\, \frac{C_\beta}{t-t_0}\,\exp\Bigl(-\beta\frac{|x-x_0|^2}{4(t-t_0)}\Bigr) \;, \tag{59}$$

for all $t > t_0 \ge 0$ and all $x, x_0 \in \mathbb{R}^2$.

ii) For any $t > t_0 \ge 0$ and any $x, x_0\in\mathbb{R}^2$, one has

$$\int_{\mathbb{R}^2}\Gamma_{\bar{u}}(t,x;t_0,x_0)\,\mathrm{d}x \,=\, 1 \;, \qquad \int_{\mathbb{R}^2}\Gamma_{\bar{u}}(t,x;t_0,x_0)\,\mathrm{d}x_0 \,=\, 1 \;. \tag{60}$$

It is very important to note that estimate (59) holds uniformly for all $t > t_0$, with a constant C_β which is independent of time. This is because $\bar{\omega}_3 = \partial_1\bar{u}_2 - \partial_2\bar{u}_1$ is uniformly bounded in $L^1(\mathbb{R}^2)$, see [16]. It follows in particular from (59), (60) that $\|S_{\bar{u}}(t,t_0)f\|_{L^2} \le \|f\|_{L^2}$ for all $t\ge t_0$, and that $S_{\bar{u}}(t,t_0)$ satisfies similar L^p–L^q estimates as the heat semigroup $e^{(t-t_0)\Delta}$.

We claim that the solution $\bar{u}_3(t)$ of (57) converges to zero in $L^2(\mathbb{R}^2)$ as $t\to\infty$. To prove that, fix any $\varepsilon > 0$, and take $t_0 > 0$ sufficiently large so that $\int_{t_0}^\infty \|N_1(s)\|_{L^2}\,\mathrm{d}s \le \varepsilon$. Then

$$\Bigl\|\int_{t_0}^t S_{\bar{u}}(t,s)N_1(s)\,\mathrm{d}s\Bigr\|_{L^2} \,\le\, \int_{t_0}^\infty \|N_1(s)\|_{L^2}\,\mathrm{d}s \,\le\, \varepsilon \;, \quad \text{for all } t\ge t_0 \;,$$

hence in the right-hand side of (58) it is sufficient to bound the first term $v(t) = S_{\bar{u}}(t,t_0)\bar{u}_3(t_0)$. Since $\bar{u}_3(t_0) \in L^2(\mathbb{R}^2)$, we can decompose $\bar{u}_3(t_0) = v_1 + v_2$ with $v_1 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $\|v_2\|_{L^2} \le \varepsilon$. Then $v(t) = v_1(t) + v_2(t)$ with

$$\|v_1(t)\|_{L^2} \;=\; \|S_{\bar{u}}(t,t_0)v_1\|_{L^2} \;\le\; \frac{C}{(t-t_0)^{1/2}}\,\|v_1\|_{L^2} \;\xrightarrow[t\to\infty]{}\; 0\;,$$

and $\|v_2(t)\|_{L^2} = \|S_{\bar{u}}(t,t_0)v_2\|_{L^2} \le \|v_2\|_{L^2} \le \varepsilon$. Thus, if $t > t_0$ is sufficiently large, we have

$$\|\bar{u}_3(t)\|_{L^2} \;\le\; \|v_1(t)\|_{L^2} + \|v_2(t)\|_{L^2} + \int_{t_0}^t \|N_1(s)\|_{L^2}\,\mathrm{d}s \;\le\; 3\varepsilon\;,$$

which proves the claim.

On the other hand, we know from (48) that $\int_0^\infty \|\nabla\bar{u}_3(t)\|_{L^2}^2\,\mathrm{d}t < \infty$, hence there exists a sequence $t_n \to \infty$ such that $\|\nabla\bar{u}_3(t_n)\|_{L^2}^2 \to 0$ as $n \to \infty$. In view of (29), we have for each n:

$$\sup_{t\ge t_n}\|\nabla\bar{u}_3(t)\|_{L^2}^2 \;\le\; \|\nabla\bar{u}_3(t_n)\|_{L^2}^2 + C\int_{t_n}^\infty \bigl(\|\nabla\bar{u}_3(s)\|_{L^2}^2\|\bar{\omega}_3(t)\|_{L^2}^2 + \|\nabla\tilde{u}(s)\|_{L^2}^3\|\Delta\tilde{u}(s)\|_{L^2}\bigr)\,\mathrm{d}s\;,$$

and the right-hand side converges to zero as $n \to \infty$. This shows that $\|\nabla\bar{u}_3(t)\|_{L^2} \to 0$ as $t \to \infty$, and we have therefore proved that $\bar{u}_3(t)$ converges to zero in $H^1(\mathbb{R}^2)$ as $t \to \infty$.

3.3 Diffusive estimates for $\bar{\omega}_3$

We now turn our attention to the third component of the two-dimensional vorticity $\bar{\omega}$, which evolves according to (15):

$$\partial_t\bar{\omega}_3 + (\bar{u}_h\cdot\nabla)\bar{\omega}_3 + N_2 \;=\; \Delta\bar{\omega}_3\;. \tag{61}$$

By (48), there exists $C_4 > 0$ such that $\|\bar{\omega}_3(t)\|_{L^1} + \|\bar{\omega}_3(t)\|_{L^2} \le C_4$ for all $t \ge 0$. To obtain sharper estimates, including decay rates in time, we use a standard method that goes back to Nash, see [5]. By the Gagliardo-Nirenberg inequality, there exists $C > 0$ such that $\|\bar{\omega}_3\|_{L^2}^2 \le C\|\bar{\omega}_3\|_{L^1}\|\nabla\bar{\omega}_3\|_{L^2}$, hence $\|\bar{\omega}_3\|_{L^2}^2 \le CC_4\|\nabla\bar{\omega}_3\|_{L^2}$. Inserting this bound into (30), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\omega}_3(t)\|_{L^2}^2 \;\le\; -C_5\|\bar{\omega}_3(t)\|_{L^2}^4 + 8\||\tilde{u}(t)||\nabla\tilde{u}(t)|\|_{L^2}\;, \tag{62}$$

where $C_5 = (CC_4)^{-2}$. Since $\||\tilde{u}(t)||\nabla\tilde{u}(t)|\|_{L^2}$ decays exponentially to zero as $t \to \infty$, it follows from (62) that

$$\sup_{t\ge 0}(1+t)\|\bar{\omega}_3(t)\|_{L^2}^2 \;=\; C_6 \;<\; \infty\;. \tag{63}$$

A similar argument can be used to estimate $\|\nabla\bar{\omega}_3\|_{L^2}$. From (61) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{\omega}_3\|_{L^2}^2 \;=\; -\int_{\mathbb{R}^2}|\Delta\bar{\omega}_3|^2\,\mathrm{d}x + \int_{\mathbb{R}^2}(\Delta\bar{\omega}_3)(\bar{u}_h\cdot\nabla)\bar{\omega}_3\,\mathrm{d}x + \int_{\mathbb{R}^2}(\Delta\bar{\omega}_3)N_2\,\mathrm{d}x\;.$$

Integrating by parts and using the fact that $\|\nabla\bar{u}_h\|_{L^2} = \|\bar{\omega}_3\|_{L^2}$, we find

$$\begin{aligned}\Bigl|\int_{\mathbb{R}^2}(\Delta\bar{\omega}_3)(\bar{u}_h\cdot\nabla)\bar{\omega}_3\,\mathrm{d}x\Bigr| \;&\le\; \||\nabla\bar{\omega}_3|\,|\nabla\bar{u}_h|\,|\nabla\bar{\omega}_3|\|_{L^1} \;\le\; \|\nabla\bar{\omega}_3\|_{L^4}^2\|\bar{\omega}_3\|_{L^2}\\ &\le\; C\|\Delta\bar{\omega}_3\|_{L^2}\|\nabla\bar{\omega}_3\|_{L^2}\|\bar{\omega}_3\|_{L^2} \;\le\; C\|\Delta\bar{\omega}_3\|_{L^2}^{3/2}\|\bar{\omega}_3\|_{L^2}^{3/2}\;,\end{aligned}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar{\omega}_3(t)\|_{L^2}^2 \;\le\; -\|\Delta\bar{\omega}_3(t)\|_{L^2}^2 + C(\|\bar{\omega}_3(t)\|_{L^2}^6 + \|N_2(t)\|_{L^2}^2)\;.$$

As $\|\nabla\bar\omega_3\|_{L^2}^2 \le \|\bar\omega_3\|_{L^2}\|\Delta\bar\omega_3\|_{L^2} \le C_6^{1/2}(1+t)^{-1/2}\|\Delta\bar\omega_3\|_{L^2}$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\bar\omega_3(t)\|_{L^2}^2 \;\le\; -C_6^{-1}(1+t)\|\nabla\bar\omega_3(t)\|_{L^2}^4 + C(\|\bar\omega_3(t)\|_{L^2}^6 + \|N_2(t)\|_{L^2}^2)\ . \tag{64}$$

Now, since $\|\bar\omega_3(t)\|_{L^2}^6 \le C_6^3(1+t)^{-3}$, and since $\|N_2(t)\|_{L^2}^2$ decays exponentially to zero as $t\to\infty$, the differential inequality (64) implies that $\|\nabla\bar\omega_3(t)\|_{L^2}^2$ decreases at least like t^{-2} as $t\to\infty$. Taking into account the fact that $\bar\omega_3(0)\in L^2(\mathbb{R}^2)$, we arrive at

$$\sup_{t\ge 0} t(1+t)\|\nabla\bar\omega_3(t)\|_{L^2}^2 \;=\; C_7 \;<\; \infty\ . \tag{65}$$

3.4 Compactness of the rescaled solution

To show that the solution $\bar\omega_3(t,x)$ of (61) converges to Oseen's vortex as $t\to\infty$, it is convenient to introduce self-similar variables. Following [7, 8], we define

$$\begin{aligned}\bar\omega_3(t,x) \;&=\; \frac{1}{1+t}\,w\left(\log(1+t)\,,\,\frac{x}{\sqrt{1+t}}\right)\ ,\\ \bar u_h(t,x) \;&=\; \frac{1}{\sqrt{1+t}}\,v\left(\log(1+t)\,,\,\frac{x}{\sqrt{1+t}}\right)\ .\end{aligned} \tag{66}$$

We also denote

$$\xi \;=\; \frac{x}{\sqrt{1+t}}\ , \qquad \tau \;=\; \log(1+t)\ .$$

Then the rescaled vorticity $w(\tau,\xi)$ satisfies the equation

$$\partial_\tau w + (v\cdot\nabla_\xi)w + \tilde N_2 \;=\; \Delta_\xi w + \frac{1}{2}(\xi\cdot\nabla_\xi)w + w\ , \tag{67}$$

where $\tilde N_2(\tau,\xi) = e^{2\tau}N_2(e^\tau - 1, \xi\, e^{\tau/2})$, and $v(\tau,\xi)$ coincides with the two-dimensional velocity field obtained from $w(\tau,\xi)$ via the Biot-Savart law (81). It is clear that

$$\int_0^\infty \|\tilde N_2(\tau)\|_{L^1}\,\mathrm{d}\tau \;=\; \int_0^\infty e^\tau\|N_2(e^\tau-1)\|_{L^1}\,\mathrm{d}\tau \;=\; \int_0^\infty \|N_2(t)\|_{L^1}\,\mathrm{d}t \;<\; \infty\ ,$$

hence the term $\tilde N_2(\tau,\xi)$ in (67) will be negligible for large times. The solution of (67) can be represented as

$$w(\tau) \;=\; \tilde S_v(\tau,\tau_0)w(\tau_0) - \int_{\tau_0}^{\tau}\tilde S_v(\tau,s)\tilde N_2(s)\,\mathrm{d}s\ , \quad \tau\ge\tau_0\ge 0\ , \tag{68}$$

where in analogy with (58) we denote by $\tilde S_v(\tau,\tau_0)$ the two-parameter evolution operator associated to the linear equation $\partial_\tau w + (v\cdot\nabla)w = \Delta w + \frac{1}{2}(\xi\cdot\nabla)w + w$ (note that $\tilde S_v$ depends on the velocity field $v(\tau,\xi)$, which is considered here as given). Using the same notations as in Section 3.2, we find that

$$(\tilde S_v(\tau,\tau_0)f)(\xi) \;=\; \int_{\mathbb{R}^2} e^\tau\Gamma_{\bar u}(e^\tau-1,\xi\,e^{\tau/2}; e^{\tau_0}-1,\xi_0\,e^{\tau_0/2})f(\xi_0)\,\mathrm{d}\xi_0\ . \tag{69}$$

The aim of this paragraph is to prove the following basic result:

Lemma 3.1 *The solution $\{w(\tau)\}_{\tau\ge 0}$ of (67) is relatively compact in $L^1(\mathbb{R}^2)$.*

Proof. By construction $w \in C^0([0,\infty), L^1(\mathbb{R}^2))$ and $\|w(\tau)\|_{L^1} \le C_4$ for all $\tau \ge 0$. To prove compactness, we use the Riesz criterion [18] and proceed in two steps:
i) We first show that

$$\sup_{\tau\ge 0} \int_{|\xi|\ge R} |w(\tau,\xi)|\,\mathrm{d}\xi \xrightarrow[R\to\infty]{} 0 \ . \tag{70}$$

Indeed, fix $\varepsilon > 0$ and take $\tau_0 \ge 0$ large enough so that $\int_{\tau_0}^\infty \|\tilde{N}_2(\tau)\|_{L^1}\,\mathrm{d}\tau \le \varepsilon/2$. Then choose $R_1 \ge 0$ large enough so that

$$\sup_{\tau\in[0,\tau_0]} \int_{|\xi|\ge R_1} |w(\tau,\xi)|\,\mathrm{d}\xi \,\le\, \varepsilon \ .$$

This is clearly possible, because the finite-time trajectory $\{w(\tau)\,|\,0 \le \tau \le \tau_0\}$ is compact in $L^1(\mathbb{R}^2)$. For $\tau \ge \tau_0$ the solution of (67) can be represented as in (68), where the second term in the right-hand side satisfies

$$\Big\|\int_{\tau_0}^{\tau} \tilde{S}_v(\tau,s)\tilde{N}_2(s)\,\mathrm{d}s\Big\|_{L^1} \,\le\, \int_{\tau_0}^{\tau} \|\tilde{N}_2(s)\|_{L^1}\,\mathrm{d}s \,\le\, \varepsilon/2 \ .$$

As for the first term $w_1(\tau) = \tilde{S}_v(\tau,\tau_0)w(\tau_0)$, it can be estimated by a direct calculation, using the representation formula (69) and the bounds (59) on the kernel $\Gamma_{\bar{u}}$. Proceeding exactly as in the proof of [8, Lemma 2.5], one finds $R_2 \ge 0$ such that

$$\sup_{\tau\ge\tau_0} \int_{|\xi|\ge R_2} |w_1(\tau,\xi)|\,\mathrm{d}\xi \,\le\, \frac{\varepsilon}{2} \ .$$

If we now choose $R = \max(R_1,R_2)$, we see that $\int_{|\xi|\ge R} |w(\tau,\xi)|\,\mathrm{d}\xi \le \varepsilon$ for all $\tau \ge 0$, which proves (70).
ii) Our second task is to verify that

$$\sup_{\tau\ge 0}\sup_{|\eta|\le\delta} \int_{\mathbb{R}^2} |w(\tau,\xi-\eta) - w(\tau,\xi)|\,\mathrm{d}\xi \xrightarrow[\delta\to 0]{} 0 \ . \tag{71}$$

By compactness of the finite-time trajectory, it is sufficient to check (71) for $\tau \ge 1$. Using the definitions (66) and the bound (65) established in Section 3.3, we find

$$\sup_{\tau\ge 1} \|\nabla w(\tau)\|_{L^2} \,=\, C_8 \,<\, \infty \ .$$

Fix $\varepsilon > 0$. By the first step, there exists $R \ge 1$ such that

$$\sup_{\tau\ge 1} \int_{|\xi|\ge R-1} |w(\tau,\xi)|\,\mathrm{d}\xi \,\le\, \frac{\varepsilon}{3} \ .$$

Take $\delta \in (0,1]$ such that $C_8\delta\pi^{1/2}(R+1) \le \varepsilon/3$. If $\eta \in \mathbb{R}^2$ satisfies $|\eta| \le \delta$, we have

$$\int_{|\xi|\ge R} |w(\tau,\xi-\eta) - w(\tau,\xi)|\,\mathrm{d}\xi \,\le\, 2\int_{|\xi|\ge R-1} |w(\tau,\xi)|\,\mathrm{d}\xi \,\le\, \frac{2\varepsilon}{3} \ .$$

On the other hand, by Fubini's theorem and Hölder's inequality,

$$\begin{aligned}\int_{|\xi|\le R} |w(\tau,\xi-\eta) - w(\tau,\xi)|\,\mathrm{d}\xi \,&\le\, \int_{|\xi|\le R}\int_0^1 |\eta\cdot\nabla w(\tau,\xi-r\eta)|\,\mathrm{d}r\,\mathrm{d}\xi \\ &\le\, |\eta|\int_{|\xi|\le R+1} |\nabla w(\tau,\xi)|\,\mathrm{d}\xi \,\le\, C_8|\eta|\pi^{1/2}(R+1) \,\le\, \frac{\varepsilon}{3} \ ,\end{aligned}$$

hence $\int_{\mathbb{R}^2} |w(\tau,\xi-\eta) - w(\tau,\xi)|\,\mathrm{d}\xi \le \varepsilon$ for all $\tau \ge 1$ whenever $|\eta| \le \delta$. This proves (71). By the Riesz criterion, (70) and (71) together imply that the trajectory $\{w(\tau)\}_{\tau\ge 0}$ is relatively compact in $L^1(\mathbb{R}^2)$. ■

3.5 Determination of the ω-limit set

We know from Lemma 3.1 that the solution $\{w(\tau)\}_{\tau\ge 0}$ of (67) lies in a compact subset of $L^1(\mathbb{R}^2)$. Let Ω_∞ be the ω-limit set of this solution, namely

$$\Omega_\infty \;=\; \Big\{ w_\infty \in L^1(\mathbb{R}^2) \,\Big|\, \exists \tau_n \to \infty \text{ such that } w(\tau_n) \xrightarrow[n\to\infty]{L^1} w_\infty \Big\} \;.$$

Since $\int_{\mathbb{R}^2} w(\tau,\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^2} \bar\omega_3(e^\tau - 1, x)\,\mathrm{d}x = \alpha$ for all $\tau \ge 0$, where α is given by (8), it is clear that

$$\int_{\mathbb{R}^2} w_\infty(\xi)\,\mathrm{d}\xi \;=\; \alpha \;, \quad \text{for all } w_\infty \in \Omega_\infty \;. \tag{72}$$

Our goal is to show that $\Omega_\infty = \{\alpha g\}$, where $g(\xi) = (4\pi)^{-1} e^{-|\xi|^2/4}$. This will imply that $\|w(\tau) - \alpha g\|_{L^1} \to 0$ as $\tau \to \infty$, which is equivalent to (10).

Let $\hat\Phi(\tau)$ denote the semiflow defined by the limiting equation

$$\partial_\tau \hat w + \hat v \cdot \nabla_\xi \hat w \;=\; \Delta_\xi \hat w + \frac{1}{2}\xi \cdot \nabla_\xi \hat w + \hat w \;, \tag{73}$$

where $\hat v$ is the velocity field obtained from $\hat w$ via the Biot-Savart law (81). Note that (73) is just the ordinary two-dimensional vorticity equation expressed in self-similar variables. We shall prove that the ω-limit set of the solution $w(\tau)$ of (67) is totally invariant under the evolution defined by (73):

Lemma 3.2 *The ω-limit set Ω_∞ satisfies $\hat\Phi(\tau)\Omega_\infty = \Omega_\infty$ for all $\tau \ge 0$.*

Using [8, Proposition 3.5], we deduce that $\Omega_\infty \subset \{\alpha' g \,|\, \alpha' \in \mathbb{R}\}$, hence $\Omega_\infty = \{\alpha g\}$ in view of (72). This is the desired result, which completes the proof of Theorem 1.1.

Proof of Lemma 3.2. Let $\{S(\tau)\}_{\tau\ge 0}$ denote the C_0-semigroup generated by the Fokker-Planck operator $\Delta + \frac{1}{2}\xi\cdot\nabla + 1$, see [7]. If $w \in L^1(\mathbb{R}^2)$, then for any $p \in [1,\infty)$ we have the following estimates:

$$\|S(\tau)w\|_{L^p} \;\le\; \frac{\|w\|_{L^1}}{4\pi a(\tau)^{1-\frac{1}{p}}} \;, \qquad \|\nabla S(\tau) w\|_{L^p} \;\le\; \frac{C\|w\|_{L^1}}{a(\tau)^{\frac{3}{2}-\frac{1}{p}}} \;, \quad \tau > 0 \;, \tag{74}$$

where $a(\tau) = 1 - e^{-\tau}$. Moreover $\|S(\tau)w\|_{L^p} \le e^{\tau(1-\frac{1}{p})}\|w\|_{L^p}$ for all $\tau \ge 0$ if $w \in L^p(\mathbb{R}^2)$.

Let $w_\infty \in \Omega_\infty$, and take a sequence $\tau_n \to \infty$ such that $\|w(\tau_n) - w_\infty\|_{L^1} \to 0$ as $n \to \infty$. Since the trajectory $\{w(\tau)\}_{\tau\ge 0}$ is bounded in $L^2(\mathbb{R}^2)$ by (63), (66), we have $w_\infty \in L^2(\mathbb{R}^2)$ and (up to extracting a subsequence) we can assume that $\|w(\tau_n) - w_\infty\|_{L^p} \to 0$ as $n \to \infty$ for any $p \in [1,2)$. For each $n \in \mathbb{N}$, let $w_n(\tau) = w(\tau + \tau_n)$ and $v_n(\tau) = v(\tau + \tau_n)$. Then $w_n(\tau)$ satisfies the integral equation

$$w_n(\tau) \;=\; S(\tau) w(\tau_n) - \int_0^\tau S(\tau - s)\Big(v_n(s)\cdot\nabla w_n(s) + \tilde N_2(\tau_n + s)\Big)\,\mathrm{d}s \;. \tag{75}$$

On the other hand, if we denote $\hat w(\tau) = \hat\Phi(\tau) w_\infty$, we have

$$\hat w(\tau) \;=\; S(\tau) w_\infty - \int_0^\tau S(\tau - s)\hat v(s)\cdot\nabla\hat w(s)\,\mathrm{d}s \;. \tag{76}$$

Subtracting (76) from (75) and using the bounds (74) on the semigroup $S(\tau)$, we obtain for any $p \in [1,2)$:

$$\begin{aligned} \|w_n(\tau) - \hat{w}(\tau)\|_{L^p} \;&\le\; e^{\tau(1-\frac{1}{p})}\|w(\tau_n) - w_\infty\|_{L^p} + \int_0^\tau \frac{C}{a(\tau-s)^{1-\frac{1}{p}}}\,\|\tilde{N}_2(\tau_n+s)\|_{L^1}\,\mathrm{d}s \\ &+ \int_0^\tau \frac{C\,e^{-\frac{1}{2}(\tau-s)}}{a(\tau-s)^{\frac{3}{2}-\frac{1}{p}}}(\|w_n(s)\|_{L^{4/3}} + \|\hat{w}(s)\|_{L^{4/3}})\|w_n(s) - \hat{w}(s)\|_{L^{4/3}}\,\mathrm{d}s\;. \end{aligned} \tag{77}$$

Here we have used the fact that $S(\tau)v\cdot\nabla w = S(\tau)\nabla\cdot(vw) = e^{-\tau/2}\nabla\cdot S(\tau)(vw)$, and the bound $\|vw\|_{L^1} \le \|v\|_{L^4}\|w\|_{L^{4/3}} \le C\|w\|^2_{L^{4/3}}$ which holds in view of Proposition A.1. We first choose $p = 4/3$ and consider equation (77) for τ in some compact interval $[0,T]$. The first line in the right-hand side converges uniformly to zero as $n \to \infty$, and in the second line we know that $\|w_n(s)\|_{L^{4/3}} + \|\hat{w}(s)\|_{L^{4/3}}$ is uniformly bounded for all $n \in \mathbb{N}$ and all $\tau \in [0,T]$. Thus it follows from Gronwall's lemma [11] that

$$\sup_{\tau\in[0,T]} \|w_n(\tau) - \hat{w}(\tau)\|_{L^{4/3}} \xrightarrow[n\to\infty]{} 0\;. \tag{78}$$

Setting now $p = 1$ in (77) and using (78), we conclude that $\|w_n(\tau) - \hat{w}(\tau)\|_{L^1} \to 0$ as $n \to \infty$, for all $\tau \in [0,T]$. In other words $w(\tau+\tau_n)$ converges to $\hat{\Phi}(\tau)w_\infty$ as $n \to \infty$, which means that $\hat{\Phi}(\tau)w_\infty \in \Omega_\infty$ for all $\tau \in [0,T]$. Since $T > 0$ was arbitrary, we have shown that $\hat{\Phi}(\tau)\Omega_\infty \subset \Omega_\infty$ for all $\tau \ge 0$.

To prove the converse inclusion, we fix $\tau \ge 0$ and take again $w_\infty \in \Omega_\infty$. If $w(\tau_n) \to w_\infty$ in $L^1(\mathbb{R}^2)$ as $n \to \infty$, then after extracting a subsequence we can assume that $w(\tau_n - \tau)$ converges as $n \to \infty$ to some $w_0 \in \Omega_\infty$. Using exactly the same arguments as before, we can prove that $w_\infty = \hat{\Phi}(\tau)w_0$. This shows that $\Omega_\infty \subset \hat{\Phi}(\tau)\Omega_\infty$, for any $\tau \ge 0$. ■

A Appendix : The Biot-Savart Law in $\mathbb{R}^2 \times \mathbb{T}^1$

In this appendix we give explicit formulas for the Biot-Savart law in the domain $\mathbb{D} = \mathbb{R}^2 \times \mathbb{T}^1$, and we collect a few estimates for the velocity field u in terms of the vorticity ω which are used throughout the paper. All these results are well-known (see [19]) and are reproduced here for the reader's convenience.

Let $u : \mathbb{D} \to \mathbb{R}^3$ be a divergence-free velocity field, and denote by $\omega = \mathrm{curl}\, u$ the associated vorticity field. As is explained in the introduction, it is convenient to decompose

$$u(x,z) \;=\; \bar{u}(x) + \tilde{u}(x,z)\;, \quad \omega(x,z) \;=\; \bar{\omega}(x) + \tilde{\omega}(x,z)\;, \quad x \in \mathbb{R}^2\;, \quad z \in \mathbb{T}^1\;,$$

where $\bar{u} = Qu$, $\bar{\omega} = Q\omega$, and Q is the vertical average operator defined by (4). Then it is straightforward to verify that $\bar{\omega} = \mathrm{curl}\,\bar{u}$ and $\tilde{\omega} = \mathrm{curl}\,\tilde{u}$. Moreover, the four vector fields $\bar{u}$, $\tilde{u}$, $\bar{\omega}$, $\tilde{\omega}$ are all divergence-free. Thus we can consider separately the Biot-Savart law for the two-dimensional part $(\bar{u},\bar{\omega})$ and for the three-dimensional fluctuation $(\tilde{u},\tilde{\omega})$.

A.1 The Biot-Savart law for $(\bar{u},\bar{\omega})$.

Since the vector fields $\bar{u}$, $\bar{\omega}$ do not depend on the vertical variable z, the relations $\mathrm{div}\,\bar{u} = 0$ and $\mathrm{curl}\,\bar{u} = \bar{\omega}$ can be written in the following equivalent form:

$$(a)\;\begin{cases} \bar{\omega}_1 = \partial_2\bar{u}_3\;, \\ \bar{\omega}_2 = -\partial_1\bar{u}_3\;, \end{cases} \qquad (b)\;\begin{cases} \partial_1\bar{u}_2 - \partial_2\bar{u}_1 = \bar{\omega}_3\;, \\ \partial_1\bar{u}_1 + \partial_2\bar{u}_2 = 0\;. \end{cases} \tag{79}$$

To solve the first system (a), we observe that $\Delta \bar{u}_3 = \partial_2 \bar{\omega}_1 - \partial_1 \bar{\omega}_2$ and we use the fundamental solution of the Laplacian operator in $\mathbb{R}^2$. After integrating by parts, we obtain

$$\bar{u}_3(x) \;=\; -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)}{|x-y|^2} \wedge \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix}(y)\,\mathrm{d}y \;, \quad x \in \mathbb{R}^2 \;. \tag{80}$$

On the other hand, the solution of system (b) is just the ordinary Biot-Savart law in $\mathbb{R}^2$:

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\bar{\omega}_3(y)\,\mathrm{d}y \;, \quad x \in \mathbb{R}^2 \;. \tag{81}$$

Here, if $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $x^\perp = (-x_2, x_1)$. In particular, we see from (81) that the horizontal part $\bar{u}_h = (\bar{u}_1, \bar{u}_2)$ of the velocity field $\bar{u}$ can be reconstructed from the third component $\bar{\omega}_3$ of the vorticity $\bar{\omega}$, an observation that is used many times in the previous sections.

In both formulas (80) and (81), the velocity field is expressed in terms of the vorticity through a convolution with a singular integral kernel, which is homogeneous of degree -1. Thus we can apply the classical Hardy-Littlewood-Sobolev inequality [14] to both cases, and obtain the following result:

Proposition A.1 *Let $\bar{u}$ be the velocity field obtained from $\bar{\omega}$ via the Biot-Savart law* (80), (81). *Assume that $1 < p < 2 < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. If $\bar{\omega} \in L^p(\mathbb{R}^2)^3$, then $\bar{u} \in L^q(\mathbb{R}^2)^3$, and there exists a constant $C > 0$ (depending only on p) such that*

$$\|\bar{u}\|_{L^q(\mathbb{R}^2)} \;\le\; C\|\bar{\omega}\|_{L^p(\mathbb{R}^2)} \;.$$

Moreover, using Calderón-Zygmund's theory, one can show that $\|\nabla \bar{u}\|_{L^p} \le C\|\bar{\omega}\|_{L^p}$ for $1 < p < \infty$. In the particular case $p = 2$, we even have $\|\nabla \bar{u}\|_{L^2} = \|\bar{\omega}\|_{L^2}$.

A.2 The Biot-Savart law for $(\tilde{u}, \tilde{\omega})$

The relation between $\tilde{u}$ and $\tilde{\omega}$ is most conveniently expressed in Fourier variables. Using the same notations as in (17), we can write

$$\tilde{u}(x,z) \;=\; \int_{\mathbb{R}^2} \sum_{n\in\mathbb{Z}^*} \tilde{u}_n(k)\, e^{i(k\cdot x + 2\pi n z)}\,\frac{\mathrm{d}k}{2\pi} \;, \quad \tilde{\omega}(x,z) \;=\; \int_{\mathbb{R}^2} \sum_{n\in\mathbb{Z}^*} \tilde{\omega}_n(k)\, e^{i(k\cdot x + 2\pi n z)}\,\frac{\mathrm{d}k}{2\pi} \;. \tag{82}$$

Observe that the sums here are taken over $n \in \mathbb{Z}^* \equiv \mathbb{Z} \setminus \{0\}$, because $\tilde{u}$ and $\tilde{\omega}$ have zero average with respect to the vertical variable. Since $\operatorname{div} \tilde{u} = 0$ and $\operatorname{curl} \tilde{u} = \tilde{\omega}$, we have $-\Delta \tilde{u} = \operatorname{curl} \tilde{\omega}$, hence

$$\tilde{u}_n(k) \;=\; \frac{1}{|k|^2 + 4\pi^2 n^2} \begin{pmatrix} 0 & -2\pi i n & i k_2 \\ 2\pi i n & 0 & -i k_1 \\ -i k_2 & i k_1 & 0 \end{pmatrix} \tilde{\omega}_n(k) \;, \quad n \in \mathbb{Z}^* \;, \quad k \in \mathbb{R}^2 \;. \tag{83}$$

Since $n \neq 0$ in (83), it follows that $\|\tilde{u}\|_{H^{s+1}} \le C\|\tilde{\omega}\|_{H^s}$ for any $s \ge 0$, see (19). In particular, taking $s = 0$ and using the Sobolev embedding $H^1(\mathbb{D}) \hookrightarrow L^q(\mathbb{D})$ for $q \in [2,6]$, we obtain:

Proposition A.2 *Let $\tilde{u}$ be the velocity field obtained from $\tilde{\omega}$ via the Biot-Savart law* (83). *If $\tilde{\omega} \in L^2(\mathbb{D})$, then $\tilde{u} \in L^q(\mathbb{D})$ for any $q \in [2,6]$, and there exists $C > 0$ (depending only on q) such that*

$$\|\tilde{u}\|_{L^q(\mathbb{D})} \;\le\; C\|\tilde{\omega}\|_{L^2(\mathbb{D})} \;.$$

A.3 The Leray projector

In the Fourier variables defined by (17), (18), the Leray projector $\mathbb{P}$ has the following simple expression

$$(\mathbb{P}f)_n(k) \;=\; f_n(k) + \frac{\xi\cdot f_n(k)}{|k|^2+4\pi^2n^2}\,\xi\;, \quad \text{where} \quad \xi \;=\; \begin{pmatrix} ik \\ 2\pi in\end{pmatrix} \in \mathbb{R}^3\;. \tag{84}$$

Clearly $\mathbb{P}$ commutes with the vertical average operator Q, which satisfies $(Qf)_n(k) \;=\; f_n(k)\delta_{n,0}$. If $\bar f = Qf$, we see from (84) that $e_3\cdot(\mathbb{P}\bar f) = e_3\cdot \bar f$. In other words, the Leray projector $\mathbb{P}$ acts trivially on the third component of z-independent vector fields.

B Appendix: Dispersive estimates

This final section is devoted to the proof of Proposition 2.3. The arguments here follow closely the analysis of [4, Chap. 5], and were already published in [20] in a slightly different form.

Proof of proposition 2.3: If $\tilde u(t,x,z)$ is a divergence-free solution of the linear Rossby equation (9), we first observe that the Fourier transform $\tilde u_n(t,k)$, which is defined as in (82), satisfies

$$\partial_t \tilde u_n(t,k) + M_n^\Omega(k)\tilde u_n(t,k) \;=\; 0\;, \quad k\in\mathbb{R}^2\;, \quad n\in\mathbb{Z}^*\;,$$

where $M_n^\Omega(k)$ is the 3×3 matrix defined by

$$M_n^\Omega(k) \;=\; (|k|^2+4\pi^2n^2)\mathbf{1} + \frac{2i\pi n\Omega}{|k|^2+4\pi^2n^2}\begin{pmatrix} 0 & -2\pi in & ik_2 \\ 2\pi in & 0 & -ik_1 \\ -ik_2 & ik_1 & 0\end{pmatrix}\;. \tag{85}$$

Indeed, the first term in (85) corresponds to $-(\Delta\tilde u)_n(k) = (|k|^2+4\pi^2n^2)\tilde u_n(k)$. On the other hand, if $\tilde\omega = \operatorname{curl}\tilde u$, we have from (83)

$$\tilde\omega_n(k) \;=\; \xi\wedge\tilde u_n(k)\;, \quad \tilde u_n(k) \;=\; \frac{\xi\wedge\tilde\omega_n(k)}{|\xi|^2}\;, \quad \text{where} \quad \xi \;=\; \begin{pmatrix} ik \\ 2\pi in\end{pmatrix}\;.$$

It follows that

$$e_3\wedge\tilde u_n(k) \;=\; \frac{1}{|\xi|^2}\,e_3\wedge(\xi\wedge\tilde\omega_n(k)) \;=\; \frac{1}{|\xi|^2}\Big((e_3\cdot\tilde\omega_n(k))\xi - (e_3\cdot\xi)\tilde\omega_n(k)\Big)\;.$$

The last member is the sum of two terms, one of which is proportional to ξ (gradient term) and the other orthogonal to ξ (divergence-free term). Thus

$$-\mathbb{P}(e_3\wedge\tilde u_n(k)) \;=\; \frac{1}{|\xi|^2}\,(e_3\cdot\xi)\tilde\omega_n(k) \;=\; \frac{2\pi in}{|\xi|^2}\,\xi\wedge\tilde u_n(k)\;,$$

which gives the second term in (85).

As is easily verified, the eigenvalues of $M_n^\Omega(k)$ are $|\xi|^2$ and $|\xi|^2\pm i\Omega\eta$, where

$$|\xi| \;=\; |\xi(k,n)| \;=\; \sqrt{|k|^2+4\pi^2n^2}\;, \quad \text{and} \quad \eta \;=\; \eta(k,n) \;=\; \frac{2\pi n}{\sqrt{|k|^2+4\pi^2n^2}}\;. \tag{86}$$

Moreover, the eigenvector corresponding to $|\xi|^2$ is proportional to ξ, whereas the normalized eigenvectors $w_n^\pm(k)$ corresponding to $|\xi|^2\pm i\Omega\eta$ are orthogonal to ξ. Since $\tilde u$ is divergence-free, we can forget about the first eigenvector, and we obtain the representation formula

$$\tilde u_n(t,k) \;=\; e^{-t|\xi|^2}\Big(e^{-it\Omega\eta}\langle\tilde u_n^0(k), w_n^+(k)\rangle + e^{it\Omega\eta}\langle\tilde u_n^0(k), w_n^-(k)\rangle\Big)\;, \quad t\geq 0\;, \tag{87}$$

where $\tilde{u}_n^0(k) = \tilde{u}_n(0,k)$ and $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{C}^3$.

To estimate the norm of $\tilde{u}$ in the space $L^1(\mathbb{R}_+, L^\infty(\mathbb{D}))$, we proceed as in [4]. Using standard approximation arguments, it is easy to show that

$$\|\tilde{u}\|_{L^1(\mathbb{R}_+,L^\infty(\mathbb{D}))} \;=\; \sup_{\phi\in\mathcal{E}} < \tilde{u}, \phi >_{L^2(\mathbb{R}_+,L^2(\mathbb{D}))} \quad ,$$

where $\mathcal{E} = \{\phi \in C_0^\infty(\mathbb{D}) \,|\, \|\phi\|_{L^\infty(\mathbb{R}_+,L^1(\mathbb{D}))} \le 1\}$. By the Parseval relation, we thus have

$$\|\tilde{u}\|_{L^1(\mathbb{R}_+,L^\infty(\mathbb{D}))} \;=\; \sup_{\phi\in\mathcal{E}} \int_0^\infty \int_{\mathbb{R}^2} \sum_{n\in\mathbb{Z}^*} \tilde{u}_n(t,k)\, \overline{\phi_n(t,k)}\, \mathrm{d}k\, \mathrm{d}t \;, \tag{88}$$

where $\phi_n(t,k)$ denotes of course the Fourier transform of $\phi(t,\cdot)$. The idea is now to replace (87) into (88), and to estimate the right-hand side. Before doing that, we recall that the initial data $\tilde{u}_n^0(k)$ were assumed to vanish outside a finite ball $\mathcal{B}_R$ in Fourier space, see Proposition 2.3. In view of (87), the same property holds for $\tilde{u}_n(t,k)$ for all $t \ge 0$. Thus $\tilde{u}_n(t,k) \equiv \psi_n(k)\tilde{u}_n(t,k)$, where

$$\psi_n(k) \;=\; (1-\delta_{n,0})\,\chi\Big(\frac{\sqrt{|k|^2+4\pi^2n^2}}{2R}\Big) \;, \quad k\in\mathbb{R}^2 \;, \quad n\in\mathbb{Z} \;. \tag{89}$$

Here χ is as in (24), and δ is the Kronecker symbol.

Given any $A \ge 0$ and any $B \in \mathbb{R}$, we denote by $K[A,B] \in C^\infty(\mathbb{D})$ the function defined in Fourier variables by

$$K[A,B]_n(k) \;=\; \frac{1}{2\pi}\, e^{-A|\xi|^2 + iB\eta}\, \psi_n(k)^2 \;, \quad k\in\mathbb{R}^2 \;, \quad n\in\mathbb{Z} \;, \tag{90}$$

where $|\xi|$ and η are as in (86). The following estimate will be crucial:

Lemma B.1 *For any $R > 0$ there exists $C_R > 0$ such that, for any $A \ge 0$ and any $B \in \mathbb{R}$, the function $K[A,B] \in C^\infty(\mathbb{D})$ defined by (90) satisfies*

$$\|K[A,B]\|_{L^\infty(\mathbb{D})} \;\le\; C_R \frac{e^{-4\pi^2 A}}{\sqrt{|B|}} \;.$$

We postpone the proof of this lemma and first conclude the proof of Proposition 2.3. After replacing (87) into (88), we have to estimate for each $\phi \in \mathcal{E}$ the quantity $M_+ + M_-$, where

$$M_\pm \;=\; \int_{\mathbb{R}^2} \sum_{n\in\mathbb{Z}^*} \langle \tilde{u}_n^0(k), w_n^\pm(k)\rangle \Big\{ \int_0^\infty e^{-t|\xi|^2 \mp it\Omega\eta} \psi_n(k)\, \overline{\phi_n(t,k)}\, \mathrm{d}t \Big\}\, \mathrm{d}k \;.$$

Since the eigenvectors $w_n^\pm(k)$ are normalized, the Cauchy-Schwarz inequality and the Parseval relation imply that $|M_\pm| \le \|\tilde{u}_0\|_{L^2(\mathbb{D})} N_\pm$, where

$$\begin{aligned}
N_\pm^2 \;&=\; \int_{\mathbb{R}^2} \sum_{n\in\mathbb{Z}^*} \Big| \int_0^\infty e^{-t|\xi|^2 \mp it\Omega\eta}\, \psi_n(k)\, \overline{\phi_n(t,k)}\, \mathrm{d}t \Big|^2 \mathrm{d}k \\
&=\; \int_{\mathbb{R}^2} \sum_{n\in\mathbb{Z}^*} \int_0^\infty \int_0^\infty e^{-(t+s)|\xi|^2 \pm i(s-t)\Omega\eta}\, \psi_n(k)^2\, \overline{\phi_n(t,k)}\, \phi_n(s,k)\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}k \\
&=\; \int_0^\infty \int_0^\infty < K[t+s, \pm\Omega(s-t)] * \phi(s,\cdot)\,,\, \phi(t,\cdot) >_{L^2(\mathbb{D})}\, \mathrm{d}t\, \mathrm{d}s \;.
\end{aligned}$$

In the last line we have used the definition (90) of $K[A,B]$ and the Parseval relation again. Now, since $\phi \in \mathcal{E}$, it follows from Young's inequality that

$$\begin{aligned} |< K[A,B] * \phi(s,\cdot)\,,\, \phi(t,\cdot) >_{L^2(\mathbb{D})}| \;&\le\; \|K[A,B]\|_{L^\infty(\mathbb{D})}\, \|\phi(t,\cdot)\|_{L^1(\mathbb{D})}\, \|\phi(s,\cdot)\|_{L^1(\mathbb{D})} \\ &\le\; \|K[A,B]\|_{L^\infty(\mathbb{D})}\ . \end{aligned}$$

Thus, setting $A = t + s$, $B = \pm\Omega(s - t)$, we obtain from Lemma B.1

$$N_\pm^2 \;\le\; C_R \int_0^\infty\!\!\int_0^\infty \frac{e^{-4\pi^2(t+s)}}{\sqrt{|\Omega||t-s|}}\,\mathrm{d}t\,\mathrm{d}s \;\le\; \frac{C_R}{|\Omega|^{1/2}}\ .$$

Summarizing, we have shown that $|M_\pm| \le C_R|\Omega|^{-1/4}\|\tilde u_0\|_{L^2(\mathbb{D})}$ for all $\phi \in \mathcal{E}$, which in turn implies $\|\tilde u\|_{L^1(\mathbb{R}_+,L^\infty(\mathbb{D}))} \le C_R|\Omega|^{-1/4}\|\tilde u_0\|_{L^2(\mathbb{D})}$. This concludes the proof of Proposition 2.3. ■

Proof of lemma B.1: Given $A \ge 0$ and $B \in \mathbb{R}$, we have to estimate the expression

$$K[A,B](x,z) \;=\; \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}^*} e^{-A|\xi(k,n)|^2 + iB\eta(k,n)}\,\psi_n(k)^2\, e^{i(k\cdot x + 2\pi nz)}\,\mathrm{d}k\ ,$$

where $|\xi|, \eta$ are defined in (86) and $\psi_n(k)$ is given by (89). Here again, we follow the approach presented in [4, Chap. 5]. As $K[A,B](x,z)$ is a radially symmetric function of $x \in \mathbb{R}^2$, we can assume without loss of generality that $x_2 = 0$. Clearly, we can also suppose that $B \ge 0$. Let L be the first-order differential operator defined by

$$L \;=\; \frac{1}{1 + B\alpha(k,n)^2}(1 + i\alpha(k,n)\partial_{k_2})\ , \quad \text{where} \quad \alpha(k,n) = -\partial_{k_2}\eta(k,n)\ .$$

Then $L(e^{iB\eta(k,n)}) = e^{iB\eta(k,n)}$, and integrating by parts (over the variable $k \in \mathbb{R}^2$) we find

$$K[A,B]((x_1,0),z) \;=\; \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}^*} e^{iB\eta(k,n)}\, e^{i(k_1x_1 + 2\pi nz)} L^t\Big(e^{-A|\xi(k,n)|^2}\,\psi_n(k)^2\Big)\,\mathrm{d}k\ ,$$

where L^t denotes the formal adjoint of L. A direct calculation gives

$$\begin{aligned} L^t\Big(e^{-A|\xi(k,n)|^2}\,\psi_n(k)^2\Big) \;&=\Big(\frac{1}{1+B\alpha^2} - i(\partial_{k_2}\alpha)\frac{1-B\alpha^2}{(1+B\alpha^2)^2}\Big)e^{-A|\xi(k,n)|^2}\,\psi_n(k)^2 \\ &\quad - \frac{i\alpha}{1+B\alpha^2}\,\partial_{k_2}\Big(e^{-A|\xi(k,n)|^2}\,\psi_n(k)^2\Big)\ . \end{aligned}$$

We have to estimate this quantity for $(k,n) \in \mathcal{B}_{2R}$ and $n \ne 0$, because $\psi_n(k) = 0$ if $(k,n) \notin \mathcal{B}_{2R}$ or $n = 0$. We first observe that

$$|\xi(k,n)| \;\ge\; 2\pi\ , \quad \text{and} \quad |\alpha(k,n)| \;=\; \frac{2\pi|n||k_2|}{(|k|^2 + 4\pi^2n^2)^{3/2}} \;\ge\; \frac{\pi|k_2|}{4R^3}\ .$$

Moreover, there exists $C_R > 0$ such that $|\alpha(k,n)| + |\partial_{k_2}\alpha(k,n)| \le C_R$. As a consequence, we have

$$\frac{1}{1+B\alpha^2} + \frac{|1-B\alpha^2|}{(1+B\alpha^2)^2} + \frac{|\alpha|}{1+B\alpha^2} \;\le\; \frac{C_R}{1+Bk_2^2}\ ,$$

so that $|L^t(e^{-A|\xi(k,n)|^2}\,\psi_n(k)^2)| \le C_R\, e^{-4\pi^2A}\psi_n(k)(1+Bk_2^2)^{-1}$. We conclude that

$$\begin{aligned} \|K[A,B]\|_{L^\infty(\mathbb{D})} \;&\le\; \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{Z}^*}|L^t(e^{-A|\xi(k,n)|^2}\,\psi_n(k)^2)|\,\mathrm{d}k \\ &\le\; C_R\, e^{-4\pi^2A}\int_{\mathbb{R}}\frac{\mathrm{d}k_2}{1+Bk_2^2} \;\le\; C_R\,\frac{e^{-4\pi^2A}}{\sqrt{B}}\ , \end{aligned}$$

which is the desired estimate. ■

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