On the linear stability of vortex columns in the energy space

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Abstract

We investigate the linear stability of inviscid columnar vortices with respect to finite energy perturbations. For a large class of vortex profiles, we show that the linearized evolution group has a sub-exponential growth in time, which means that the associated growth bound is equal to zero. This implies in particular that the spectrum of the linearized operator is entirely contained in the imaginary axis. This contribution complements the results of our previous work [10], where spectral stability was established for the linearized operator in the enstrophy space.

1 Introduction

It is well known that radially symmetric vortices in two-dimensional incompressible and inviscid fluids are stable if the vorticity distribution is a monotone function of the distance to the vortex center [3, 14]. In a three-dimensional framework, this result exactly means that *columnar vortices* with no axial flow are stable with respect to two-dimensional perturbations, provided Arnold's monotonicity condition is satisfied. Vortex columns play an important role in nature, especially in atmospheric flows, and are also often observed in laboratory experiments [2]. It is therefore of great interest to determine their stability with respect to arbitrary perturbations, with no particular symmetry, but this question appears to be very difficult and the only rigorous results available so far are sufficient conditions for *spectral stability*.

In a celebrated paper [18], Lord Kelvin considered the particular case of Rankine's vortex and proved that the linearized operator has a countable family of eigenvalues on the imaginary axis. The corresponding eigenfunctions, which are now referred to as *Kelvin's vibration modes*, have been extensively studied in the literature, also for more general vortex profiles [8, 13, 17]. An important contribution was made by Lord Rayleigh in [15], who gave a simple condition for spectral stability with respect to axisymmetric perturbations. Rayleigh's criterion, which requires that the angular velocity Ω and the vorticity W have the same sign everywhere, is actually implied by Arnold's monotonicity condition for localized vortices. In the non-axisymmetric case, the only stability result one can obtain using the techniques introduced by Rayleigh is restricted to perturbations in a particular subspace, where the angular Fourier mode m and the vertical wave number k are fixed. In that subspace, we have a sufficient condition for spectral stability, involving a quantity that can be interpreted as a local Richardson number. However, as is emphasized by Howard and Gupta [12], that criterion always fails when the ratio k^2/m^2 is sufficiently small, and therefore does note provide any unconditional stability result.

In a recent work [10], we perform a rigorous mathematical study of the linearized operator at a columnar vortex, using the vorticity formulation of the Euler equations. We assume that the unperturbed vorticity profile satisfies Arnold's monotonicity condition, hence Rayleigh's criterion as well, and we impose an additional condition which happens to be satisfied in all classical examples and may only be technical. We work in the enstrophy space, assuming periodicity (with arbitrary period) in the vertical direction. In this framework, we prove that the spectrum of the linearized operator is entirely contained in the imaginary axis of the complex plane, which gives the first spectral stability result for columnar vortices with smooth velocity profile. More precisely, in any Fourier subspace characterized by its angular mode $m \neq 0$ and its vertical wave number $k \neq 0$, we show that the spectrum of the linearized operator consists of an essential part that fills an interval of the imaginary axis, and of a countable family of imaginary eigenvalues which accumulate only on the essential spectrum (the latter correspond to Kelvin's vibration modes). The most difficult part of our analysis is to preclude the existence of isolated eigenvalues with nonzero real part, which can eventually be done by combining Howard and Gupta's criterion, a homotopy argument, and a detailed analysis of the eigenvalue equation when critical layers occur.

The goal of the present paper is to extend the results of [10] in several directions. First, we use the velocity formulation of the Euler equations, and assume that the perturbations have finite energy. This functional framework seems more natural than the enstrophy space used in [10], but part of the analysis becomes more complicated. In particular, due to the pressure term in the velocity formulation, it is not obvious that the linearized operator in a given Fourier sector is the sum of a (nearly) skew-symmetric principal part and a compact perturbation. This decomposition, however, is the starting point of our approach, as it shows that the spectrum outside the imaginary axis is necessarily discrete. Also, unlike in [10], we do not have to assume periodicity in the vertical direction, so that our result applies to localized perturbations as well. Finally, we make a step towards linear stability by showing that the evolution group generated by the linearized operator has a mild, sub-exponential growth as $|t| \to \infty$. This is arguably the strongest way to express spectral stability.

We now present our result in more detail. We consider the incompressible Euler equations in the whole space $\mathbb{R}^3$:

$$\partial_t u + (u \cdot \nabla) u \,=\, -\nabla p\,, \qquad \operatorname{div} u \,=\, 0\,, \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^3$ denotes the velocity of the fluid at point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$, and $p = p(x,t) \in \mathbb{R}$ is the associated pressure. The solutions we are interested in are perturbations of flows with axial symmetry, and are therefore conveniently described using cylindrical coordinates (r, θ, z) defined by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $x_3 = z$. The velocity field is decomposed as

$$u \,=\, u_r(r,\theta,z,t) e_r + u_\theta(r,\theta,z,t) e_\theta + u_z(r,\theta,z,t) e_z\,,$$

where e_r, e_θ, e_z are unit vectors in the radial, azimuthal, and vertical directions, respectively. The evolution equation in (1.1) is then written in the equivalent form

$$\begin{aligned}
\partial_t u_r + (u\cdot\nabla)u_r - \frac{u_\theta^2}{r} \,&=\, -\partial_r p\,,\\
\partial_t u_\theta + (u\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} \,&=\, -\frac{1}{r}\partial_\theta p\,,\\
\partial_t u_z + (u\cdot\nabla)u_z \,&=\, -\partial_z p\,,
\end{aligned} \tag{1.2}$$

where $u\cdot\nabla = u_r\partial_r + \frac{1}{r}u_\theta\partial_\theta + u_z\partial_z$, and the incompressibility condition becomes

$$\operatorname{div} u \,=\, \frac{1}{r}\partial_r(r u_r) + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z \,=\, 0\,. \tag{1.3}$$

Columnar vortices are described by stationary solutions of (1.2), (1.3) of the following form

$$u \,=\, V(r)\,e_\theta\,, \qquad p \,=\, P(r)\,, \tag{1.4}$$

where the velocity profile $V : \mathbb{R}_+ \to \mathbb{R}$ is arbitrary, and the pressure $P : \mathbb{R}_+ \to \mathbb{R}$ is determined by the centrifugal balance $rP'(r) = V(r)^2$. Other physically relevant quantities that characterize the vortex are the angular velocity Ω and the vorticity W:

$$\Omega(r) \,=\, \frac{V(r)}{r}\,, \qquad W(r) \,=\, \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\bigl(rV(r)\bigr) \,=\, r\Omega'(r) + 2\Omega(r)\,. \tag{1.5}$$

To investigate the stability of the vortex (1.4), we consider perturbed solutions of the form

$$u(r,\theta,z,t) \,=\, V(r)\,e_\theta + \tilde{u}(r,\theta,z,t)\,, \qquad p(r,\theta,z,t) \,=\, P(r) + \tilde{p}(r,\theta,z,t)\,.$$

Inserting this Ansatz into (1.2) and neglecting the quadratic terms in $\tilde{u}$, we obtain the linearized evolution equations

$$\begin{aligned} \partial_t u_r + \Omega\partial_\theta u_r - 2\Omega u_\theta \,&=\, -\partial_r p\,, \\ \partial_t u_\theta + \Omega\partial_\theta u_\theta + W u_r \,&=\, -\frac{1}{r}\partial_\theta p\,, \\ \partial_t u_z + \Omega\partial_\theta u_z \,&=\, -\partial_z p\,, \end{aligned} \tag{1.6}$$

where we have dropped all tildes for notational simplicity. Remark that the incompressibility condition (1.3) still holds for the velocity perturbations. Thus, taking the divergence of both sides in (1.6), we see that the pressure p satisfies the second order elliptic equation

$$-\partial_r^*\partial_r p - \frac{1}{r^2}\partial_\theta^2 p - \partial_z^2 p \,=\, 2\bigl(\partial_r^*\Omega\bigr)\partial_\theta u_r - 2\partial_r^*\bigl(\Omega\, u_\theta\bigr)\,, \tag{1.7}$$

where we introduced the shorthand notation $\partial_r^* f = \frac{1}{r}\partial_r(rf) = \partial_r f + \frac{1}{r}f$.

We want to solve the evolution equation (1.6) in the Hilbert space

$$X \,=\, \Bigl\{u = (u_r, u_\theta, u_z) \in L^2(\mathbb{R}^3)^3 \,\Big|\, \partial_r^* u_r + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z = 0\Bigr\}\,,$$

equipped with the standard L^2 norm. Note that the definition of X incorporates the incompressibility condition (1.3). In Section 3 we shall verify that, for any $u \in X$, the elliptic equation (1.7) has a unique solution (up to an irrelevant additive constant) that satisfies $\nabla p \in L^2(\mathbb{R}^3)^3$. Denoting that solution by $p = P[u]$, we can write Eq. (1.6) in the abstract form $\partial_t u = Lu$, where L is the integro-differential operator in X defined by

$$Lu \,=\, \begin{pmatrix} -\Omega\partial_\theta u_r + 2\Omega u_\theta - \partial_r P[u] \\ -\Omega\partial_\theta u_\theta - W u_r - \frac{1}{r}\partial_\theta P[u] \\ -\Omega\partial_\theta u_z - \partial_z P[u] \end{pmatrix}\,. \tag{1.8}$$

If the angular velocity Ω and the vorticity W are, for instance, bounded and continuous functions on $\mathbb{R}_+$, it is not difficult to verify that the operator L generates a strongly continuous group of bounded linear operators in X, see Section 2. Our goal is to show that, under additional assumptions on the vortex profile, the norm of this evolution group has a mild growth as $|t| \to \infty$. Following [10], we make the following assumptions.

Assumption H1: *The vorticity profile* $W : \overline{\mathbb{R}}_+ \to \mathbb{R}_+$ *is a* $\mathcal{C}^2$ *function satisfying* $W'(0) = 0$, $W'(r) < 0$ *for all* $r > 0$, $r^3 W'(r) \to 0$ *as* $r \to \infty$, *and*

$$\Gamma \,:=\, \int_0^\infty W(r) r \,\mathrm{d}r \,<\, \infty\,. \tag{1.9}$$

According to (1.5), the angular velocity Ω can be expressed in terms of the vorticity W by the formula

$$\Omega(r) \,=\, \frac{1}{r^2}\int_0^r W(s) s \,\mathrm{d}s\,, \qquad r > 0\,, \tag{1.10}$$

and the derivative Ω' satisfies

$$\Omega'(r) \,=\, \frac{W(r) - 2\Omega(r)}{r} \,=\, \frac{1}{r^3}\int_0^r W'(s) s^2 \,\mathrm{d}s\,, \qquad r > 0\,. \tag{1.11}$$

Thus $\Omega \in \mathcal{C}^2(\overline{\mathbb{R}}_+) \cap C^3(\mathbb{R}_+)$ is a positive function satisfying $\Omega(0) = W(0)/2$, $\Omega'(0) = 0$, $\Omega'(r) < 0$ for all $r > 0$, and $r^2\Omega(r) \to \Gamma$ as $r \to \infty$. Moreover, since W is nonincreasing, it follows from (1.9) that $r^2 W(r) \to 0$ as $r \to \infty$, and this implies that $r^3\Omega'(r) \to -2\Gamma$ as $r \to \infty$. Similarly $r^4\Omega''(r) \to 6\Gamma$ as $r \to \infty$. Finally, assumption H1 implies that the *Rayleigh function* is positive:

$$\Phi(r) \,=\, 2\Omega(r) W(r) \,>\, 0\,, \qquad r \ge 0\,. \tag{1.12}$$

Assumption H2: *The* $\mathcal{C}^1$ *function* $J : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$J(r) \,=\, \frac{\Phi(r)}{\Omega'(r)^2}\,, \qquad r > 0\,, \tag{1.13}$$

satisfies $J'(r) < 0$ *for all* $r > 0$ *and* $rJ'(r) \to 0$ *as* $r \to \infty$.

The reader is referred to the previous work [10] for a discussion of these hypotheses. We just recall here that assumptions H1, H2 are both satisfied in all classical examples that can be found in the physical literature. In particular, they hold for the Lamb-Oseen vortex:

$$\Omega(r) \,=\, \frac{1}{r^2}\Bigl(1 - e^{-r^2}\Bigr)\,, \qquad W(r) \,=\, 2\,e^{-r^2}\,, \tag{1.14}$$

and for the Kaufmann-Scully vortex:

$$\Omega(r) \,=\, \frac{1}{1+r^2}\,, \qquad W(r) \,=\, \frac{2}{(1+r^2)^2}\,. \tag{1.15}$$

Our main result can now be stated as follows:

Theorem 1.1. *Assume that the vorticity profile* W *satisfies assumptions H1, H2 above. Then the linear operator* L *defined in* (1.8) *is the generator of a strongly continuous group* $(e^{tL})_{t\in\mathbb{R}}$ *of bounded linear operators in* X. *Moreover, for any* $\epsilon > 0$, *there exists a constant* $C_\epsilon \ge 1$ *such that*

$$\|e^{tL}\|_{X\to X} \,\le\, C_\epsilon\, e^{\epsilon|t|}\,, \qquad \textit{for all } t \in \mathbb{R}\,. \tag{1.16}$$

Remark 1.2. Estimate (1.16) means that *growth bound* of the group e^{tL} is equal to zero, see [7, Section I.5]. Equivalently, the spectrum of e^{tL} is contained in the unit circle $\{z \in \mathbb{C} \,|\, |z| = 1\}$ for all $t \in \mathbb{R}$. Invoking the Hille-Yosida theorem, we deduce from (1.16) that the spectrum of the generator L is entirely contained in the imaginary axis of the complex plane, and that the following resolvent bound holds for any $a > 0$:

$$\sup\Bigl\{\|(z - L)^{-1}\|_{X\to X} \,\Big|\, z \in \mathbb{C}\,,\; |\mathrm{Re}(z)| \ge a\Bigr\} \,<\, \infty\,. \tag{1.17}$$

In fact, since X is a Hilbert space, the Gearhart-Prüss theorem [7, Section V.1] asserts that the resolvent bound (1.17) is also equivalent to the group estimate (1.16).

Remark 1.3. The constant C_ϵ in (1.16) may of course blow up as $\epsilon \to 0$, but unfortunately our proof does not give any precise information. It is reasonable to expect that $C_\epsilon = \mathcal{O}(\epsilon^{-N})$ for some $N > 0$, which would imply that $\|e^{tL}\| = \mathcal{O}(|t|^N)$ as $|t| \to \infty$, but proving such an estimate is an open problem.

Remark 1.4. In (1.14), (1.15), and in all what follows, we always assume that the vortex profile is normalized so that $W(0) = 2$, hence $\Omega(0) = 1$. The general case can be easily deduced by a rescaling argument.

The rest of this paper is organized as follows. In Section 2, we describe the main steps in the proof of Theorem 1.1. In particular, we show that the linearized operator (1.8) is the generator of a strongly continuous group in the Hilbert space X, and we reduce the linearized equations to a family of one-dimensional problems using a Fourier series expansion in the angular variable θ and a Fourier transform with respect to the vertical variable z. For a fixed value of the angular Fourier mode $m \in \mathbb{Z}$ and of the vertical wave number $k \in \mathbb{R}$, we show that the restricted linearized operator $L_{m,k}$ is the sum of a (nearly) skew-symmetric part A_m and of a compact perturbation $B_{m,k}$. Actually, proving compactness of $B_{m,k}$ requires delicate estimates on the pressure, which are postponed to Section 3. We then invoke the result of [10] to show that $L_{m,k}$ has no eigenvalue, hence no spectrum, outside the imaginary axis. The last step in the proof consists in showing that, for any $a \neq 0$, the resolvent norm $\|(s - L_{m,k})^{-1}\|$ is uniformly bounded for all $m \in \mathbb{Z}$, all $k \in \mathbb{R}$, and all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = a$. This crucial bound is obtained in Section 4 using a priori estimates for the resolvent equation, which give explicit bounds in some regions of the parameter space, combined with a contradiction argument which takes care of the other regions. The proof of Theorem 1.1 is thus concluded at the end of Section 2, taking for granted the results of Sections 3 and 4 which are the main original contributions of this paper.

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2 Main steps of the proof

The proof of Theorem 1.1 can be divided into four main steps, which are detailed in the following subsections. The first two steps are rather elementary, but the remaining two require more technical calculations which are postponed to Sections 3 and 4.

2.1 Splitting of the linearized operator

The linearized operator (1.8) can be decomposed as $L = A + B$, where A is the first order differential operator

$$Au \;=\; -\Omega(r)\partial_\theta u + r\Omega'(r)u_r\, e_\theta\,, \tag{2.1}$$

and B is the nonlocal operator

$$Bu \;=\; -\nabla P[u] + 2\Omega u_\theta e_r - 2(r\Omega)' u_r e_\theta\,. \tag{2.2}$$

We recall that $W = r\Omega' + 2\Omega$, and that $P[u]$ denotes the solution p of the elliptic equation (1.7). As is easily verified, both operators A and B preserve the incompressibility condition $\mathrm{div}\, u = 0$, and this is precisely the reason for which we included the additional term $r\Omega'(r)u_r\, e_\theta$ in the definition (2.1) of the advection operator A.

Lemma 2.1. *Under assumption H1, the linear operator A is the generator of a strongly continuous group in the Hilbert space X, and B is a bounded linear operator in X.*

Proof. The evolution equation $\partial_t u = Au$ is equivalent to the system

$$\partial_t u_r + \Omega(r)\partial_\theta u_r \,=\, 0\,, \quad \partial_t u_\theta + \Omega(r)\partial_\theta u_\theta \,=\, r\Omega' u_r\,, \quad \partial_t u_z + \Omega(r)\partial_\theta u_z \,=\, 0\,,$$

which has the explicit solution

$$\begin{aligned} u_r(r,\theta,z,t) \,&=\, u_r\bigl(r,\theta-\Omega(r)t,z,0\bigr)\,, \\ u_\theta(r,\theta,z,t) \,&=\, u_\theta\bigl(r,\theta-\Omega(r)t,z,0\bigr) + r\Omega'(r)t\, u_r\bigl(r,\theta-\Omega(r)t,z,0\bigr)\,, \\ u_z(r,\theta,z,t) \,&=\, u_z\bigl(r,\theta-\Omega(r)t,z,0\bigr)\,, \end{aligned} \tag{2.3}$$

for any $t \in \mathbb{R}$. Under assumption H1, the functions Ω and $r \mapsto r\Omega'(r)$ are bounded on $\mathbb{R}_+$. With this information at hand, it is straightforward to verify that the formulas (2.3) define a strongly continuous group $(e^{tA})_{t\in\mathbb{R}}$ of bounded operators in X. Moreover, there exists a constant $C > 0$ such that $\|e^{tA}\|_{X\to X} \le C(1+|t|)$ for all $t \in \mathbb{R}$.

On the other hand, in view of definition (1.7), the pressure $p = P[u]$ satisfies the energy estimate

$$\|\partial_r p\|^2_{L^2(\mathbb{R}^3)} + \|\frac{1}{r}\partial_\theta p\|^2_{L^2(\mathbb{R}^3)} + \|\partial_z p\|^2_{L^2(\mathbb{R}^3)} \,\le\, C\Bigl(\|u_r\|^2_{L^2(\mathbb{R}^3)} + \|u_\theta\|^2_{L^2(\mathbb{R}^3)}\Bigr)\,, \tag{2.4}$$

which is established in Section 3, see Remark 3.2 below. This shows that B is a bounded linear operator in X. □

It follows from Lemma 2.1 and standard perturbation theory [7, Section III.1] that the linear operator $L = A + B$ is the generator of a strongly continuous group of bounded operators in X. Our goal is to show that, under appropriate assumptions on the vortex profile, this evolution group has a mild (i.e., sub-exponential) growth as $|t| \to \infty$, as specified in (1.16).

2.2 Fourier decomposition

To fully exploit the symmetries of the linearized operator (1.8), whose coefficients only depend on the radial variable r, it is convenient to look for velocities and pressures of the following form

$$u(r,\theta,z,t) \,=\, u_{m,k}(r,t)\, e^{im\theta}\, e^{ikz}\,, \qquad p(r,\theta,z,t) \,=\, p_{m,k}(r,t)\, e^{im\theta}\, e^{ikz}\,, \tag{2.5}$$

where $m \in \mathbb{Z}$ is the angular Fourier mode and $k \in \mathbb{R}$ is the vertical wave number. Of course, we assume that $\overline{u_{m,k}} = u_{-m,-k}$ and $\overline{p_{m,k}} = p_{-m,-k}$ so as to obtain real-valued functions after summing over all possible values of m, k. When restricted to the Fourier sector

$$X_{m,k} \,=\, \Bigl\{u = (u_r, u_\theta, u_z) \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)^3 \,\Big|\, \partial_r^* u_r + \frac{im}{r}u_\theta + iku_z = 0\Bigr\}\,, \tag{2.6}$$

the linear operator (1.8) reduces to the one-dimensional operator

$$L_{m,k}u \,=\, \begin{pmatrix} -im\Omega u_r + 2\Omega u_\theta - \partial_r P_{m,k}[u] \\ -im\Omega u_\theta - W u_r - \frac{im}{r} P_{m,k}[u] \\ -im\Omega u_z - ikP_{m,k}[u] \end{pmatrix}\,, \tag{2.7}$$

where $P_{m,k}[u]$ denotes the solution p of the following elliptic equation on $\mathbb{R}_+$:

$$-\partial_r^*\partial_r p + \frac{m^2}{r^2}\, p + k^2 p \,=\, 2im\bigl(\partial_r^*\Omega\bigr)u_r - 2\partial_r^*\bigl(\Omega\, u_\theta\bigr)\,. \tag{2.8}$$

As in Section 2.1, we decompose $L_{m,k} = A_m + B_{m,k}$, where

$$A_m u \;=\; \begin{pmatrix} -im\Omega u_r \\ -im\Omega u_\theta + r\Omega' u_r \\ -im\Omega u_z \end{pmatrix}, \qquad B_{m,k} u \;=\; \begin{pmatrix} -\partial_r P_{m,k}[u] + 2\Omega u_\theta \\ -\frac{im}{r} P_{m,k}[u] - 2(r\Omega)' u_r \\ -ik P_{m,k}[u] \end{pmatrix}. \tag{2.9}$$

The following result is the analog of [10, Proposition 2.1] in the present context.

Proposition 2.2. *Assume that the vorticity profile W satisfies assumption H1 and the normalization condition $W(0) = 2$. For any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$,*
1) The linear operator A_m defined by (2.9) *is bounded in $X_{m,k}$ with spectrum given by*

$$\sigma(A_m) \;=\; \Big\{ z \in \mathbb{C} \,\Big|\, z = -imb \text{ for some } b \in [0,1] \Big\}; \tag{2.10}$$

2) The linear operator $B_{m,k}$ defined by (2.9) *is compact in $X_{m,k}$.*

Proof. Definition (2.9) shows that A_m is essentially the multiplication operator by the function $-im\Omega$, whose range is precisely the imaginary interval (2.10) since the angular velocity is normalized so that $\Omega(0) = 1$. So the first assertion in Proposition 2.2 is rather obvious, and can be established rigorously by studying the resolvent operator $(z - A_m)^{-1}$, see [10, Proposition 2.1]. The proof of the second assertion requires careful estimates on a number of quantities related to the pressure, and is postponed to Section 3.2. □

2.3 Control of the discrete spectrum

For any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$, it follows from Proposition 2.2 and Weyl's theorem [6, Theorem I.4.1] that the *essential spectrum* of the operator $L_{m,k} = A_m + B_{m,k}$ is the purely imaginary interval (2.10), whereas the rest of the spectrum entirely consists of isolated eigenvalues with finite multiplicities[1]. To prove spectral stability, it is therefore sufficient to show that $L_{m,k}$ has no *eigenvalue* outside the imaginary axis. Given any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$, the eigenvalue equation $(s - L_{m,k})u = 0$ is equivalent to the system

$$\begin{aligned} \gamma(r)u_r - 2\Omega(r)u_\theta \;&=\; -\partial_r p\,, \\ \gamma(r)u_\theta + W(r)u_r \;&=\; -\tfrac{im}{r} p\,, \\ \gamma(r)u_z \;&=\; -ikp\,, \end{aligned} \qquad \partial_r^* u_r + \frac{im}{r}u_\theta + iku_z \;=\; 0\,, \tag{2.11}$$

where $\gamma(r) = s + im\Omega(r)$. If $(m,k) \neq (0,0)$, one can eliminate the pressure p and the velocity components u_θ, u_z from system (2.11), which then reduces to a scalar equation for the radial velocity only:

$$-\partial_r\Big(\frac{r^2\partial_r^* u_r}{m^2 + k^2r^2}\Big) + \Big\{1 + \frac{1}{\gamma(r)^2}\frac{k^2r^2\Phi(r)}{m^2+k^2r^2} + \frac{imr}{\gamma(r)}\partial_r\Big(\frac{W(r)}{m^2+k^2r^2}\Big)\Big\}u_r \;=\; 0\,, \tag{2.12}$$

where $\Phi = 2\Omega W$ is the Rayleigh function. The derivation of (2.12) is standard and can be found in many textbooks, see e.g. [4, Section 15]. It is reproduced in Section 4.1 below in the more general context of the resolvent equation.

The main result of our previous work on columnar vortices can be stated as follows.

[1] It is not difficult to verify that, in the present case, the various definitions of the essential spectrum given e.g. in [6, Section I.4] are all equivalent.

Proposition 2.3. **[10]** *Under assumptions H1 and H2, the elliptic equation* (2.12) *has no nontrivial solution* $u_r \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ *if* $\mathrm{Re}(s) \neq 0$.

Corollary 2.4. *Under assumptions H1 and H2, the operator* $L_{m,k}$ *in* $X_{m,k}$ *has no eigenvalue outside the imaginary axis.*

Proof. Assume that $u \in X_{m,k}$ satisfies $L_{m,k}u = su$ for some complex number s with $\mathrm{Re}(s) \neq 0$. If $m = k = 0$, the incompressibility condition shows that $\partial_r^* u_r = 0$, hence $u_r = 0$, and since $\gamma(r) = s \neq 0$ the second and third relations in (2.11) imply that $u_\theta = u_r = u_z = 0$. If $(m, k) \neq (0, 0)$, the radial velocity u_r satisfies (2.12), and Proposition 2.3 asserts that $u_r = 0$. Using the relations (4.6), (4.7) below (with $f = 0$), we conclude that $u_\theta = u_z = 0$. $\square$

2.4 Uniform resolvent estimates

Under assumptions H1, H2, it follows from Proposition 2.2 and Corollary 2.4 that the spectrum of the linear operator $L_{m,k} = A_m + B_{m,k}$ is entirely located on the imaginary axis. Equivalently, for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$, the resolvent $(s - L_{m,k})^{-1}$ is well defined as a bounded linear operator in $X_{m,k}$. The main technical result of the present paper, whose proof is postponed to Section 4 below, asserts that the resolvent bound is uniform with respect to the Fourier parameters m and k, and to the spectral parameter $s \in \mathbb{C}$ if $\mathrm{Re}(s)$ is fixed.

Proposition 2.5. *Assume that the vortex profile satisfies assumptions H1, H2. Then for any real number* $a \neq 0$*, one has*

$$\sup_{\mathrm{Re}(s)=a}\ \sup_{m\in\mathbb{Z}}\ \sup_{k\in\mathbb{R}} \big\|(s - L_{m,k})^{-1}\big\|_{X_{m,k}\to X_{m,k}} < \infty\,. \tag{2.13}$$

Equipped with the uniform resolvent estimate given by Proposition 2.5, it is now straightforward to conclude the proof of our main result.

End of the proof of Theorem 1.1. We know from Lemma 2.1 that the operator L defined by (1.8) is the generator of a strongly continuous group of bounded linear operators in the Hilbert space X. For any $a \neq 0$, we set

$$F(a) = \sup_{\mathrm{Re}(s)=a} \big\|(s - L)^{-1}\big\|_{X\to X} \le \sup_{\mathrm{Re}(s)=a}\ \sup_{m\in\mathbb{Z}}\ \sup_{k\in\mathbb{R}} \big\|(s - L_{m,k})^{-1}\big\|_{X_{m,k}\to X_{m,k}}\,, \tag{2.14}$$

where the last inequality follows from Parseval's theorem. The function $F : \mathbb{R}^* \to (0, \infty)$ defined by (2.14) is even by symmetry, and a straightforward perturbation argument shows that

$$\frac{F(a)}{1 + |b|F(a)} \le F(a + b) \le \frac{F(a)}{1 - |b|F(a)}\,,$$

for all $a \neq 0$ and all $b \in \mathbb{R}$ with $|b|F(a) < 1/2$, so that F is continuous. Moreover, the Hille-Yosida theorem [7, Theorem II.3.8] asserts that $F(a) = \mathcal{O}(|a|^{-1})$ as $|a| \to \infty$, and it follows that the resolvent bound (1.17) holds for any $a > 0$. In particular, given any $\epsilon > 0$, the semigroup $\big(e^{t(L-\epsilon)}\big)_{t\ge 0}$ satisfies the assumptions of the Gearhart-Prüss theorem [7, Theorem V.1.11], and is therefore uniformly bounded. This gives the desired bound (1.16) for positive times, and a similar argument yields the corresponding estimate for $t \le 0$. The proof of Theorem 1.1 is thus complete. $\square$

3 Estimates for the pressure

In this section, we give detailed estimates on the pressure $p = P_{m,k}[u]$ satisfying (2.8). That quantity appears in all components of the vector-valued operator $B_{m,k}$ introduced in (2.9), and our ultimate goal is to prove the last part of Proposition 2.2, which asserts that $B_{m,k}$ is a compact operator in the space $X_{m,k}$.

We assume henceforth that the vorticity profile W satisfies assumption H1 in Section 1. To derive energy estimates, it is convenient in a first step to suppose that the divergence-free vector field $u \in X_{m,k}$ is smooth and has compact support in $(0, +\infty)$. As is shown in Proposition 5.2 in the Appendix, the family of all such vector fields is dense in $X_{m,k}$, and the estimates obtained in that particular case remain valid for all $u \in X_{m,k}$ by a simple continuity argument. With this observation in mind, we now proceed assuming that u is smooth and compactly supported.

Equation (2.8) has a unique solution p such that the quantities $\partial_r p$, mp/r, and kp all belong to $L^2(\mathbb{R}_+, r\,\mathrm{d}r)$; the only exception is the particular case $m = k = 0$ where uniqueness holds up to an additive constant. One possibility to justify this claim is to return to the cartesian coordinates and to consider the elliptic equation (1.7) for the pressure $p : \mathbb{R}^3 \to \mathbb{R}$, which can be written in the form

$$-\Delta p \,=\, 2r\Omega'(r)\big(e_r, (\nabla u)e_\theta\big) - 2\Omega(r)(\mathrm{curl}\, u) \cdot e_z\,, \tag{3.1}$$

where $r = (x_1^2 + x_2^2)^{1/2}$. Note that the right-hand side is regular (of class $\mathcal{C}^2$ under assumption H1) and has compact support in the horizontal variables. Eq. (3.1) thus holds in the classical sense, and uniqueness up to a constant of a bounded solution p is a consequence of Liouville's theorem for harmonic functions in $\mathbb{R}^3$. In our framework, however, using (3.1) is not the easiest way to prove existence, because according to (2.5) we are interested in solutions which depend on the variables θ, z in a specific way, and do not decay to zero in the vertical direction.

If we restrict ourselves to the Fourier sector indexed by (m, k), existence of a solution to (2.8) is conveniently established using the explicit representation formulas collected in Lemma 5.4 below. As can be seen from these expressions, the solution p of (2.8) is smooth near the origin and satisfies the homogeneous Dirichlet condition at the artificial boundary $r = 0$ if $|m| \geq 1$, and the homogeneous Neumann condition if $m = 0$ or $|m| \geq 2$. As $r \to \infty$, it follows from (5.8), (5.9) that $p(r)$ decays to zero exponentially fast if $k \neq 0$, and behaves like $r^{-|m|}$ if $m \neq 0$ and $k = 0$. In the very particular case where $m = k = 0$, the pressure vanishes near infinity if u has compact support. Boundary conditions and decay properties for the derivatives of p can be derived in a similar way, and will (often implicitly) be used in the proofs below to neglect boundary terms when integrating by parts.

For functions or vector fields defined on $\mathbb{R}_+$, we always use in the sequel the notation $\|\cdot\|_{L^2}$ to denote the Lebesgue L^2 norm with respect to the measure $r\,\mathrm{d}r$. The corresponding Hermitian inner product will be denoted by $\langle\cdot,\cdot\rangle$.

3.1 Energy estimates

Throughout this section, we assume that $u \in X_{m,k}$ and we denote by $p = P_{m,k}[u]$ the solution of (2.8) given by Lemma 5.4. We begin with a standard L^2 energy estimate.

Lemma 3.1. *For any $u \in X_{m,k}$ we have*

$$\|\partial_r p\|_{L^2}^2 + \Big\|\frac{mp}{r}\Big\|_{L^2}^2 + \|kp\|_{L^2}^2 \,\leq\, C\big(\|u_r\|_{L^2}^2 + \|u_\theta\|_{L^2}^2\big)\,, \tag{3.2}$$

where the constant $C > 0$ depends only on Ω.

Proof. By density, it is sufficient to prove (3.2) under the additional assumption that u is smooth and compactly supported in $(0,+\infty)$. To do so, we multiply both sides of (2.8) by $r\bar{p}$ and integrate the result over $\mathbb{R}_+$. Integrating by parts and using Hölder's inequality, we obtain

$$\begin{aligned}
\|\partial_r p\|_{L^2}^2 + \Big\|\frac{mp}{r}\Big\|_{L^2}^2 + \|kp\|_{L^2}^2 \;&=\; \int_0^\infty \bar{p}\Big(\frac{2im}{r}(r\Omega)' u_r - 2\partial_r^*(\Omega u_\theta)\Big) r\,\mathrm{d}r \\
&=\; \int_0^\infty \Big(\frac{2im}{r}(r\Omega)'\bar{p}u_r + 2(\partial_r\bar{p})\Omega u_\theta\Big) r\,\mathrm{d}r \\
&\le\; 2\|(r\Omega)'\|_{L^\infty}\Big\|\frac{mp}{r}\Big\|_{L^2}\|u_r\|_{L^2} + 2\|\Omega\|_{L^\infty}\|\partial_r p\|_{L^2}\|u_\theta\|_{L^2}\,,
\end{aligned}$$

hence

$$\|\partial_r p\|_{L^2}^2 + \Big\|\frac{mp}{r}\Big\|_{L^2}^2 + \|kp\|_{L^2}^2 \;\le\; 4\|(r\Omega)'\|_{L^\infty}^2\|u_r\|_{L^2}^2 + 4\|\Omega\|_{L^\infty}^2\|u_\theta\|_{L^2}^2\,.$$

Note that, by assumption H1, $\|(1+r)^2\Omega\|_{L^\infty} + \|(1+r)^3\Omega'\|_{L^\infty} + \|(1+r)^4\Omega''\|_{L^\infty} < \infty$. $\square$

Remark 3.2. The integrated pressure bound (2.4) can be established by an energy estimate as in the proof of Lemma 3.1, or can be directly deduced from (3.2) using Parseval's theorem.

For later use, we also show that the solution $p = P_{m,k}[u]$ of (2.8) depends continuously on the parameter k as long as $k \neq 0$.

Lemma 3.3. *Assume that $u_1 \in X_{m,k_1}$ and $u_2 \in X_{m,k_2}$, where $m \in \mathbb{Z}$ and $k_1, k_2 \neq 0$. If we denote $p = P_{m,k_1}[u_1] - P_{m,k_2}[u_2]$, we have the estimate*

$$\|\partial_r p\|_{L^2}^2 + \Big\|\frac{mp}{r}\Big\|_{L^2}^2 + \|k_1 p\|_{L^2}^2 \;\le\; C\bigg(\|u_1{-}u_2\|_{L^2}^2 + \|u_2\|_{L^2}^2\Big|\frac{k_1}{k_2} - \frac{k_2}{k_1}\Big|^2\bigg)\,, \tag{3.3}$$

where the constant $C > 0$ depends only on Ω.

Proof. In view of (2.8), the difference $p = p_1 - p_2 \equiv P_{m,k_1}[u_1] - P_{m,k_2}[u_2]$ satisfies the equation

$$-\partial_r^*\partial_r p + \frac{m^2}{r^2}\,p + k_1^2 p \;=\; \frac{2im}{r}(r\Omega)'(u_{1,r}{-}u_{2,r}) - 2\partial_r^*\big(\Omega\,(u_{1,\theta}{-}u_{2,\theta})\big) + (k_2^2 - k_1^2)p_2\,.$$

As in the proof of Lemma 3.1, we multiply both sides by $r\bar{p}$ and we integrate over $\mathbb{R}_+$. Integrating by parts and using Hölder's inequality, we easily obtain

$$\|\partial_r p\|_{L^2}^2 + \Big\|\frac{mp}{r}\Big\|_{L^2}^2 + \|k_1 p\|_{L^2}^2 \;\le\; C\bigg(\|u_1{-}u_2\|_{L^2}^2 + \|k_2 p_2\|_{L^2}^2\Big|\frac{k_1}{k_2} - \frac{k_2}{k_1}\Big|^2\bigg)\,,$$

where the constant $C > 0$ depends only on $\|\Omega\|_{L^\infty}$ and $\|(r\Omega)'\|_{L^\infty}$. As $\|k_2 p_2\|_{L^2} \le C\|u_2\|_{L^2}$ by (3.2), this gives the desired result. $\square$

Finally, we derive a weighted estimate which allows us to control the pressure $p = P_{m,k}[u]$ in the far-field region where $r \gg 1$.

Lemma 3.4. *Assume that $k \neq 0$ or $|m| \ge 2$. If $u \in X_{m,k}$ and $p = P_{m,k}[u]$, then*

$$\|r\partial_r p\|_{L^2}^2 + \|mp\|_{L^2}^2 + \|krp\|_{L^2}^2 \;\le\; 3\|p\|_{L^2}^2 + C\big(\|u_r\|_{L^2}^2 + \|u_\theta\|_{L^2}^2\big)\,, \tag{3.4}$$

where the constant $C > 0$ depends only on Ω. If $|m| \ge 2$ the first term in the right-hand side can be omitted.

Proof. We multiply both sides of (2.8) by $r^3\bar{p}$ and integrate the result over $\mathbb{R}_+$. Note that the integrand decays to zero exponentially fast if $k \neq 0$, and like $r^{1-2|m|}$ if $k = 0$, so that the integral converges if we assume that $k \neq 0$ or $|m| \geq 2$. After integrating by parts, we obtain the identity

$$\|r\partial_r p\|_{L^2}^2 + \|mp\|_{L^2}^2 + \|krp\|_{L^2}^2 \;=\; 2\|p\|_{L^2}^2 + \mathrm{Re}\big(I_1 + I_2\big)\,, \tag{3.5}$$

where $I_1 = 2im\langle rp, (r\Omega)' u_r\rangle$ and $I_2 = 2\langle \partial_r(r^2 p), \Omega u_\theta\rangle = 2\langle r\partial_r p, r\Omega u_\theta\rangle + 4\langle p, r\Omega u_\theta\rangle$. We observe that

$$\begin{aligned}
|I_1| \;&\leq\; 2\|r(r\Omega)'\|_{L^\infty}\|mp\|_{L^2}\|u_r\|_{L^2} \;\leq\; \frac{1}{4}\|mp\|_{L^2}^2 + 4\|r(r\Omega)'\|_{L^\infty}^2\|u_r\|_{L^2}^2\,,\\
|I_2| \;&\leq\; 2\|r\Omega\|_{L^\infty}\Big(\|r\partial_r p\|_{L^2} + 2\|p\|_{L^2}\Big)\|u_\theta\|_{L^2} \;\leq\; \frac{1}{4}\Big(\|r\partial_r p\|_{L^2}^2 + \|p\|_{L^2}^2\Big) + 20\|r\Omega\|_{L^\infty}^2\|u_\theta\|_{L^2}^2\,,
\end{aligned}$$

and replacing these estimates into (3.5) we obtain (3.4). If $|m| \geq 2$, then $3\|p\|_{L^2}^2 \leq \frac{3}{4}\|mp\|_{L^2}^2$, so that the first term in the right-hand side of (3.4) can be included in the left-hand side. $\square$

Corollary 3.5. *For any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$, the linear map $u \mapsto kP_{m,k}[u]$ from $X_{m,k}$ into $L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ is compact.*

Proof. We can of course assume that $k \neq 0$. If u lies in the unit ball of $X_{m,k}$, it follows from estimates (3.2) and (3.4) that $\|k\partial_r p\|_{L^2}^2 + \|krp\|_{L^2}^2 \leq C(k,\Omega)$ for some constant $C(k,\Omega)$ independent of u. Applying Lemma 5.3, we conclude that the map $u \mapsto kp$ is compact. $\square$

3.2 Compactness results

The aim of this section is to complete the proof of Proposition 2.2, by showing the compactness of the linear operator $B_{m,k}$ defined in (2.9). In view of Corollary 3.5, which already settles the case of the third component $B_{m,k,z}u := -ikP_{m,k}[u]$, we are left to prove that the linear mappings

$$\begin{aligned}
u \mapsto B_{m,k,r}u \;&:=\; -\partial_r P_{m,k}[u] + 2\Omega u_\theta\,, \qquad \text{and}\\
u \mapsto B_{m,k,\theta}u \;&:=\; -\frac{im}{r}P_{m,k}[u] - 2(r\Omega)' u_r\,,
\end{aligned}$$

are compact from $X_{m,k}$ to $L^2(\mathbb{R}_+, r\,\mathrm{d}r)$, for any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$. In the sequel, to simplify the notation, we write B_r, B_θ, B_z instead of $B_{m,k,r}u$, $B_{m,k,\theta}u$, $B_{m,k,z}u$, respectively.

We first treat the simple particular case where $m = 0$.

Lemma 3.6. *If $m = 0$ and $u \in X_{0,k}$, then*

$$\|\partial_r^* B_r\|_{L^2} + \|\partial_r^* B_\theta\|_{L^2} + \|rB_r\|_{L^2} + \|rB_\theta\|_{L^2} \;\leq\; C(k,\Omega)\|u\|_{L^2}\,, \tag{3.6}$$

where the constant $C(k,\Omega)$ depends only on k and Ω.

Proof. If $m = 0$ and $k = 0$, then $\partial_r^* u_r = \operatorname{div} u = 0$, and this implies that $u_r = 0$. Similarly, the incompressibility condition for the vector B implies that $B_r = 0$, and in view of (2.9) it follows that B vanishes identically. Thus estimate (3.6) is trivially satisfied in that case. If $m = 0$ and $k \neq 0$, we deduce from (3.2) and (3.4) that

$$\|rB_r\|_{L^2} \;\leq\; \|r\partial_r p\|_{L^2} + 2\|r\Omega\|_{L^\infty}\|u_\theta\|_{L^2} \;\leq\; C(k,\Omega)\big(\|u_r\|_{L^2} + \|u_\theta\|_{L^2}\big)\,,$$

and

$$\|rB_\theta\|_{L^2} \;=\; 2\|r(r\Omega)' u_r\|_{L^2} \;\leq\; 2\|r(r\Omega)'\|_{L^\infty}\|u_r\|_{L^2}\,.$$

As for the derivatives, we observe that $\partial_r^* B_r = -\partial_r^*\partial_r p + 2\partial_r^*(\Omega u_\theta) = -k^2 p$ in view of (2.8), and therefore we deduce from (3.2) that

$$\|\partial_r^* B_r\|_{L^2} \,=\, \|k^2 p\|_{L^2} \,\le\, C(k,\Omega)\big(\|u_r\|_{L^2} + \|u_\theta\|_{L^2}\big)\,.$$

Finally, as $\partial_r^* u_r + iku_z = 0$, we have $\partial_r^* B_\theta = -2(r\Omega)'' u_r - 2(r\Omega)'\partial_r^* u_r = -2(r\Omega)'' u_r + 2ik(r\Omega)' u_z$, and it follows that

$$\|\partial_r^* B_\theta\|_{L^2} \,\le\, C(k,\Omega)\big(\|u_r\|_{L^2} + \|u_z\|_{L^2}\big)\,.$$

Collecting these estimates, we arrive at (3.6). ◻

When $m \neq 0$, useful estimates on the vector $B_{m,k}u$ can be deduced from an elliptic equation satisfied by the radial component B_r, which also involves the quantities R_1, R_2 defined by

$$R_1 \,=\, 2\big(-(r\Omega)'' u_r + im\Omega' u_\theta + ik(r\Omega)' u_z\big)\,, \quad \text{and} \quad R_2 \,=\, \frac{2}{r}p + 2\Omega u_\theta\,. \tag{3.7}$$

To derive that equation, we first observe that, in view of the definitions (2.9) of B_r, B_θ and of the incompressibility condition for u, the following relation holds

$$\partial_r\big(rB_\theta\big) \,=\, -im\partial_r p - 2\partial_r\big(r(r\Omega)' u_r\big) \,=\, imB_r + rR_1\,. \tag{3.8}$$

Next, we have the incompressibility condition for B, which is equivalent to (2.8) :

$$\partial_r^* B_r + \frac{im}{r}B_\theta + ikB_z \,=\, 0\,. \tag{3.9}$$

If we multiply both members of (3.9) by r^2 and differentiate the resulting identity with respect to r, we obtain using (3.8) the desired equation

$$-\partial_r^2 B_r - \frac{3}{r}\partial_r B_r + \Big(\frac{m^2-1}{r^2} + k^2\Big)B_r \,=\, \frac{im}{r}R_1 + k^2 R_2\,. \tag{3.10}$$

If $u \in X_{m,k}$ is smooth and compactly supported in $(0,+\infty)$, it is clear from the definition (2.9) that the radial component B_r satisfies exactly the same boundary conditions at $r = 0$ as the pressure derivative $\partial_r p$, and has also the same decay properties at infinity. In particular B_r decays to zero exponentially fast as $r \to \infty$ if $k \neq 0$, and behaves like $r^{-1-|m|}$ if $k = 0$ and $m \neq 0$. These observations also apply to the azimuthal component B_θ.

We now exploit (3.10) to estimate B_r and B_θ, starting with the general case where $|m| \ge 2$.

Lemma 3.7. *If $|m| \ge 2$ and $u \in X_{m,k}$, then*

$$\|\partial_r B_r\|_{L^2} + \|\partial_r^* B_\theta\|_{L^2} + \|rB_r\|_{L^2} + \|rB_\theta\|_{L^2} \,\le\, C(m,k,\Omega)\|u\|_{L^2}\,, \tag{3.11}$$

where the constant $C(m,k,\Omega)$ depends only on m, k and Ω.

Proof. We first observe that $\|R_1\|_{L^2} + \|r^2 R_1\|_{L^2} \le C\|u\|_{L^2}$, where the constant depends on m, k, and Ω. Similarly, in view of (3.2) and (3.4), we have $\|krR_2\|_{L^2} + \|kr^2 R_2\|_{L^2} \le C\|u\|_{L^2}$. Now, we multiply (3.10) by $r\bar{B}_r$ and integrate by parts. This leads to the identity

$$\|\partial_r B_r\|_{L^2}^2 + (m^2-1)\Big\|\frac{B_r}{r}\Big\|_{L^2}^2 + \|kB_r\|_{L^2}^2 \,=\, \mathrm{Re}\langle B_r, \frac{im}{r}R_1 + k^2 R_2\rangle\,. \tag{3.12}$$

To control the right-hand side, we use the estimates

$$\Big|\langle B_r, \frac{im}{r}R_1\rangle\Big| \,\le\, \Big\|\frac{mB_r}{r}\Big\|_{L^2}\|R_1\|_{L^2} \,\le\, C(m,k,\Omega)\Big\|\frac{mB_r}{r}\Big\|_{L^2}\|u\|_{L^2}\,,$$

and

$$\left|\langle B_r, k^2 R_2\rangle\right| \le 2\left\|\frac{B_r}{r}\right\|\|k^2 r R_2\|_{L^2} \le C(m,k,\Omega)\left\|\frac{B_r}{r}\right\|\|u\|_{L^2}\,.$$

Inserting these bounds into (3.12) and using Young's inequality together with the assumption that $|m| \ge 2$, we easily obtain

$$\|\partial_r B_r\|_{L^2}^2 + m^2\left\|\frac{B_r}{r}\right\|_{L^2}^2 + k^2\|B_r\|_{L^2}^2 \le C(m,k,\Omega)\|u\|_{L^2}^2\,. \tag{3.13}$$

In exactly the same way, if we multiply (3.10) by $r^3\bar{B}_r$ and integrate by parts, we arrive at the weighted estimate

$$\|r\partial_r B_r\|_{L^2}^2 + m^2\|B_r\|_{L^2}^2 + k^2\|rB_r\|_{L^2}^2 \le C(m,k,\Omega)\|u\|_{L^2}^2\,. \tag{3.14}$$

When $k = 0$, estimates (3.13), (3.14) remain valid but they do not provide the desired control on $\|rB_r\|_{L^2}$. In that case, we multiply (3.10) by $r^5\bar{B}_r$ to derive the additional identity

$$\|r^2\partial_r B_r\|_{L^2}^2 + (m^2-1)\|rB_r\|_{L^2}^2 = -2\,\mathrm{Re}\langle rB_r, r^2\partial_r B_r\rangle + \mathrm{Re}\langle r^4 B_r, \frac{im}{r}R_1\rangle\,.$$

To estimate the right-hand side, we use the following bounds

$$\begin{aligned}
2\left|\langle rB_r, r^2\partial_r B_r\rangle\right| &\le \frac{3}{4}\|r^2\partial_r B_r\|_{L^2}^2 + \frac{4}{3}\|rB_r\|_{L^2}^2\,,\\
\left|\langle rB_r, imr^2 R_1\rangle\right| &\le \|mrB_r\|_{L^2}\|r^2 R_1\|_{L^2} \le C(m,\Omega)\|mrB_r\|_{L^2}\|u\|_{L^2}\,.
\end{aligned}$$

Taking into account the assumption that $|m| \ge 2$, so that $\frac{4}{3} \le \frac{m^2-1}{2}$, we deduce that, for $k = 0$,

$$\|r^2\partial_r B_r\|_{L^2}^2 + m^2\|rB_r\|_{L^2}^2 \le C(m,\Omega)\|u\|_{L^2}^2\,. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15) (when $k = 0$), we obtain in particular the inequality

$$\|\partial_r B_r\|_{L^2} + \|rB_r\|_{L^2} \le C(m,k,\Omega)\|u\|_{L^2}\,. \tag{3.16}$$

It remains to estimate the azimuthal component B_θ, which satisfies $\partial_r^* B_\theta = \frac{im}{r}B_r + R_1$ by (3.8). Using inequalities (3.13) and (3.4) (in the case where $|m| \ge 2$), we easily obtain

$$\|\partial_r^* B_\theta\|_{L^2} + \|rB_\theta\|_{L^2} \le C(m,k,\Omega)\|u\|_{L^2}\,, \tag{3.17}$$

and estimate (3.11) follows by combining (3.16) and (3.17). ☐

The case where $m = \pm 1$ requires a slightly different argument, because an essential term in the elliptic equation (3.10) vanishes when $m^2 = 1$. It can be shown that this phenomenon is related to the translation invariance of the Euler equation in the original, cartesian coordinates.

Lemma 3.8. *Assume that $m = \pm 1$, $k \ne 0$ and $u \in X_{m,k}$. Then*

$$\|\partial_r B_r\|_{L^2} + \|\partial_r^* D\|_{L^2} + \|rB_r\|_{L^2} + \|rD\|_{L^2} \le C(k,\Omega)\|u\|_{L^2}\,, \tag{3.18}$$

where $D = B_r + imB_\theta$ and the constant $C(k,\Omega)$ depends only on k and Ω.

Proof. We multiply both sides of (3.10) by $r^2\partial_r\bar{B}_r$, take the real part, and integrate by parts. We obtain the identity

$$2\|\partial_r B_r\|_{L^2}^2 + k^2\|B_r\|_{L^2}^2 \;=\; -\operatorname{Re}\langle\partial_r B_r, imR_1 + k^2 rR_2\rangle\,,$$

where the right-hand side is estimated as in the previous lemma. This yields the bound

$$\|\partial_r B_r\|_{L^2}^2 + k^2\|B_r\|_{L^2}^2 \;\le\; C(k,\Omega)\|u\|_{L^2}^2\,. \tag{3.19}$$

In exactly the same way, multiplying (3.10) by $r^4\partial_r\bar{B}_r$, we arrive at

$$\|r\partial_r B_r\|_{L^2}^2 + k^2\|rB_r\|_{L^2}^2 \;\le\; C(k,\Omega)\|u\|_{L^2}^2\,. \tag{3.20}$$

In particular, combining (3.2), (3.4), (3.19) and (3.20), we find

$$\|D\|_{L^2} + \|rD\|_{L^2} \;\le\; C(k,\Omega)\|u\|_{L^2}\,. \tag{3.21}$$

In addition, using the identity $\partial_r^* D = \partial_r B_r + \frac{1}{r}B_r + im(\frac{im}{r}B_r + R_1) = \partial_r B_r + imR_1$, we obtain

$$\|\partial_r^* D\|_{L^2} \;\le\; C(k,\Omega)\|u\|_{L^2}\,. \tag{3.22}$$

Estimate (3.18) follows directly from (3.19)–(3.22). □

Lemma 3.9. *Assume that $m = \pm 1$, $k = 0$, and $u \in X_{m,0}$. Then, for any $\alpha \in (0,1)$,*

$$\|\partial_r B_r\|_{L^2} + \|\partial_r^* D\|_{L^2} + \|r^\alpha B_r\|_{L^2} + \|r^\alpha D\|_{L^2} \;\le\; C(\alpha,\Omega)\|u\|_{L^2}\,, \tag{3.23}$$

where $D = B_r + imB_\theta$ and the constant $C(\alpha,\Omega)$ depends only on α and Ω.

Proof. If $|m| = 1$ and $k = 0$, equation (3.10) reduces to

$$-\frac{1}{r^3}\partial_r\left(r^3\partial_r B_r\right) \;=\; \frac{im}{r}R_1\,,$$

which can be explicitly integrated to give

$$\partial_r B_r(r) \;=\; -\frac{im}{r^3}\int_0^r s^2 R_1(s)\,\mathrm{d}s\,, \qquad r > 0\,, \tag{3.24}$$

and finally

$$B_r(r) \;=\; \frac{im}{2}\int_0^r R_1(s)\frac{s^2}{r^2}\,\mathrm{d}s + \frac{im}{2}\int_r^\infty R_1(s)\,\mathrm{d}s\,, \qquad r > 0\,. \tag{3.25}$$

Since $|m| = 1$ and $k = 0$, it follows from (3.7) that $\|r^{-1}R_1\|_{L^2} + \|r^3 R_1\|_{L^2} \le C\|u\|_{L^2}$, where the constant depends only on Ω. Using that information, it is straightforward to deduce from the representations (3.24) and (3.25) that

$$\|\partial_r B_r\|_{L^2} + \|r^\alpha B_r\|_{L^2} \;\le\; C(\alpha,\Omega)\|u\|_{L^2}\,,$$

for any $\alpha < 1$. Note that $rB_r \notin L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ in general, because the first term in the right-hand side of (3.25) decays exactly like r^{-2} as $r \to \infty$.

On the other hand, it follows from (3.9) that $imB_\theta = -r\partial_r B_r - B_r$, which implies that $D = -r\partial_r B_r$. Moreover, as in the previous lemma, we have $\partial_r^* D = \partial_r B_r + imR_1$. So, using the estimates above on R_1, we easily obtain the bound $\|\partial_r^* D\|_{L^2} + \|r^\alpha D\|_{L^2} \le C\|u\|_{L^2}$, which concludes the proof. □

End of the Proof of Proposition 2.2. When $|m| \neq 1$, in view of Lemmas 3.6 and 3.7, the compactness of the maps $u \mapsto B_{m,k,r}u$ and $u \mapsto B_{m,k,\theta}u$ is a direct consequence of Lemma 5.3 in the Appendix. When $m = \pm 1$, Lemma 5.3, combined with Lemma 3.8 or Lemma 3.9, shows that the maps $u \mapsto B_{m,k,r}u$ and $u \mapsto B_{m,k,r}u + imB_{m,k,\theta}u$ are compact, and so is the map $u \mapsto B_{m,k,\theta}u$. $\square$

Remark 3.10. It is also possible to obtain explicit representation formulas for the components of the vector-valued operator $B_{m,k}$ defined in (2.9), and to use them to prove that the map $u \mapsto B_{m,k}u$ is compact in $X_{m,k}$. The computations, however, are rather cumbersome. That approach was followed in a previous version of this work [11].

4 Resolvent bounds on vertical lines

This final section is entirely devoted to the proof of Proposition 2.5. Let $a \neq 0$ be a nonzero real number. For any value of the angular Fourier mode $m \in \mathbb{Z}$, of the vertical wave number $k \in \mathbb{R}$, and of the spectral parameter $s \in \mathbb{C}$ with $\mathrm{Re}(s) = a$, we consider the resolvent equation $(s - L_{m,k})u = f$, which by definition (2.7) is equivalent to the system

$$\begin{aligned} \gamma(r)u_r - 2\Omega(r)u_\theta \;&=\; -\partial_r p + f_r\,, \\ \gamma(r)u_\theta + W(r)u_r \;&=\; -\tfrac{im}{r}p + f_\theta\,, \\ \gamma(r)u_z \;&=\; -ikp + f_z\,, \end{aligned} \tag{4.1}$$

where $\gamma(r) = s + im\Omega(r)$ and $p = P_{m,k}[u]$ is the solution of (2.8) given by Lemma 5.4. We recall that u, f are divergence-free:

$$\partial_r^* u_r + \frac{im}{r}u_\theta + iku_z \;=\; 0\,, \qquad \partial_r^* f_r + \frac{im}{r}f_\theta + ikf_z \;=\; 0\,. \tag{4.2}$$

Our goal is to show that, given any $f \in X_{m,k}$, the (unique) solution $u \in X_{m,k}$ of (4.1) satisfies $\|u\|_{L^2} \le C\|f\|_{L^2}$, where the constant $C > 0$ depends only on the spectral abscissa a and on the angular velocity profile Ω. In particular, the constant C is independent of m, k, and s provided $\mathrm{Re}(s) = a$.

Remark 4.1. It is interesting to observe how the resolvent system (4.1), (4.2) is transformed under the action of the following isometries:

$$\begin{aligned} \mathcal{I}_1 \,:\, X_{m,k} \to X_{-m,k}\,, &\qquad u \mapsto \tilde u \,:=\, (u_r, -u_\theta, u_z)\,, \\ \mathcal{I}_2 \,:\, X_{m,k} \to X_{m,-k}\,, &\qquad u \mapsto \hat u \,:=\, (u_r, u_\theta, -u_z)\,, \\ \mathcal{I}_3 \,:\, X_{m,k} \to X_{-m,-k}\,, &\qquad u \mapsto \bar u \,:=\, (\bar u_r, \bar u_\theta, \bar u_z)\,, \end{aligned}$$

where (as usual) $\bar u$ denotes the complex conjugate of u. If $u, f \in X_{m,k}$ and $s \in \mathbb{C}$, the resolvent equation $(s - L_{m,k})u = f$ is equivalent to any of the following three relations:

$$\bigl(s + L_{-m,k}\bigr)\tilde u \;=\; \tilde f\,, \qquad \bigl(s - L_{m,-k}\bigr)\hat u \;=\; \hat f\,, \qquad \bigl(\bar s - L_{-m,-k}\bigr)\bar u \;=\; \bar f\,.$$

This implies in particular that the spectrum of the operator $L_{m,k}$ in $X_{m,k}$ satisfies

$$\sigma(L_{m,k}) \;=\; \sigma(L_{m,-k}) \;=\; -\sigma(L_{-m,k})\,, \qquad \text{and} \quad \sigma(L_{m,k}) \;=\; -\overline{\sigma(L_{m,k})}\,. \tag{4.3}$$

As the spectrum $\sigma(L_{m,k})$ is symmetric with respect to the imaginary axis, due to the last relation in (4.3), we can assume in what follows that the spectral abscissa a is positive. Also, thanks to the first two relations, we can suppose without loss of generality that $m \in \mathbb{N}$ and $k \ge 0$.

4.1 The scalar resolvent equation

A key ingredient in the proof of Proposition 2.5 is the observation that the resolvent system (4.1) is equivalent to a second order differential equation for the radial velocity u_r.

Lemma 4.2. *Assume that $(k, m) \neq (0, 0)$. If $u \in X_{m,k}$ is the solution of the resolvent equation (4.1) for some $f \in X_{m,k}$, the radial velocity u_r satisfies, for all $r > 0$,*

$$-\partial_r\big(\mathcal{A}(r)\partial_r^* u_r\big) + \Big(1 + \frac{k^2}{\gamma^2}\,\mathcal{A}(r)\Phi(r) + \frac{imr}{\gamma}\partial_r\Big(\frac{W(r)}{m^2 + k^2r^2}\Big)\Big)\,u_r \;=\; \mathcal{F}(r)\,, \tag{4.4}$$

where $\mathcal{A}(r) = r^2/(m^2 + k^2r^2)$ and

$$\mathcal{F}(r) \;=\; \frac{1}{\gamma}f_r + \mathcal{A}\Big(\frac{2ik\Omega}{\gamma^2} + \frac{2km}{\gamma}\frac{1}{m^2 + k^2r^2}\Big)\big(-ikf_\theta + \frac{im}{r}f_z\big) + \frac{im}{\gamma r}\mathcal{A}\partial_r^* f_\theta + \frac{ik}{\gamma}\mathcal{A}\partial_r f_z\,. \tag{4.5}$$

In addition, the azimuthal and vertical velocities are expressed in terms of u_r by

$$u_\theta \;=\; \frac{im\mathcal{A}}{r}\partial_r^* u_r - \frac{k^2\mathcal{A}}{\gamma}\big(Wu_r - f_\theta\big) - \frac{mk\mathcal{A}}{\gamma r}f_z\,, \tag{4.6}$$

$$u_z \;=\; ik\mathcal{A}\partial_r^* u_r + \frac{mk\mathcal{A}}{\gamma r}\big(Wu_r - f_\theta\big) + \frac{m^2\mathcal{A}}{\gamma r^2}f_z\,. \tag{4.7}$$

Proof. If we eliminate the pressure p from the last two lines in (4.1), we obtain

$$kWu_r + k\gamma u_\theta - \frac{\gamma m}{r}u_z \;=\; kf_\theta - \frac{m}{r}f_z\,. \tag{4.8}$$

This first relation can be combined with the incompressibility condition in (4.2) to eliminate the azimuthal velocity u_θ. This gives

$$k\Big(\partial_r^* - \frac{imW}{\gamma r}\Big)u_r + i\Big(k^2 + \frac{m^2}{r^2}\Big)u_z \;=\; g_1 \;:=\; \frac{im^2}{\gamma r^2}f_z - \frac{imk}{\gamma r}f_\theta\,, \tag{4.9}$$

which is (4.7). As is easily verified, if in the previous step we eliminate the vertical velocity u_z from (4.8) and (4.2), we arrive at (4.6) instead of (4.7).

Alternatively, we can eliminate the pressure from the first and the last line in (4.1). This gives the second relation

$$ik\gamma u_r - 2ik\Omega u_\theta - \partial_r(\gamma u_z) \;=\; ikf_r - \partial_r f_z\,, \tag{4.10}$$

which can in turn be combined with (4.8) to eliminate the azimuthal velocity u_θ. Using the relations $\gamma' = im\Omega'$ and $W = r\Omega' + 2\Omega$, we obtain in this way

$$\gamma^2\Big(\partial_r + \frac{imW}{\gamma r}\Big)u_z - ik\big(\gamma^2 + \Phi\big)u_r \;=\; g_2 \;:=\; 2i\Omega\Big(\frac{m}{r}f_z - kf_\theta\Big) + \gamma\Big(\partial_r f_z - ikf_r\Big)\,, \tag{4.11}$$

where $\Phi = 2\Omega W$ is the Rayleigh function.

Now, we multiply the equality (4.9) by $\mathcal{A} = r^2/(m^2+k^2r^2)$ and apply the differential operator $\partial_r + \frac{imW}{\gamma r}$ to both members of the resulting expression. In view of (4.11), we find

$$k\Big(\partial_r + \frac{imW}{\gamma r}\Big)\mathcal{A}\Big(\partial_r^* - \frac{imW}{\gamma r}\Big)u_r - k\Big(1 + \frac{\Phi}{\gamma^2}\Big)u_r \;=\; \Big(\partial_r + \frac{imW}{\gamma r}\Big)\mathcal{A}g_1 - \frac{i}{\gamma^2}g_2\,. \tag{4.12}$$

If $k \neq 0$, this equation is equivalent to (4.4), as is easily verified by expanding the expressions in both sides of (4.12) and performing elementary simplifications. In the particular case where $k = 0$ (and $m \neq 0$), equation (4.4) still holds but the derivation above is not valid anymore. Instead, one must eliminate the pressure p from the first two lines in (4.1), and then express the azimuthal velocity u_θ using the incompressibility condition. The details are left to the reader. $\square$

Remark 4.3. If $f = 0$, then $\mathcal{F} = 0$ and Eq. (4.4) reduces to the eigenvalue equation (2.12).

Corollary 4.4. *Under the assumptions of Lemma 4.2, we have the estimate*

$$\max\bigl\{\|u_\theta\|_{L^2}, \|u_z\|_{L^2}\bigr\} \;\le\; \|\mathcal{A}^{\frac{1}{2}}\partial_r^* u_r\|_{L^2} + \frac{1}{a}\Bigl(\|W\|_{L^\infty}\|u_r\|_{L^2} + \|f_\theta\|_{L^2} + \|f_z\|_{L^2}\Bigr)\,. \tag{4.13}$$

Proof. As $|\gamma(r)| \ge \mathrm{Re}(s) = a$ and

$$0 \;<\; \mathcal{A}(r) \;\le\; \min\Bigl\{\frac{1}{k^2}, \frac{r^2}{m^2}\Bigr\}\,, \tag{4.14}$$

estimate (4.13) follows immediately from the representations (4.6), (4.7). □

4.2 Explicit resolvent estimates in particular cases

We first establish the resolvent bound in the relatively simple case where $m = 0$, which corresponds to axisymmetric perturbations of the columnar vortex.

Lemma 4.5. *Assume that $m = 0$. For any $f \in X_{0,k}$, the solution $u \in X_{0,k}$ of (4.1) satisfies*

$$\|u\|_{L^2} \;\le\; C_0\Bigl(\frac{1}{a} + \frac{1}{a^4}\Bigr)\|f\|_{L^2}\,, \tag{4.15}$$

where the constant $C_0 > 0$ depends only on Ω.

Proof. When $k = 0$, the incompressibility condition (4.2) implies that $u_r = 0$, and since $\gamma(r) = s$ we deduce from the last two lines in (4.1) that $u_\theta = f_\theta/s$ and $u_z = f_z/s$. As $|s| \ge \mathrm{Re}(s) = a$, we thus have $\|u\|_{L^2} \le \|f\|_{L^2}/a$, which is the desired conclusion.

If $k \neq 0$, we assume without loss of generality that $k > 0$. Since $m = 0$, equation (4.4) satisfied by the radial velocity u_r reduces to

$$-\partial_r\partial_r^* u_r + k^2\Bigl(1 + \frac{\Phi(r)}{s^2}\Bigr)u_r \;=\; \frac{k^2}{s}f_r + \frac{2k^2\Omega(r)}{s^2}f_\theta + \frac{ik}{s}\partial_r f_z\,.$$

We multiply both sides by $sr\bar{u}_r$ and integrate the resulting equality over $\mathbb{R}_+$. After taking the real part, we obtain the identity

$$a\int_0^\infty \Bigl\{|\partial_r^* u_r|^2 + k^2\Bigl(1 + \frac{\Phi(r)}{|s|^2}\Bigr)|u_r|^2\Bigr\}r\,\mathrm{d}r \;=\; \mathrm{Re}\int_0^\infty \bar{u}_r\Bigl(k^2 f_r + \frac{2k^2\Omega(r)}{s}f_\theta + ik\partial_r f_z\Bigr)r\,\mathrm{d}r\,.$$

As $\Phi(r) \ge 0$ by assumption H1, we easily deduce that

$$a\Bigl(\|\partial_r^* u_r\|_{L^2}^2 + k^2\|u_r\|_{L^2}^2\Bigr) \;\le\; k^2\|u_r\|_{L^2}\Bigl(\|f_r\|_{L^2} + \frac{2\|\Omega\|_{L^\infty}}{a}\|f_\theta\|_{L^2}\Bigr) + k\|\partial_r^* u_r\|_{L^2}\|f_z\|_{L^2}\,,$$

and applying Young's inequality we obtain

$$\frac{1}{k^2}\,\|\partial_r^* u_r\|_{L^2}^2 + \|u_r\|_{L^2}^2 \;\le\; \frac{C}{a^2}\bigl(\|f_r\|_{L^2}^2 + \|f_z\|_{L^2}^2\bigr) + \frac{C}{a^4}\|f_\theta\|_{L^2}^2\,, \tag{4.16}$$

where the constant $C > 0$ depends only on Ω.

With estimate (4.16) at hand, we deduce from the second line in (4.1) that

$$\|u_\theta\|_{L^2} \;\le\; \frac{1}{|s|}\Bigl(\|W\|_{L^\infty}\|u_r\|_{L^2} + \|f_\theta\|_{L^2}\Bigr) \;\le\; C\Bigl(\frac{1}{a} + \frac{1}{a^3}\Bigr)\|f\|_{L^2}\,. \tag{4.17}$$

Similarly, using the third line in (4.1) and estimate (3.2) for the pressure, we obtain

$$\|u_z\|_{L^2} \;\le\; \frac{1}{|s|}\bigl(\|kp\|_{L^2} + \|f_z\|_{L^2}\bigr) \;\le\; \frac{C}{a}\bigl(\|u_r\|_{L^2} + \|u_\theta\|_{L^2} + \|f_z\|_{L^2}\bigr) \;\le\; C\Bigl(\frac{1}{a} + \frac{1}{a^4}\Bigr)\|f\|_{L^2}\,. \tag{4.18}$$

Combining (4.16), (4.17), and (4.18), we arrive at (4.15). □

In the rest of this section, we consider the more difficult case where $m \neq 0$. In that situation, given any $s \in \mathbb{C}$ with $\mathrm{Re}(s) = a$, there exists a unique $b \in \mathbb{R}$ such that

$$s \,=\, a - imb\,, \qquad \text{hence} \quad \gamma(r) \,=\, a + im\big(\Omega(r) - b\big)\,. \tag{4.19}$$

Our goal is to obtain a resolvent bound that is uniform in the parameters m, k, and b. In view of Remark 4.1, we can assume without loss of generality that $m \geq 1$ and $k \geq 0$.

Unlike in the axisymmetric case, we are not able to obtain here an explicit resolvent bound of the form (4.15) for all values of the parameters m, k, and b. In some regions, we will have to invoke Proposition 2.3, which was established in [10] using a contradiction argument that does not provide any explicit estimate of the resolvent operator. Nevertheless, our strategy is to obtain explicit bounds in the largest possible region of the parameter space, and to rely on Proposition 2.3 only when the direct approach does not work.

We begin with the following elementary observation:

Lemma 4.6. *If $u, f \in X_{m,k}$ satisfy (4.1), then for any $M > 0$ we have the estimate*

$$\|1_{\{|\gamma| \geq M\}}\, u\|_{L^2} \,\leq\, \frac{C_1}{M}\big(\|u\|_{L^2} + \|f\|_{L^2}\big)\,, \tag{4.20}$$

where the constant C_1 depends only on Ω.

Proof. We multiply all three equations in (4.1) by $\gamma(r)^{-1} 1_{\{|\gamma| \geq M\}}$ and take the L^2 norm of the resulting expression. Using estimate (3.2) to control the pressure, we arrive at (4.20). $\square$

To obtain more general resolvent estimates, we exploit the differential equation (4.4) satisfied by the radial velocity u_r. As a preliminary step, we prove the following result.

Lemma 4.7. *If $u, f \in X_{m,k} \cap H^1(\mathbb{R}_+, r\,\mathrm{d}r)$ and $\mathcal{F}$ is defined by (4.5), we have*

$$\Big|\int_0^\infty \mathcal{F}(r)\bar{u}_r r\,\mathrm{d}r\Big| \,\leq\, \frac{2}{a}\,\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}\|f\|_{L^2} + C_2\Big(\frac{1}{a} + \frac{1}{a^2}\Big)\|u_r\|_{L^2}\|f\|_{L^2}\,, \tag{4.21}$$

where the constant C_2 depends only on Ω.

Proof. We split the integral $\int_0^\infty \mathcal{F}(r)\bar{u}_r r\,\mathrm{d}r$ into four pieces, according to the expression of $\mathcal{F}$ in (4.5). As $|\gamma(r)| \geq \mathrm{Re}(s) = a$, the first term is easily estimated:

$$\Big|\int_0^\infty \frac{1}{\gamma(r)}\,\bar{u}_r f_r r\,\mathrm{d}r\Big| \,\leq\, \frac{1}{a}\,\|u_r\|_{L^2}\|f_r\|_{L^2}\,.$$

As for the second term, we observe that $|k\mathcal{A}(-ikf_\theta + \frac{im}{r}f_z)| \leq |f_\theta| + |f_z|$ by (4.14), so that

$$\Big|\int_0^\infty \mathcal{A}\Big(\frac{2ik\Omega}{\gamma^2} + \frac{2km}{\gamma}\,\frac{1}{m^2 + k^2r^2}\Big)\big(-ikf_\theta + \frac{im}{r}f_z\big)\bar{u}_r r\,\mathrm{d}r\Big| \,\leq\, \Big(\frac{2}{a^2} + \frac{2}{am}\Big)\|u_r\|_{L^2}\big(\|f_\theta\|_{L^2} + \|f_z\|_{L^2}\big)\,.$$

The third term is integrated by parts as follows:

$$\int_0^\infty \bar{u}_r im\frac{\mathcal{A}}{\gamma r^2}\partial_r(rf_\theta)\,r\,\mathrm{d}r \,=\, -\int_0^\infty (\partial_r^*\bar{u}_r) im\frac{\mathcal{A}}{\gamma r} f_\theta\, r\,\mathrm{d}r - \int_0^\infty im\bar{u}_r\partial_r\Big(\frac{\mathcal{A}}{\gamma r^2}\Big) r f_\theta\, r\,\mathrm{d}r\,.$$

Since $|m\mathcal{A}^{\frac12}/r| \leq 1$ by (4.14), we have on the one hand

$$\Big|\int_0^\infty (\partial_r^*\bar{u}_r) im\frac{\mathcal{A}}{\gamma r}\, f_\theta\, r\,\mathrm{d}r\Big| \,\leq\, \frac{1}{a}\,\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}\|f_\theta\|_{L^2}\,,$$

and on the other hand

$$\Big|mr\partial_r\Big(\frac{\mathcal{A}}{\gamma r^2}\Big)\Big| \;=\; \Big|\frac{im^2\Omega'\mathcal{A}}{\gamma^2 r} + \frac{2mk^2\mathcal{A}^2}{\gamma r^2}\Big| \;\le\; \frac{\|r\Omega'\|_{L^\infty}}{a^2} + \frac{2}{am}\,,$$

so that

$$\Big|\int_0^\infty im\partial_r\Big(\frac{\mathcal{A}}{\gamma r^2}\Big) r f_\theta \bar{u}_r\, r\,\mathrm{d}r\Big| \;\le\; C\Big(\frac{1}{a^2} + \frac{1}{am}\Big)\|u_r\|_{L^2}\|f_\theta\|_{L^2}\,.$$

In a similar way, the fourth and last term is integrated by parts:

$$\int_0^\infty \bar{u}_r\frac{ik}{\gamma}\,\mathcal{A}\partial_r f_z\, r\,\mathrm{d}r \;=\; -\int_0^\infty (\partial_r^*\bar{u}_r)\frac{ik}{\gamma}\,\mathcal{A} f_z\, r\,\mathrm{d}r - \int_0^\infty ik\bar{u}_r\partial_r\Big(\frac{\mathcal{A}}{\gamma}\Big) f_z\, r\,\mathrm{d}r\,. \tag{4.22}$$

Since $|k\mathcal{A}^{\frac12}| \le 1$ by (4.14), we have

$$\Big|\int_0^\infty (\partial_r^*\bar{u}_r)\frac{ik}{\gamma}\mathcal{A} f_z\, r\,\mathrm{d}r\Big| \;\le\; \frac{1}{a}\,\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}\|f_z\|_{L^2}\,.$$

Moreover, using the relations $r\mathcal{A}' = 2\mathcal{A}(1-k^2\mathcal{A})$ and $\gamma' = im\Omega'$, we can estimate the last integral in (4.22) as follows:

$$\begin{aligned}\Big|\int_0^\infty ik\bar{u}_r\partial_r\Big(\frac{\mathcal{A}}{\gamma}\Big) f_z\, r\,\mathrm{d}r\Big| \;&\le\; \Big\|\frac{2k\mathcal{A}}{\gamma r}(1-k^2\mathcal{A}) - imk\frac{\Omega'\mathcal{A}}{\gamma^2}\Big\|_{L^\infty}\|u_r\|_{L^2}\|f_z\|_{L^2} \\ &\le\; \Big(\frac{2}{am} + \frac{\|r\Omega'\|_{L^\infty}}{a^2}\Big)\|u_r\|_{L^2}\|f_z\|_{L^2}\,.\end{aligned}$$

Collecting all estimates above and recalling that $m \ge 1$, we arrive at (4.21). $\square$

We next establish an explicit estimate that will be useful when the vertical wave number k is small compared to the angular Fourier mode m.

Lemma 4.8. *If $m \ge 1$ and $u, f \in X_{m,k}$ satisfy (4.1), we have the estimate*

$$\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}^2 + \|u_r\|_{L^2}^2 \;\le\; C_3\Big(\frac{1}{a^2} + \frac{1}{a^4}\Big)\frac{k^2}{m^2+k^2}\,\|u_r\|_{L^2}^2 + C_3\Big(\frac{1}{a^2} + \frac{1}{a^6}\Big)\|f\|_{L^2}^2\,, \tag{4.23}$$

where the constant $C_3 > 0$ depends only on Ω.

Proof. We start from the scalar resolvent equation (4.4) satisfied by the radial velocity u_r. Multiplying both sides by $r\bar{u}_r$ and integrating the resulting expression over $\mathbb{R}_+$, we obtain the following identity:

$$\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}^2 + \|u_r\|_{L^2}^2 + I_1 + I_2 \;=\; \int_0^\infty \mathcal{F}(r)\bar{u}_r r\,\mathrm{d}r\,, \tag{4.24}$$

where $\mathcal{F}(r)$ is defined in (4.5) and

$$I_1 \;=\; \int_0^\infty \frac{k^2}{\gamma^2}\,\mathcal{A}\Phi\,|u_r|^2 r\,\mathrm{d}r\,, \qquad I_2 \;=\; \int_0^\infty \frac{imr}{\gamma}\partial_r\Big(\frac{W}{m^2+k^2r^2}\Big)|u_r|^2 r\,\mathrm{d}r\,. \tag{4.25}$$

The right-hand side of (4.24) is estimated in Lemma 4.7. On the other hand, using (4.14) and the fact that $|\gamma(r)| \ge \mathrm{Re}(s) = a$, we can bound

$$\Big|\frac{k^2}{\gamma^2}\mathcal{A}\Phi\Big| \;\le\; \min\Big\{\frac{\|\Phi\|_{L^\infty}}{a^2}\,,\, \frac{k^2}{a^2m^2}\,\|r^2\Phi\|_{L^\infty}\Big\}, \quad \text{so that} \quad |I_1| \;\le\; \frac{C}{a^2}\,\frac{k^2}{m^2+k^2}\,\|u_r\|_{L^2}^2\,. \tag{4.26}$$

Moreover, we have

$$\Big|\frac{mr}{\gamma}\partial_r\Big(\frac{W}{m^2+k^2r^2}\Big)\Big| \,\le\, \frac{1}{am}\big(\|rW'\|_{L^\infty}+2\|W\|_{L^\infty}\big)\,, \quad \text{so that} \quad |I_2| \,\le\, \frac{C}{am}\|u_r\|_{L^2}^2\,. \qquad (4.27)$$

Combining (4.24), (4.25), (4.26), (4.21) and using Young's inequality, we obtain the preliminary estimate

$$\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}^2 + \|u_r\|_{L^2}^2 \,\le\, C_4\Big(\frac{k^2}{a^2(m^2+k^2)}+\frac{1}{am}\Big)\|u_r\|_{L^2}^2 + C_4\Big(\frac{1}{a^2}+\frac{1}{a^4}\Big)\|f\|_{L^2}^2\,, \qquad (4.28)$$

where the constant $C_4 > 0$ depends only on Ω.

If $ma \ge 2C_4$, it is clear that (4.28) implies (4.23). In the rest of the proof, we assume therefore that $ma \le 2C_4$. To obtain the improved bound (4.23), the idea is to control the integral term I_2 in a different way. Denoting

$$Z(r) \,=\, -r\partial_r\Big(\frac{W(r)}{m^2+k^2r^2}\Big) \,>\, 0\,,$$

we observe that

$$I_2 \,=\, -\int_0^\infty \frac{im}{\gamma}\,Z(r)|u_r|^2 r\,\mathrm{d}r \,=\, \int_0^\infty \frac{m^2(b-\Omega)-iam}{|\gamma|^2}\,Z(r)|u_r|^2 r\,\mathrm{d}r\,. \qquad (4.29)$$

As $\Omega(r) \le 1$ for all r, a lower bound on $\mathrm{Re}\, I_2$ is obtained if we replace $b-\Omega$ by $b-1$ in (4.29). Thus, taking the real part of (4.24), we obtain the bound

$$\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}^2 + \|u_r\|_{L^2}^2 + (b-1)\int_0^\infty \frac{m^2}{|\gamma|^2}\,Z(r)|u_r|^2 r\,\mathrm{d}r \,\le\, |I_1| + |I_3|\,, \qquad (4.30)$$

where $I_3 = \int_0^\infty \mathcal{F}(r)\bar{u}_r r\,\mathrm{d}r$. If $b \ge 1$, we can drop the integral in the left-hand side, and using the estimates (4.26), (4.21) on $|I_1|$, $|I_3|$ we arrive at (4.23). If $b < 1$, we consider also the imaginary part of (4.24), which gives the inequality

$$\int_0^\infty \frac{am}{|\gamma|^2}\,Z(r)|u_r|^2 r\,\mathrm{d}r \,\le\, |I_1| + |I_3|\,. \qquad (4.31)$$

Combining (4.30), (4.31) so as to eliminate the integral term, we obtain

$$\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}^2 + \|u_r\|_{L^2}^2 \,\le\, \Big(1+\frac{m(1-b)}{a}\Big)\big(|I_1|+|I_3|\big)\,. \qquad (4.32)$$

If $b \ge -1$, then $m(1-b)/a \le 2m/a \le 4C_4/a^2$. If $b \le -1$, we can assume that $m(1-b) \le 4C_1$, because in the converse case we have

$$|\gamma(r)| \,\ge\, m(\Omega(r)-b) \,\ge\, -mb \,\ge\, \frac{m(1-b)}{2} \,\ge\, 2C_1\,, \quad \text{for all } r > 0\,,$$

so that we can apply Lemma 4.6 with $M = 2C_1$ and deduce (4.23) from (4.20) and (4.28). So, in all relevant cases, the right-hand side of (4.32) is smaller than $C(1+a^{-2})\big(|I_1|+|I_3|\big)$, and using the estimates (4.26), (4.21) on $|I_1|$, $|I_3|$ we obtain (4.23). □

Remark 4.9. Estimate (4.28) implies in particular that

$$\|\mathcal{A}^{\frac12}\partial_r^* u_r\|_{L^2}^2 \,\le\, C_4\Big(\frac{1}{a}+\frac{1}{a^2}\Big)\,\|u_r\|_{L^2}^2 + C_4\Big(\frac{1}{a^2}+\frac{1}{a^4}\Big)\|f\|_{L^2}^2\,. \qquad (4.33)$$

In view of Corollary 4.4, this shows that controlling the quantity $\|u_r\|_{L^2}$ in terms of $\|f\|_{L^2}$ is equivalent to the full resolvent estimate, because the azimuthal and vertical velocities can be estimated using (4.13), (4.33). As an aside, we also observe that (4.28) provides an explicit resolvent estimate if $a > 0$ is sufficiently large, for instance if $a \ge 2C_4 + 1$. Thus we may assume in the sequel that a is bounded from above by a constant depending only on Ω.

To estimate the radial velocity u_r in the regime where k is large compared to m, it is convenient to introduce the auxiliary function $v(r) = \gamma(r)^{-1/2}u_r(r)$ (this idea already used in [10] is borrowed from [12]). The new function v satisfies the differential equation

$$-\partial_r\big(\mathcal{A}(r)\gamma(r)\partial_r^* v\big) + \mathcal{E}(r)v \,=\, \gamma(r)^{1/2}\mathcal{F}(r)\,, \qquad r>0\,, \tag{4.34}$$

where $\mathcal{A}(r)$, $\mathcal{F}(r)$ are as in (4.4) and

$$\mathcal{E}(r) \,=\, \gamma(r) + \frac{k^2}{\gamma(r)}\,\mathcal{A}(r)\Phi(r) + \frac{imr}{2}\partial_r\Big(\frac{W(r)+2\Omega(r)}{m^2+k^2r^2}\Big) - \frac{m^2\Omega'(r)^2}{4\gamma(r)}\,\mathcal{A}(r)\,.$$

Lemma 4.10. *If $m \ge 1$ and $u, f \in X_{m,k}$ satisfy* (4.1)*, there exists a constant $C_5 > 0$ depending only on Ω such that the function $v(r) = \gamma(r)^{-1/2}u_r(r)$ satisfies the estimate*

$$\|\mathcal{A}^{1/2}\partial_r^* v\|_{L^2}^2 + \|v\|_{L^2}^2 \,\le\, \frac{C_5}{a^2}\,\frac{m^2}{m^2+k^2}\,\|1_B v\|_{L^2}^2 + C_5\Big(\frac{1}{a^3}+\frac{1}{a^5}\Big)\|f\|_{L^2}^2\,, \tag{4.35}$$

where 1_B is the indicator function of the set $B = \{r>0\,;\, |\gamma(r)| \le r|\Omega'(r)|\}$.

Proof. Multiplying both sides of (4.34) by $r\bar v$, integrating the resulting expression over $\mathbb{R}_+$ and taking the real part, we obtain the identity

$$a\int_0^\infty\Big\{\mathcal{A}|\partial_r^* v|^2 + \Big(1+\frac{k^2}{|\gamma|^2}\mathcal{A}\Phi\Big)|v|^2\Big\}r\,\mathrm{d}r = \mathrm{Re}\int_0^\infty \bar v\gamma^{\frac12}\mathcal{F}r\,\mathrm{d}r + \frac{a}{4}\int_0^\infty m^2\Omega'^2\frac{\mathcal{A}}{|\gamma|^2}|v|^2 r\,\mathrm{d}r\,. \tag{4.36}$$

Since $\Phi \ge 0$, the left-hand side of (4.36) is bounded from below by $a\big(\|\mathcal{A}^{\frac12}\partial_r^* v\|_{L^2}^2 + \|v\|_{L^2}^2\big)$. On the other hand, repeating the proof of Lemma 4.7, we can estimate the first integral in the right-hand side as follows:

$$\begin{aligned}\Big|\int_0^\infty \bar v\gamma^{\frac12}\mathcal{F}r\,\mathrm{d}r\Big| \,&\le\, \frac{2}{a^{1/2}}\,\|\mathcal{A}^{\frac12}\partial_r^* v\|_{L^2}\|f\|_{L^2} + C\Big(\frac{1}{a^{1/2}}+\frac{1}{a^{3/2}}\Big)\|v\|_{L^2}\|f\|_{L^2}\\ &\le\, \frac{a}{2}\big(\|\mathcal{A}^{\frac12}\partial_r^* v\|_{L^2}^2 + \|v\|_{L^2}^2\big) + C\Big(\frac{1}{a^2}+\frac{1}{a^4}\Big)\|f\|_{L^2}^2\,,\end{aligned} \tag{4.37}$$

where the constant $C>0$ depends only on Ω. It remains to estimate the second integral in the right-hand side of (4.36). Defining $\mathcal{G}(r) = m^2\Omega'(r)^2\frac{\mathcal{A}(r)}{|\gamma(r)|^2}$, we observe that

$$\begin{aligned}\frac{a}{4}\int_0^\infty m^2\Omega'^2\,\frac{\mathcal{A}}{|\gamma|^2}\,|v|^2 r\,\mathrm{d}r \,&\le\, \frac{a}{4}\,\|1_{\{\mathcal{G}\le 1\}}v\|_{L^2}^2 + \frac{a}{4}\,\|\mathcal{G}\|_{L^\infty}\|1_{\{\mathcal{G}\ge 1\}}v\|_{L^2}^2\\ &\le\, \frac{a}{4}\,\|v\|_{L^2}^2 + \frac{1}{4a}\,\frac{m^2}{m^2+k^2}\,\|(1+r^2)\Omega'^2\|_{L^\infty}\|1_{\{\mathcal{G}\ge 1\}}v\|_{L^2}^2\,,\end{aligned} \tag{4.38}$$

where the upper bound on the quantity $\|\mathcal{G}\|_{L^\infty}$ is obtained using the definition of $\mathcal{A}$ and the fact that $|\gamma(r)| \ge \mathrm{Re}(s) = a$. Now, if $\mathcal{G}(r) \ge 1$, then $|\gamma(r)|^2 \le m^2\Omega'(r)^2\mathcal{A}(r) \le r^2\Omega'(r)^2$, so that the set $\{\mathcal{G}\ge 1\}$ is contained in $B = \{r>0\,;\,|\gamma(r)| \le r|\Omega'(r)|\}$. Thus, combining (4.36), (4.37), and (4.38), we obtain (4.35). □

The following result is a rather direct consequence of Lemmas 4.6 and 4.10:

Lemma 4.11. *If $m \ge 1$ and $u, f \in X_{m,k}$ satisfy* (4.1)*, there exists a constant $C_6 > 0$, depending only on Ω, such that the inequality*

$$\|u\|_{L^2} \,\le\, C_6\Big(\frac{1}{a}+\frac{1}{a^{7/2}}\Big)\|f\|_{L^2} \tag{4.39}$$

holds in each of the following three situations:

i) $ak \ge C_6 m$, ii) $am \ge C_6$ and $C_6(1-b) \le a$, iii) $am \ge C_6$ and $C_6 b \le a$.

Proof. Applying Lemma 4.6 with $M = 3C_1$, we deduce from (4.20) that

$$\|1_{\{|\gamma|\geq M\}}u\|_{L^2} \,\leq\, \frac{1}{2}\|1_{\{|\gamma|\leq M\}}u\|_{L^2} + \frac{1}{2}\|f\|_{L^2}\,. \tag{4.40}$$

In the sequel, we may thus focus our attention to the region where $|\gamma| \leq M$. Our strategy is to use Lemma 4.10, which requires a good control on the term involving $1_B v$ in the right-hand side of (4.35). We consider three cases separately.

i) If $ak \geq m\sqrt{2C_5}$, we simply observe that

$$\frac{C_5}{a^2}\,\frac{m^2}{m^2+k^2}\,\|1_B v\|_{L^2}^2 \,\leq\, \frac{1}{2}\|1_B v\|_{L^2}^2 \,\leq\, \frac{1}{2}\|v\|_{L^2}^2\,. \tag{4.41}$$

ii) By definition, for any $r > 0$, we have

$$r \in B \quad \text{if and only if} \quad a^2 + m^2(\Omega(r) - b)^2 \,\leq\, r^2\Omega'(r)^2\,. \tag{4.42}$$

Clearly $B = \emptyset$ if $a > \|r\Omega'\|_{L^\infty}$, hence we may assume that $a \leq \|r\Omega'\|_{L^\infty}$. Since $\Omega'(r) = \mathcal{O}(r)$ as $r \to 0$ by assumption H1, there exists a small constant $\epsilon > 0$ (depending only on Ω) such that inequality (4.42) cannot be satisfied if $r \leq \epsilon a^{1/2}$. On the other hand, if $r \geq \epsilon a^{1/2}$, then $\Omega(r) \leq \Omega(\epsilon a^{1/2}) \leq 1 - 2\delta a$ for some sufficiently small $\delta > 0$. Thus, if we assume that $b \geq 1 - \delta a$ and $m\delta a > \|r\Omega'\|_{L^\infty}$, we see that $m(b - \Omega(r)) \geq m\delta a > \|r\Omega'\|_{L^\infty}$, so that inequality (4.42) is not satisfied either. Summarizing, we have $B = \emptyset$ if $ma \geq C$ and $C(1-b) \leq a$ for some sufficiently large $C > 0$.

iii) Similarly, since $r\Omega'(r) = \mathcal{O}(r^{-2})$ as $r \to \infty$ by assumption H1, there exists a large constant $\rho > 0$ (depending only on Ω) such that (4.42) cannot be satisfied if $r \geq \rho a^{-1/2}$. If $r \leq \rho a^{-1/2}$, we have $\Omega(r) \geq \Omega(\rho a^{-1/2}) \geq 2\sigma a$ for some $\sigma > 0$. Thus, if we assume that $b \leq \sigma a$ and $m\sigma a > \|r\Omega'\|_{L^\infty}$, inequality (4.42) is never satisfied, so that $B = \emptyset$.

In all three cases, we deduce from (4.35) the estimate

$$\|\mathcal{A}^{1/2}\partial_r^* v\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 \,\leq\, C_5\Big(\frac{1}{a^3} + \frac{1}{a^5}\Big)\|f\|_{L^2}^2\,. \tag{4.43}$$

As $u_r(r) = \gamma(r)^{1/2}v(r)$, we have $\|1_{\{|\gamma|\leq M\}}u_r\|_{L^2} \leq M^{1/2}\|1_{\{|\gamma|\leq M\}}v\|_{L^2} \leq M^{1/2}\|v\|_{L^2}$, and

$$\big\|1_{\{|\gamma|\leq M\}}\mathcal{A}^{\frac{1}{2}}\partial_r^* u_r\big\|_{L^2} \,\leq\, M^{1/2}\big\|1_{\{|\gamma|\leq M\}}\mathcal{A}^{\frac{1}{2}}\partial_r^* v\big\|_{L^2} + \frac{\|r\Omega'\|_{L^\infty}}{2a^{1/2}}\,\|v\|_{L^2}\,.$$

Thus, using the representations (4.6), (4.7) of the azimuthal and vertical velocities, we deduce from (4.43) that

$$\|1_{\{|\gamma|\leq M\}}u_r\|_{L^2} + \|1_{\{|\gamma|\leq M\}}u_\theta\|_{L^2} + \|1_{\{|\gamma|\leq M\}}u_z\|_{L^2} \,\leq\, C\Big(\frac{1}{a} + \frac{1}{a^{7/2}}\Big)\|f\|_{L^2}\,.$$

Finally, invoking (4.40) to bound $\|1_{\{|\gamma|\geq M\}}u\|_{L^2}$ in terms of $\|1_{\{|\gamma|\leq M\}}u\|_{L^2}$, and recalling that we can assume $a \leq 2C_4 + 1$ by Remark 4.9, we arrive at (4.39). $\square$

Remark 4.12. Alternatively, one can obtain the resolvent estimate in case iii) by the following argument. If $m \geq 1$ is large and $b > 0$ is small, the inequality $|\gamma(r)| \leq M := 3C_1$ can be satisfied only if $r \gg 1$. In that region, the coefficients $\Omega(r)$ and $W(r)$ in (4.1) are very small, and so is the pressure p in view of Lemma 3.4. It is thus easy to estimate $\|1_{\{|\gamma|\leq M\}}u\|_{L^2}$ in terms $\|f\|_{L^2}$ directly from (4.1). Combining this observation with Lemma 4.6 gives the desired result.

4.3 End of the proof of Proposition 2.5

If we combine Lemma 4.5, Lemma 4.8, Remark 4.9, and Lemma 4.11, we obtain the following statement which specifies the regions in the parameter space where we could obtain a uniform resolvent estimate, with explicit (or at least computable) constant.

Corollary 4.13. *Assume that $m \in \mathbb{N}$, $k \geq 0$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) = a > 0$. There exists a constant $C > 0$, depending only on Ω, such that the resolvent estimate*

$$\big\|(s - L_{m,k})^{-1}\big\|_{X_{m,k}\to X_{m,k}} \leq C\Big(\frac{1}{a} + \frac{1}{a^4}\Big) \tag{4.44}$$

holds in each of the following cases:

$$\begin{array}{lll} 1)\ m = 0\,, & 2)\ a \geq C\,, & 3)\ ma^2 \geq Ck\,, \\ 4)\ ak \geq Cm\,, & 5)\ am \geq C \text{ and } C(1-b) \leq a\,, & 6)\ am \geq C \text{ and } Cb \leq a\,. \end{array} \tag{4.45}$$

We recall that b is defined by (4.19) when $m \neq 0$.

To conclude the proof of Proposition 2.5, we use a contradiction argument to establish a resolvent estimate in the regions that are not covered by Corollary 4.13. More precisely, if we consider a sequence of values of the parameters m, k, s (with $\mathrm{Re}(s) = a$) such that none of the conditions 1)–6) in (4.45) is satisfied, two possibilities can occur. Either the angular Fourier mode m goes to infinity, as well as the vertical wave number k, and the parameter b remains in the interval $[a/C, 1-a/C] \subset (0,1)$. In that case, after extracting a subsequence, we can assume that b converges to some limit. So, to establish the resolvent estimate, we have to prove that, for any $b \in (0,1)$,

$$\sup_{\mathrm{Re}(s)=a}\ \limsup_{\substack{m\to+\infty,\\ \mathrm{Im}(s)/m\to -b}} \big\|(s - L_{m,k})^{-1}\big\|_{X_{m,k}\to X_{m,k}} < \infty\,. \tag{4.46}$$

The other possibility is that the angular Fourier mode $m \geq 1$ stays bounded, as well as the vertical wave number $k \geq k_0 := a^2/C$. In that case, we have to prove that, for all $N \geq 1$,

$$\sup_{\mathrm{Re}(s)=a}\ \sup_{\substack{1\leq m\leq N,\\ k_0\leq k\leq N}} \big\|(s - L_{m,k})^{-1}\big\|_{X_{m,k}\to X_{m,k}} < \infty\,. \tag{4.47}$$

Proof of estimate (4.46)**:**
We argue by contradiction and assume the existence of sequences $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(u^n)_{n\in\mathbb{N}}$, $(f^n)_{n\in\mathbb{N}}$ in X_{m_n,k_n} with the following properties: u^n, f^n are solutions of the resolvent system $(s_n - L_{m_n,k_n})u^n = f^n$ where $s_n = a - im_nb_n$, $\|u^n\|_{L^2} = 1$ $\forall n \in \mathbb{N}$, and we have $\|f^n\|_{L^2} \to 0$, $m_n \to +\infty$, and $b_n \to b$ as $n \to +\infty$. Without loss of generality we may assume that $b_n \in (0,1)$ for all $n \in \mathbb{N}$, and we define $r_n = \Omega^{-1}(b_n)$; in particular $r_n \to \bar{r} := \Omega^{-1}(b)$ as $n \to +\infty$. We also denote by $(p_n)_{n\in\mathbb{N}}$ the sequence of pressures associated to u^n, namely $p_n = P_{m_n,k_n}[u^n]$, and we set $\gamma_n(r) = a + im_n(\Omega(r) - b_n)$.

In view of inequalities (4.13) and (4.33), the normalization condition $\|u^n\|_{L^2} = 1$ and the assumption that $\|f^n\|_{L^2} \to 0$ as $n \to \infty$ imply that the quantity $\|u_r^n\|_{L^2}$ is bounded from below for large values of n, namely

$$I_r := \liminf_{n\to+\infty} \|u_r^n\|_{L^2}^2 > 0\,. \tag{4.48}$$

Setting $M = C_1\sqrt{2/I_r}$, we deduce from (4.48) and Lemma 4.6 that

$$\liminf_{n\to+\infty} \int_{\{|\gamma_n|\leq M\}} |u_r^n(r)|^2\, r\,\mathrm{d}r \geq \frac{I_r}{2} > 0\,. \tag{4.49}$$

As the angular velocity Ω is continuously differentiable and strictly decreasing on $\mathbb{R}_+$, the set $\{|\gamma_n| \le M\}$ is asymptotically contained in the interval $[r_n - R/m_n, r_n + R/m_n]$, where $R > 0$ is a constant that depends only on Ω and I_r (one may take $R = 2M|\Omega'(\overline{r})|^{-1}$). Since the length of that interval shrinks to zero as $n \to \infty$, it is useful to introduce rescaled vector fields and functions by setting

$$u^n(r) = m_n^{1/2}\, \tilde u^n(m_n(r{-}r_n))\,, \quad f^n(r) = m_n^{1/2}\, \tilde f^n(m_n(r{-}r_n))\,, \quad p_n(r) = m_n^{-1/2}\, \tilde p_n(m_n(r{-}r_n))\,.$$

Note that the new variable $y := m_n(r{-}r_n)$ is defined on the n-dependent domain $(-m_n r_n, \infty)$. Likewise, we set $\Omega(r) = \tilde\Omega_n(m_n(r{-}r_n))$, $W(r) = \tilde W_n(m_n(r{-}r_n))$ and $\gamma_n(r) = \tilde\gamma_n(m_n(r{-}r_n))$. The system (4.1) may then be rewritten as

$$\begin{aligned}
\tilde\gamma_n(y)\tilde u_r^n - 2\tilde\Omega_n(y)\tilde u_\theta^n \;&=\; -\partial_y \tilde p_n + \tilde f_r^n\,,\\
\tilde\gamma_n(y)\tilde u_\theta^n + \tilde W_n(y)\tilde u_r^n \;&=\; -\tfrac{i}{r_n + y/m_n}\tilde p_n + \tilde f_\theta^n\,,\\
\tilde\gamma_n(y)\tilde u_z^n \;&=\; -i\tfrac{k_n}{m_n}\tilde p_n + \tilde f_z^n\,,
\end{aligned} \tag{4.50}$$

and the incompressibility condition becomes

$$\partial_y \tilde u_r^n + \tfrac{i}{r_n+y/m_n}\tilde u_\theta^n + i\tfrac{k_n}{m_n}\tilde u_z^n \;=\; -\tfrac{1}{r_n m_n + y}\tilde u_r^n\,. \tag{4.51}$$

After this change of variables, inequality (4.49) implies the lower bound

$$\liminf_{n\to+\infty} \int_{-R}^{R} |\tilde u_r^n(y)|^2 \,\mathrm{d}y \;\ge\; \frac{I_r}{2\overline{r}} \;>\; 0\,. \tag{4.52}$$

Since, by assumption, inequalities 3) and 4) in (4.45) are not satisfied, we can suppose without loss of generality that $k_n/m_n \to \delta \in (0, +\infty)$ as $m \to +\infty$. By construction, we also have $\tilde\Omega_n(y) \to \Omega(\bar r)$, $\tilde W_n(y) \to W(\bar r)$ and $\tilde\gamma_n(y) \to \overline{\gamma}(y) := a + i\Omega'(\bar r)y$ as $n \to +\infty$, uniformly on any compact subset of $\mathbb{R}$.

Using the normalization condition for u^n, we observe that

$$1 \;=\; \int_{-m_n r_n}^{\infty} |\tilde u^n(y)|^2 \Big(r_n + \frac{y}{m_n}\Big)\,\mathrm{d}y \;\ge\; \frac{r_n}{2}\int_{-m_n\frac{r_n}{2}}^{\infty} |\tilde u^n(y)|^2 \,\mathrm{d}y\,.$$

Extracting a subsequence if needed, we may therefore assume that $\tilde u^n \rightharpoonup U$ in $L^2(K)$ for each compact subset $K \subset \mathbb{R}$, where $U \in L^2(\mathbb{R})$ and $\|U\|_{L^2}^2 \le 2/\overline{r}$. Similarly, using the uniform bounds on the pressure given by Lemma 3.1, we may assume that $\tilde p_n \to P$ and $\partial_y \tilde p_n \rightharpoonup P'$ in $L^2(K)$, for each compact subset $K \subset \mathbb{R}$, where $P \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $P' \in L^2(\mathbb{R})$. The radial velocities $\tilde u_r^n$ have even better convergence properties. Indeed, it follows from (4.33) that the quantity $\|\mathcal{A}_n^{1/2}\partial_r^* u_r^n\|_{L^2}$ is uniformly bounded for n large, and since $\|\mathcal{A}_n^{1/2} r^{-1} u_r^n\|_{L^2} \le 1/m_n \to 0$ we deduce that $\|\mathcal{A}_n^{1/2}\partial_r u_r^n\|_{L^2}$ is uniformly bounded too. After the change of variables, this implies that

$$C \;\ge\; \int_{-m_n r_n}^{\infty} \frac{m_n^2 r^2}{m_n^2 + k_n^2 r^2}\,|\partial_y \tilde u_r^n(y)|^2\, r\,\mathrm{d}y \;\ge\; \frac{r_n}{2}\,\frac{r_n^2}{4+\delta_n^2 r_n^2}\int_{-m_n\frac{r_n}{2}}^{\infty} |\partial_y \tilde u_r^n(y)|^2 \,\mathrm{d}y\,,$$

where $r = r_n + y/m_n$ and $\delta_n = k_n/m_n$. Thus $U_r \in H^1(\mathbb{R})$, and extracting a further subsequence if necessary we can assume that $\partial_y \tilde u_r^n \rightharpoonup U_r'$ and $\tilde u_r^n \to U_r$ in $L^2(K)$, for each compact subset $K \subset \mathbb{R}$. In particular, we deduce from (4.52) that U_r is not identically zero. Moreover, passing to the limit in (4.50), (4.51), we obtain the asymptotic system

$$\begin{aligned}
&(a + i\Omega'(\overline{r})y)U_r - 2\Omega(\overline{r})U_\theta \;=\; -P'\,,\\
&(a + i\Omega'(\overline{r})y)U_\theta + W(\overline{r})U_r \;=\; -\tfrac{i}{\overline{r}}P\,, \qquad\qquad U_r' + \tfrac{i}{\overline{r}}U_\theta + i\delta U_z \;=\; 0\,,\\
&(a + i\Omega'(\overline{r})y)U_z \;=\; -i\delta P\,,
\end{aligned} \tag{4.53}$$

where equalities hold almost everywhere. We claim that system (4.53) does not possess any solution such that $U \in L^2_{\rm loc}(\mathbb{R})$, $P \in H^1_{\rm loc}(\mathbb{R})$ and such that $U_r \in H^1(\mathbb{R})$ is nontrivial. This will provide the desired contradiction.

Indeed, if we repeat the proof of Lemma 4.2 (with $f = 0$), we can extract from system (4.53) a second-order differential equation for the radial velocity U_r. Eliminating the pressure P and the azimuthal velocity U_θ, we obtain as in (4.9), (4.11) :

$$\delta\Big(U_r' - \frac{iW(\overline{r})}{\overline{r}\,\overline{\gamma}(y)}U_r\Big) + i\Big(\delta^2 + \frac{1}{\overline{r}^2}\Big)U_z \,=\, 0\,, \qquad U_z' + \frac{iW(\overline{r})}{\overline{r}\,\overline{\gamma}(y)}U_z - i\delta\Big(1 + \frac{\Phi(\overline{r})}{\overline{\gamma}(y)^2}\Big) \,=\, 0\,,$$

and combining these relations we arrive at

$$-U_r'' + \left[\Big(\delta^2 + \frac{1}{\overline{r}^2}\Big) + \frac{\Phi(\overline{r})\delta^2}{\overline{\gamma}(y)^2}\right]U_r \,=\, 0\,, \qquad y \in \mathbb{R}\,, \tag{4.54}$$

where $\Phi(\overline{r}) = 2\Omega(\overline{r})W(\overline{r}) > 0$. If we observe that $\overline{\gamma}(y) = a + i\Omega'(\overline{r})y = i\Omega'(\overline{r})(y + ic)$, where $c = -a/\Omega'(\overline{r})$, we can write (4.54) in the equivalent form

$$-U_r'' + \Big(\kappa^2 - \frac{J(\overline{r})\delta^2}{(y+ic)^2}\Big)U_r \,=\, 0\,, \qquad y \in \mathbb{R}\,, \tag{4.55}$$

where $\kappa^2 = 1/\overline{r}^2 + \delta^2$ and $J(\overline{r}) = \Phi(\overline{r})/\Omega'(\overline{r})^2$. Up to a multiplicative constant, the unique solution of (4.55) that belongs to $L^2(\mathbb{R}_+)$ is

$$U_r(y) \,=\, (y+ic)^{1/2}K_\nu\big(\kappa(y+ic)\big)\,, \qquad y \in \mathbb{R}\,, \tag{4.56}$$

where K_ν is the modified Bessel function, see [1, Section 9.6], and $\nu \in \mathbb{C}$ is determined, up to an irrelevant sign, by the relation $\nu^2 = \frac{1}{4} - J(\overline{r})\delta^2$. In fact, any linearly independent solution of (4.55) grows like $\exp(\kappa y)$ as $y \to +\infty$. Now, it is well known that the function $K_\nu(\kappa(y+ic))$ has itself an exponential growth as $y \to -\infty$, see [1, Section 9.7], and this implies that (4.55) has no nontrivial solution in $L^2(\mathbb{R})$.

Proof of estimate (4.47) **:**
This is the only place where we use our assumption H2 on the vorticity profile. According to Proposition 2.3, which is the main result of [10], the resolvent operator $(s - L_{m,k})^{-1}$ is well defined as a bounded linear operator in $X_{m,k}$ for any $m \in \mathbb{N}$, any $k \in \mathbb{R}$, and any $s \in \mathbb{C}$ with $\mathrm{Re}(s) \neq 0$. To prove (4.47), it remains to show that, for any fixed m, the resolvent estimate holds uniformly in k on compact subsets of $\mathbb{R}_+ = (0,\infty)$, and uniformly in s on vertical lines. Actually, we can assume that the spectral parameter lies in a compact set too, because if m is fixed and $|\,\mathrm{Im}(s)| \geq m + 2C_1$, we have $|\gamma(r)| \geq |\,\mathrm{Im}(s)| - m \geq 2C_1$ and the resolvent bound follows from estimate (4.20) with $M = 2C_1$. So the only missing step is :

Lemma 4.14. *For any $m \in \mathbb{Z}$, the resolvent norm $\|(s - L_{m,k})^{-1}\|_{X_{m,k}\to X_{m,k}}$ is uniformly bounded in the neighborhood of any point $(k,s) \in \mathbb{R}\times\mathbb{C}$ with $k \neq 0$ and* $\mathrm{Re}(s) > 0$.

Proof. Since the function space $X_{m,k}$ changes when k is varied, due to the incompressibility condition, the result does not immediately follow from standard perturbation theory. However, it is easy to reformulate the problem so that perturbation theory can be applied. It is sufficient to note that, for any fixed $k^* \neq 0$, the mappings

$$M_k : X_{m,k^*} \to X_{m,k}\,, \qquad (u_r, u_\theta, u_z) \,\mapsto\, \Big(u_r, u_\theta, \frac{k^*}{k}u_z\Big)\,,$$

are linear homeomorphisms that depend continuously on k in a neighborhood of k^*. Given $s \in \mathbb{C}$, $m \in \mathbb{Z}$, and $k \in \mathbb{R}$ close to k^*, the resolvent equation $(s - L_{m,k})u = f$ for $u, f \in X_{m,k}$ is equivalent to the conjugated equation $(s - \mathcal{L}_{m,k})v = g$, where $u = M_k v$, $f = M_k g$, and

$$\mathcal{L}_{m,k} \,=\, M_k^{-1} L_{m,k} M_k \,:\, X_{m,k^*} \to X_{m,k^*}\,. \tag{4.57}$$

Now, using in particular estimate (3.3) in Lemma 3.3, it is straightforward to verify that the operator $\mathcal{L}_{m,k}$ depends continuously on k as a bounded linear operator in X_{m,k^*}, as long as $k \neq 0$. This implies that the resolvent norm $\|(s - \mathcal{L}_{m,k})^{-1}\|_{X_{m,k^*} \to X_{m,k^*}}$ depends continuously on the parameters s and k, when k stays in a neighborhood of k^*, and the conclusion easily follows. $\square$

5 Appendix: analysis in $X_{m,k}$

We collect here various auxiliary results that are useful for our analysis in Section 3. We first show that smooth and compactly supported divergence-free vector fields are dense in the space $X_{m,k}$ defined by (2.6), and we give simple criteria for compactness in that space. Finally, we establish explicit representations formulas for the pressure p satisfying (2.8).

5.1 Approximation in $X_{m,k}$

Truncating divergence-free vector fields is not straightforward, and a general solution to that problem involves the so-called Bogovskii operator, see e.g. [9]. However, in the particular case of the space $X_{m,k}$ introduced in (2.6), localization can be performed in a rather elementary way, which we now describe.

Lemma 5.1. *For any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$, the set of all $u \in X_{m,k}$ with compact support in $(0, +\infty)$ is dense in $X_{m,k}$.*

Proof. Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}$ be smooth, monotonic functions such that

$$\phi(r) \,=\, \begin{cases} 0 & \text{if } r \le \frac{1}{2}\,, \\ 1 & \text{if } r \ge 1\,, \end{cases} \qquad \text{and} \qquad \psi(r) \,=\, \begin{cases} 1 & \text{if } r \le 1\,, \\ 0 & \text{if } r \ge 2\,. \end{cases}$$

Given $\epsilon \in (0,1)$, we define $\chi_\epsilon(r) = \min\{\phi(r/\epsilon), \psi(\epsilon r)\}$. By construction χ_ϵ is smooth and satisfies $\chi_\epsilon(r) = 0$ if $r \le \epsilon/2$ or $r \ge 2/\epsilon$, and $\chi_\epsilon(r) = 1$ if $\epsilon \le r \le 1/\epsilon$.

Assume first that $m \neq 0$. Given $u \in X_{m,k}$, we define $v_\epsilon = u\chi_\epsilon + w_\epsilon e_\theta$, where

$$w_\epsilon(r) \,=\, \frac{i}{m}\, r\chi_\epsilon'(r) u_r(r)\,, \quad r > 0\,.$$

The corrector w_ϵ is tailored so that $\operatorname{div} v_\epsilon = (\operatorname{div} u)\chi_\epsilon + u_r \chi_\epsilon' + \frac{im}{r} w_\epsilon = 0$. Moreover w_ϵ is supported in the set $[\epsilon/2, 2/\epsilon]$ by construction. Since $\chi_\epsilon(r) \to 1$ as $\epsilon \to 0$ for any $r > 0$, it is clear that $\|u\chi_\epsilon - u\|_{L^2} \to 0$ as $\epsilon \to 0$. Moreover

$$\begin{aligned} \|w_\epsilon\|_{L^2}^2 \,&=\, \frac{1}{m^2}\int_{\epsilon/2}^{\epsilon} \frac{r^2}{\epsilon^2}\,|\phi'(r/\epsilon)|^2\,|u_r(r)|^2 r\,\mathrm{d}r + \frac{1}{m^2}\int_{1/\epsilon}^{2/\epsilon} \epsilon^2 r^2 |\psi'(\epsilon r)|^2\,|u_r(r)|^2 r\,\mathrm{d}r \\ &\le\, \frac{C}{m^2}\Big(\int_0^{\epsilon} |u_r(r)|^2 r\,\mathrm{d}r + \int_{1/\epsilon}^{\infty} |u_r(r)|^2 r\,\mathrm{d}r\Big) \xrightarrow[\epsilon \to 0]{} 0\,. \end{aligned}$$

Thus $\|v_\epsilon - u\|_{L^2} \to 0$ as $\epsilon \to 0$, which is the desired result.

Next we assume that $m = 0$ and $k \neq 0$. Given any $u \in X_{0,k}$, the divergence-free condition $\partial_r^* u_r + iku_z = 0$ implies that

$$u_r(r) \;=\; -\frac{ik}{r}\int_0^r u_z(s)s\,\mathrm{d}s\,, \qquad \text{hence} \quad |u_r(r)|^2 \;\le\; \frac{k^2}{2}\int_0^r |u_z(s)|^2 s\,\mathrm{d}s\,, \tag{5.1}$$

for any $r > 0$. We now define $\tilde v_\epsilon = u\chi_\epsilon + \tilde w_\epsilon e_z$, where

$$\tilde w_\epsilon(r) \;=\; \frac{i}{k}\,\chi_\epsilon'(r)u_r(r)\,, \quad r > 0\,.$$

As before $\tilde v_\epsilon$ is divergence-free and supported in $[\epsilon/2, 2/\epsilon]$. Moreover, using (5.1), we find

$$\begin{aligned}\|\tilde w_\epsilon\|_{L^2}^2 \;&=\; \frac{1}{k^2}\int_{\epsilon/2}^{\epsilon}\frac{1}{\epsilon^2}\,|\phi'(r/\epsilon)|^2\,|u_r(r)|^2 r\,\mathrm{d}r + \frac{1}{k^2}\int_{1/\epsilon}^{2/\epsilon}\epsilon^2|\psi'(\epsilon r)|^2\,|u_r(r)|^2 r\,\mathrm{d}r \\ &\le\; C\int_0^{\epsilon}|u_z(r)|^2 r\,\mathrm{d}r + \frac{C\epsilon^2}{k^2}\int_{1/\epsilon}^{\infty}|u_r(r)|^2 r\,\mathrm{d}r \;\xrightarrow[\epsilon\to 0]{}\; 0\,,\end{aligned}$$

and this shows that $\|\tilde v_\epsilon - u\|_{L^2} \to 0$ as $\epsilon \to 0$.

Finally, if $u \in X_{0,0}$, the divergence-free condition asserts that $\partial_r^* u_r = 0$, hence $u_r = 0$. It follows that $u\chi_\epsilon$ is divergence-free, and we know that $\|u\chi_\epsilon - u\|_{L^2} \to 0$ as $\epsilon \to 0$. □

Using Lemma 5.1 and a standard regularization procedure, we obtain:

Proposition 5.2. *For any $m \in \mathbb{Z}$ and any $k \in \mathbb{R}$, the set of all smooth, divergence-free vector fields with compact support in $(0, +\infty)$ is dense in $X_{m,k}$.*

Proof. According to Lemma 5.1, it is sufficient to prove that any $u \in X_{m,k}$ with compact support can be approximated by smooth, divergence-free and compactly supported vector fields. Assume thus that $u \in X_{m,k}$ is such that $u(r) = 0$ for $r \le r_1$ and $r \ge r_2$, with $0 < r_1 < r_2 < \infty$. We consider the vector field $U = (U_1, U_2, U_3)$ in $\mathbb{R}^3$ defined by

$$U(r\cos\theta, r\sin\theta, z) \;=\; \Big(u_r(r)e_r(\theta) + u_\theta(r)e_\theta(\theta) + u_z(r)e_z\Big)\,e^{im\theta}\,e^{ikz}\,, \tag{5.2}$$

where $r > 0$, $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, and $z \in \mathbb{R}$. Then $\operatorname{div} U = 0$ and, for any fixed $x_3 \in \mathbb{R}$, the map $(x_1, x_2) \mapsto U(x_1, x_2, x_3)$ belongs to $L^2(\mathbb{R}^2, \mathbb{C}^3)$, because $\|U(\cdot,\cdot,x_3)\|_{L^2(\mathbb{R}^2)}^2 = 2\pi\|u\|_{L^2}^2 < \infty$. Given $\epsilon > 0$, we define the approximation

$$U^\epsilon(x_1, x_2, x_3) \;=\; \frac{1}{\epsilon^2}\int_{\mathbb{R}^2}\chi\Big(\frac{x_1 - y_1}{\epsilon}, \frac{x_2 - y_2}{\epsilon}\Big)\,U(y_1, y_2, x_3)\,\mathrm{d}y_1\,\mathrm{d}y_2\,,$$

where $\chi : \mathbb{R}^2 \to \mathbb{R}_+$ is smooth, *radially symmetric*, supported in the unit ball, and normalized so that $\int \chi\,\mathrm{d}x_1\,\mathrm{d}x_2 = 1$. By construction, the vector field U^ϵ is smooth, divergence-free, and close to U in the sense that $\|U^\epsilon(\cdot,\cdot,x_3) - U(\cdot,\cdot,x_3)\|_{L^2(\mathbb{R}^2)} \to 0$ as $\epsilon \to 0$ for any $x_3 \in \mathbb{R}$. If $\epsilon \le r_1/2$, we also have $U^\epsilon(x_1, x_2, x_3) = 0$ whenever $r := (x_1^2 + x_2^2)^{1/2} \le r_1/2$ or $r \ge r_1 + r_2$. Under this assumption, since χ is radially symmetric, we can represent U^ϵ as

$$U^\epsilon(r\cos\theta, r\sin\theta, z) \;=\; \Big(u_r^\epsilon(r)e_r(\theta) + u_\theta^\epsilon(r)e_\theta(\theta) + u_z^\epsilon(r)e_z\Big)\,e^{im\theta}\,e^{ikz}\,, \tag{5.3}$$

for some *smooth* vector field $u^\epsilon = u_r^\epsilon e_r + u_\theta^\epsilon e_\theta + u_z^\epsilon e_z \in X_{m,k}$, which is supported in the compact interval $[r_1/2, r_2 + r_1] \subset (0, \infty)$. Here the condition on the support is essential, because the unit vectors e_r, e_θ are smooth only away from the axis $r = 0$. From (5.2), (5.3) we deduce that

$$\|u^\epsilon - u\|_{L^2(\mathbb{R}_+, r\,\mathrm{d}r)}^2 \;=\; \frac{1}{2\pi}\|U^\epsilon(\cdot,\cdot,0) - U(\cdot,\cdot,0)\|_{L^2(\mathbb{R}^2)}^2 \;\xrightarrow[\epsilon\to 0]{}\; 0\,,$$

and this gives the desired result. □

5.2 Compactness criteria

We next mention two simple compactness criteria in the space $X = L^2(\mathbb{R}_+, r\,\mathrm{d}r)$.

Lemma 5.3. *For any $\alpha > 0$ and any $M > 0$, the sets*

$$\begin{aligned} E_{M,\alpha} &= \bigl\{f \in X\,;\, \|\partial_r f\|_{L^2} \le M\,,\ \|r^\alpha f\|_{L^2} \le M\bigr\}\,, \quad \textit{and} \\ E^*_{M,\alpha} &= \bigl\{f \in X\,;\, \|\partial^*_r f\|_{L^2} \le M\,,\ \|r^\alpha f\|_{L^2} \le M\bigr\}\,, \end{aligned}$$

are compact in X. We recall that $\partial^*_r = \partial_r + \frac{1}{r}$.

Proof. If $f \in X$, we define $F : \mathbb{R}^2 \to \mathbb{C}$ by $F(x) = (2\pi)^{-1/2} f(|x|)$ for all $x \in \mathbb{R}^2$. The linear map $f \mapsto F$ is an isometric embedding of X into $L^2(\mathbb{R}^2)$, and the image of $E_{M,\alpha}$ under that map is included in the set

$$\bigl\{F \in L^2(\mathbb{R}^2)\,;\, \|\nabla F\|_{L^2} \le M\,,\ \||x|^\alpha F\|_{L^2} \le M\bigr\}\,,$$

which is known to be compact in $L^2(\mathbb{R}^2)$ by Rellich's criterion, see [16, Theorem XIII.65]. This shows that the closed subset $E_{M,\alpha} \subset X$ is relatively compact, hence compact.

Compactness of $E^*_{M,\alpha}$ can be established by a variant of the previous argument, but for a change we give here a direct proof based on the Arzelà-Ascoli theorem. If $f \in E^*_{M,\alpha}$, we observe that

$$f(r) \,=\, \frac{1}{r}\int_0^r \partial^*_r f(s) s\,\mathrm{d}s\,, \qquad \text{for all } r > 0\,. \tag{5.4}$$

This shows that $|f(r)| \le \|\partial^*_r f\|_{L^2} \le M$ for all $r > 0$, and we deduce that

$$\int_0^\epsilon |f(r)|^2 r\,\mathrm{d}r \,\le\, M^2\epsilon^2\,, \qquad \int_L^\infty |f(r)|^2 r\,\mathrm{d}r \,\le\, \frac{1}{L^{2\alpha}}\|r^\alpha f\|^2_{L^2} \,\le\, \frac{M^2}{L^{2\alpha}}\,,$$

for any $\epsilon > 0$ and any $L > 0$. In particular, the set $E^*_{M,\alpha}$ is bounded in X, and its elements are uniformly small near the origin and at infinity. Moreover, it follows from (5.4) and Hölder's inequality that

$$|r_1 f(r_1) - r_2 f(r_2)| \,\le\, M|r_1 - r_2|^{1/2}\,, \qquad \text{for all } r_1, r_2 > 0\,,$$

which means that the elements of $E^*_{M,\alpha}$ are *uniformly equicontinuous* on any compact interval $[\epsilon, L] \subset (0,\infty)$. These properties altogether imply that $E^*_{M,\alpha}$ is a compact subset of X. □

5.3 Representation formulas

Finally we give explicit representation formulas for the pressure p satisfying (2.8), in terms of solutions of the homogeneous equation

$$-\partial^*_r\partial_r p(r) + \frac{m^2}{r^2}p(r) + k^2 p(r) \,=\, 0\,. \tag{5.5}$$

If $k \neq 0$, a pair of linearly independent solutions of (5.5) is given by the modified Bessel functions $I_m(|k|r)$ and $K_m(|k|r)$, see e.g. [1, Section 9.6]. For later use, we recall that $I_{-m}(r) = I_m(r)$, $K_{-m}(r) = K_m(r)$, and $K_m(r)I'_m(r) - K'_m(r)I_m(r) = 1/r$ for all $r > 0$. Moreover, if $m \ge 1$, then

$$I_m(r) \sim \frac{1}{m!}\Bigl(\frac{r}{2}\Bigr)^m\,, \qquad K_m(r) \sim \frac{(m-1)!}{2}\Bigl(\frac{2}{r}\Bigr)^m\,, \qquad \text{as } r \to 0\,, \tag{5.6}$$

whereas $I_0(r) \to 1$ and $K_0(r) \sim -\log(r)$ as $r \to 0$. For all $m \in \mathbb{Z}$, we also have

$$I_m(r) \sim \frac{1}{\sqrt{2\pi}} \frac{e^r}{\sqrt{r}}\,, \qquad K_m(r) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-r}}{\sqrt{r}}\,, \qquad \text{as } r \to +\infty\,. \tag{5.7}$$

When $k = 0$ linearly independent solutions of (5.5) are $r^{\pm m}$ if $m \neq 0$, and $\{1, \log(r)\}$ if $m = 0$.

Lemma 5.4. *Assume that the vorticity profile W satisfies assumption H1. For any $m \in \mathbb{Z}$, $k \in \mathbb{R}$, and $u \in X_{m,k}$, the elliptic equation (2.8) has a unique solution $p = P_{m,k}[u]$ such that $p(r) = \mathcal{O}(|\log r|^{1/2})$ as $r \to 0$ and $p(r) \to 0$ as $r \to +\infty$. If $k \neq 0$, we have $p = 2imp_1 + 2|k|p_2$ where*

$$\begin{aligned} p_1(r) \,&=\, K_m(|k|r) \int_0^r I_m(|k|s)(s\Omega)' u_r(s)\,\mathrm{d}s + I_m(|k|r) \int_r^\infty K_m(|k|s)(s\Omega)' u_r(s)\,\mathrm{d}s\,, \\ p_2(r) \,&=\, K_m(|k|r) \int_0^r I_m'(|k|s)\Omega(s) u_\theta(s) s\,\mathrm{d}s + I_m(|k|r) \int_r^\infty K_m'(|k|s)\Omega(s) u_\theta(s) s\,\mathrm{d}s\,. \end{aligned} \tag{5.8}$$

If $k = 0$ and $m \neq 0$, then $p = \sigma p_1 + p_2$ where $\sigma = m/|m|$ and

$$\begin{aligned} p_1(r) \,&=\, \frac{i}{r^{|m|}} \int_0^r s^{|m|}(s\Omega)'(s) u_r(s)\,\mathrm{d}s + i r^{|m|} \int_r^\infty \frac{1}{s^{|m|}}(s\Omega)'(s) u_r(s)\,\mathrm{d}s\,, \\ p_2(r) \,&=\, \frac{1}{r^{|m|}} \int_0^r s^{|m|}\Omega(s) u_\theta(s)\,\mathrm{d}s - r^{|m|} \int_r^\infty \frac{1}{s^{|m|}}\Omega(s) u_\theta(s)\,\mathrm{d}s\,. \end{aligned} \tag{5.9}$$

Finally, if $k = m = 0$, then $p(r) = -2\int_r^\infty \Omega(s) u_\theta(s)\,\mathrm{d}s$.

Proof. In view of (2.8) we can suppose without loss of generality that $k \geq 0$. If $k > 0$, we first assume that $u \in X_{m,k} \cap C_c^1(\mathbb{R}_+)$ and we consider the linear elliptic equation

$$-\partial_r^* \partial_r p(r) + \frac{m^2}{r^2}\, p(r) + k^2 p(r) \,=\, f(r)\,, \qquad r > 0\,, \tag{5.10}$$

where $f = 2im(\partial_r^* \Omega) u_r - 2\partial_r^*(\Omega\, u_\theta)$. The unique solution of (5.10) that is regular at the origin and decays to zero at infinity is

$$p(r) \,=\, K_m(kr) \int_0^r I_m(ks) f(s) s\,\mathrm{d}s + I_m(kr) \int_r^\infty K_m(ks) f(s) s\,\mathrm{d}s\,, \qquad r > 0\,. \tag{5.11}$$

Replacing f by its expression and integrating by parts, we easily obtain the representation (5.8). The general case where u is an arbitrary function in $X_{m,k}$ follows by a density argument, using Proposition 5.2.

If $k = 0$ and $m \neq 0$, the solutions of the homogeneous equation (5.5) are $r^{|m|}$ and $r^{-|m|}$, instead of $I_m(|k|r))$ and $K_m(|k|r)$. Proceeding exactly as above, we thus arrive at (5.9) instead of (5.8). Finally, if $k = m = 0$, any solution of (2.8) such that $\partial_r p \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ satisfies $\partial_r p = 2\Omega u_\theta$, hence $p(r) = -2\int_r^\infty \Omega(s) u_\theta(s)\,\mathrm{d}s$. In all cases, the solution of (2.8) given by the above formulas satisfies $p(r) = \mathcal{O}(|\log r|^{1/2})$ as $r \to 0$ and $p(r) \to 0$ as $r \to +\infty$, and is unique in that class. $\square$

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